Brownian motion on manifolds and volume growth

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Setup

Let M be always a Riemannian manifold that is geodesically complete and non-compact. Recall that the Laplace-Beltrami operator Δ on M is given in the local coordinates $x_1, ..., x_n$ as follows:

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left(\sqrt{\det g} g^{ij} \frac{\partial}{\partial x_j} \right)$$

where $g = (g_{ij})$ is the Riemannian metric tensor and $(g^{ij}) = (g_{ij})^{-1}$. This operator is symmetric with respect to the Riemannian measure

$$d\mu = \sqrt{\det g}\, dx$$

and, moreover, allows a unique extension to a self-adjoint operator in $L^2(M, \mu)$.

Besides, Δ is the generator of a diffusion process $\left((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in M}\right)$ on M that is called Brownian motion on M. The transition density of this process with respect to measure μ is denoted by $p_t(x, y)$ (where $t > 0$, $x, y \in M$) and is called the *heat kernel*. Equivalently, the heat kernel is the minimal positive fundamental solution of the heat equation $\partial_t u = \Delta u$ on M and the integral kernel of the heat semigroup $\left\{e^{t\Delta}\right\}$.

In this talk we collect some results relating the properties of Brownian motion to the volume growth of M.

Recurrence

If Brownian motion of M is recurrent then manifold M is called *parabolic*. For example, $\mathbb{R}^n$ is parabolic if and only if $n \leq 2$ (theorem of Polya, 1921).

It is known that the following properties are equivalent:

- manifold M is parabolic;
- any positive superharmonic function on M is constant (a function $u \in C^2(M)$ is called superharmonic if $\Delta u \leq 0$);
- there exists no positive fundamental solution of $-\Delta$;
- $\int^\infty p_t(x,y)\,dt = \infty$ for all/some $x, y \in M$;
- the capacity of any compact set is zero.

Now let us relate the parabolicity with a volume growth. Let $d(x,y)$ denote the geodesic distance on M. Consider geodesic balls

$$B(x,r) = \{y \in M : d(x,y) < r\},$$

and their volumes $V(x,r) = \mu(B(x,r))$.

Let us fix a reference point x_0 and denote $V(r) = V(x_0, r)$.

Theorem 1 *(Cheng-Yau, '75) If, for all large enough r,*

$$V(r) \leq Cr^2 \tag{1}$$

then M is parabolic.

Theorem 2 *(AG '83, Karp '82, Varopoulos '83) If*

$$\int^{\infty} \frac{rdr}{V(r)} = \infty \tag{2}$$

then M is parabolic.

For example, (2) is satisfied if

$$V(r) \leq Cr^2 \log r.$$

The condition (2) is sharp: if $f(r)$ is a smooth convex function such that $f'(r) > 0$ and

$$\int^{\infty} \frac{rdr}{f(r)} < \infty,$$

then there is a non-parabolic manifold such that $V(r) = f(r)$ for large r.

Stochastic completeness

Manifold M is called *stochastically complete* if the lifetime of Brownian motion on M is ∞ almost surely. Equivalent characterizations of the stochastic completeness:

- for all $x \in M$ and $t > 0$,
$$\int_M p_t(x,y)d\mu(y) = 1.$$
- For some/any $\lambda > 0$, any bounded solution u to $\Delta u - \lambda u = 0$ on M is identical zero.
- For some/any $T \in (0, \infty]$, the Cauchy problem
$$\begin{cases} \partial_t u = \Delta u & \text{in } M \times (0,T) \\ u\big|_{t=0} = 0 \end{cases} \tag{3}$$

has the only bounded solution $u \equiv 0$.

Theorem 3 *(Karp and Li '83, Takeda '89, Davies '92) If*
$$V(r) \le \exp\left(Cr^2\right) \tag{4}$$
then M is stochastically complete.

Theorem 4 *(AG '86) If*

$$\int^{\infty} \frac{r dr}{\log V(r)} = \infty \tag{5}$$

then M is stochastically complete.

