Sharp long distance upper bounds for solutions of Leibenson's equation on Riemannian manifolds

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Abstract

We consider on Riemannian manifolds the Leibenson equation $\partial_t u = \Delta_p u^q$ that is also known as a doubly nonlinear evolution equation. We prove sharp upper estimates of weak subsolutions to this equation on Riemannian manifolds with non-negative Ricci curvature in the whole range of $p > 1$ and $q > 0$ satisfying $q(p-1) < 1$. In this way, we improve the result of [17] and prove Conjecture 1.2 from [17].

Contents

1 Introduction

Let M be an arbitrary Riemannian manifold. We consider solutions of the non-linear evolution equation

$$\partial_t u = \Delta_p u^q, \tag{1.1}$$

where $p > 1$, $q > 0$, $u = u(x,t)$ is an unknown non-negative function of $x \in M$, $t \geq 0$, and Δ_p is the Riemannian p-Laplacian $\Delta_p v = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right)$.

The equation (1.1) is frequently referred to as a *doubly non-linear parabolic equation*. For the physical meaning of this equation see [18, 22, 23].

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When $M = \mathbb{R}^n$, G. I. Barenblatt [3] constructed for all $p > 1, q > 0$ spherically symmetric self-similar solutions of (1.1), that are nowadays called *Barenblatt solutions*.

Let us assume that

$$D := 1 - q(p-1) > 0. \tag{1.2}$$

If in addition

$$\beta := p - nD > 0,$$

then the Barenblatt solution satisfies the estimate

$$u(x,t) \simeq \frac{1}{t^{n/\beta}} \left(1 + \frac{|x|}{t^{1/\beta}}\right)^{-\frac{p}{D}} \tag{1.3}$$

(cf. Section 7.1 in [17]), where the symbol "$\simeq$" means that the ratio of the terms is bounded from above and below by a positive constant.

In [17] two of the authors proved upper bounds for solutions of the Leibenson equation (1.1) on geodesically complete Riemannian manifolds in a subcase of (1.2). However, the long distance estimate obtained in this paper was not optimal.

The purpose of the present paper is to obtain sharp estimates for solutions of (1.1) on Riemannian manifolds in the full range of p and q satisfying (1.2).

We understand solutions of (1.1) in $M \times \mathbb{R}_+$ in a certain weak sense (see Section 2 for the definition).

Denote by μ the *Riemannian measure* on M, by d the *geodesic distance* and by $B(x,r)$ the *geodesic ball* of radius r centered at x.

The main result of the present paper is as follows (cf. **Theorem 6.1**).

Theorem 1.1. *Let M satisfy a relative Faber-Krahn inequality (see Section 3 for definition) and assume that, for all $x \in M$ and all $R \geq 1$,*

$$\mu(B(x,R)) \geq cR^\alpha, \tag{1.4}$$

for some $c, \alpha > 0$. Assume that (1.2) holds and that

$$\beta := p - \alpha D > 0. \tag{1.5}$$

Let u be a bounded non-negative solution of (1.1) in $M \times [0,\infty)$ with initial function $u_0 = u(\cdot,0) \in L^1(M) \cap L^\infty(M)$. Set $A = \operatorname{supp} u_0$ and denote $|x| = d(x,A)$. Then, for all $t > 0$ and all $x \in M$, we have

$$\|u(\cdot,t)\|_{L^\infty(B(x,\frac{1}{2}|x|))} \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{|x|}{t^{1/\beta}}\right)^{-\frac{p}{D}}, \tag{1.6}$$

where the positive constants C and γ depend on $c, \alpha, p, q, ||u_0||_{L^\infty(M)}$, $||u_0||_{L^1(M)}$ and on the constants in the relative Faber-Krahn inequality.

In particular, if the solution u is continuous then the left hand side of (1.6) can be replaced by $u(x,t)$.

The relative Faber-Krahn inequality is satisfied if, for example, M has non-negative Ricci curvature (see [7, 14, 24]).

Comparing the upper bound (1.6) from Theorem 1.1 with the estimate (1.3) of the Barenblatt solution, we see that the estimate (1.6) is sharp in $\mathbb{R}^n$. A similar comparison takes place for some class of spherically symmetric manifolds (model manifolds) satisfying the relative Faber-Krahn inequality (see Remark 6.3).

In particular, our Theorem 1.1 improves the result of [17] and implies Conjecture 1.2 from that paper. In this paper the following was proved. Let M satisfy the relative Faber-Krahn inequality and (1.4). Assume that

$$1 < p < 2 \quad \text{and} \quad 1 \le q < \frac{1}{p-1} \tag{1.7}$$

(note that (1.7) implies (1.2)) and (1.5) hold. Then it was proved in [17] that

$$\|u(\cdot,t)\|_{L^\infty(B(x,\frac{1}{2}|x|))} \le \frac{C}{t^{\alpha/\beta}}\Phi\left(1+\frac{|x|}{t^{1/\beta}}\right) \tag{1.8}$$

where

$$\Phi(s) = s^{-\frac{p}{D}}\log^\gamma(1+s),$$

where γ is a positive constant. Hence, Theorem 1.1 improves the result of [17] in two ways. We prove Theorem 1.1 in the whole range of p and q satisfying (1.2). In particular, in contrast to the result in [17], our Theorem 1.1 also holds when $p=2$, that is, when equation (1.1) becomes the *porous medium equation* and (1.2) amounts to $q<1$. Secondly, we prove the estimate (1.8) with $\Phi(s)$ without the logarithmic term $\log^\gamma(1+s)$.

Let us discuss the differences in the methods of the proof of Theorem 1.1 and the result in [17].

The main technical lemma (Lemma 4.4) in the present paper about the long distance decay of solutions of (1.1) says the following. Let u be a bounded non-negative solution of (1.1) in $M\times[0,\infty)$. Let $B=B(x_0,R)$ be a ball such that the initial function $u(\cdot,0)=u_0$ satisfies

$$u_0 = 0 \text{ in } B.$$

Then, for all $t>0$,

$$\|u\|_{L^\infty\left(\frac{1}{2}B\times[0,t]\right)} \le C_B\left(\frac{t}{R^p}\right)^{\frac{1}{D}}, \tag{1.9}$$

where the positive constant C_B depends on the intrinsic geometry of B and γ depends on p,q and on the constants in the relative Faber-Krahn inequality.

