NOTES ON THE ASSOCIATOR

MARKUS ROST

Contents

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Preface

You are looking at the text "Notes on the associator" [pdf].

The original purpose of these notes was to request references on the 5-term relation for associators and its relation with the associahedron.

Quickly I got a hint to the field of alternative algebras. Here the 5-term relation appears in many papers as a trivial preliminary.

Concerning the chain complex of the associahedron I had a hard time: So many papers, overwhelming!

After some time I figured out myself how to write down the chain complex of the associahedron and prove its acyclicity, see Section §6.

The crucial step was to find a way to control the signs in the differential of the complex. I first tried with parenthesized expressions, but gave up. With unrooted plane trees however, things became simpler and I had the chain complex in my hands. After some (lengthy) explicit computations in low degrees, I found the pretty short description of the contracting chain homotopy.

But then there should be a corresponding way to establish the associahedron as contractible CW-complex.

And it does: The associahedron is not a polytope! It's a cube.

See §7 for my beautiful drawings of the associahedron on a cube. (Move some vertices out a little if you insist on a true associahedral polytope.)

Somewhat later (February 2025) I realized that my cube presentation of the associahedron is exactly the "Tamari polytope" in Loday 2012 [17, p. 75]. Some cube drawings and diagrams can be found here:

- Jean-Louis Loday / Images (archived home page): Polytope de Stasheff (associahedron) de dimension 4, version cubique (mars 2002)
- Loday 2011 [16, 5 Appendix 1: Drawing a Stasheff polytope on a cube, pp. 288–289]
- Loday 2012 [17, 8.3 Realizing of the associahedron, p. 75]
- Saneblidze-Umble 2004 [27, p. 400]
- Saneblidze-Umble 2024 [28, p. 148]

The first 4 sections stem essentially from April 2024 and, after adding references for alternative algebras, May 2024. The sections starting with §5 have been added since December 2024. In §6 we actually prove something. §7 contains the cubical description of the 3-dimensional and to some extent of the 4-dimensional associahedron. §8 has a TEX version of the original Tamari diagram.

At the moment I am in a process of digesting things. At some point I might try to write some kind of exposition, but who knows.

§1. A relation for associators

Let

$$\mu\colon M \otimes M \to M$$
$$\mu(x, y) = xy$$

be a bilinear product on some R-module M. For μ there are no assumptions on associativity, commutativity or unitality (but the base ring R has these properties).

Let

$$(\ ,\ ,\ )\colon M^{\otimes 3} \to M$$
$$(x, y, z) = (xy)z - x(yz)$$

denote the associator of the algebra (M, μ).

There is the following 5-term relation

$$(*) \qquad x(y, z, t) + (x, y, z)t = (xy, z, t) - (x, yz, t) + (x, y, zt)$$

It follows easily by expanding the associator expressions (this is spelled out in [26, Proof of Lemma 1]).

Relation $(*)$ plays a basic role in texts on alternative algebras. It appears in Zorn 1931 [43, (2), p. 125] where it is called "Viereridentität" and used to prove Artin's theorem on alternative algebras. Other places are Schafer 1961 [29, (12), p. 10][1], Schafer 1966 [30, (2.4), p. 13] (with 3 other applications), Kurosh 1965 (1962) [12, (3), p. 244], Bourbaki 1970 [2, Lemme 1, (2), p. A III.173], Bourbaki 1974 (1970) [3, Lemma 1, (2), p. 612] (thanks to J.-P. Tignol for the hint), Zhevlakov et al. 1982 (1978) [42, Chap. 7, (5), p. 136], Garibaldi-Petersson-Racine 2024 [8, 7.5, (2), p. 44].

Our first application of $(*)$ was related with parametrization of algebras. The relation can also be used to establish the associativity of the multiplication in the free product of groups. (To give details would lead to far here.)

§2. More general relations for the associator?

I got aware of relation $(*)$ many years ago.

Since then I was wondering occasionally whether there are more relations for the associator, valid for any M with product μ.

Only recently I realized that the question is closely related to monoidal categories (see for instance Mac Lane 1998 (1971) [20]). In this context the 5-term relation $(*)$ (and its proof) appears in the form of the pentagon axiom.

In fact, a closer look at Mac Lane 1963 [19, Theorem 3.1, p. 33] reveals that the 5-term relation $(*)$ is the main relation for the associator (in this context).

There are further relations which stem from expanding 2 associators. In the simplest case, a parenthesized expression like

$$(a, b, c) \ldots (x, y, z)$$

yields the 4-term relation

$$(ab)c \ldots (x, y, z) - a(bc) \ldots (x, y, z) = (a, b, c) \ldots (xy)z - (a, b, c) \ldots x(yz)$$

---

[1][29] are lecture notes by Schafer from 1961 with the same title as the 1966 book [30]. On gutenberg.com there are scans, the page with the relation is p0012.png.

The 2 associators could also be nested, like

$$(\dots(x,y,z)\dots,b,c),\quad (a,\dots(x,y,z)\dots,c),\quad (a,b,\dots(x,y,z)\dots)$$

giving rise to further 4-term relations.

The 4-term relations are sort of obvious and less sophisticated than the 5-term relation $(*)$. In the context of monoidal categories the 4-term relations are hidden in the setup.

§3. Higher relations for the associator?

Once the question for general relations for the associator is settled, one may ask for higher relations, that is, relations among the relations etc.

An answer is provided by the exactness of the chain complex of the associahedron.

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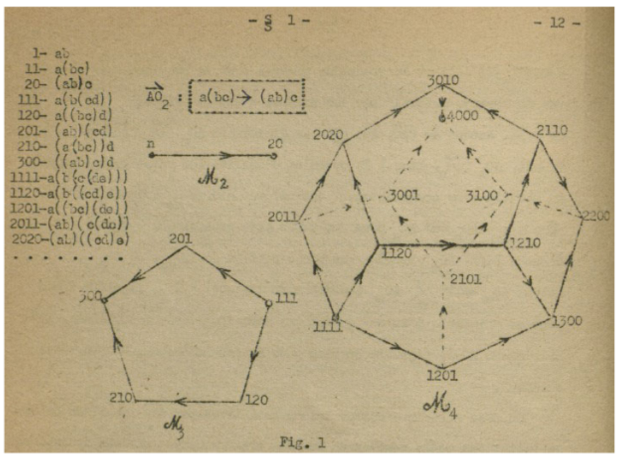


Fig. 1 Tamari's associahedron

The vertices (0-cells) of the n-dimensional associahedron T_n are the parenthesized expressions in $n+2$ variables $x_1,\dots,x_{n+2}\in M$ (each of which appears once and in the given order, as in $(*)$). The edges (1-cells) are given by associators appearing in nested ways, the 2-cells correspond to relations among these associators, etc.

The associahedron T_2 (labeled $\mathcal{M}_3$ in the picture) is a pentagon with its 2-cell representing the 5-term relation.

The associahedron T_3 (labeled $\mathcal{M}_4$) has as faces besides pentagons some quadrilaterals, each of which representing a 4-term relation.

The chain complex of the associahedron can be described in a combinatorial manner using less and less parenthesized expressions as free generators (this is not the place to give details). The crucial fact then is that this chain complex is the chain complex of a polytope (namely the associahedron). Thus the chain complex is acyclic and we know all general higher relations for the associator (in this context).

3.1. **Notes and references.** The image above of Tamari's associahedron has been taken from Stasheff 2012 [37, p. 46] in the Tamari memorial Festschrift 2012 [22]. See also Loday 2012 [17, 8 Realizing of the associahedron, p. 74] in the same book.

There seem to be hundreds of millions of related papers, I downloaded tens of thousands of them but read (almost) none. Nevertheless, here are a few further references I want to mention:

Cayley 1891 [5] (always a pleasure to refer to Caley).

Tamari 1951 [39]: Tamari's thesis contains the image above.
Tamari 1954 [40]: Article with the same title. Doesn't have the drawing.

Stasheff 1963 [35].

Mac Lane 1963 [19], Joyal-Street 1993 [10].

Stasheff 1970 [34], Boardman-Vogt 1973 [1].

Lee 1989 [13], Gelfand-Kapranov-Zelevinsky 1994 [9, Ch. 7].

Shnider-Sternberg 1993 [33], Stasheff 1997 [36], Markl-Shnider-Stasheff 2002 [21].

Loday 2004 [15], Postnikov 2009 [25].

Leinster 2004 [14], Loday-Vallette 2012 [18].

Tamari memorial Festschrift 2012 [22].

Ceballos-Santos-Ziegler 2015 [7], Pilaud-Santos-Ziegler 2023 [23].

§4. Questions and Remarks

- In spite of the (hopefully correct) discussion of the associahedron above, I haven't much understood about it. I don't have a grasp yet on the associahedral chain complex, despite the many papers on it.

  For monoidal/associahedral people:

  What is your answer to the question on general (higher) relations for the associator tensor $(\ ,\ ,\ )\colon M^{\otimes 3} \to M$ associated to multiplication tensors $M^{\otimes 2} \to M$?

  What is the genuine way to pin down the associahedral chain complex and settle its acyclicity? (Or at least: what is the state of the art?)
- A first problem is actually to formulate precise questions for relations for the associator tensor. We discussed here the relations among particular parenthesized expressions. One may think of other expressions involving associators. However instead of wildly speculating, one should probably rather look for concrete problems where associators are involved.
- What about relations for associators in some specific cases? I am thinking here of the cases where M is a locally free R-module of some fixed finite rank n.

  I am pretty sure that the cases $n = 2, 3$ are worthwhile to look at (the cases $n \geq 4$ might get very complicated).

  One should include here the unital cases, then of rank 3, 4.

  This touches the question of parametrizing algebras of finite rank. The set of equations for a multiplication tensor given by associativity is really a non-trivial one. It is long known that for associative, commutative and unital algebras of higher rank the etale algebras are not dense (the corresponding Hilbert scheme is not irreducible). See e.g. Poonen 2007 [24, Remark 1.2, p. 818].

§5. Further notes and remarks (December 2024)

The previous part stems essentially from April/May 2024. Here are newer notes.

5.1. **An A_∞ question.** Meanwhile I looked somewhat into the book Loday-Vallette 2012 "Algebraic operads" [18], in particular near [18, 9.2.5 The Associahedron (Stasheff Polytope), p. 340]. Proposition 9.2.3 identifies chain complexes

$$(A_\infty)_n = C_\bullet(\mathcal{K}^{n-2})$$

and just before that it is argued that $C_\bullet(\mathcal{K}^n)$ is acyclic, since $\mathcal{K}^n$ is contractible.

