NOTES ON ASSOCIATOR IDENTITIES

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Contents

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Pre-Introduction

This text consists of first notes on (well known) associator identities. A goal is to make presentations and proofs transparent and explicit. Examples are Lemma 1, Lemma 2 and Lemma 7. A more systematic approach is started in [14].

A starting point is Bruck and Kleinfeld 1951 [4] where one finds a bunch of identities. See also Kleinfeld 1953 [6], Smiley 1957 [16].

We list some of the identities in [4], including the numbering.

The commutator and associator in an algebra are denoted by

$$(x, y) = [x, y] = xy - yx$$
$$(x, y, z) = A(x, y, z) = (xy)z - x(yz)$$

First there is the 6-term identity

(2.2) $$(xy, z) - x(y, z) - (x, z)y = (x, y, z) - (x, z, y) + (z, x, y)$$

which relates the commutator and associator. Then the 5-term identity

(2.3) $$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z$$

for the associator alone. It is called "Viereridentität" in Zorn 1931 [17, (2), p. 125].

Now one assumes that the algebra is alternative.

By definition, this means that the associator is alternating. Equivalently, A factors as

$$A\colon \Lambda^3 V \to V$$

where V denotes the given algebra.

A remarkable observation of Bruck and Kleinfeld 1951 [4] is the existence of an alternating 4-linear morphism

$$f\colon \Lambda^4 V \to V$$

such that

(2.7) $$(wx, y, z) = (x, y, z)w + x(w, y, z) + f(w, x, y, z)$$

(2.8) $$3f(w, x, y, z) = (w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y))$$

(2.9) $$f(w, x, y, z) = ((w, x), y, z) + ((y, z), w, x)$$

We will prove [4, (2.3), (2.7), (2.8), (2.9)], see (1.1), Corollary 4 and Corollary 9.

Introduction

For an alternative algebra we derive the 4-variable identity

$$(*) \qquad A(x, yz, t) - A(x, z, t)y - zA(x, y, t) = A(x, z, [y, t]) + A([x, z], y, t)$$

for its associator $A(x, y, z) = (xy)z - x(yz)$, see (2.1).

It appears in Bruck and Kleinfeld 1951 [4, (2.7), (2.9), p. 880].

One may consider ($*$) as a lift of the 3-variable Moufang identities (of degrees 2, 1, 1). It immediately implies the Moufang identities, in the form of Zorn 1931 [17] and Moufang 1935 [11], see Corollary 5 resp. Corollary 6.

One feature of our presentation is a separation of the roles of the various tensors involved.

Section 4 contains a quote of Zorn on Artin and some brief remarks on Artin's theorem on alternative algebras.

Section 3 is a late addition to this text.

A broader picture is the question for all general relations for the associator (with the 5-term relation (1.1) as basic example) and the relation with the associahedron. See Remark 2.

§1. The 5-term relation (Vireridentität)

Let V be a module over some base ring (which is associative, commutative and unital).

We consider morphisms

$$\mu\colon V^{\otimes 2} \to V$$
$$A\colon V^{\otimes 3} \to V$$
$$R\colon V^{\otimes 4} \to V$$

taken as multi-linear maps $V^n \to V$ or as elements of $\mathrm{Hom}(V^{\otimes n}, V)$.

Given μ, we consider the pair (V, μ) as non-associative[1], non-unital algebra over the base ring with product

$$xy = \mu(x, y)$$

Given μ, the *associator* $A = A(\mu)$ of μ is defined as

$$A(x, y, z) = (xy)z - x(yz)$$

or, in tensor notation,

$$A(\mu) = \mu \circ \big(\mu \otimes \mathrm{id} - \mathrm{id} \otimes \mu\big)$$

As a polynomial in μ, $A(\mu)$ is of degree 2.

Given μ and A, define $R = R(\mu, A)$ (of degree 1 in μ and in A):

$$R(x, y, z, t) = A(x, y, z)t + A(x, yz, t) + xA(y, z, t)$$
$$- A(x, y, zt) - A(xy, z, t)$$

[1] The prefix “non-” stands for “not required”.

We may plug μ into A to get $R(\mu, A(\mu))$, a cubic polynomial in μ. It vanishes identically:

Lemma 1. For any μ one has $R(\mu, A(\mu)) = 0$.