In the proof of Lemma 4.4 we use a certain mean value inequality that is stated in Lemma 4.1 that we borrowed from [28].

In contrast to (1.9), it was proved in [17] under the above assumptions and under the additional restrictions (1.7) that

$$\|u\|_{L^\infty\left(\frac{1}{2}B\times[0,t]\right)} \le C_B\left(\frac{t}{R^p}\right)^{\frac{1}{D}}\log^\gamma\left(2+\left(\frac{R^p}{t}\right)^{\frac{1}{D}}\frac{\|u_0\|_{L^1(M)}}{\mu(B)}\right).$$

The additional restrictions (1.7) in [17] came from the mean value inequality in [16] which was used in the proof.

Let us also discuss the differences in the method of the proof of the main lemmas about the long time decay of solutions of (1.1) and how the estimates hold for different ranges for p and q. The main ingredient in both proofs is a non-linear mean inequality (see Lemma 5.2) which says the following. Let u be a non-negative bounded solution in $Q=B\times[0,T]$, $B=B(x_0,R)$, $T>0$. Then, for the cylinder

$$Q' = \frac{1}{2}B\times[\frac{1}{2}T,T],$$

we have

$$\|u\|_{L^\infty(Q')} \le \left(\frac{C_B S}{\mu(B)}\int_Q u^\sigma\right)^{1/(\sigma+D)},$$

where

$$S = \frac{\|u\|^D_{L^\infty(Q)}}{T} + \frac{1}{R^p},$$

$\sigma > 0$ is any and the constant C_B depends on p, q, σ and the intrinsic geometry of the ball B (in fact, on the Faber-Krahn inequality in B).

Even though, in both papers, the estimate of the long time decay follows from this mean value inequality and a modification of the classical De Giorgi iteration argument [9], the ranges of p and q for which the mean value inequality holds are different.

The mean value inequality in [16] is proved in the case (1.7). This is because the proof uses the fact that, if u is a non-negative subsolution of (1.1), then the function

$$(u^a - \theta)_+^{1/a} \tag{1.10}$$

is also a subsolution of (1.1), provided $\theta \geq 0$ and

$$a := \frac{q(p-1)-1}{p-2} \in (0, 1]. \tag{1.11}$$

In particular, the condition $a \in (0, 1]$ in (1.11) is satisfied provided (1.7) holds.

The second ingredient in this proof is a *Caccioppoli-type inequality*, which says the following. Assume that u is a subsolution of (1.1) in $B \times I$. Let $\eta(x, t)$ be a locally Lipschitz non-negative bounded function. Then for any $t_1, t_2 \in I$ such that $t_1 < t_2$ and any σ large enough,

$$\left[\int_B u^{\sigma+D}\eta^p\right]_{t_1}^{t_2} + c_1 \int_{B\times[t_1,t_2]} \left|\nabla\left(u^{\sigma/p}\eta\right)\right|^p \leq \int_{B\times[t_1,t_2]} pu^{\sigma+D}\partial_t\eta\eta^{p-1} + c_2u^\sigma |\nabla\eta|^p.$$

Then the aforementioned property allows to apply this inequality to the subsolutions

$$u_k = \left(u^a - \left(1 - 2^{-k}\right)\theta\right)_+^{1/a}, \quad k \geq 0,$$

for some fixed $\theta > 0$, where a is given by (1.11).

However, in the present paper we prove the mean value inequality for the whole range $p > 1, q > 0$. This is because we use in the proof instead a Caccioppoli type inequality of the form

$$\left[\int_B u_{k+1}^\lambda\eta^p\right]_{t_1}^{t_2} + A^k \int_{B\times[t_1,t_2]} \left|\nabla\left(u_{k+1}^{\sigma/p}\eta\right)\right|^p \leq \int_{B\times[t_1,t_2]} pu_k^{\sigma+D}\partial_t\eta\eta^{p-1} + B^k u_k^\sigma |\nabla\eta|^p.$$

where

$$u_k = \left(u - \left(1 - 2^{-k}\right)\theta\right)_+, \quad k \geq 0,$$

and A, B are positive constants (cf. Lemma 2.2).

The structure of the present paper is as follows.

In Section 2 we define the notion of a weak solution of the equation (1.1).

In Section 3 the aforementioned relative Faber-Krahn inequality is discussed.

In Section 4 we prove the main technical lemma (Lemma 4.4) about the long distance decay of solutions of (1.1).

In Section 5 we prove the main lemma (Lemma 5.4) about the long time decay of solutions of (1.1).

For further qualitative properties for solutions (1.1) in the case (1.2) in various settings, we refer to [6, 12, 13, 25, 27].

In the case $D < 0$ the Barenblatt solution has a finite propagation speed, and the same phenomenon occurs on arbitrary Riemannian manifolds (see [5, 10, 11, 16, 18, 19, 29]).

In the borderline case $D = 0$, the Barenblatt solutions is positive but decays exponentially in distance. Similar sub-Gaussian upper bounds of solutions of (1.1) on Riemannian manifolds were proved in the case $D = 0$ in [26].

We denote by c, c', C, C' positive constants whose value might change at each occurance.

Data availability. This article has no associated data.

2 Weak subsolutions

We consider in what follows the following evolution equation on a Riemannian manifold M:

$$\partial_t u = \Delta_p u^q. \tag{2.12}$$

By a *subsolution* of (2.12) we mean a non-negative function u satisfying

$$\partial_t u \leq \Delta_p u^q \tag{2.13}$$

in a certain weak sense as explained below.

We assume throughout that

$$p > 1 \quad \text{and} \quad q > 0.$$

Set

$$D = 1 - q(p-1).$$

Let μ denote the Riemannian measure on M. For simplicity of notation, we frequently omit in integrations the notation of measure. All integration in M is done with respect to $d\mu$, and in $M \times \mathbb{R}$ – with respect to $d\mu dt$, unless otherwise specified.

Let Ω be an open subset of M and I be an interval in $[0, \infty)$.