Am I correct to understand that this is the state-of-the-art argument for the acyclicity? I had hoped that there is purely operadic argument. (Not that I know what an operad is.)

Dear reader: Please comment.

5.2. **Dictionaries.** One dictionary is

- Non-crossing diagonals in (simple) polygons.
- Finite trees together with, for each vertex P, a cyclic order on the set of edges starting at P.

I like to call such trees "vertex cyclic trees". They are just ribbon graphs which happen to be a tree. After adding the ribbons, one gets planar trees (unrooted).

The basic translation is to associate to a polygon fully triangulated by diagonals the tree with vertices the 2-cells and segments and with edges the diagonals and segments (this is not the place to give more details, but see Gelfand-Kapranov-Zelevinsky [9, Ch. 7. Triangulations and Secondary Polytopes, 3B, pp. 237ff.]).

Under this translation the leaves of an unrooted planar tree correspond to the segments of the polygon and the points of the polygon correspond to neighbored pair of leaves (which in turn define a path in the tree).

Another dictionary is

- Planar rooted trees.
- Partially parenthesized words, or "multi-magma monomials", as I like to call them. These are the elements of the set M defined as follows:

$$\begin{aligned}
M_1 &= \{*\} \\
M_n &= \coprod_{1 \le i_k < n;\ \sum_k i_k = n} \prod_k M_{i_k} \\
M &= \coprod_{n \ge 1} M_n
\end{aligned}$$

If one restricts to binary planar rooted trees one gets "magma monomials" or non-associative words: The elements of the free magma on one element $*$, cf. [32, p. 18] (obtained by restricting k to a 2-element set in the definition of M above).

(Again, this is not the place to give more details.)

The language of multi-magmas makes the phrase "partially parenthesized words" precise in a formal way.

5.3. **Lee subdivision.** One of earliest proofs that the combinatorial associahedron is a polytope seems to be in Lee [13]. For each dimension a certain triangulation of the sphere (obtained by an explicit series of stellar subdivisions of $\partial\Delta^n$) is identified with the dual of the associahedron (minus the big cell).

The recipe itself is really simple.

Not only that: [13] has also a "symmetrical realization". It preserves the full automorphism group of a polygon (the Dieder group). In terms of planar trees this means roughly that no root has to be chosen. (Disclosure: I haven't yet worked through the details of that and of Gelfand-Kapranov-Zelevinsky [9, Ch. 7. Triangulations and Secondary Polytopes, pp. 214].)

I think the latter changes the point of view: The basic objects are the ones from the first dictionary above (diagonals in a polygon, unrooted planar trees). For the Stasheff polytope and things like the 5-term relations, multi-magmas, etc. one chooses then a root (an edge resp. point).

5.4. **Loday realization.** The Loday realization [15] yields the associahedron as intersection of half spaces in a very particular and beautiful way.

However I had some difficulties finding my way through [15]. A particular worrisome spot for me is [15, 2.5. Recollection on the Stasheff polytope, p. 271]:

> It is shown by J. Stasheff and S. Shnider in [36] Appendix, that the Stasheff polytope can be obtained from the standard simplex by truncating along the hyperplanes corresponding to the admissible shuffles.

Maybe I am blind, but I just couldn't find this in [36, Appendix B].

The functions c_k Loday uses **do not** satisfy the conditions formulated in [36, Appendix B] (with basic example $c_k = 3^k$). The do however satisfy something like

$$c(I_1) + c(I_2) < c(I_1 \cup I_2) + c(I_1 \cap I_2)$$

which was enough for me to establish the Loday realization by myself.

I am confused since one finds the same reference in other papers by other authors.

Dear reader: Can you help me with this?

At the moment, the only complete reference for the Loday realization I know of would be via Postnikov 2009 [25, Corollary 8.2, p. 1051-1052] (something I haven't worked through yet).

§6. Optimistic announcement

Maybe recently (November 2024) I could resolve my original worries:

(1) Pre-Theorem. For $n \geq 0$ (actually $n \geq -1$) there is a complex of the form

$$C(n)_i = \bigoplus_{\Gamma \in X_{n,i}} \mathbf{L}_\Gamma$$

with differential

$$d\colon C(n)_i \to C(n)_{i-1}$$

Here $X_{n,i}$ is the set of planar trees (unrooted) with given $n+3$ leaves (they are same for all trees) and $n+1-i$ inner nodes. Moreover the $\mathbf{L}_\Gamma$ are free cyclic groups (so $\mathbf{L}_\Gamma \simeq \mathbf{Z}$, but not canonically). The components $\mathbf{L}_\Gamma \to \mathbf{L}_{\Gamma'}$ of the differential d are an isomorphism (to be described below) if Γ is obtained from Γ' by contracting an inner edge and else trivial.

One may think of the $\mathbf{L}_\Gamma \simeq \mathbf{Z}$ as a coefficient system. Their purpose is to get the signs right in the definition of the differential so that $dd = 0$.

A toy model is the chain complex of Δ^n which can be described as

$$\Lambda^\bullet \mathbf{Z}^N = \bigoplus_{I \subset N} \Lambda^{|I|} \mathbf{Z}^I$$

Here $N = \{0, \ldots, n\}$ and the differential is given by omitting an element from a subset. A compact way to describe this differential is as composition

$$\Lambda^\bullet \mathbf{Z}^N \xrightarrow{\delta} \Lambda^1 \mathbf{Z}^N \otimes \Lambda^\bullet \mathbf{Z}^N \xrightarrow{\varepsilon \otimes \mathrm{id}} \Lambda^\bullet \mathbf{Z}^N$$

The groups $\mathbf{L}_\Gamma$ are defined as

$$\mathbf{L}_\Gamma = \Lambda^{|V|} \mathbf{Z}^V \otimes \Lambda^{|E|} \mathbf{Z}^E$$

where V is the set of vertices and E is the set of (oriented) edges of the tree. (We use the definition of a graph as in Serre 1980 (1977, 1968) [31], with the edges coming in pairs.)

The differential d is defined as follows: the inverse $\mathbf{L}_{\Gamma'} \to \mathbf{L}_\Gamma$ of a nonzero component of the differential is given by

$$\begin{gathered} P \wedge Q \wedge \alpha \otimes (x \wedge \overline{x}) \wedge \beta \\ \to \\ PQ \wedge \alpha \otimes \beta \end{gathered}$$

where x is the edge of Γ' to be contracted, with x starting at P and ending at Q, with $\overline{x}$ the reversed edge, and with PQ the combined vertex of Γ. Note that the first expression is invariant under exchanging P with Q (and so x with $\overline{x}$). Moreover α, β represent the remaining vertices and edges of Γ' resp. their images in Γ. For the second expression one may instead choose generally $\alpha \wedge PQ \otimes \beta$ which would result in a global sign change $(-1)^i$ on $C(n)_i$.

(6.1) Lemma. *For $\Gamma \in X_{n,0}$, the group $\mathbf{L}_\Gamma$ is canonically isomorphic to $\mathbf{Z}$.*

Proof: For $i = 0$ there are $n+1$ inner nodes Q_k, all with valency 3, and $2(2n+3)$ oriented edges. There is the canonical term (independent of the order of the Q_k)

$$Q_0 \wedge \cdots \wedge Q_n \otimes (x_{0,1} \wedge x_{0,2} \wedge x_{0,3}) \wedge \cdots \wedge (x_{n,1} \wedge x_{n,2} \wedge x_{n,3})$$

where $x_{k,1}, x_{k,2}, x_{k,3}$ are the cyclically ordered edges starting in Q_k.

Another canonical term is given by the leaves (the nodes with valency 1):

$$P_1 \wedge \cdots \wedge P_{n+3} \otimes x_1 \wedge \cdots \wedge x_{n+3}$$

where the x_i are the leave edges with start point P_i. The product of the two canonical terms is a basis of $\mathbf{L}_\Gamma$.

(The same argument works for any Γ with all nodes of odd valency since $y_1 \wedge \cdots \wedge y_{2h+1}$ is invariant under a cyclic shift of indices. An example is the pentagon corolla in $X_{2,2}$.)

□

Extend the complex $C(n)$ by

$$C(n)_0 \xrightarrow{d_0} C(n)_{-1} \to 0$$

where $C(n)_{-1} = \mathbf{Z}$ and d_0 is componentwise the isomorphism from Lemma (6.1).

The resulting complex is called the augmented complex and denoted by $C'(n)$.

(2) Pre-Theorem. The augmented complex $C'(n)$ is chain homotopy equivalent to 0:

After choosing a root for the planar trees, there is a contracting homotopy,

$$H\colon C'(n)_i \to C'(n)_{i+1}$$

with

$$Hd + dH = \mathrm{id}$$

Here $H|C(n)_{-1}$ sends 1 to the "first" rooted planar binary tree Γ_0. That is the tree with parenthetical expression $((\cdots((**)*)*)\cdots)*)*$, obtained by iterated multiplication of $*$ from the right.

Sketch of proof of $dd = 0$: Let me sketch a proof of $dd = 0$ (actually for the dual of d). One has to look at two edges x, x' to be contracted and compare the results for the different orders of contractions.

If x, x' are disjoint, say x is $P \to Q$ and x' is $P' \to Q'$, one gets (first contract x, then x')

$$\begin{aligned}
&P \wedge Q \wedge P' \wedge Q' \wedge \alpha \otimes (x \wedge \overline{x}) \wedge (x' \wedge \overline{x}') \wedge \beta \\
&\to \\
&PQ \wedge P' \wedge Q' \wedge \alpha \otimes (x' \wedge \overline{x}') \wedge \beta \\
=\ &P' \wedge Q' \wedge PQ \wedge \alpha \otimes (x' \wedge \overline{x}') \wedge \beta \\
&\to \\
&P'Q' \wedge PQ \wedge \alpha \otimes \beta
\end{aligned}$$

and (first contract x', then x)

$$\begin{aligned}
& P \wedge Q \wedge P' \wedge Q' \wedge \alpha \otimes (x \wedge \overline{x}) \wedge (x' \wedge \overline{x}') \wedge \beta \\
= \; & P' \wedge Q' \wedge P \wedge Q \wedge \alpha \otimes (x' \wedge \overline{x}') \wedge (x \wedge \overline{x}) \wedge \beta \\
& \to \\
& P'Q' \wedge P \wedge Q \wedge \alpha \otimes (x \wedge \overline{x}) \wedge \beta \\
= \; & P \wedge Q \wedge P'Q' \wedge \alpha \otimes (x \wedge \overline{x}) \wedge \beta \\
& \to \\
& PQ \wedge P'Q' \wedge \alpha \otimes \beta
\end{aligned}$$

The two resulting expressions cancel.