Proof: Simply expand $A(\mu)$:

$$\begin{aligned} R(x,y,z,t) &= A(x,y,z)t + A(x,yz,t) + xA(y,z,t) - A(x,y,zt) - A(xy,z,t) \\ &= ((xy)z)t - (x(yz))t \\ &\qquad + (x(yz))t - x((yz)t) \\ &\qquad\qquad + x((yz)t) - x(y(zt)) \\ &\qquad\qquad\qquad + x(y(zt)) - (xy)(zt) \\ &\qquad\qquad\qquad\qquad + (xy)(zt) - ((xy)z)t \end{aligned}$$

□

Remark 1. Lemma 1 means that in any algebra the associator A is subject to the relation

(1.1) $$A(x,y,z)t + A(x,yz,t) + xA(y,z,t) - A(x,y,zt) - A(xy,z,t) = 0$$

We call this identity the (4-variable) 5-term relation for the associator. Zorn calls it "Viereridentität" [17, (2), p. 125]. It appears also in Bruck and Kleinfeld 1951 [4, (2.3), p. 879]. More places are listed in my text [13] (Notes on the associator, [pdf]), see also the list of references for Artin's theorem in Section 4.

Remark 2. The proof of Lemma 1 reveals a 5-cycle of parenthesized expressions, with each consecutive pair related by an associator. In monoidal categories this sequence appears in the form of the pentagon axiom which requires that the composition of the 5 associativity constraints

$$((A \otimes B) \otimes C) \otimes D \simeq (A \otimes (B \otimes C)) \otimes D \simeq \cdots$$

is the identity. References:

- Mac Lane 1998 (1971) [10], Mac Lane 1963 [9], Joyal and Street 1993 [5].
- for the pentagon: [9, (3.5), p. 33], [5, p. 23]; see also my text [13] (Notes on the associator, [pdf]).
- for the hexagon (related to the 6-term identity [4, (2.2), p. 879]): [9, (4.5), p. 38], [5, p. 33].

The 5-cycle appears as well in a tensor categorical take on vector product algebras, see page 8 in my text [12] (On Vector Product Algebras, [pdf]).

§2. Alternativity

We keep the notations of the previous section.

The tensor A is alternating if

$$A(x,x,y) = 0, \qquad A(x,y,y) = 0$$

Equivalently, A factors as

$$A\colon V^{\otimes 3} \to \Lambda^3 V \to V$$

and one may write A as

$$A(x,y,t) = A(x \otimes y \otimes z) = A(x \wedge y \wedge z)$$

The algebra (V, μ) is called *alternative* if $A(\mu)$ is alternating.

For the commutator we write as usual

$$[x, y] = xy - yx$$

Here is an important computation:

Lemma 2. Let A be alternating and let $R = R(\mu, A)$. Then

$$\begin{aligned} R(x,y,z,t) - R(z,x,y,t) - R(x,z,t,y) = \\ A(x,yz,t) - A(x,z,t)y - zA(x,y,t) \\ + A(z,x,[y,t]) + A([z,x],y,t) \end{aligned}$$

Proof: By expansion, naturally. The 3 R-terms on the left yield $3 \cdot 5 = 15$ A-terms, more than half of them (namely 8) cancel because of alternativity. In detail:

$$\begin{aligned} &+ \underline{A(x,y,z)t}_1 + A(x,yz,t) + \underline{xA(y,z,t)}_2 - \underline{\underline{A(x,y,zt)}}_3 - \underline{\underline{A(xy,z,t)}}_4 \\ &- \underline{A(z,x,y)t}_1 - \underline{\underline{A(z,xy,t)}}_4 - zA(x,y,t) + A(z,x,yt) + A(zx,y,t) \\ &- A(x,z,t)y - \underline{\underline{A(x,zt,y)}}_3 - \underline{xA(z,t,y)}_2 + A(x,z,ty) + A(xz,t,y) \end{aligned}$$

The 4 canceling pairs are underlined. In the bottom right corner the remaining 2-columns can be merged by commutators (using again that A is alternating). □

And here are important implications:

Corollary 3. Let (V, μ) be alternative. Then, with $A = A(\mu)$,

$$A(x,yz,t) - A(x,z,t)y - zA(x,y,t) = A(x,z,[y,t]) + A([x,z],y,t) \tag{2.1}$$

Proof: In Lemma 2 pass to the (alternating) associator $A(\mu)$ and apply the 5-term relation (Lemma 1). □