Definition 2.1. We say that a non-negative function $u = u(x,t)$ is a *weak subsolution* of (2.12) in $\Omega \times I$, if

$$u \in C\left(I; L^{1+q}(\Omega)\right) \quad \text{and} \quad u^q \in L^p_{loc}\left(I; W^{1,p}(\Omega)\right) \tag{2.14}$$

and (2.13) holds weakly in $\Omega \times I$, which means that for all $t_1, t_2 \in I$ with $t_1 < t_2$, and all non-negative *test functions*

$$\psi \in W^{1,1+\frac{1}{q}}_{loc}\left(I; L^{1+\frac{1}{q}}(\Omega)\right) \cap L^p_{loc}\left(I; W^{1,p}_0(\Omega)\right), \tag{2.15}$$

we have

$$\left[\int_\Omega u\psi\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega -u\partial_t\psi + |\nabla u^q|^{p-2}\langle\nabla u^q, \nabla\psi\rangle \leq 0. \tag{2.16}$$

Existence results for weak solutions of (2.12) were obtained in [1, 4, 8, 21, 20] in the euclidean setting and in [2, 10, 28] on manifolds.

If u is of the class (2.14) then $\nabla(u^q)$ is defined as an element of $L^p(\Omega)$. Then we define ∇u as follows:

$$\nabla u := \begin{cases} q^{-1}u^{1-q}\nabla(u^q), & u > 0, \\ 0, & u = 0. \end{cases}$$

Lemma 2.2. *[28] Let $u = u(x,t)$ be a non-negative bounded subsolution to (2.12) in a cylinder $\Omega \times (0,T)$. Let $\eta(x,t)$ be a locally Lipschitz non-negative bounded function in $\Omega \times [0,T]$ such that $\eta(\cdot,t)$ has compact support in Ω for all $t \in [0,T]$. Fix some real σ such that*

$$\sigma \geq \max(p, pq) \tag{2.17}$$

and set

$$\lambda = \sigma + D \quad \text{and} \quad \alpha = \frac{\sigma}{p}. \tag{2.18}$$

Choose $0 \leq t_1 < t_2 \leq T$ and set $Q = \Omega \times [t_1, t_2]$. Then, for any $\theta_1 > \theta_0 > 0$,

$$\left[\int_\Omega (u-\theta_1)_+^\lambda \eta^p\right]_{t_1}^{t_2} + c_1 \left(\frac{\theta_1}{\theta_1-\theta_0}\right)^{-(q-1)(p-1)_-} \int_Q \left|\nabla\left((u-\theta_1)_+^\alpha \eta\right)\right|^p \tag{2.19}$$

$$\leq \int_Q \left[p(u-\theta_0)_+^\lambda \eta^{p-1}\partial_t \eta + c_2\left(\frac{\theta_1}{\theta_1-\theta_0}\right)^{(q-1)(p-1)_+} (u-\theta_0)_+^\sigma |\nabla\eta|^p\right], \tag{2.20}$$

where c_1, c_2 are positive constants depending on p, q, λ.

Let us recall for later that

$$v^\alpha = (u-\theta_1)_+^\alpha \in L^p\left((0,T); W_0^{1,p}(\Omega)\right). \tag{2.21}$$

Indeed, using $\alpha \geq q$, we get that the function $\Phi(s) = s^{\frac{\alpha}{q}}$ is Lipschitz on any bounded interval in $[0,\infty)$. Thus, $u^\alpha = \Phi(u^q) \in W_0^{1,p}(\Omega)$ and $|\nabla u^\alpha| = |\Phi'(u^q)\nabla u^q| \leq C\,|\nabla u^q|$, whence

$$\int_Q u^{\alpha p} + |\nabla u^\alpha|^p \leq \left(||u||_{L^\infty(Q)}^{\sigma-pq} + C\right)\int_Q u^{pq} + |\nabla u^q|^p. \tag{2.22}$$

Therefore, $|\nabla v^\alpha| = \alpha v^{\alpha-1}|\nabla v| \leq \alpha u^{\alpha-1}|\nabla u| = |\nabla u^\alpha|$, since $\alpha \geq 1$ by (2.17) and $v \leq u$.

Lemma 2.3. [16] *Let M be geodesically complete and $v = v(x,t)$ be a bounded non-negative subsolution to (2.12) in $M \times I$. For any $\lambda \in [1,\infty]$, the function*

$$t \mapsto \|v(\cdot,t)\|_{L^\lambda(M)}$$

is monotone decreasing in I.

3 Faber-Krahn inequality

Let M be a connected Riemannian manifold of dimension n and d be the geodesic distance on M. For any $x \in M$ and $r > 0$, denote by $B(x,r)$ the geodesic ball of radius r centered at x, that is,

$$B(x,r) = \{y \in M : d(x,y) < r\}.$$

Let the ball B be precompact. Then the following *Faber-Krahn inequality* in B of order $p \geq 1$ holds: if $w \in W_0^{1,p}(B)$ is non-negative,

$$E = \{w > 0\}$$

and $r(B)$ denotes the radius of the ball B, then

$$\int_B |\nabla w|^p \geq \frac{1}{r(B)^p}\left(\iota(B)\frac{\mu(B)}{\mu(E)}\right)^\nu \int_B w^p, \tag{3.23}$$

where $\nu > 0$ and $\iota(B)$ is a positive constant that depends on the geometry of B. The value of ν is independent of B and can be chosen as follows:

$$\nu = \begin{cases} \frac{p}{n}, & \text{if } n > p, \\ \text{any number} \in (0,1), & \text{if } n \leq p. \end{cases} \tag{3.24}$$

Choosing $\iota(B)$ to be an optimal constant in (3.23) we obtain that the function

$$B \mapsto \frac{(\iota(B)\mu(B))^{\nu}}{r(B)^p} \tag{3.25}$$

is monotone decreasing with respect to the partial order $\subset$ on balls.

We say that M satisfies a *relative Faber-Krahn inequality* of order p if (3.23) holds with $\iota(B) \geq \text{const} > 0$ for all geodesic balls B. For example, this holds if M is complete, non-compact and satisfies $Ricci_M \geq 0$ (see [7, 14, 24]).