The condition (5) is sharp: if $f(r)$ is a smooth convex function such that $f'(r) > 0$ and

$$\int^{\infty} \frac{r dr}{f(r)} < \infty$$

then there exists a stochastically incomplete manifold with $V(r) = \exp(f(r))$.

Escape rate

Let us fix $x_0 \in M$ and set $|x| = d\,(x, x_0)$ for all $x \in M$. An increasing function $R(t)$ is called *an upper rate function* (*upper radius*) for Brownian motion if

$$\mathbb{P}_{x_0}\left(|X_t| < R(t) \text{ for all large enough } t\right) = 1.$$

Similarly, an increasing function $r\,(t)$ is called a *lower rate function* (*lower radius*) if

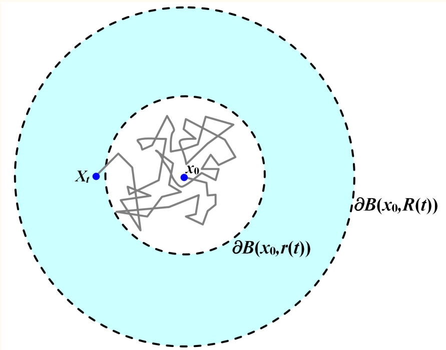
$$\mathbb{P}_{x_0}\left(|X_t| > r(t) \text{ for all large enough } t\right) = 1.$$

Hence, for all large enough t,

X_t is contained in the annulus

$$B\left(x_0, R\left(t\right)\right) \setminus B\left(x_0, r\left(t\right)\right)$$

almost surely.



Observe that, for the existence of an upper rate function it is necessary for M to be stochastically complete, and for the existence of a positive lower rate function Brownian motion must be transient (that is M must be non-parabolic).

For example, in $\mathbb{R}^n$ the following upper rate function is known:

$$R\left(t\right)=\sqrt{\left(4+\varepsilon\right)t\log\log t}\quad \text{(Khinchin's law of iterated log).}$$

If $n>2$ and $r\left(t\right)/\sqrt{t}$ is decreasing then $r\left(t\right)$ is a lower rate function in $\mathbb{R}^n$ if and only if

$$\int^{\infty}\left(\frac{r(t)}{\sqrt{t}}\right)^{n-2}\frac{dt}{t}<\infty \tag{6}$$

(Dvoretzky and Erdös, '50). Here is an example of such a function: $r\left(t\right)=\frac{C\sqrt{t}}{\log^{\frac{1+\varepsilon}{n-2}}t}$.

Theorem 5 *(AG and M.Kelbert '98, AG '99) Assume that, for all r large enough,*

$$V\left(r\right)\leq Cr^{\alpha}\,, \tag{7}$$

with some $\alpha, C>0$. Then the following function is an upper rate function:

$$R(t)=\sqrt{2\alpha t\log t}. \tag{8}$$

Under assumption (7), the upper rate function (8) is optimal (AG, Kelbert '00). Similar result for simple random walks on graphs was proved by Hardy–Littlewood (1914, for $\mathbb{Z}$) and by M.Barlow–E.Perkins '89 for arbitrary graphs.

Consider now a more general situation when $V(r)$ can be superpolynomial.

Recall that M is called a *Cartan-Hadamard* manifold if M is simply connected and has non-positive sectional curvature.

Theorem 6 *(AG and E.P. Hsu 2009) Let M be a Cartan-Hadamard manifold and assume that*

$$\int^{\infty} \frac{r dr}{\log V(r)} = \infty. \tag{9}$$

Define a function $\varphi(t)$ as follows:

$$t = \int_{r_0}^{\varphi(t)} \frac{r dr}{\log V(r)}. \tag{10}$$

Then $R(t) = \varphi(Ct)$ is an upper rate function.

Example. If $V(r) = Cr^{\alpha}$ then (10) yields

$$t \simeq \frac{\varphi^2(t)}{\log \varphi(t)}$$

whence $R(t) \simeq \varphi(t) \simeq \sqrt{t \log t}$ that matches (8).