Suppose the edges x, x' meet, say x is $P \to Q$ and x' is $Q \to R$. Then one gets (first contract x, then x')

$$\begin{aligned}
& P \wedge Q \wedge R \wedge \alpha \otimes (x \wedge \overline{x}) \wedge (x' \wedge \overline{x}') \wedge \beta \\
& \to \\
& PQ \wedge R \wedge \alpha \otimes (x' \wedge \overline{x}') \wedge \beta \\
& \to \\
& PQR \wedge \alpha \otimes \beta
\end{aligned}$$

and (first contract x', then x)

$$\begin{aligned}
& P \wedge Q \wedge R \wedge \alpha \otimes (x \wedge \overline{x}) \wedge (x' \wedge \overline{x}') \wedge \beta \\
= \; & Q \wedge R \wedge P \wedge \alpha \otimes (x' \wedge \overline{x}') \wedge (x \wedge \overline{x}) \wedge \beta \\
& \to \\
& QR \wedge P \wedge \alpha \otimes (x \wedge \overline{x}) \wedge \beta
\end{aligned}$$

x is now $P \to QR$, so we have to change the order of points:

$$\begin{aligned}
= \; & -P \wedge QR \wedge \alpha \otimes (x \wedge \overline{x}) \wedge \beta \\
& \to \\
& -PQR \wedge \alpha \otimes \beta
\end{aligned}$$

Again the two resulting expressions cancel.

$\square$

6.0.1. **On graph complexes.** Soon after having typed the preceding lines, I found some references.

First let me mention that $\mathbf{L}_\Gamma$ appears more or less at the very end of Loday-Vallette 2012 [18, Handling Signs in Graph Complexes, p. 607]. (As for $dd = 0$, there is a reference to Burgunder 2010 [4], but I couldn't find that there.)

Let Γ be a finite graph, with set of vertices V and with set of oriented edges E. Let $F = \{\, \{x, \overline{x}\} \mid x \in E \,\}$ be the set of unoriented edges. Consider the complex of free abelian groups

$$0 \to \mathbf{Z}[F] \xrightarrow{[x] \mapsto x + \overline{x}} \mathbf{Z}[E] \xrightarrow{x \mapsto t(x) - o(x)} \mathbf{Z}[V] \to 0$$

Its homology is $H_1(\Gamma, \mathbf{Z})$, $H_0(\Gamma, \mathbf{Z})$, since $\mathbf{Z}[E]/\mathbf{Z}[F] \to \mathbf{Z}[V]$ is the chain complex of the graph Γ. If Γ is connected, then $H_0(\Gamma, \mathbf{Z}) = \mathbf{Z}$ and (by the theory of elimination) one gets an identification of determinant line bundles:

$$\mathbf{L}_\Gamma = \mathbf{L}'_\Gamma$$

where

$$\mathbf{L}_\Gamma = \Lambda^{|V|}\mathbf{Z}[V] \otimes \Lambda^{|E|}\mathbf{Z}[E]$$
$$\mathbf{L}'_\Gamma = \Lambda^{|F|}\mathbf{Z}[F] \otimes \Lambda^{\max} H_1(\Gamma, \mathbf{Z})$$

(Note that a free cyclic group is canonically isomorphic to its $\mathbf{Z}$-dual.)

The coefficient system $\mathbf{L}'_\Gamma$ appears in Kontsevich 1993 [11, p. 175].

We have just identified the two approaches $\mathbf{L}_\Gamma$, $\mathbf{L}'_\Gamma$ to define "graph complexes".

Note that the proof of $dd = 0$ becomes a triviality when working with Kontsevich's $\mathbf{L}'_\Gamma$. Namely, to check what happens with $\mathbf{L}'_\Gamma$ under a contraction $\Gamma \to \Gamma/x$, one doesn't really have to look at the vertices and for two contractions there is no need for the case distinction whether the 2 edges are disjoint or not (as for $\mathbf{L}_\Gamma$ above).

On the other hand, $\mathbf{L}_\Gamma$ is useful for the proof of Lemma (6.1).

6.1. **A decomposition into paths (chains).**

Language: By a *proper path* in a graph we mean a path without self-crossings. In other words: A proper path is a directed subgraph isomorphic to a subdivided interval. It is also an injective morphism of an abstract path Path_n (Serre [31, p. 14]) to the graph. In a tree, a proper path is a geodesic, Serre [31, Proposition 8, p. 18].

More language: A *rooted planar tree* is a planar tree without vertices of valency 2 and with one vertex of valency 1 selected, called the root. In a rooted planar tree, the *output-valency* of a vertex (different from the root) is its valency minus 1. Moreover, the *root node* is the vertex just next to the root.

A key observation is: If the root node has output-valency ≥ 3, then the tree "de-contracts" to exactly 2 trees with root node of output-valency 2 (the root node is "binary").

In terms of parenthetical expressions: Putting a pair of parenthesis into

$$\alpha_1 \cdots \alpha_r \qquad (r \geq 3)$$

yields

$$\alpha_1(\alpha_2 \cdots \alpha_r), \qquad (\alpha_1 \cdots \alpha_{r-1})\alpha_r$$

as only binary words, with the others of the form

$$\alpha_1 \cdots (\cdots) \cdots \alpha_r, \qquad \alpha_1 \cdots \widetilde{\alpha}_k \cdots \alpha_r$$

involving at least 3 factors on top level.

Fix an ordered $(n+3)$-element set Z and let $R = R(Z) \in Z$ be its first element. A Z-tree is a rooted planar tree with root R and Z its (automatically) ordered set of leaves.

Let $\Omega_n = \Omega(Z)$ be the oriented graph with vertices resp. oriented edges the Z-trees with root node of output-valency 2 resp. ≥ 3. Moreover, an edge $\alpha_1 \cdots \alpha_r$ $(r \geq 3)$ starts in the binary word $\alpha_1(\alpha_2 \cdots \alpha_r)$ and ends in the binary word $(\alpha_1 \cdots \alpha_{r-1})\alpha_r$.

Consider a binary word $\alpha\beta$. If α is composed, say $\alpha = \alpha_1 \cdots \alpha_k$ $(k \geq 2)$, then $\alpha\beta$ ends the edge $\alpha_1 \cdots \alpha_k\beta$. Similarly, if β is composed, $\beta = \beta_1 \cdots \beta_k$ $(k \geq 2)$, then $\alpha\beta$ starts the edge $\alpha\beta_1 \cdots \beta_k$.

It follows that Ω_n decomposes into directed proper paths, starting from terms $*\beta$ and ending in terms $\alpha*$, where $*$ denotes the "atom" (representing a rooted tree with 1 non-root leaf).

(6.2) Definition. Let Γ be a Z-tree with output-valency 2 of the root node.

The *(immediate) chain* of Γ is the directed subpath of $\Omega(Z)$ from Γ to $\Gamma'*$ (with Γ' a Z'-tree, $Z' = Z \setminus \{\text{last element}\}$).

The *full chain* of Γ is the immediate chain of Γ followed by the immediate chain of Γ', followed by the immediate chain of Γ'', etc., down to $** \in X_{0,0}$.

Note that consecutive elements in a chain are connected by a unique tree with output-valency 3 of the root node.

The longest chains are related with *root shift*: Suppose $Z = (1, \ldots, m)$ and let Γ be a Z-tree. Then Γ is also a $Z[-1]$-tree with $Z[-1] = (m, 1, \ldots, m-1)$ (the next leave to the right of the root is the new root). The unique order preserving map $Z[-1] \to Z$ (here: $k \mapsto k+1 \bmod m$) yields a new Z-tree Γ' (the tree with shifted root). It turns out that the chain from $*\Gamma$ ends in $\Gamma'*$ (this is a not too difficult exercise).

For $\Gamma \in X_{n,0}$ the full chain is the "canonical directed path" in Mac Lane 1963 [19, p. 34], obtained "by moving an outermost parenthesis toward the front".

See §7 for illustrations.

One somehow recognizes $C(n)$ itself as a homotopy from $*C(n-1)$ to $C(n-1)*$ (with respect to a tree rotation). By induction it follows that $C(n)$ is contractible.

These considerations explain the basic reason why the following simple definition of the chain homotopy H works out.

6.2. **The homotopy.** One defines H inductively like this:

$$H\big((\alpha)*\big) = H(\alpha)*$$

$$H(\alpha_1 \cdots \alpha_r) = 0 \qquad r \geq 3$$

$$H\big(\alpha_1(\alpha_2 \cdots \alpha_r) - (\alpha_1 \cdots \alpha_{r-1})\alpha_r\big) = \alpha_1 \cdots \alpha_r \qquad r \geq 3$$

or rather more precisely and correctly,

$$H\big(d(\alpha_1 \cdots \alpha_r)\big) = \alpha_1 \cdots \alpha_r \qquad r \geq 3$$

For the latter note that $d(\alpha_1 \cdots \alpha_r)$ is a sum of

$$\pm\alpha_1(\alpha_2 \cdots \alpha_r), \qquad \pm(\alpha_1 \cdots \alpha_{r-1})\alpha_r$$

and elements $\beta_1 \cdots \beta_s$ ($s \geq 3$) which vanish under H.

Further, $Hd + dH = \mathrm{id}$ follows by looking at the generators

(6.3) $\quad (\alpha)*$

(6.4) $\quad \alpha_1 \cdots \alpha_r \qquad r \geq 3$

(6.5) $\quad d(\alpha_1 \cdots \alpha_r) \qquad r \geq 3$

Proof: First note that the elements (6.3)–(6.5) are indeed generators: The elements (6.5) are modulo the elements (6.4) differences of binary words. These form chains from an arbitrary binary word down to an element in (6.3) (called above the immediate chain).

Now let us check $Hd + dH = \mathrm{id}$ on an element β in (6.3)–(6.5):

If $\beta = (\alpha)*$ is from (6.3), use induction on n.

If $\beta = \alpha_1 \cdots \alpha_r$ is from (6.4), one finds

(6.6) $$Hd(\beta) + dH(\beta) = \beta + 0$$

by the very definition of H.

If $\beta = d(\gamma)$, $\gamma = \alpha_1 \cdots \alpha_r$ is from (6.5), one finds

$$Hdd(\gamma) + dHd(\gamma) = 0 + d(\gamma - dH(\gamma)) = 0 + \beta - 0$$

using (6.6) for γ.

□

Simple enough!

At the beginning I was suspicious, as the proof looks too simple. However, I think the picture about the graphs Ω_n makes the situation transparent: There is a natural filtration on the complex $C(n)$ whose graded terms are direct sums of chain complexes of subdivided intervals and these define a homotopy retraction from $C(n)$ to $C(n-1)$ (graded versions).