4 Long distance decay

From now on we always assume that

$$D = 1 - q(p-1) > 0. \tag{4.26}$$

Lemma 4.1. *Let the ball $B = B(x_0, R)$ be precompact. Let u be a non-negative bounded subsolution in $Q = B \times [0,t]$ such that*

$$u(\cdot, 0) = 0 \text{ in } B.$$

Let σ and λ be reals such that

$$\sigma > 0 \quad \text{and} \quad \lambda = \sigma + D. \tag{4.27}$$

Then, for the cylinder $Q' = \frac{1}{2}B \times [0,t]$, we have

$$\|u\|_{L^\infty(Q')} \leq \left(\frac{C}{\iota(B)\mu(B)R^p} \int_Q u^\sigma \right)^{1/\lambda}, \tag{4.28}$$

where $\iota(B)$ is the Faber-Krahn constant in B, and the constant C depends on p, q, λ and the Faber-Krahn exponent ν.

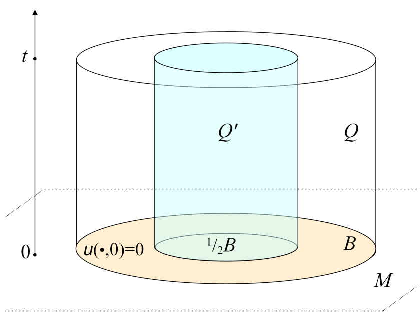

Figure 1: Cylinders Q and Q'

Remark 4.2. In [17] Lemma 4.1 was taken from [16] where it was proved under the additional condition

$$p < 2 \quad \text{and} \quad q \geq 1.$$

Proof. This Lemma is already proved in [28]. However, since the convergence of the constant C in (4.28) for $\sigma \to 0+$ is not adressed there and we need this for later usage, we give a shortened proof here.

It was shown in the first part of the proof of Lemma 4.5 in [28] that for any large enough σ, say $\sigma \geq \sigma_0$, with σ_0 only depending on p and q, that

$$\|u\|_{L^\infty(Q')} \leq \left(\frac{C}{\iota(B)\mu(B)R^p}\int_Q u^\sigma\right)^{1/\lambda}, \tag{4.29}$$

where $\lambda = \sigma + D$ and C depends on p, q, ν and σ.

Now we prove (4.29) for any $\sigma > 0$ (cf. [16]). Let $\sigma < \sigma_0$ and denote

$$\lambda_0 = \sigma_0 + D \quad \text{and} \quad \lambda = \sigma + D$$

so that $\lambda < \lambda_0$.

For simplicity of notation, for any set $E \subset M$, denote $E^t = E \times [0, t]$.

In particular, (4.29) implies that, for any precompact ball B of radius R,

$$\|u\|^{\lambda_0}_{L^\infty(\frac{1}{2}B^t)} \leq \frac{C(\sigma_0)}{\chi(B)R^p}\int_{B^t} u^{\sigma_0}, \tag{4.30}$$

where $C(\sigma_0)$ depends on p, q, ν and σ_0, $\chi(B) := \iota(B)\mu(B)$ and according to our notation $B^t = B \times [0, t]$. Consider for $k \geq 0$, a sequence

$$R_k = \left(1 - \frac{1}{2^{k+1}}\right)R,$$

so that $R_0 = \frac{1}{2}R$ and $R_k \uparrow R$ as $k \to \infty$, and set $B_k = B(x_0, R_k)$. Denoting also $B = B(x_0, R)$, we see that

$$\frac{1}{2}B \subset B_k \subset B \quad \text{and} \quad B_k \uparrow B$$

as $k \to \infty$. Set also $\rho_k = R_{k+1} - R_k = \frac{1}{2^{k+2}}R$.

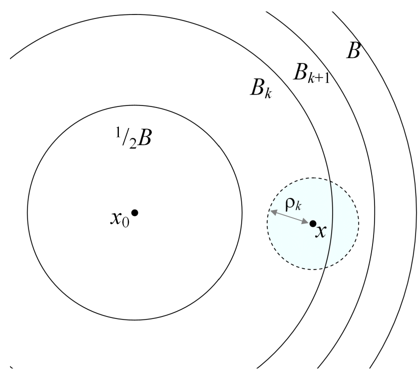

Figure 2: Balls B_k and $B(x, \rho_k)$

For any point $x \in B_k$, applying (4.30) in the ball $B(x, \rho_k)$, we obtain

$$\|u\|^{\lambda_0}_{L^\infty(B^t(x,\frac{1}{2}\rho_k))} \leq \frac{C(\sigma_0)}{\chi(B(x,\rho_k))\,\rho_k^p}\int_{B^t(x,\rho_k)} u^{\sigma_0}$$

$$\le \frac{C(\sigma_0)}{\chi\left(B(x,\rho_k)\right)\rho_k^p}\,\|u\|_{L^\infty(B^t(x,\rho_k))}^{\sigma_0-\sigma}\int_{B^t(x,\rho_k)}u^\sigma .$$

Since $B\left(x,\rho_k\right)\subset B_{k+1}\subset B$, we have by the monotonicity of (3.25)

$$\frac{\chi(B(x,\rho_k))}{\rho_k^{p/\nu}}\ge\frac{\chi\left(B\right)}{R^{p/\nu}}$$

whence

$$\frac{1}{\chi(B(x,\rho_k))}\le\frac{(R/\rho_k)^{p/\nu}}{\chi(B)}=\frac{2^{(k+2)p/\nu}}{\chi(B)}.$$

Hence, we obtain

$$\|u\|_{L^\infty(B^t(x,\frac{1}{2}\rho_k))}^{\lambda_0}\le\frac{C(\sigma_0)2^{kp(\nu^{-1}+1)}}{\chi\left(B\right)r^p}\,\|u\|_{L^\infty(B_{k+1}^t)}^{\lambda_0-\lambda}\int_{B^t}u^\sigma .$$

Covering B_k by a sequence of balls $B(x,\frac{1}{2}\rho_k)$ with $x\in B_k$, we obtain

$$\|u\|_{L^\infty(B_k^t)}^{\lambda_0}\le\frac{C(\sigma_0)2^{kp(\nu^{-1}+1)}}{\chi\left(B\right)R^p}\,\|u\|_{L^\infty(B_{k+1}^t)}^{\lambda_0-\lambda}\int_{B^t}u^\sigma . \tag{4.31}$$