Example. If $V(r) = \exp(r^{\alpha})$ where $0 < \alpha < 2$ then

$$t \simeq \varphi(t)^{2-\alpha}$$

whence $R(t) = Ct^{\frac{1}{2-\alpha}}$.

Example. If $V(r) = \exp(r^2)$ then

$$t \simeq \log \varphi(t)$$

whence $R(t) = \exp(Ct)$.

Theorem 7 *(E.P. Hsu and G.Qin '10) On any complete manifold M, satisfying (9), define function $\varphi(t)$ as follows:*

$$t = \int_{r_0}^{\varphi(t)} \frac{r dr}{\log V(r) + \log \log r}.$$

Then $R(t) = C\varphi(Ct)$ is an upper rate function.

Example. Let $V(r) \leq C \log r$. Then $t \simeq \frac{\varphi^2(t)}{\log \log \varphi(t)}$ and we obtain an upper rate function

$$R(t) = C\sqrt{t \log \log t}.$$

In order to state the results about a lower rate function, we need, apart from the volume function, the following notion. We say that M satisfies a *relative Faber-Krahn inequality* if, for any ball $B(x,r)$ and any open set $\Omega \subset B(x,r)$,

$$\lambda_1(\Omega) \geq \frac{c}{r^2}\left(\frac{V(x,r)}{\mu(\Omega)}\right)^{2/n} \tag{11}$$

where $n = \dim M$, $c > 0$ is a constant, and $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Δ in Ω.

For example, in $\mathbb{R}^n$ we have $V(x,r) = c_n r^n$ and (11) becomes

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-2/n},$$

which is true by the theorem of Faber-Krahn that $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$ where Ω^* is a ball of the same volume as Ω.

It is known that the relative Faber-Krahn inequality holds on any complete manifold with non-negative Ricci curvature. It is also known that the relative Faber-Krahn inequality implies the volume doubling condition so that the volume function $V(r)$ is at most polynomial in this case. Besides, it implies that the condition $\int^\infty \frac{rdr}{V(r)} = \infty$ is not only sufficient for the parabolicity of M but is also necessary.

Theorem 8 *(AG '99) Assume that a relative Faber-Krahn inequality holds on M and that*

$$\int^{\infty} \frac{rdr}{V(r)} < \infty$$

so that M is non-parabolic and X_t is transient. Denote

$$\gamma(r) := \left(\int_r^{\infty} \frac{sds}{V(s)} \right)^{-1}. \tag{12}$$

Let $r(t)$ be an increasing positive function on $(0, \infty)$ such that

$$\int^{\infty} \frac{\gamma(r(t))}{V(\sqrt{t})} dt < \infty. \tag{13}$$

Then $r(t)$ is a lower rate function for Brownian motion on M.

Example. Let $V(x, r) \simeq r^{\alpha}$ for large r and some $\alpha > 2$. We obtain from (12) $\gamma(t) \simeq t^{\alpha-2}$, and (13) amounts to

$$\int^{\infty} \frac{r^{\alpha-2}(t)dt}{t^{\alpha/2}} < \infty,$$

which coincides with the Dvoretzky–Erdös condition (6).

Biparabolic manifolds

A function $u \in C^4(M)$ is called *bi-superharmonic* if $\Delta u \leq 0$ and $\Delta^2 u \geq 0$.

For example, let M be nonparabolic and consider the Green operator

$$Gf = \int_M g(x,y) f(y) \, d\mu(y)$$

where $g(x,y) = \int_0^\infty p_t(x,y) \, dt$ is the Green function. Then $u = Gf$ is bi-superharmonic if u is finite and f is non-negative and superharmonic.

