CHAIN LEMMA FOR SPLITTING FIELDS OF SYMBOLS

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Introduction

In this paper p always is a prime, k is a field with $\operatorname{char} k \neq p$ and $K_n^{\mathrm{M}} k$ denotes Milnor's n-th K-group of k. Let

$$h_{(n,p)}\colon K_n^{\mathrm{M}} k/p \to H^n_{\text{ét}}(k, \mu_p^{\otimes n}),$$
$$\{a_1, \dots, a_n\} \mapsto (a_1, \dots, a_n).$$

be the norm residue homomorphism.

Voevodsky's theorem. V. Voevodsky announced in October 1996 the following theorem:

Theorem (Voevodsky). *Let p be a prime and let m be a natural number.*

Suppose that for every subfield $k \subset \mathbb{C}$ containing the p-th roots of unity and for every sequence of elements $a_1, \dots, a_n \in k^*$, $2 \leq n \leq m$, there exists a smooth projective variety X over k such that:

(V_1) $\{a_1, \dots, a_n\}_{k(X)} = 0$ *in* $K_n^{\mathrm{M}} k(X)/p$.

(V_2) *X has dimension $d = p^{n-1} - 1$.*

(V_3) *In the category of Chow motifs over $k(X)$ with $\mathbb{Z}_{(p)}$-coefficients there exist an effective object Y such that*

$$X_{k(X)} = L^{\otimes 0} \oplus (Y \otimes L).$$

Here L denotes the Tate motive.

(V_4) *The characteristic number $s_d\big(X(\mathbb{C})\big) \in \mathbb{Z}$ is not divisible by p^2.*

(V_5) *The sequence*

$$\coprod_{x \in X_{(1)}} K_2 \kappa(x) \xrightarrow{d} \coprod_{x \in X_{(0)}} K_1 \kappa(x) \xrightarrow{N_X} K_1 k$$

is exact. Here $N_X = \sum \mathrm{N}_{\kappa(x)|k}$.

Then for all fields k with $\operatorname{char} k \neq p$ one has:

(BK) *The Bloch-Kato conjecture holds in weight m and $\operatorname{mod} p$, i.e., the norm residue homomorphism $h_{(m,p)}$ is bijective.*

(S) *For $n \leq m$, for elements $a_1, \dots, a_n \in k^*$, and for a smooth projective variety X satisfying (V_1)–(V_5), the sequence*

$$\coprod_{x \in X_{(0)}} K_1 \kappa(x) \xrightarrow{N} K_1 k \xrightarrow{b \mapsto (a_1, \dots, a_n, b)} H^{n+1}_{\text{ét}}(k, \mu_p^{\otimes (n+1)})$$

is exact.

Interpretation of Voevodsky's conditions. Condition (V_1) identifies X as a splitting variety of the symbol $\{a_1, \dots, a_n\}$ $(\operatorname{mod} p)$. The dimension in (V_2) is presumably the minimal dimension of a *generic* splitting variety of a mod p-symbol of length n.

Condition (V_3) can be interpreted as follows. For a somewhat reasonable generic splitting variety X of a symbol, one should expect that for any field F/k, over which the symbol splits, the variety X_F is "split". For example, Brauer-Severi varieties have the property that they become rational over any field which split the corresponding algebra. One may think of condition (V_3) as a kind of weak "rationality condition". In fact, if $X_{k(X)}$ is rational, then (V_3) holds. Condition (V_3) has the following consequence for the Chow group of zero cycles:

$(\mathrm{V}_3)'$ For any field F/k with $X(F) \neq \varnothing$ the degree map

$$\deg\colon\ \mathrm{CH}_0(X_F) \to \mathbb{Z}$$

is bijective.

In the discussions of this paper we will need from condition (V_3) only this implication.

Condition (V_4) is the most exciting one. Recall the complex cobordism ring

$$\Omega_*^{\mathbb{C}} = \mathbb{Z}[x_1, x_2, \ldots], \quad \dim_{\mathbb{R}}(x_i) = 2i.$$

Let $P \in \Omega_{2d}^{\mathbb{C}}$ be a polynomial and write P as

$$P = a_P x_d + P'(x_1, \ldots, x_{d-1}).$$

The Chern numbers of (complex) dimension d define homomorphisms $\Omega_{2d}^{\mathbb{C}} \to \mathbb{Z}$ and for s_d one has

$$s_d(P) = \begin{cases} \pm p a_P & \text{if } d = p^m - 1 \text{ for some prime } p \text{ and } m > 0, \\ \pm a_P & \text{else.} \end{cases}$$

Therefore condition (V_4) is equivalent to saying that $X(\mathbb{C})$ is a ring generator of the complex cobordism ring mod p.