Setting $J_k=\|u\|_{L^\infty(B_k^t)}^{-(\lambda_0-\lambda)}$, we rewrite (4.31) as follows:

$$J_{k+1}\le\frac{A^k}{\Theta}J_k^{\frac{\lambda_0}{\lambda_0-\lambda}}=\frac{A^k}{\Theta}J_k^{1+\omega},$$

where

$$A=2^{p(\nu^{-1}+1)},\quad \Theta^{-1}=\frac{C(\sigma_0)}{\chi\left(B\right)R^p}\int_{B^t}u^\sigma\quad\text{and}\quad\omega=\frac{\lambda_0}{\lambda_0-\lambda}-1=\frac{\lambda}{\lambda_0-\lambda}.$$

Applying Lemma 5.2 from [16], we obtain

$$J_k\le\left(\frac{J_0}{\left(A^{-1/\omega}\Theta\right)^{1/\omega}}\right)^{(1+\omega)^k}\left(A^{-1/\omega}\Theta\right)^{1/\omega},$$

that is,

$$J_0\ge\left(A^{-1/\omega}\Theta\right)^{1/\omega}\left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega}J_k\right)^{\frac{1}{(1+\omega)^k}}.$$

Since $J_k\ge\|u\|_{L^\infty(B^t)}^{-(\lambda_0-\lambda)}>\text{const}>0$, we see that

$$\liminf_{k\to\infty}\left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega}J_k\right)^{\frac{1}{(1+\omega)^k}}\ge 1,$$

whence

$$J_0\ge\left(A^{-1/\omega}\Theta\right)^{1/\omega}.$$

It follows that $J_0^{-1}\le A^{1/\omega^2}\Theta^{-1/\omega}$, that is,

$$\|u\|_{L^\infty(B_0^t)}^{\lambda_0-\lambda}\le A^{1/\omega^2}\left(\frac{C(\sigma_0)}{\chi\left(B\right)R^p}\int_{B^t}u^\sigma\right)^{1/\omega},$$

and thus,

$$\|u\|_{L^\infty(\frac{1}{2}B\times[0,t])}\le\left(\frac{A^{\frac{\lambda_0-\lambda}{\lambda}}C(\sigma_0)}{\iota(B)\mu(B)R^p}\int_{B\times[0,t]}u^\sigma\right)^{1/\lambda}, \tag{4.32}$$

which proves (4.28) with $C=A^{\frac{\lambda_0-\lambda}{\lambda}}C(\sigma_0)$. ■

Remark 4.3. Using the notation from the proof of Lemma 4.1, we see from (4.32) that the constant $C = A^{\frac{\lambda_0-\lambda}{\lambda}} C(\sigma_0)$ in (4.28) converges to $A^{\frac{\sigma_0}{D}} C(\sigma_0)$ as $\sigma \to 0+$. Hence, the limit of C as $\sigma \to 0+$ is a positive constant that depends only on p, q and the Faber-Krahn exponent ν.

The next lemma is the main result of this section.

Lemma 4.4. *Assume that M is geodesically complete and let u be a bounded non-negative subsolution in $M \times [0, T]$. Let $B = B(x_0, R)$ be a ball such that*

$$u_0 = u(\cdot, 0) = 0 \ \textit{in } B.$$

Then, for all $t \in [0, T]$,

$$\|u\|_{L^\infty\left(\frac{1}{2}B \times [0,t]\right)} \le C \left(\frac{t}{\iota(B) R^p} \right)^{\frac{1}{D}}, \tag{4.33}$$

where the positive constant C depends on p, q and the Faber-Krahn exponent ν.

Remark 4.5. In [17] the estimate (4.33) was proved with additional factor

$$\ln^\gamma \left(2 + \left(\frac{\iota(B) R^p}{t} \right)^{\frac{1}{D}} \frac{||u_0||_{L^1(M)}}{\mu(B)} \right),$$

where γ is a positive constant.

Proof. Let $Q = B \times [0, t]$ and $Q' = \frac{1}{2}B \times [0, t]$. Then it follows from Lemma 4.1 with

$$\sigma > 0 \quad \text{and} \quad \lambda = \sigma + D$$

and the monotonicity of $t \mapsto ||u(\cdot, t)||_{L^\infty(M)}$ from Lemma 2.3 that

$$\begin{aligned} \|u\|_{L^\infty(Q')} &\le \left(\frac{C}{\iota(B)\mu(B) r^p} \int_Q u^\sigma \right)^{1/\lambda} \\ &\le \left(\frac{C}{\iota(B)\mu(B) R^p} t\mu(B) ||u_0||^\sigma_{L^\infty(M)} \right)^{1/\lambda} \\ &= \left(\frac{Ct}{\iota(B) R^p} ||u_0||^\sigma_{L^\infty(M)} \right)^{1/\lambda}. \end{aligned} \tag{4.34}$$

By Remark 4.3 the constant C converges to some positive constant that depends only on p, q and the Faber-Krahn exponent ν when $\sigma \to 0+$. Therefore, sending $\sigma \to 0+$ in (4.34) we conclude the claim. ■

5 Long time decay

The main result of this section is Lemma 5.4.

5.1 Comparison in two cylinders

Lemma 5.1. *Consider two balls $B_0 = B(x_0, r_0)$ and $B_1 = B(x_0, r_1)$ with $0 < r_1 < r_0$ where B_0 is precompact. Assuming $0 < t_0 < t_1 < T$, consider two cylinders $Q_i = B_i \times [t_i, T]$, $i = 0, 1$. Let v_0 be non-negative bounded subsolution in Q_0. For $\theta_1 > \theta_0 > 0$ set*

$$v_i = (u - \theta_i)_+ .$$

Let σ and λ be reals satisfying (2.17) and (2.18). Set

$$J_i = \int_{Q_i} v_i^\sigma d\mu dt.$$

Then

$$J_1 \leq \frac{C r_0^p S^\nu}{\left(\iota(B_0)\mu(B_0)\right)^\nu (\theta_1-\theta_0)^{\lambda\nu} (r_0-r_1)^p} \left(\frac{\theta_1}{\theta_1-\theta_0}\right)^{|q-1|(p-1)} J_0^{1+\nu}, \tag{5.35}$$

where

$$S = \frac{\|v_0\|_{L^\infty(Q_0)}^D}{t_1-t_0} + \left(\frac{\theta_1}{\theta_1-\theta_0}\right)^{(q-1)(p-1)_+} \frac{1}{(r_0-r_1)^p},$$

ν is the Faber-Krahn exponent, $\iota(B_0)$ is the Faber-Krahn constant in B_0, and C depends on p, q and λ.