Corollary 4. Let (V, μ) be alternative. There exist a morphism

$$f \colon \Lambda^4 V \to V$$

such that, with $A = A(\mu)$,

$$\begin{aligned} f(x,z,y,t) &= A(x,yz,t) - A(x,z,t)y - zA(x,y,t) \\ &= A(x,z,[y,t]) + A([x,z],y,t) \end{aligned}$$

Proof: The two right hand sides coincide by (2.1) and define f as a 4-linear morphism. In the first line, each A-term is alternating in x, t. In the second, each A-term is alternating in x, z and alternating in y, t. Thus f is alternating in each of (z, x), (x, t), (t, y) and therefore is alternating in all 4 variables. □

Corollary 5. Let (V, μ) be alternative. Then, with $A = A(\mu)$,

$$A(x,yx,t) = xA(x,y,t) \tag{2.2}$$

$$A(x,yz,y) = A(x,z,y)y \tag{2.3}$$

Proof: In (2.1) just set $z = x$ resp. $t = y$. □

A convenient abbreviation in an alternative algebra is the notation

$$xyx = (xy)x = x(yx)$$

It is unambiguous because of $A(x, y, x) = A(x \wedge y \wedge x) = 0$.

Corollary 6 (Moufang identities). Let (V, μ) be alternative. Then

$$(xyx)t = x(y(xt)) \tag{2.4}$$

$$x(yzy) = ((xy)z)y \tag{2.5}$$

$$(xt)(yx) = x(ty)x \tag{2.6}$$

Proof: Using alternativity, one reformulates (2.2), (2.3) in appropriate ways and expands:

$$0 = A(x, yx, t) + xA(y, x, t) = (x(yx))t - x(\underline{(yx)t}) + x\big(\underline{(yx)t} - y(xt)\big)$$

$$0 = A(x, yz, y) + A(x, y, z)y = (\underline{x(yz)})y - x((yz)y) + \big((xy)z - \underline{x(yz)}\big)y$$

$$0 = A(x, t, yx) - xA(t, y, x) = (xt)(yx) - x(\underline{t(yx)}) - x\big((ty)x - \underline{t(yx)}\big)$$

□

Remark 3. Corollary 4 is contained in Bruck and Kleinfeld 1951 [4, Lemma 2.1, p. 880]. Our proof is different. (A proof of [4, Lemma 2.1, p. 880] is completed with Corollary 9.)

Identity (2.2) appears in Zorn [17, p. 142]. Perhaps it goes back to Artin (see Section 4).

Moufang [11, p. 419] uses (2.2) to derive what are called the Moufang identities. Corollary 6 and its proof just display Moufang's remarks.

Remark 4. The opposite of an alternative algebra is also alternative.

Zorn ("reziproke Aussage"), Moufang (taking inverses in alternative fields) and other authors use this fact to derive from an identity its opposite variant.

Note that (2.1) is self-opposite. Identities (2.2) and (2.3) are opposites of each other. As for the Moufang identites: (2.4) and (2.5) are opposites, (2.6) is self-opposite.

§3. More computations

Here is another computation.

Lemma 7. Let A be alternating and let $R = R(\mu, A)$. Then

$$\begin{aligned} R(x,y,z,t) + R(t,z,y,x) = {} & [A(x,y,z),t] + [x, A(y,z,t)] \\ & + A(x,[y,z],t) - A(x,y,[z,t]) - A([x,y],z,t) \end{aligned}$$

Proof: Expansion of the R-terms yields the terms

$$\begin{aligned} & + \underline{A(x,y,z)t}_1 + \underline{\underline{A(x,yz,t)}}_3 + \underline{xA(y,z,t)}_2 - \underline{\underline{A(x,y,zt)}}_4 - \underline{\underline{A(xy,z,t)}}_5 \\ & + \underline{A(t,z,y)x}_2 + \underline{\underline{A(t,zy,x)}}_3 + \underline{tA(z,y,x)}_1 - \underline{\underline{A(t,z,yx)}}_5 - \underline{\underline{A(tz,y,x)}}_4 \end{aligned}$$

which combine by commutators (using that A is alternating). □

Corollary 8. Let (V, μ) be alternative. Then, with $A = A(\mu)$,

$$f(x,y,z,t) = A(x,[y,z],t) + [x, A(y,z,t)] - [t, A(x,y,z)] \tag{3.1}$$

Proof: Immediate from Corollary 4 (second description of f). □

Corollary 9.

$$3f(x,y,z,t) = [x, A(y,z,t)] - [y, A(z,t,x)] + [z, A(x,y,t)] - [t, A(x,y,z)] \tag{3.2}$$

Proof: Take (3.1) after the permutation of variables $(x,t) \leftrightarrow (y,z)$,

$$f(y,x,t,z) = A(y,[x,t],z) + [y, A(x,t,z)] - [z, A(y,x,t)]$$

add that to (3.1), use the alternativity of f (and A) and again Corollary 4. □

Remark 5. Equation (3.2) is [4, (2.8), p. 880].