Further, I did not expect such a simplicity and found the construction of H via hard explicit work for small n.

I am also wondering now whether the basic argument has been used elsewhere. Comments are welcome!

I plan to write the details in a forthcoming text.

6.3. **Further ideas.**

- Define the n-dimensional associahedron as a topological space Y_n.

  If one thinks about it, the idea of chains leads almost immediately to the construction: Once Y_n is build, construct $Y_{n+1} = Y_n \times [0,1]$ realizing the homotopy between Y_n and a root shift of Y_n (which is the same space, only the references to trees are changed).

  One takes the CW-complex

  $$Y_n = [0,1]^n$$

  (the standard cube) together with a (special kind of) cellular subdivision along the chains of the graphs Ω_m.

  The 0-cells are given by an obvious embedding

  $$X_{n,0} \to Y_n = [0,1]^n$$

  Then the i-cells are the convex hulls of their 0-cells in the boundary (determined via full de-contractions of trees). A particular feature is that every i-cell is itself a cuboid (the convex hull of 2^i points), with the boundary itself subdivided into cuboidal cells.

  See (and enjoy) §7 for illustrations.

- After choosing a root, trivialize the coefficient system L_Γ (select appropriate isomorphisms $L_\Gamma = \mathbf{Z}$).

  Then get a general formula like

  $$d(\alpha_1 \cdots \alpha_r) = \sum_{i=1}^{r} (-1)^? \alpha_1 \cdots d(\alpha_1) \cdots \alpha_r + \sum_I \partial_I(\alpha_1, \ldots, \alpha_r)$$

  Here the first sum corresponds to de-contractions away from the root node, the second sum corresponds to de-contractions at the root node.

  The signs in the first sum should be easy to figure out. The second sum should be given by the following formula

  $$\partial(m_n) = - \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k,q\geq 2}} (-1)^{p+qr} m_k \circ (\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r})$$

  in Loday-Vallette 2012 [18, 9.2.4 The Operad A_∞, p. 339].

- Recover coherence for monoidal categories. We refer to the (partial) diagram for the 4-dimensional associahedron Y_4 on page 22 in §7. If we have a simple loop in the 1-skeleton of Y_n and it leaves $Y_{n-1}*$ somewhere, then it must walk a rung in one of the ladders. But two neighbored rungs in the same ladder span a quadrangle (as for a normal ladder) or a pentagon (in a pretty dangerous ladder). In any case one may shorten the loop by a simple homotopy along a 2-cell (quadrangle or pentagon). Repeating this, we may arrange that the loop is in Y_{n-1} and we are done by induction.

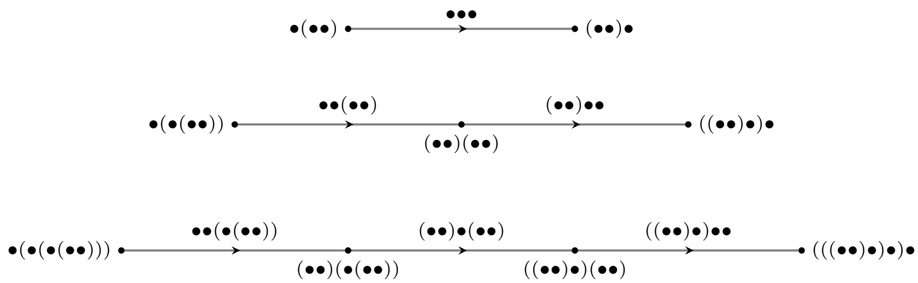
  Note that this recipe reflects *precisely* the arguments in Mac Lane [19].

  Here one does not need the full cube Y_n, just its 2-skeleton $Y_n^{(2)}$, which can be easily build from scratch. As by-result one gets $\pi_1(Y_n^{(2)}) = 1$.

- Identify the complex $C(n)$ with the chain complex of a polytope realization. For example take the Loday realization or the Lee subdivision. Also for the unrooted case, taking a "symmetrical" realization.

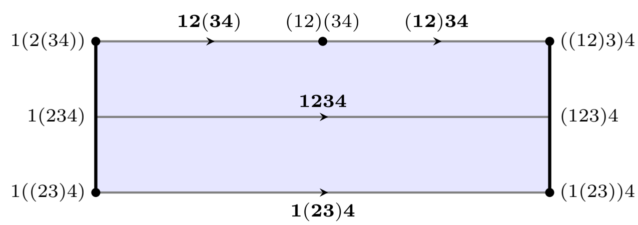
§7. **Diagrams**

Here are the longest chains in the graphs Ω_n for $n = 1, 2, 3$.



The graph Ω_1 consists of just 1 chain.

The graph Ω_2 decomposes into 1 chain of length 2 and 2 chains of length 1. In the next diagram, the vertices of Ω_2 are the 5 points together with the 2 vertical segments. The edges of Ω_2 establish a homotopy between $*\Omega_1'$ on the left (note the root shift) and Ω_1* on the right. One recognizes a pentagon, with the 2-cell given by the arrow between the 2 vertical segments.



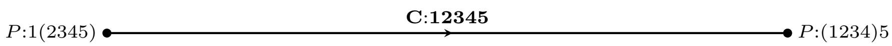
[blank space for better formatting of next page]

The graph Ω_3 decomposes into 11 chains (11=number of cells in the pentagon).

The vertices and edges of the graph Ω_3 make up the cells (faces of any dimension) of the 3-dimensional associahedron.

In the following P indicates a pentagon, Q a quadrilateral, C the corolla (the 3-cell).

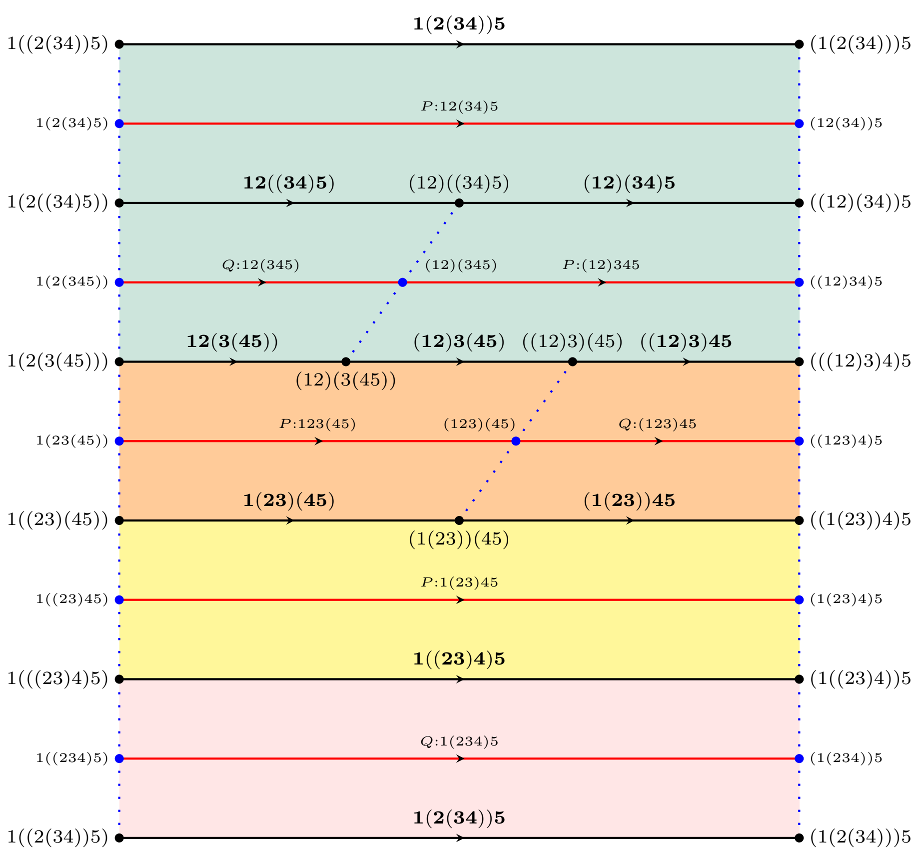
One path is



Its vertices correspond in the associahedron to 2 disjoint pentagons, its edge to the 3-cell. After removing these from the associahedron, there remains a cylinder.

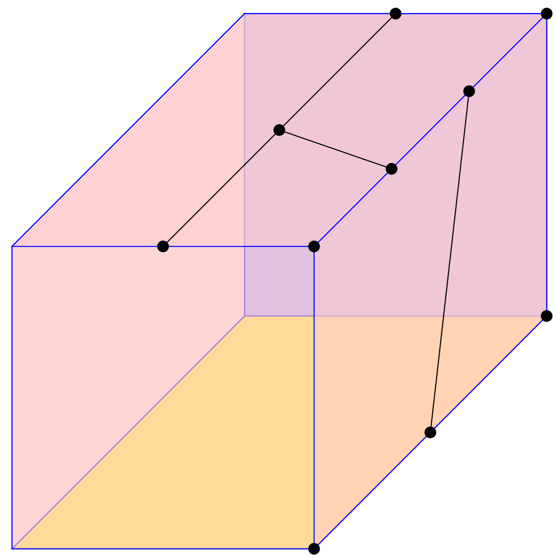
The remaining 10 chains of the graph Ω_3 are depicted in the next diagram.

Here the top chain (of length 1) is repeated on the bottom. The dotted lines are irrelevant for the graph itself. However they indicate 1-cells in the associahedron. The red arrows between them correspond to 2-cells. These are the remaining 4 pentagons for P-arrows and the 3 disjoint quadrilaterals for Q-arrows. After identifying the top line with the bottom line, one recognizes the cylinder. The vertical sides describe the removed "caps" of the cylinder: the pentagons $*\Omega_2'$, Ω_2*.

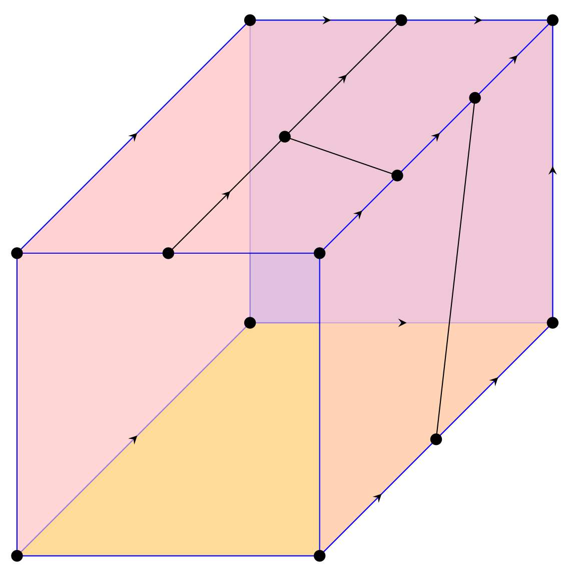


It can be that simple. Why would one ever want to draw a polytopal realization?