Another example of bi-superharmonic functions in a precompact domain $\Omega \subset M$: if f is a non-negative continuous function on $\partial\Omega$ then

$$u(x) = \mathbb{E}_x \left(\tau_\Omega f\left(X_{\tau_\Omega} \right) \right),$$

where τ_Ω is the first exit time from Ω, solves the following boundary value problem

$$\begin{cases} \Delta^2 u = 0 \ \text{ in } \Omega \\ -\Delta u|_{\partial\Omega} = f, \\ u|_{\partial\Omega} = 0, \end{cases}$$

and, hence, is bi-superharmonic in Ω.

A manifold M is called *biparabolic*, if any positive bi-superharmonic function on M is harmonic, that is $\Delta u = 0$.

Note that the notion of parabolicity also admits a similar equivalent definition: M is parabolic if and only if any positive superharmonic function on M is harmonic.

One can show that $\mathbb{R}^n$ is biparabolic if and only if $n \leq 4$. For example, if $n > 4$ then $u(x) = |x|^{-(n-4)}$ is bi-superharmonic but not harmonic.

Theorem 9 *(S.Faraji and AG, 2019) Manifold M is biparabolic provided*

$$V(r) \leq C \frac{r^4}{\log r} \quad \textit{for large } r. \tag{14}$$

The condition (14) is not far from optimal in the following sense: for any $\beta > 1$ there exists a manifold M with

$$V(r) \leq C\, r^4 \log^\beta r$$

that is not biparabolic.

Conjecture. *If*

$$V(r) \leq C r^4 \log r \quad \textit{or} \quad \textit{even} \quad \int^\infty \frac{r^3 dr}{V(r)} = \infty,$$

then M is biparabolic.

Heat kernel lower bounds

Theorem 10 *(Coulhon and AG, '97) Assume that, for all* $r \geq r_0 > 0$,

$$V(r) \leq Cr^{\alpha}, \tag{15}$$

for some $C, \alpha > 0$. *Then, for all large enough* t,

$$p_t(x_0, x_0) \geq \frac{1/4}{V(\sqrt{Kt \log t})}, \tag{16}$$

where $K = K(x_0, r_0, C, \alpha) > 0$. *Consequently,*

$$p_t(x_0, x_0) \geq \frac{c}{(t \log t)^{\alpha/2}} \tag{17}$$

The best possible lower bound would be

$$p_t(x_0, x_0) \geq \frac{c}{V(\sqrt{t})}$$

that is valid on manifolds of non-negative Ricci curvature (Li and Yau '85). However, under the hypothesis (15), the lower bound (16) is optimal (AG and Kelbert, '00).

Theorem 11 *(Coulhon and AG, '97) Assume that the function $V(r)$ is doubling, that is,*

$$V(2r) \leq CV(r),$$

and that, for all $t > 0$,

$$p_t(x_0, x_0) \leq \frac{C}{V(\sqrt{t})}.$$

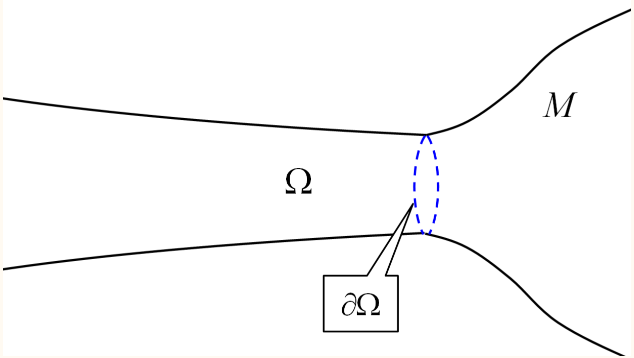
Then, for all $t > 0$,

$$p_t(x_0, x_0) \geq \frac{c}{V(\sqrt{t})}.$$

Lower bound on a manifolds with an end

Let Ω be an *end* of M, that is, an open connected proper subset of M such that $\overline{\Omega}$ is non-compact but $\partial\Omega$ is compact. Assume that $\partial\Omega$ is smooth hypersurface.

We consider the closure $\overline{\Omega}$ as a manifold with boundary $\partial\Omega$. Then we say that $\overline{\Omega}$ is parabolic if Brownian motion in Ω with reflecting boundary condition on $\partial\Omega$ is recurrent.