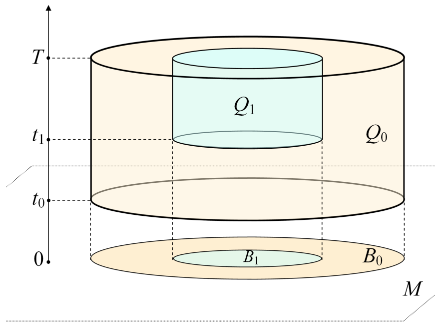

Figure 3: Cylinders Q_0 and Q_1

Proof. Let $\eta(x,t) = \eta(x)$ be a bump function of B_1 in $B_{1/2} := B\left(x_0, \frac{r_0+r_1}{2}\right)$. Recall that by (2.21), $v_1^\alpha \eta \in L^p\left([t_0,T]; W_0^{1,p}(B)\right)$, where α is defined by (2.18), that is $\alpha = \frac{\sigma}{p}$. Hence, applying the Faber-Krahn inequality (3.23) in ball B_0 for any $t \in [t_0, T]$ we get that

$$\int_{B_1} v_1^\sigma \leq \int_{B_0} (v_1^\alpha \eta)^p \leq r_0^p \left(\frac{\mu(D_t)}{\iota(B_0)\mu(B_0)}\right)^\nu \int_{B_0} |\nabla (v_1^\alpha \eta)|^p, \tag{5.36}$$

where we used that $\alpha p = \sigma$ and $\eta = 1$ in B_1 and

$$D_t = \{v_1^\alpha \eta(\cdot,t) > 0\} = \{v_1 > 0\} \cap \{\eta > 0\} = \{u(\cdot,t) > \theta_1\} \cap B_{1/2}.$$

Also, note that $\eta_t = 0$ and $|\nabla \eta| \leq \frac{2}{r_0-r_1}$. From (2.20) we therefore obtain

$$\begin{aligned} c_1 \int_{t_1}^T \int_{B_0} |\nabla (v_1^\alpha \eta)|^p &\leq c_2 \left(\frac{\theta_1}{\theta_1-\theta_0}\right)^{|q-1|(p-1)} \int_{t_1}^T \int_{B_0} v_0^\sigma |\nabla \eta|^p \\ &\leq \frac{c_3}{(r_0-r_1)^p} \left(\frac{\theta_1}{\theta_1-\theta_0}\right)^{|q-1|(p-1)} J_0, \end{aligned} \tag{5.37}$$

where $c_3 = c_2 2^p$.

Let us now apply Lemma 2.2 to function v_0 in $B_0 \times [t_0, T]$. Take

$$\eta(x,t) = \eta_1(x)\eta_2(t),$$

where η_1 is a bump function of $B_{1/2}$ in B_0 so that

$$|\nabla\eta_1| \le \frac{2}{r_0 - r_1},$$

and η_2 is a bump function of $[t_1, T]$ in $[t_0, T]$, that is,

$$\eta_2(t) = \begin{cases} 1, & t \ge t_1 \\ \frac{t-t_0}{t_1-t_0}, & t_0 \le t \le t_1 \end{cases}$$

so that

$$|\partial_t \eta_2| \le \frac{1}{t_1 - t_0}.$$

From (2.20) we obtain

$$\begin{aligned}\left[\int_{B_0} v_0^\lambda \eta^p\right]_{t_0}^T &\le \int_{t_0}^T \int_{B_0} \left[p\eta^{p-1}\partial_t \eta v_0^\lambda + c_2 \left(\frac{\theta_1}{\theta_1 - \theta_0}\right)^{(q-1)(p-1)_+} |\nabla \eta|^p v_0^\sigma\right] \\ &= \int_{t_0}^T \int_{B_0} \left[p\eta^{p-1}\partial_t \eta v_0^D + c_2 \left(\frac{\theta_1}{\theta_1 - \theta_0}\right)^{(q-1)(p-1)_+} |\nabla \eta|^p\right] v_0^\sigma.\end{aligned}$$

Hence, for any $t \in [t_1, T]$, using that $\eta_2(t_0) = 0$ and $\eta(x,t) = 1$ for $x \in B_{1/2}$ and $t \ge t_1$,

$$\int_{B_{1/2}} v_0^\lambda(\cdot, t) \le c_4 \int_{t_0}^T \int_{B_0} \left[\frac{||v_0||_{L^\infty}^D}{t_1 - t_0} + \left(\frac{\theta_1}{\theta_1 - \theta_0}\right)^{(q-1)(p-1)_+} \frac{1}{(r_0 - r_1)^p}\right] v_0^\lambda \le c_4 S J_0,$$

where $c_4 = \max(p, c_3)$. Thus, we deduce

$$\mu(D_t) \le \frac{1}{(\theta_1 - \theta_0)^\lambda} \int_{B_{1/2} \cap \{u > \theta_1\}} v_0^\lambda(\cdot, t) \le \frac{c_4 S J_0}{(\theta_1 - \theta_0)^\lambda}.$$