Here is a minimalistic 3d image of the 3-dimensional associahedron. Only the chains of length ≥ 2 are indicated. One recognizes the 3 disjoint quadrangles.

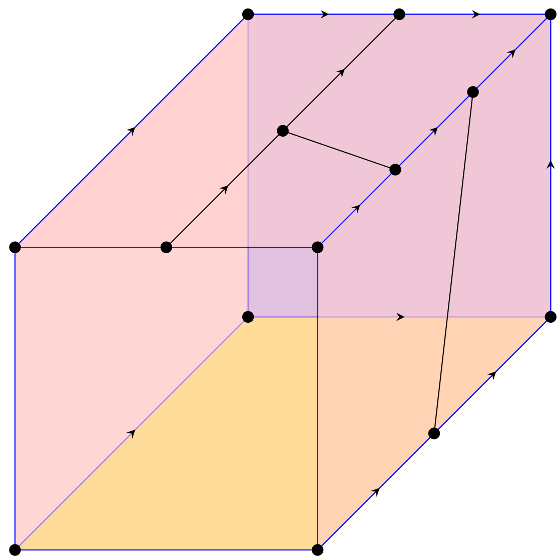


The next image shows additionally for each vertex of the associahedron its full chain down to $X_{0,0}$. These are the "canonical directed path" in Mac Lane 1963 [19, p. 34], obtained "by moving an outermost parenthesis toward the front".



The chain homotopy is a combination of following simple geometric homotopies. First contract the cuboid to the rear pentagon, then contract that to its right side, and finally contract that upwards to $X_{0,0}$ (or rather perform these simultaneously).

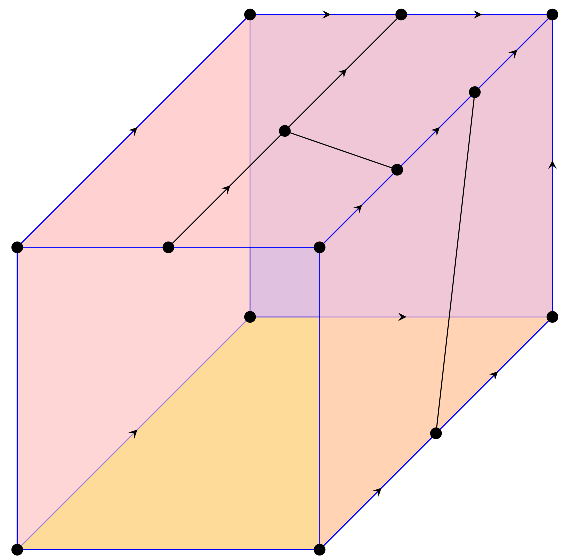
Same picture with captions appended.



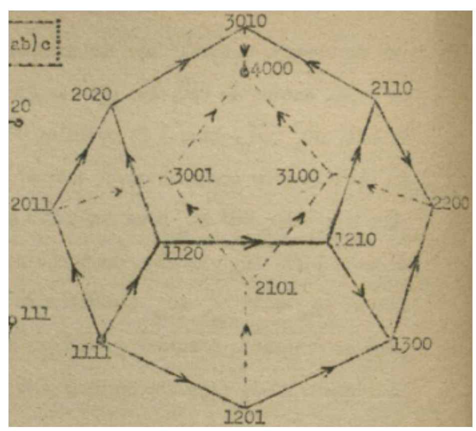
$$
\begin{tikzcd}
1((2(34))5) \arrow[r] \arrow[d, dash] & (1(2(34)))5 \arrow[dr] & & \\
1(2((34)5)) \arrow[r] \arrow[d, dash] & (12)((34)5) \arrow[r] \arrow[d, dash] & ((12)(34))5 \arrow[dr] & \\
1(2(3(45))) \arrow[r] \arrow[d, dash] & (12)(3(45)) \arrow[r] & ((12)3)(45) \arrow[r] & (((12)3)4)5 \\
1((23)(45)) \arrow[r] \arrow[d, dash] & (1(23))(45) \arrow[r] \arrow[ur, dash] & ((1(23))4)5 \arrow[ur] & \\
1(((23)4)5) \arrow[r] \arrow[d, dash] & (1((23)4))5 \arrow[ur] \arrow[d, dash] & & \\
1((2(34))5) \arrow[r] & (1(2(34)))5 & &
\end{tikzcd}
$$

The top line is duplicated on the bottom to show the left side of the cube (a quadrangle).

Same as previous page, but with the captions taken from Tamari's drawing (page 4).

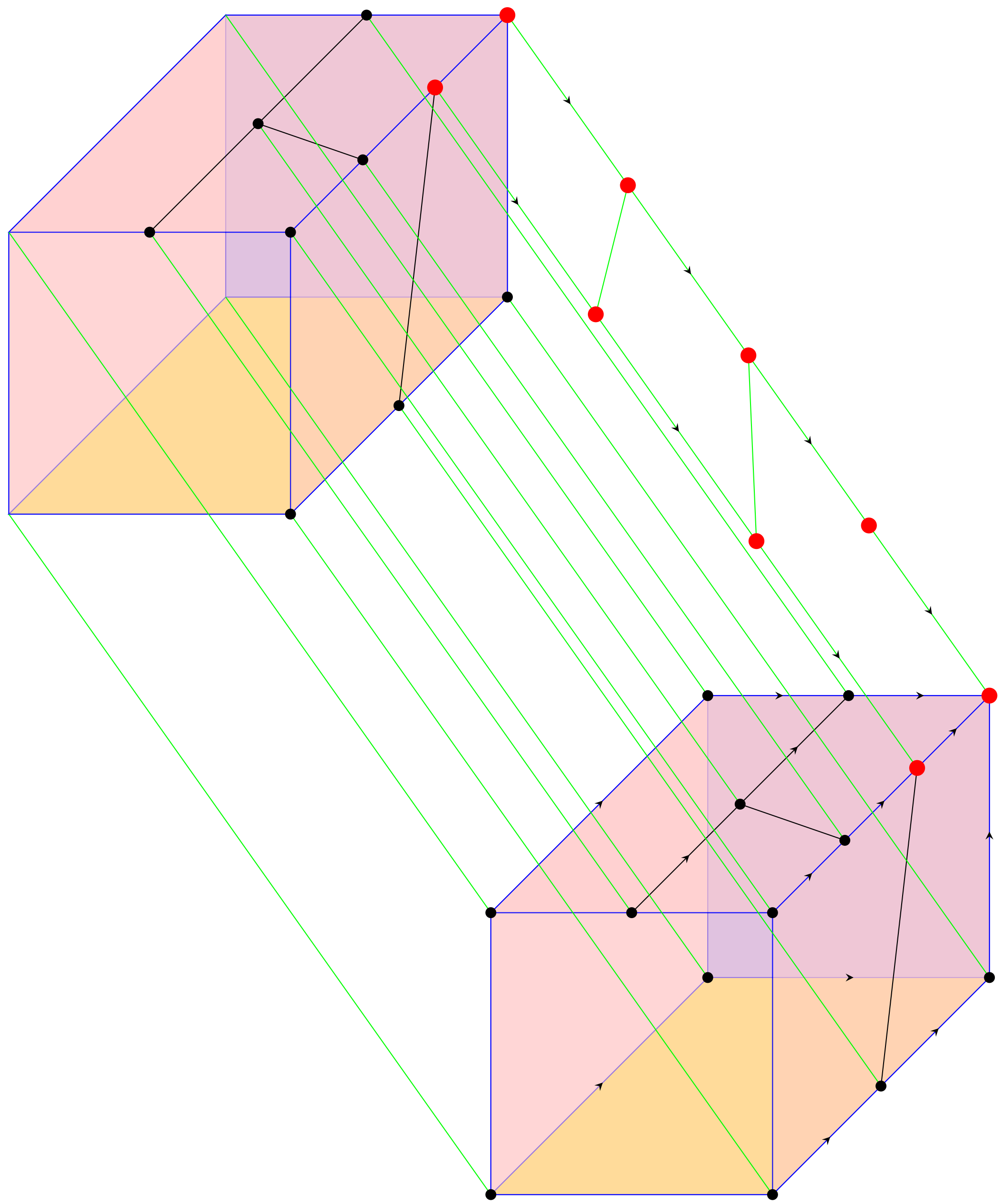


$$
\begin{tikzcd}
1210 \arrow[r] \arrow[d, dash] & 2110 \arrow[dr] & & \\
1120 \arrow[r] \arrow[d, dash] & 2020 \arrow[r] \arrow[d, dash] & 3010 \arrow[dr] & \\
1111 \arrow[r] \arrow[d, dash] & 2011 \arrow[r] & 3001 \arrow[r] & 4000 \\
1201 \arrow[r] \arrow[d, dash] & 2101 \arrow[r] \arrow[ur, dash] & 3100 \arrow[ur] & \\
1300 \arrow[r] \arrow[d, dash] & 2200 \arrow[ur] \arrow[d, dash] & & \\
1210 \arrow[r] & 2110 & &
\end{tikzcd}
$$



The top line is duplicated on the bottom to show the right quadrangle in Tamari's drawing.