All the aforementioned results about parabolicity remain true also for the manifold with boundary $\overline{\Omega}$.

In the next statement we use the following notation:

$$V_{\Omega}(r) = \mu\left(B\left(x_0, r\right) \cap \Omega\right).$$

Theorem 12 *(AG and P.Sürig, 2021) Let Ω be an end of M that satisfies the following two assumptions:*

(a) *there exist $C > 0$ and $\alpha > 2$ such that, for all large enough r,*

$$V_\Omega(r) \le Cr^\alpha \tag{18}$$

(b) *$\overline{\Omega}$ is non-parabolic.*

Then, for any $x \in M$, there exist $t_x > 0$ and $c_x > 0$ such that

$$p_t(x,x) \ge \frac{c_x}{(t \log t)^{\alpha/2}} \quad \text{for all } t \ge t_x. \tag{19}$$

In other words, the most essential part of the lower bound (19) – the rate of decay of the heat kernel in t, is determined by the geometry of Ω alone and is insensitive to the rest of the manifold $M \setminus \Omega$ although the latter can be arbitrarily large in terms of volume growth.

The reason for that is that, due to the non-parabolicity of Ω, Brownian motion on M escapes to ∞ along Ω with positive probability so that the heat kernel does not see the geometry of $M \setminus \Omega$.

In contract to that, if $\overline{\Omega}$ is parabolic then Brownian motion on M returns from Ω to the complement $M \setminus \Omega$ with probability 1 and then proceeds to $M \setminus \Omega$, so that the influence of $M \setminus \Omega$ onto the heat kernel becomes dominating.

Indeed, there is an example of a manifold M with a parabolic end $\overline{\Omega}$ satisfying (18) where the heat kernel $p_t\left(x,x\right)$ decays superpolynomially as $t \to \infty$:

$$p_t(x,x) \leq C_x \exp\left(-ct^{\varepsilon}\right).$$

In this example the complement $M \setminus \Omega$ has a superpolynomial volume growth.

Recurrence of α-process

For any $\alpha \in (0,2)$, the operator $(-\Delta)^{\alpha/2}$ is a generator of a jump process on M that is called *the α-process*. It is a natural generalization of the symmetric stable Levy process of index α in $\mathbb{R}^d$. By a general semigroup theory, the Green function $G^{(\alpha)}(x,y)$ of $(-\Delta)^{\alpha/2}$ is given by

$$G^{(\alpha)}(x,y) = \int_0^\infty t^{\alpha/2-1} p(t,x,y)dt,$$

and the recurrence of the α-process is equivalent to $G^{(\alpha)} \equiv \infty$, that is, to

$$\int^\infty t^{\alpha/2-1} p(t,x,x)dt = \infty. \tag{20}$$

Theorem 13 *(AG '99) If for all $r \geq r_0 > 0$*

$$V(r) \leq Cr^\alpha, \tag{21}$$

then the α-process is recurrent.

Indeed, by Theorem 10 we have $p_t(x_0,x_0) \geq \frac{c}{t^{\alpha/2}\log^{\alpha/2} t}$. Substituting into (20) we see that the integral diverges.

Heat kernel upper bounds

Theorem 14 *(Barlow, Coulhon, AG '01) Let M be a manifold of bounded geometry. Assume that, for all $x \in M$ and $r \geq r_0 > 0$*

$$V(x,r) \geq cr^{\alpha}.$$

Then, for all $x \in M$ and large enough t,

$$p_t(x,x) \leq Ct^{-\frac{\alpha}{\alpha+1}}. \tag{22}$$

For any $\alpha \geq 1$, there exists an example of a manifold with $V(x,r) \simeq r^{\alpha}$ and