Combining this with (5.36) and (5.37) we obtain

$$J_1 = \int_{t_1}^T \int_{B_1} v_1^\sigma \le r_0^p \left(\frac{c_4 S J_0}{\iota(B_0)\mu(B_0)(\theta_1 - \theta_0)^\lambda}\right)^\nu \frac{c_3}{c_1(r_0 - r_1)^p} \left(\frac{\theta_1}{\theta_1 - \theta_0}\right)^{|q-1|(p-1)} J_0,$$

which implies (5.35) and finishes the proof. ■

5.2 Iterations and the mean value theorem

Lemma 5.2. *Let the ball $B = B(x_0, R)$ be precompact. Let u be a non-negative bounded subsolution in $Q = B \times [0, T]$. Let σ and λ be reals such that*

$$\sigma > 0 \quad \text{and} \quad \lambda = \sigma + D. \tag{5.38}$$

Then, for the cylinder

$$Q' = \frac{1}{2}B \times [\frac{1}{2}T, T],$$

we have

$$\|u\|_{L^\infty(Q')} \le \left(\frac{CS}{\iota(B)\mu(B)} \int_Q u^\sigma\right)^{1/\lambda}, \tag{5.39}$$

where

$$S = \frac{\|u\|_{L^\infty(Q)}^D}{T} + \frac{1}{R^p}, \tag{5.40}$$

$\iota(B)$ is the Faber-Krahn constant in B, and the constant C depends on p, q, λ and ν.

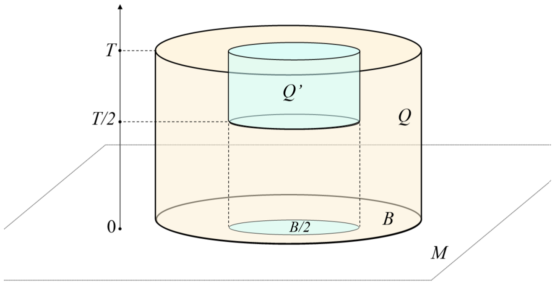

Figure 4: Cylinders Q' and Q

Remark 5.3. In [17] the same mean value inequality was proved under the condition that

$$1 < p < 2 \quad \text{and} \quad 1 \leq q < \frac{1}{p-1}. \tag{5.41}$$

Proof. Let us first prove (5.39) for σ large enough as in Lemma 2.2. Consider sequences

$$r_k = \left(\frac{1}{2} + 2^{-k-1}\right) R, \quad t_k = \left(1 - 2^{-k}\right) \frac{T}{2}$$

where $k = 0, 1, 2, ...$, so that $r_0 = R$ and $r_k \searrow \frac{1}{2}R$ as $k \to \infty$ $t_0 = 0$ and $t_k \nearrow \frac{1}{2}T$ as $k \to \infty$. Set $B_k = B\left(x_0, r_k\right)$, $Q_k = B_k \times [t_k, T]$ so that $B_0 = B$, $Q_0 = Q$ and $Q_\infty := \lim_{k\to\infty} Q_k = Q'$.

Choose some $\theta > 0$ to be specified later and define $\theta_k = \left(1 - 2^{-k}\right)\theta$ and

$$u_k = (u - \theta_k)_+ = \left(u - \left(1 - 2^{-k}\right)\theta\right)_+.$$

Set

$$J_k = \int_{Q_k} u_k^\sigma.$$

We obtain by Lemma 5.1 that

$$J_{k+1} \leq \frac{C r_k^p S_k^\nu}{\left(\iota(B_k)\mu(B_k)\right)^\nu (\theta_{k+1} - \theta_k)^{\lambda\nu} (r_k - r_{k+1})^p} \left(\frac{\theta_{k+1}}{\theta_{k+1} - \theta_k}\right)^{|q-1|(p-1)} J_k^{1+\nu},$$

where

$$S_k = \frac{\|u\|^D_{L^\infty(Q_k)}}{t_{k+1} - t_k} + \left(\frac{\theta_{k+1}}{\theta_{k+1} - \theta_k}\right)^{(q-1)(p-1)_+} \frac{1}{(r_k - r_{k+1})^p}.$$

By monotonicity of the function (3.25), we have

$$\frac{r_k^p}{\left(\iota(B_k)\mu(B_k)\right)^\nu} \leq \frac{R^p}{\left(\iota(B)\mu(B)\right)^\nu}.$$

Since $r_k - r_{k+1} = 2^{-k-2}R$, $t_{k+1} - t_k = 2^{-k-2}T$, and $\theta_{k+1} - \theta_k = 2^{-(k+1)}\theta$ it follows that

$$S_k \leq 2^{(k+2)(p+(q-1)(p-1)_+)} \left(\frac{\|u\|^D_{L^\infty(Q)}}{T} + \frac{1}{R^p}\right) = 2^{(k+2)(p+(q-1)(p-1)_+)} S.$$

Hence,

$$J_{k+1} \leq \frac{C 2^{(k+2)(\lambda\nu + p(1+\nu) + \nu(q-1)(p-1)_+ + |q-1|(p-1))} S^\nu}{(\iota(B)\mu(B))^\nu \theta^{\lambda\nu}} J_k^{1+\nu} = \frac{A^k J_k^{1+\nu}}{\Theta}$$

where

$$A = 2^{\lambda\nu + p(1+\nu) + \nu(q-1)(p-1)_+ + |q-1|(p-1)} \geq 1 \quad \text{and} \quad \Theta = c\left(\frac{\iota(B)\mu(B)\theta^\lambda}{S}\right)^\nu.$$

Now let us apply Lemma 6.1 from [18] with $\omega = \nu$: if

$$\Theta \geq A^{1/\nu} J_0^\nu, \tag{5.42}$$

then, for all $k \geq 0$,

$$J_k \leq A^{-k/\nu} J_0.$$

In terms of θ the condition (5.42) is equivalent

$$c\left(\frac{\iota(B)\mu(B)\theta^\lambda}{S}\right)^\nu \geq A^{1/\nu} J_0^\nu$$

that is,

$$\theta \geq \left(\frac{CSJ_0}{\iota(B)\mu(B)}\right)^{1/\lambda}.$$

Hence, we choose θ to have equality here. For this θ we obtain $J_k \to 0$ as $k \to \infty$, which implies that $u \leq \theta$ in Q_∞. Hence,

$$\|u\|_{L^\infty(Q')} \leq \left(\frac{CSJ_0}{\iota(B)\mu(B)}\right)^{1/\lambda},$$

which proves (5.39) for any large enough σ.

By standard methods (see the proof of Lemma 4.1 or [16]) we conclude from this that (5.39) holds for any $\sigma > 0$. ■

5.3 Optimal long time decay

The next lemma is the main result about long time decay.