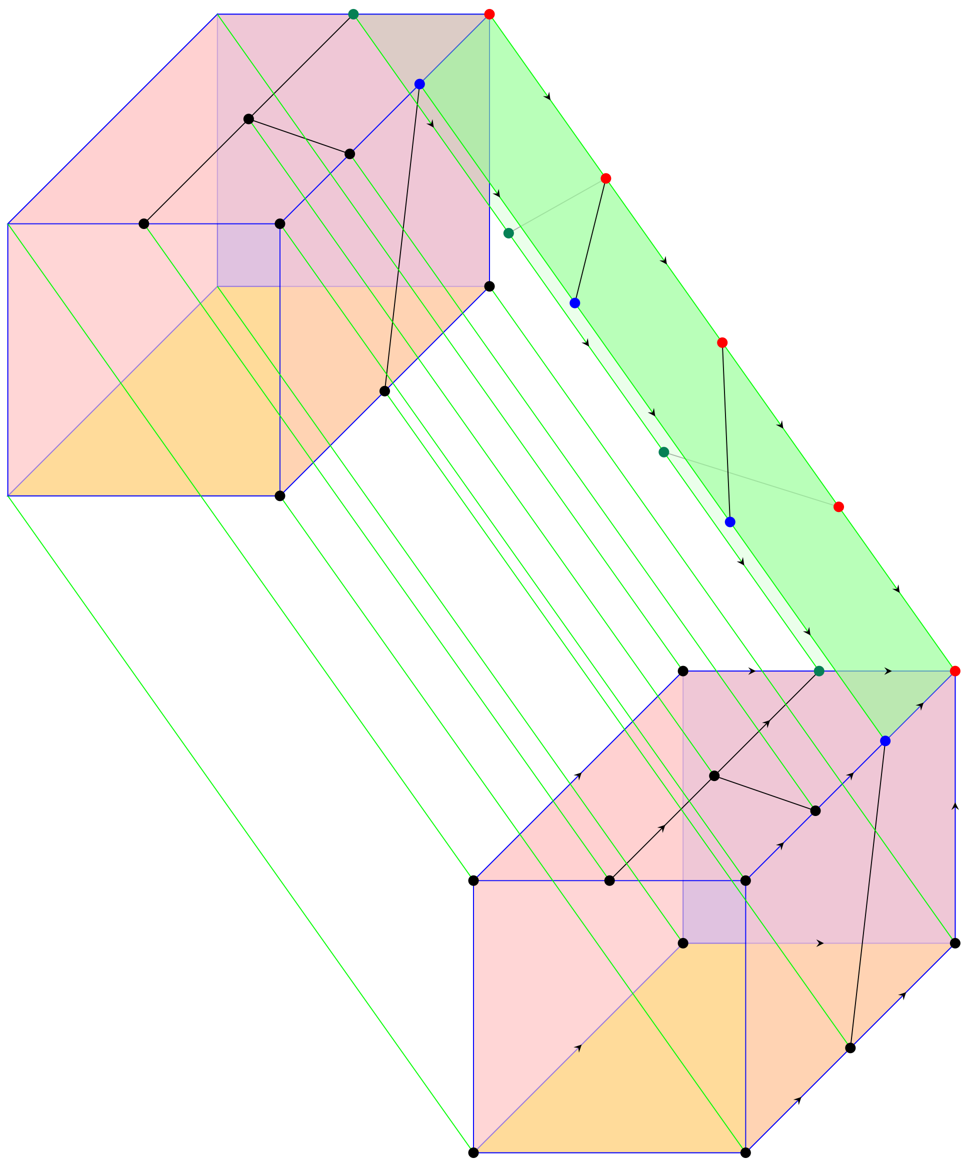
Partial image of the 4-dimensional cubical associahedron I.



Only 2 chains in the 4th dimension are drawn, the longest chain and a neighbor:

$$\begin{array}{ccccccccc}
1(2(3(4(56)))) & \longrightarrow & (12)(3(4(56))) & \longrightarrow & ((12)3)(4(56)) & \longrightarrow & (((12)3)4)(56) & \longrightarrow & ((((12)3)4)5)6 \\
| & & | & & | & & & \nearrow & \\
1(2(3((45)6))) & \longrightarrow & (12)(3((45)6)) & \longrightarrow & ((12)3)((45)6) & \longrightarrow & (((12)3)(45))6 & &
\end{array}$$

Partial image of the 4-dimensional cubical associahedron II.



Here 3 chains in the 4th dimension are drawn, the longest chain and 2 neighbors:

$$
\begin{tikzcd}
1(2((34)(56))) \arrow[r] \arrow[d, dash] & (12)((34)(56)) \arrow[r] \arrow[d, dash] & ((12)(34))(56) \arrow[r] \arrow[dr, dash] & (((12)(34))5)6 \arrow[dr] & \\
1(2(3(4(56)))) \arrow[r] \arrow[d, dash] & (12)(3(4(56))) \arrow[r] \arrow[d, dash] & ((12)3)(4(56)) \arrow[r] \arrow[d, dash] & (((12)3)4)(56) \arrow[r] & ((((12)3)4)5)6 \\
1(2(3((45)6))) \arrow[r] & (12)(3((45)6)) \arrow[r] & ((12)3)((45)6) \arrow[r] & (((12)3)(45))6 \arrow[ur] &
\end{tikzcd}
$$

Here are the ladders from the longest chain to its 3 neighbors.

The first ladder is depicted on pages 22, 23, the second ladder on page 23, the 3rd ladder you have to draw yourself (oh no, now there is page 25).

Note that all ladders are dangerous (the side rails don't match), but at different spots with each giving rise to a pentagon.

Ladder 1:

$$\begin{tikzcd}
1(2(3(4(56)))) \arrow[r] & (12)(3(4(56))) \arrow[r] & ((12)3)(4(56)) \arrow[r] & (((12)3)4)(56) \arrow[r] & ((((12)3)4)5)6 \\
1(2(3((45)6))) \arrow[r] \arrow[u, dash] & (12)(3((45)6)) \arrow[r] \arrow[u, dash] & ((12)3)((45)6) \arrow[r] \arrow[u, dash] & (((12)3)(45))6 \arrow[ur] &
\end{tikzcd}$$

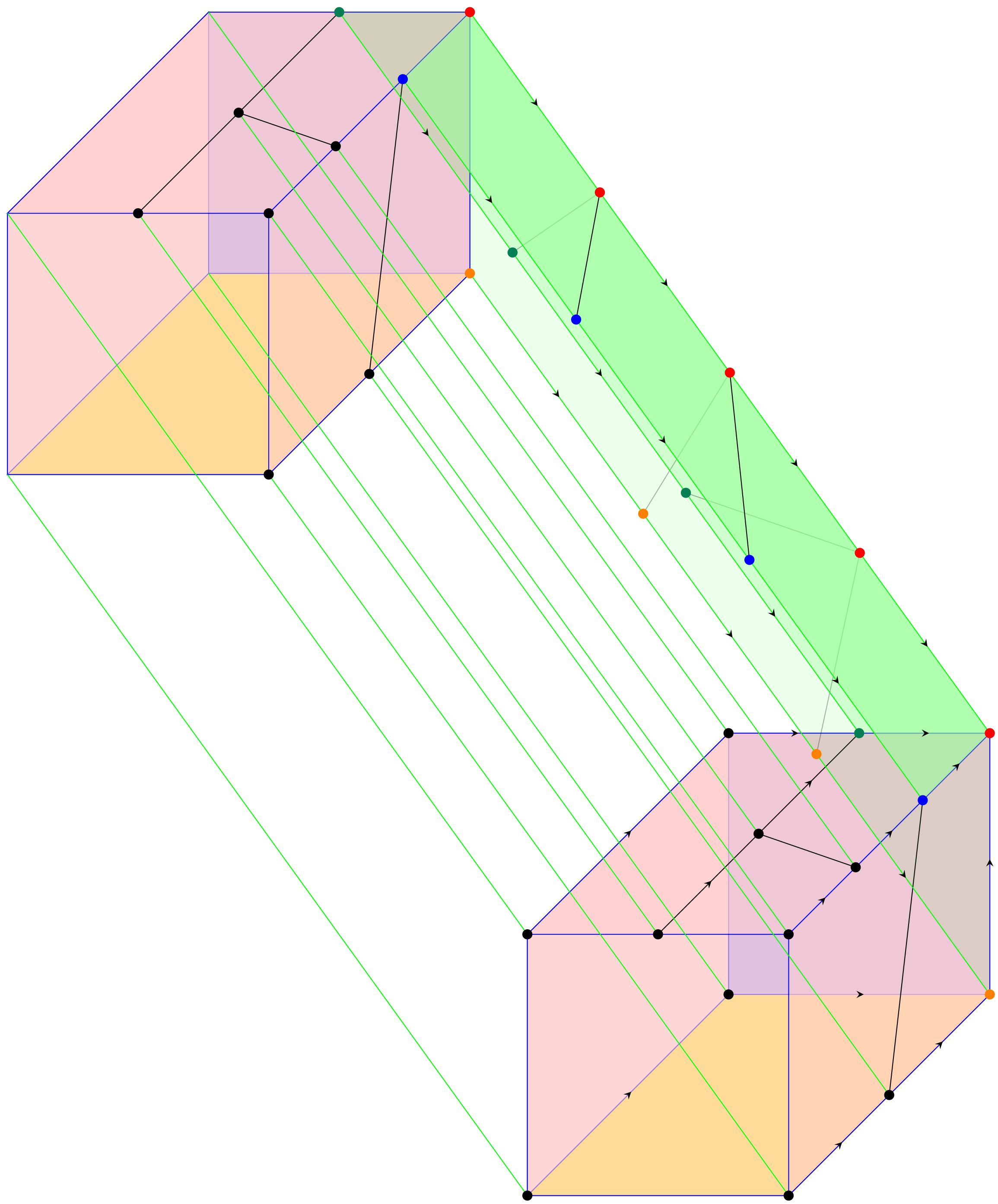
Ladder 2:

$$\begin{tikzcd}
1(2(3(4(56)))) \arrow[r] & (12)(3(4(56))) \arrow[r] & ((12)3)(4(56)) \arrow[r] & (((12)3)4)(56) \arrow[r] & ((((12)3)4)5)6 \\
1(2((34)(56))) \arrow[r] \arrow[u, dash] & (12)((34)(56)) \arrow[r] \arrow[u, dash] & ((12)(34))(56) \arrow[r] \arrow[ur, dash] & (((12)(34))5)6 \arrow[ur] &
\end{tikzcd}$$

Ladder 3:

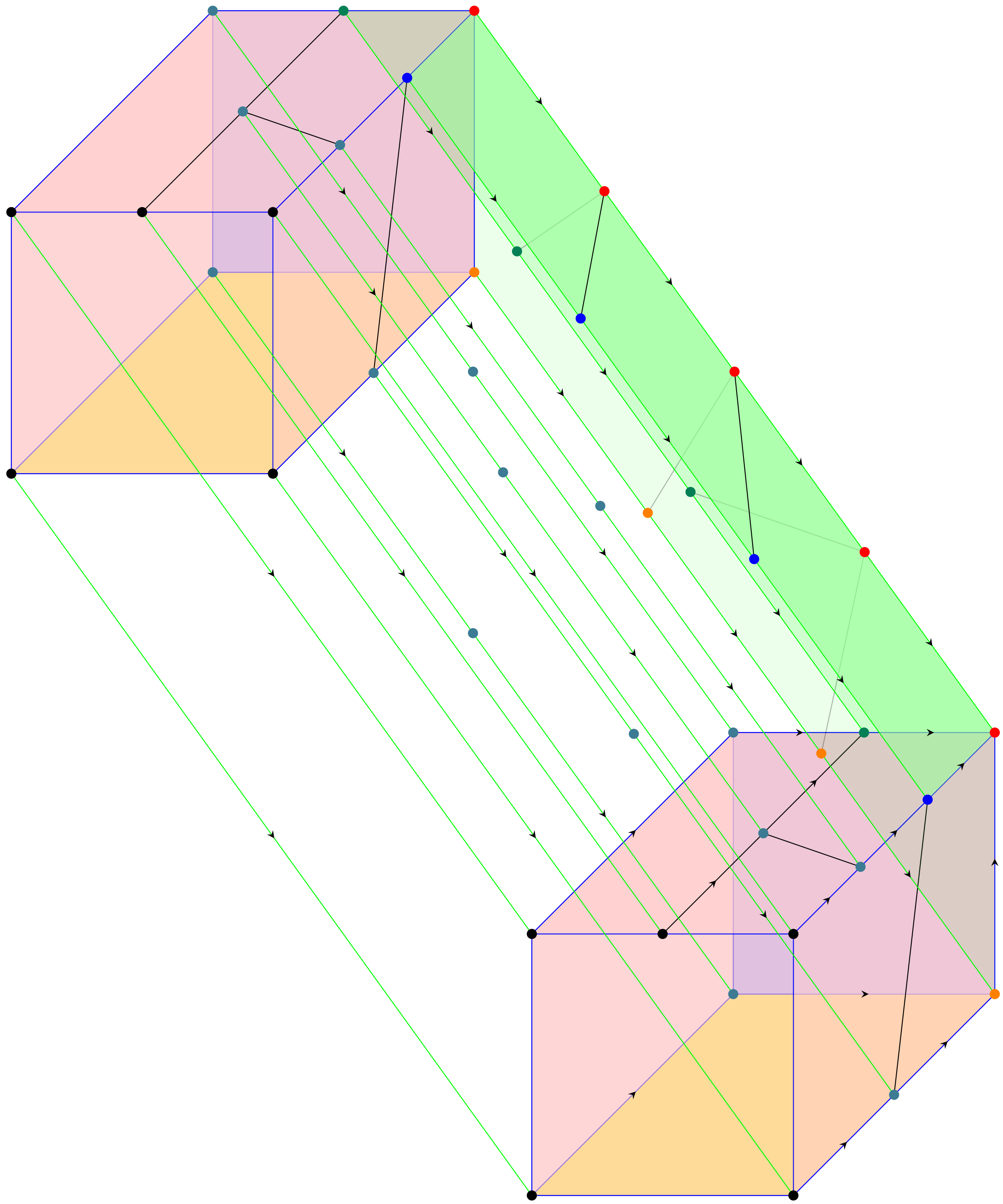
$$\begin{tikzcd}
1(2(3(4(56)))) \arrow[r] & (12)(3(4(56))) \arrow[r] & ((12)3)(4(56)) \arrow[r] & (((12)3)4)(56) \arrow[r] & ((((12)3)4)5)6 \\
1((23)(4(56))) \arrow[r] \arrow[u, dash] & (1(23))(4(56)) \arrow[r] \arrow[ur, dash] & ((1(23))4)(56) \arrow[r] \arrow[ur, dash] & (((1(23))4)5)6 \arrow[ur] &
\end{tikzcd}$$

Partial image of the 4-dimensional cubical associahedron III.



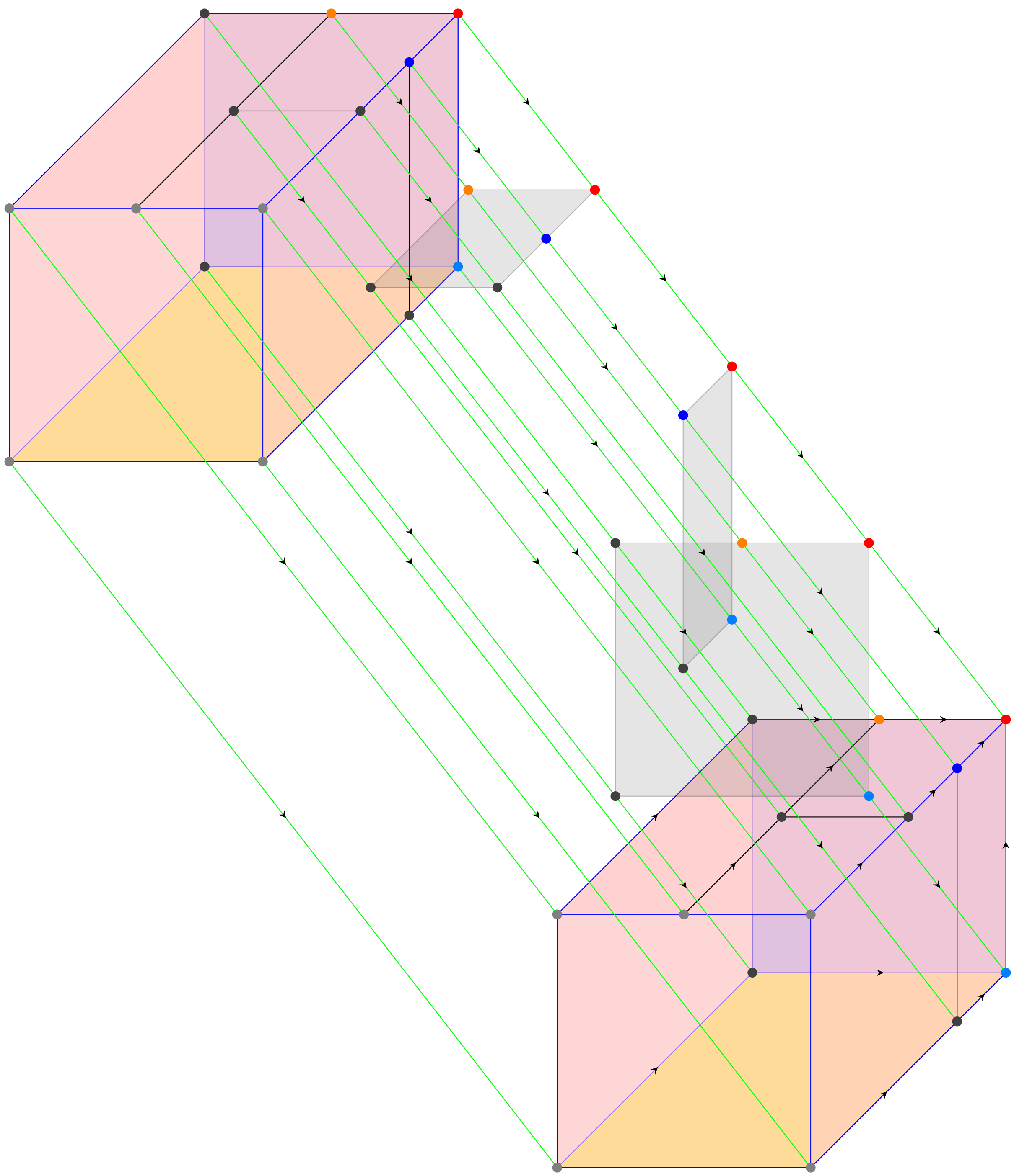
Partial image of the 4-dimensional cubical associahedron IV.

(with all 42 points and contracting arrows)



Partial image of the 4-dimensional cubical associahedron V.

Finally, here is one with all 42 points and 84 edges (June 2025).



Here are captions for the maximal path with no direction change (the straight line with the 5 red vertices) and for the path with 4 directions (starting in the gray opposite corner of the upper cube):

$$\begin{array}{ll}
\bullet\overset{1}{(}\bullet\overset{2}{(}\bullet\overset{3}{(}\bullet\overset{4}{(}\bullet\bullet\overset{4}{)}\overset{3}{)}\overset{2}{)}\overset{1}{)}\to & \qquad \bullet\overset{1}{(}\overset{2}{(}\overset{3}{(}\overset{4}{(}\bullet\bullet\overset{4}{)}\bullet\overset{3}{)}\bullet\overset{2}{)}\bullet\overset{1}{)}\to \\
\overset{1}{(}\bullet\bullet\overset{1}{)}\overset{2}{(}\bullet\overset{3}{(}\bullet\overset{4}{(}\bullet\bullet\overset{4}{)}\overset{3}{)}\overset{2}{)}\to & \qquad \overset{1}{(}\bullet\overset{2}{(}\overset{3}{(}\overset{4}{(}\bullet\bullet\overset{4}{)}\bullet\overset{3}{)}\bullet\overset{2}{)}\overset{1}{)}\bullet\to \\
\overset{2}{(}\overset{1}{(}\bullet\bullet\overset{1}{)}\bullet\overset{2}{)}\overset{3}{(}\bullet\overset{4}{(}\bullet\bullet\overset{4}{)}\overset{3}{)}\to & \qquad \overset{1}{(}\overset{2}{(}\bullet\overset{3}{(}\overset{4}{(}\bullet\bullet\overset{4}{)}\bullet\overset{3}{)}\overset{2}{)}\bullet\overset{1}{)}\bullet\to \\
\overset{3}{(}\overset{2}{(}\overset{1}{(}\bullet\bullet\overset{1}{)}\bullet\overset{2}{)}\bullet\overset{3}{)}\overset{4}{(}\bullet\bullet\overset{4}{)}\to & \qquad \overset{1}{(}\overset{2}{(}\overset{3}{(}\bullet\overset{4}{(}\bullet\bullet\overset{4}{)}\overset{3}{)}\bullet\overset{2}{)}\bullet\overset{1}{)}\bullet\to \\
\overset{4}{(}\overset{3}{(}\overset{2}{(}\overset{1}{(}\bullet\bullet\overset{1}{)}\bullet\overset{2}{)}\bullet\overset{3}{)}\bullet\overset{4}{)}\bullet & \qquad \overset{1}{(}\overset{2}{(}\overset{3}{(}\overset{4}{(}\bullet\bullet\overset{4}{)}\bullet\overset{3}{)}\bullet\overset{2}{)}\bullet\overset{1}{)}\bullet
\end{array}$$

§8. The diagram from Tamari's thesis

A TEX/Ti*k*Z diagram after Fig. 1 in Tamari's thesis [39, p. 12] (see page 4):