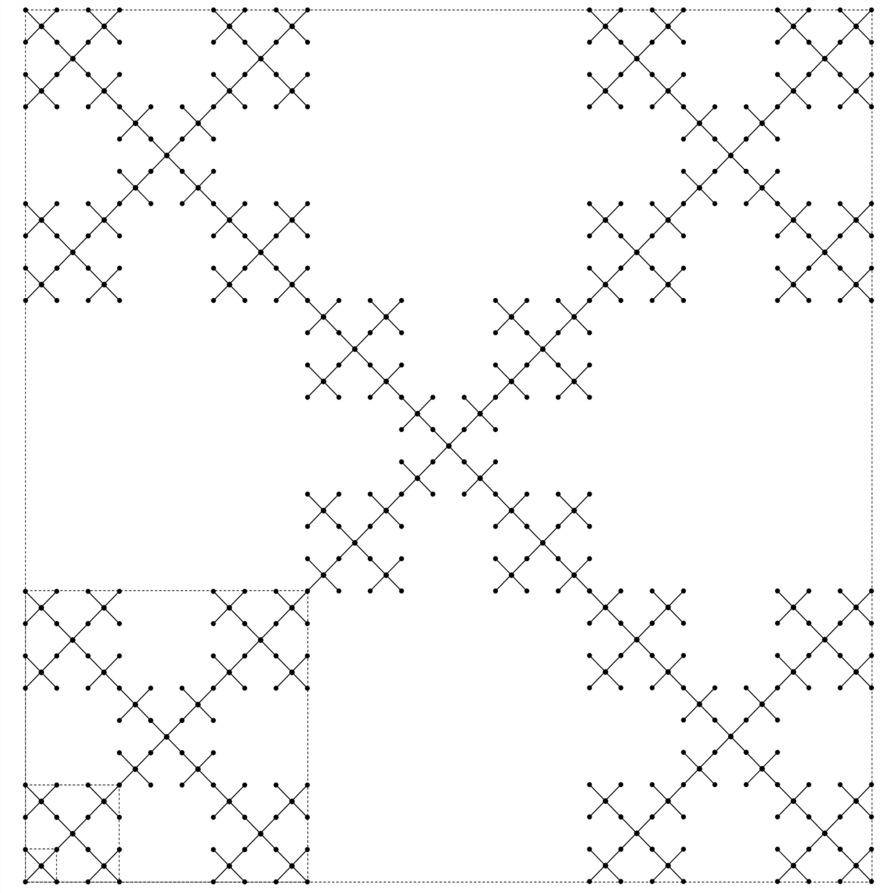
$$p_t(x,x) \simeq t^{-\frac{\alpha}{\alpha+1}}.$$

Indeed, take any fractal graph where the volume function is r^{α} and the on-diagonal decay of the heat kernel is given by $t^{-\alpha/\beta}$. It is known that such a graph exists for any α, β satisfying

$$2 \leq \beta \leq \alpha + 1$$

(Barlow '04). Choose $\beta = \alpha + 1$ and then inflate the graph into a manifold.

Here is one of such graphs, the *Vicsek tree*:



For this fractal

$$\alpha = \frac{\log 5}{\log 3}$$

and

$$\beta = \alpha + 1 = \frac{\log 15}{\log 3}$$

Liouville properties

Theorem 15 *(S.T.Yau '78) If u is a harmonic function on M and $u \in L^p(M,\mu)$ with $1 < p < \infty$ then $u \equiv \mathrm{const}$.*

In other words, any geodesically complete manifold satisfies L^p-*Liouville property* if $1 < p < \infty$. For $p = 1$ and $p = \infty$ this is not true: there are manifolds with non-trivial L^1 harmonic function as well as those with non-trivial L^∞ harmonic functions.

Theorem 16 *(AG '89) Assume that*

$$\int^{\infty} \frac{rdr}{\log V(r)} = \infty.$$

Then any positive superharmonic function $u \in L^1(M,\mu)$ is identical constant.

Open Questions:. *Find "good" conditions to ensure that*

(a) *any harmonic function $u \in L^1$ is identical constant;*

(b) *any harmonic function $u \in L^\infty$identical constant.*

Bounded solutions of Schrödinger equations

Let $Q(x)$ be a nonnegative continuous function on M, $Q \not\equiv 0$. Consider the equation

$$\Delta u - Qu = 0 \tag{23}$$

and ask if (23) has a non-trivial bounded solution (equivalently: a positive bounded solution).