Remark 6. Consider the right hand side of (3.2) without commutator brackets:

$$B(x,y,z,t) = xA(y,z,t) - yA(z,t,x) + zA(x,y,t) - tA(x,y,z)$$

Since A is 3-alternating, it is immediate that B is 4-alternating (use for instance the bijection $C_4 \times \Sigma_3 \to \Sigma_4$).

Let B^T denote the opposite of B. One finds that $B + B^T$ (the sign is correct!) equals the right hand side of (3.2). Thus we get

$$3f = B + B^T$$

Remark 7. As for opposites: For now we confine ourselves with Remark 4. A precise discussion of opposites involves the notion of polynomial functors.

Let

$$\tau_n\colon V^{\otimes n} \to V^{\otimes n}$$

$$\tau_n(x_1 \otimes \cdots \otimes x_n) = x_n \otimes \cdots \otimes x_1$$

be the transpose. For $n = 2$ this is the switch involution $\tau = \tau_2$:

$$\tau(x \otimes y) = y \otimes x$$

Note that the equation of Lemma 7 can be written as

(3.3) $$R \circ (\mathrm{id} + \tau_4) = R(A, c)$$

where

$$c\colon V^{\otimes 2} \to V$$

$$c = \mu \circ (\mathrm{id} - \tau)$$

is the commutator.

Remark 8. We find the presentation and proofs of Lemma 2, Lemma 7 appealing as they show in detail how alternativity plays out.

Remark 9. This is work in progress.

Lemma 7 was found a couple of days after Lemma 2. Maybe one can fiddle a bit to simplify even further.

The true nature of $R = R(A, \mu)$ for alternating A is not yet clear to me.

Remark 10. The tensor $R(A, \mu)$ has an underlying cyclic 5-symmetry which can be made precise using a rotation:

There is a well known dictionary from parenthesized expressions to rooted planar binary trees. In this picture expressions like $A(x, y, z)t$ are presented by a tree with one normal node and one node of valency 4 (3 inputs, 1 output). If one passes to unrooted trees, the trees given by the 5 terms of R are permuted by a rotation of order 5 (with angle 144°).

Maybe one can incorporate that to understand better the tensor $R(A, \mu)$, but I have no precise idea.

§4. Artin's theorem on alternative algebras

See Zorn [17, Satz 4 (Satz von Artin), p. 127].

Theorem. An alternative algebra generated by 2 elements is associative.

Proofs can be found in Zorn 1931 [17, Satz 4, p. 127] and other places: Schafer 1966 [15, Theorem 3.1, p. 29], Kurosh 1962 [8, § 7, p. 264], Kurosh 1965 (1962) [7, § 7, p. 243], Bourbaki 1970 [1, Proposition 1, p. A III.173], Bourbaki 1974 (1970) [2, Proposition 1, p. 612]. The 5-term relation (1.1) can be found there as well.

We plan to give a proof based on the linearization of identity (2.2). The argument we have in mind follows Schafer's 1966 book.

We conclude with a quote from the introduction of Zorn 1931 [17]. It explicitly attributes Artin's theorem to Artin. Perhaps also identities (1.1) and (2.2) go back to Artin in some way.

> (. . .) Herr ARTIN, der diese Arbeit veranlaßt hat, hat auch einen Teil der Ergebnisse, wie die Existenz des Einheitselements und die Reduzibilität in einfache Systeme unter der Annahme, daß das halbeinfache System über einem Grundkörper der Charakteristik Null endlich ist, bewiesen, ferner den schönen Satz, daß ein aus zwei Elementen erzeugtes alternatives System assoziativ ist, mit der interessanten Folgerung: Ein nullteilerfreies alternatives System mit endlich vielen Elementen ist Galoisfeld. Darüber hinaus bin ich ihm für viele Hinweise und Ratschläge zu Dank verpflichtet.
>
> § 1 zeigt das alternative Rechnen und bringt im wesentlichen den erwähnten Satz von ARTIN.
>
> (. . .)

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