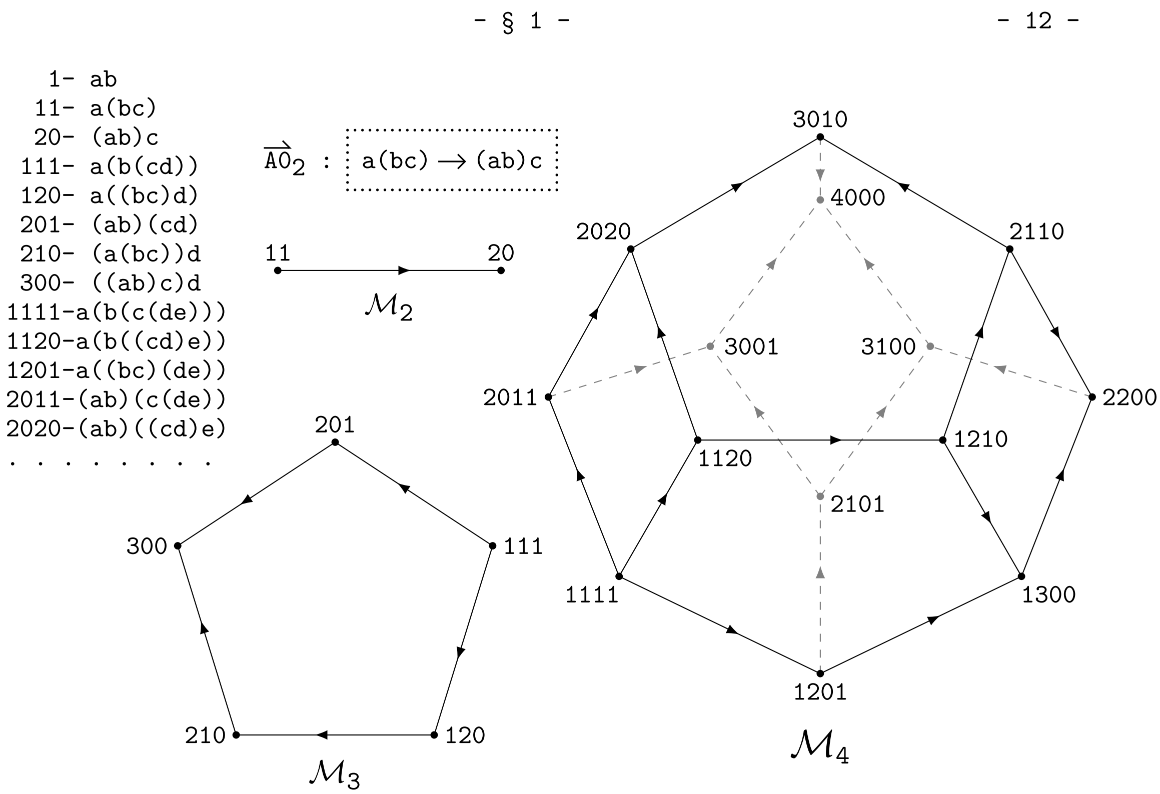
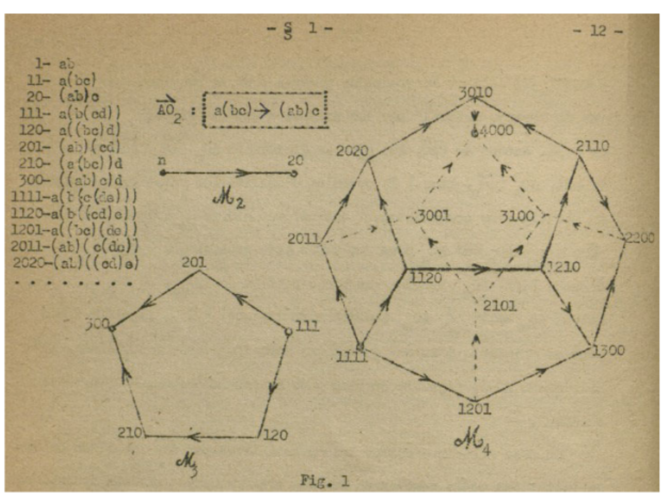


Fig. 1

Scans of the original diagram can be found here: Loday 2012 [17, p. 74], Stasheff 2012 [37, p. 46] (both in the Tamari memorial Festschrift 2012 [22]) and Stasheff 2019 [38, p. 94] (arXiv:1809.02526v2 [math.QA], p. 4).



The following colored variant has reversed arrows and perspective. It illustrates the strong deformation retractions

$$\mathcal{M}_4 \to \mathcal{M}_3 \to \mathcal{M}_2 \to \mathcal{M}_1$$

The embeddings (written above as $*\mathcal{M}_k \subset \mathcal{M}_{k+1}$)

$$\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to \mathcal{M}_4$$

are given by prepending `1` to the labels. The green/red/blue arrows indicate the homotopies. In terms of the cubical view, these retract the cuboid linearly to a side.

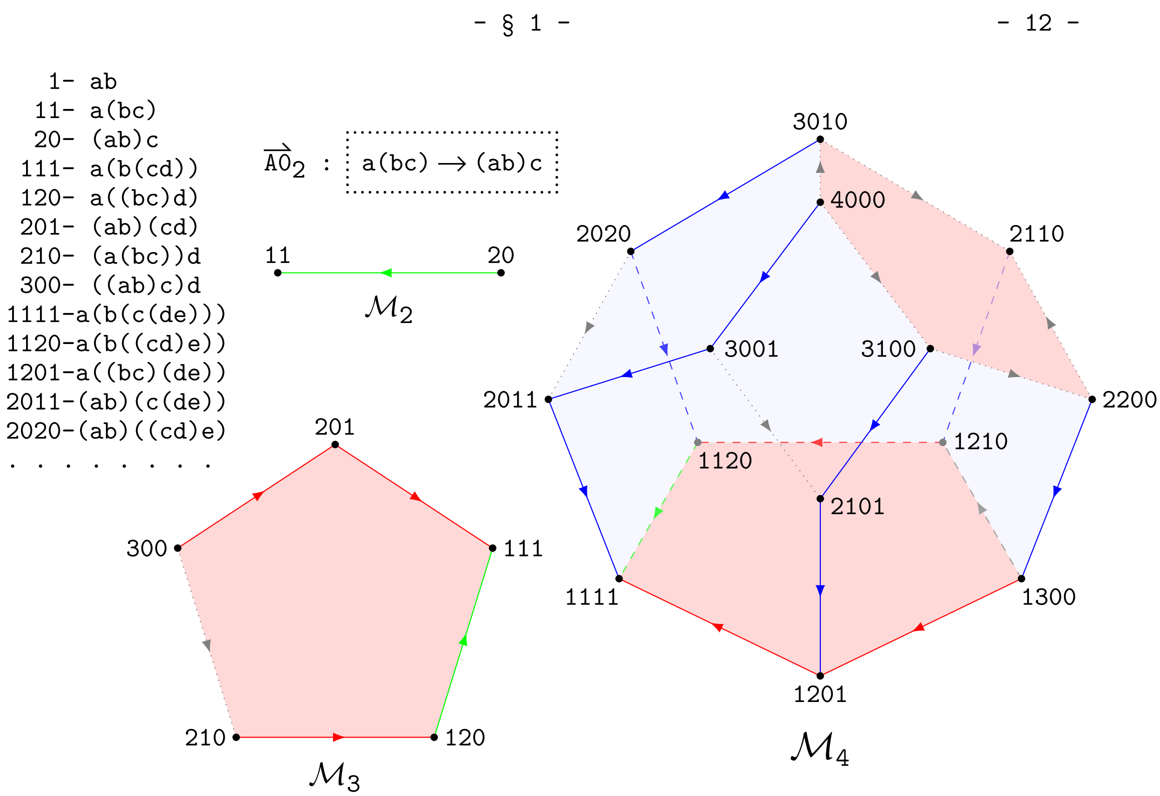


Fig. 1

The next variant shows the duals. The dual (of the boundary) of the associahedron is a simplicial complex build from non-crossing polygon diagonals. It is a triangulation of the sphere. See [13].

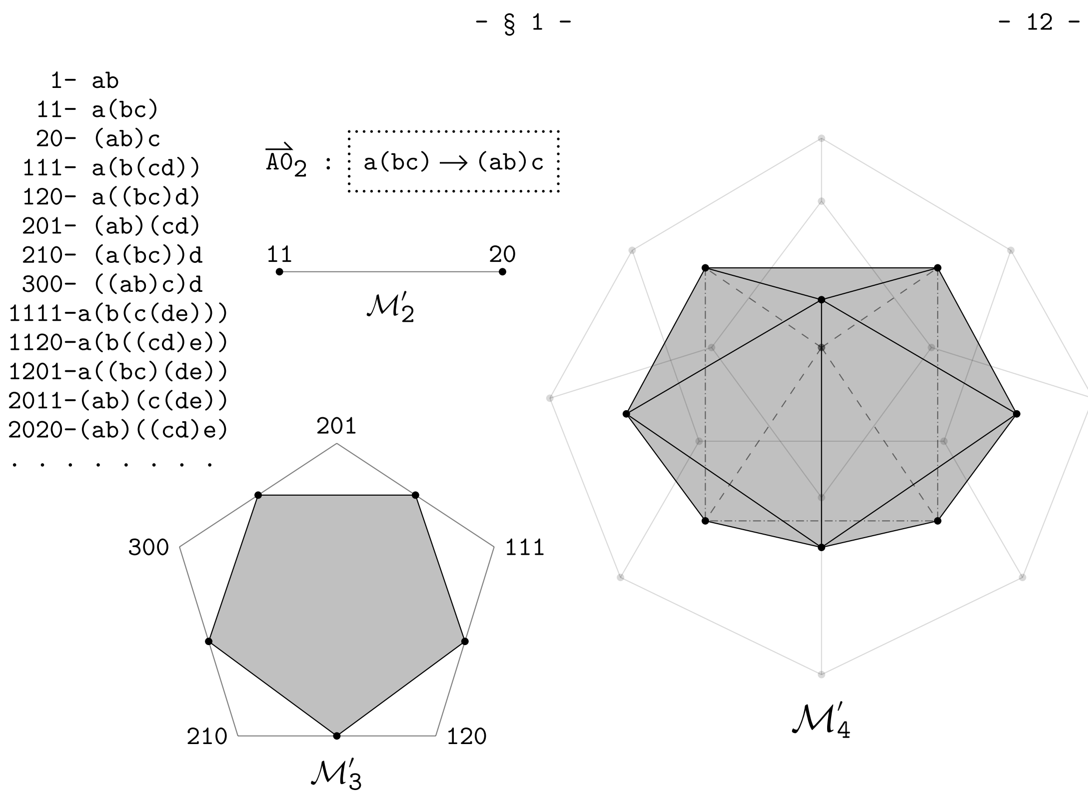


Fig. 1

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Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

Email address: rost *at* math.uni-bielefeld.de

URL: www.math.uni-bielefeld.de/~rost