Using the mean value inequality 5.2 and the method from [17] we can prove the following result, which improves the range of p and q in Lemma 5.4 from [17].

Lemma 5.4. *Assume that M is geodesically complete and satisfies the relative Faber-Krahn inequality. Assume that, for all $x \in M$ and $R \geq 1$,*

$$\mu(B(x,R)) \geq cR^\alpha, \tag{5.43}$$

for some $c, \alpha > 0$. Assume also that

$$\beta := p - D\alpha > 0.$$

Let u be a non-negative bounded subsolution in $M \times [0,\infty)$ with initial function $u_0 = u(\cdot, 0)$. Then, for all $t > 0$, we have

$$\|u(\cdot,t)\|_{L^\infty(M)} \leq \frac{C}{t^{\alpha/\beta}}\left(\|u_0\|_{L^1(M)} + \|u_0\|_{L^\infty(M)}\right)^{p/\beta}, \tag{5.44}$$

where C depends on c, α, p, q and on the constants in the relative Faber-Krahn inequality.

Remark 5.5. Note that in the result in [17] we also, besides $D > 0$, assumed that

$$p < 2 \quad \text{and} \quad q \geq 1$$

(see (5.41)).

6 Combined estimate

The following theorem is our main result (equivalent to Theorem 1.1 from the Introduction).

Theorem 6.1. *Assume that M is geodesically complete and satisfies the relative Faber-Krahn inequality. Assume that, for all $x \in M$ and $R \geq 1$,*

$$\mu(B(x,R)) \geq cR^{\alpha},$$

for some $c, \alpha > 0$. Assume that (4.26) holds and that

$$\beta := p - D\alpha > 0.$$

Let u be a bounded non-negative subsolution in $M \times [0, \infty)$ with initial function $u_0 = u(\cdot, 0) \in L^1(M) \cap L^\infty(M)$ and set $A = \operatorname{supp} u_0$. Denote $|x| = d(x, A)$. Then, for all $t > 0$ and all $x \in M$, we have

$$\|u(\cdot,t)\|_{L^\infty(B(x,\frac{1}{2}|x|))} \leq \frac{C}{t^{\alpha/\beta}}\left(1 + \frac{|x|}{t^{1/\beta}}\right)^{-\frac{p}{D}}, \tag{6.45}$$

where the positive constant C depends on $c, \alpha, p, q, ||u_0||_{L^1(M)}, ||u_0||_{L^\infty(M)}$ and on the constants in the relative Faber-Krahn inequality.

Remark 6.2. Theorem 6.1 improves Theorem 6.1 from [17] in two ways. In [17] it was proved that

$$\|u(\cdot,t)\|_{L^\infty(B(x,\frac{1}{2}|x|))} \leq \frac{C}{t^{\alpha/\beta}}\Phi\left(1 + \frac{|x|}{t^{1/\beta}}\right)$$

with

$$\Phi(s) = s^{-\frac{p}{D}} \log^{\gamma}(1+s),$$

where γ is a positive constant. Secondly, it was additionally assumed that

$$p < 2 \quad \text{and} \quad q \geq 1.$$

Proof. Let us first prove that for all $t > 0$ and all $x \in M \setminus A$, we have

$$\|u(\cdot,t)\|_{L^\infty(B(x,\frac{1}{2}|x|))} \leq \frac{C_1}{t^{\alpha/\beta}} \wedge C_2\left(\frac{t}{|x|^p}\right)^{\frac{1}{D}}, \tag{6.46}$$

where the positive constants C_1, C_2 depend on $c, \alpha, p, q, ||u_0||_{L^1(M)}, \|u_0\|_{L^\infty(M)}$ and on the constants in the relative Faber-Krahn inequality.

By Lemma 5.4 we have

$$\|u(\cdot,t)\|_{L^\infty(M)} \leq \frac{C}{t^{\alpha/\beta}}\left(||u_0||_{L^1(M)} + \|u_0\|_{L^\infty(M)}\right)^{p/\beta},$$

which gives the first term in (6.46). In order to obtain the second term in (6.46), we apply Lemma 4.4 in the ball $B_x = B(x, |x|)$ that is disjoint with $\operatorname{supp} u_0$ and deduce

$$\|u(\cdot,t)\|_{L^\infty(\frac{1}{2}B_x)} \leq C\left(\frac{t}{\iota(B_x)|x|^p}\right)^{\frac{1}{D}} \leq C\left(\frac{t}{|x|^p}\right)^{\frac{1}{D}}.$$

Now let us show how (6.46) implies (6.45). In the case when $\frac{|x|}{t^{1/\beta}} \leq C'$ for some constant $C' > 1$, we have $\left(1 + \frac{|x|}{t^{1/\beta}}\right)^{-p/D} \geq \text{const} > 0$, which yields (6.45). On the other hand, if $\frac{|x|}{t^{1/\beta}} \geq C'$, we see that

$$\frac{1}{t^{\alpha/\beta}}\left(1 + \frac{|x|}{t^{1/\beta}}\right)^{-p/D} \simeq \frac{t^{1/D}}{|x|^{p/D}},$$

because $\frac{p}{\beta D} - \frac{\alpha}{\beta} = \frac{1}{D}$, which finishes the proof of (6.45) also in this case. ■

Remark 6.3. Consider a model manifold with profile $S(r) = Cr^{\alpha-1}$, for some $\alpha \in (0, n]$ and all $r \geq r_0$, that is, $M = (0, +\infty) \times \mathbb{S}^{n-1}$ as topological spaces and M is equipped with the Riemannian metric ds^2 given by $ds^2 = dr^2 + \psi^2(r)d\theta^2$, where $\psi(r)$ is a smooth positive function on $(0, +\infty)$ and $d\theta^2$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$ and $S(r) = \psi^{n-1}(r)$ (cf. Section 7.1 in [17]). By Proposition 4.10 in [15] it satisfies the relative Faber-Krahn inequality. In [17] we constructed a solution which satisfies the estimate

$$u(r,t) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{r}{t^{1/\beta}}\right)^{-p/D},$$

which shows that our estimate (6.45) is sharp on such manifolds.

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