Set $|x| = d(x, x_0)$ and denote

$$q(r) = \inf_{|x|=r} Q(x) \quad \text{and} \quad F(r) = \int_0^{r/2} \sqrt{q(t)}dt.$$

Theorem 17 *(AG, '90) If for all large enough r*

$$V(r) \leq Cr^2 \exp\left(CF(r)^2\right) \tag{24}$$

then (23) has no bounded solution except for $u \equiv 0$.

Example. Let $Q \equiv 1$. Then (23) has no bounded solution (except for zero) if and only if M is stochastically complete. We have $q \equiv 1$, $F(r) = r/2$, and (24) becomes $V(r) \leq \exp(Cr^2)$, which coincides with the condition (4) for the stochastic completeness.

Example. Let Q have compact support. In this case (23) has no bounded solution (except for zero) if and only if M is parabolic. Since $q(r) = 0$ for large enough r, we obtain that $F(r) = \text{const}$ for large r, and (24) becomes $V(r) \leq Cr^2$, which coincides with the condition (1) for the parabolicity.

Example. Assume that, for large $|x|$

$$Q(x) \geq \frac{c}{|x|^2 \log |x|}.$$

Then

$$F(r) \geq \int_2^{r/2} \frac{c}{t\sqrt{\log t}} dt \simeq \sqrt{\log r}$$

so that (24) is satisfied provided

$$V(r) \leq Cr^N.$$

In this case (23) has no bounded solution except for zero.

On the other hand, if in $\mathbb{R}^n$

$$Q(x) \leq \frac{C}{|x|^2 \log^{1+\varepsilon} |x|}$$

then (23) has a positive solution in $\mathbb{R}^n$.

Semilinear PDE

Consider on M the inequality

$$\Delta u + u^{\sigma} \leq 0 \tag{25}$$

and ask if it has a non-negative solution u except for $u \equiv 0$. Here $\sigma > 1$ is a given parameter. Any solution of (25) is superharmonic. Hence, if M is parabolic then u must be identical zero. In particular, this is the case if $V(r) \leq Cr^2$.

Theorem 18 *(AG and Y.Sun '14) Assume that, for all large r,*

$$V(r) \leq Cr^p \log^q r, \tag{26}$$

where

$$p = \frac{2\sigma}{\sigma - 1}, \quad q = \frac{1}{\sigma - 1}. \tag{27}$$

Then any nonnegative solution of (25) is identical zero.

The values of the exponents p and q in (27) are sharp: if either $p > \frac{2\sigma}{\sigma-1}$ or $p = \frac{2\sigma}{\sigma-1}$ and $q > \frac{1}{\sigma-1}$ then there is a manifold satisfying (26) where the inequality (25) has a positive solution.

Equivalent reformulation: if, for some $\alpha > 2$

$$V(r) \leq Cr^{\alpha} \log^{\frac{\alpha-2}{2}} r, \tag{28}$$

then, for any $\sigma \leq \frac{\alpha}{\alpha-2}$, any nonnegative solution of (25) is identical zero.

For example, in $\mathbb{R}^n$ with $n > 2$ (28) holds with $\alpha = n$ which implies that, for $\sigma \leq \frac{n}{n-2}$, any nonnegative solution of (25) is identical zero.

It is known that, for any $\sigma > \frac{n}{n-2}$, (25) has a positive solution in $\mathbb{R}^n$ (Mitidieri and Pohozaev, '98).

Conjecture. *If*

$$\int^{\infty} \frac{r^{2\sigma-1} dr}{V(r)^{\sigma-1}} = \infty \tag{29}$$

then any nonnegative solution of (25) is identical zero.

In particular, the function (28) satisfies (29) with $\sigma = \frac{\alpha}{\alpha-2}$.

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