The numbers s_d receive a further important role over non-closed fields because of the following "degree formula". For a variety X we denote by $\mathbf{D}(X) \subset \mathbb{Z}$ the ideal generated by the degrees of the closed points on X.

- Let $f\colon X \to Y$ be a morphism of irreducible smooth proper varieties of dimension $d = p^m - 1$. Then

$$\frac{s_d(X)}{p} = \deg f \cdot \frac{s_d(Y)}{p} \bmod \mathbf{D}(Y).$$

Note that over an algebraically closed field this formula is vain, because then always $\mathbf{D}(Y) = \mathbb{Z}$.

In the case of curves ($p = 2$, $m = 1$) this formula is a consequence of the Riemann-Roch theorem. Also for $m = 1$ and arbitrary p one can derive it from Riemann-Roch. In the general case one uses at the moment algebraic cobordism theory. `ref`

Using resolution of singularities, the formula implies that the element

$$\widetilde{s}(X) = \frac{s_d(X)}{p} \in \mathbb{Z}/\mathbf{D}(X)$$

is a birational invariant of X (defined for proper smooth irreducible varieties of dimension $p^m - 1$). In the case $m = 1$ the element $\widetilde{s}(X)$ can also be described in terms of the Todd number. `ref`

Concerning splitting varieties of symbols, the formula has the following striking consequence.

If p is a prime, we call a field k *p-special*, if $\operatorname{char} k \neq p$ and if k has no finite field extensions of degree prime to p. A splitting variety X of a symbol mod p is called *p-generic* if for any splitting field F the variety X_F has a point of degree prime to p. In other words, X is p-generic if it is a generic splitting variety with respect to p-special splitting fields.

- Let Y be a *generic* splitting variety of a nontrivial symbol $\{a_1, \dots, a_n\} \bmod p$ of dimension $d = p^{n-1} - 1$. Moreover let X be a variety satisfying conditions (V_1), (V_2) and (V_4). Then Y satisfies (V_4). Moreover X is a p-generic splitting variety.

Here is a sketch of proof: Since Y is a generic splitting field, there exist a rational morphism $f\colon X \to Y$. Using resolution of singularities, we may assume that f is regular. The degree formula tells that Y satisfies (V_4) and $\deg f$ is prime to p. $\square$

Condition (V_5) can be thought of as a generalization of the classical Hilbert's Satz 90. Assuming that X is a generic splitting variety, then condition (V_5) computes essentially the kernel of the norm map

$$\bigoplus_F F^* \to k^*.$$

with F running through the finite splitting fields of the symbol $\{a_1, \dots, a_n\}$. In the case $n = 1$ this amounts to Hilbert's Satz 90 for a cyclic field extension L/k of degree p, i.e., the exactness of

$$K_1 L \xrightarrow{1-\sigma} K_1 L \xrightarrow{N_{L/k}} K_1 k.$$

For the purpose of this introduction we use the notation

$$A(X) = \operatorname{coker}\Big(\coprod_{x \in X_{(1)}} K_2 \kappa(x) \xrightarrow{d} \coprod_{x \in X_{(0)}} K_1 \kappa(x) \Big).$$

The group $A(X)$ is also known as $H^d(X, \mathcal{K}_{d+1})$. It is a birational invariant of X, see [11]. Let

$$N_X \colon A(X) \to k^*$$

be the norm map. Condition (V_5) means that N_X is injective.

Computations of the characteristic numbers s_d. In the case $n = 2$ the splitting varieties one considers are Brauer-Severi varieties for algebras of degree p. In this case (V_4) is well known: $s_d(\mathbb{P}^d) = d + 1$.

In the case $p = 2$ one uses quadrics. Again the numbers s_d are well known in this case. In fact it is not difficult to compute s_d for a smooth projective hypersurface
ref of any degree.

For $n \geq 3$ or $p \geq 3$, the computation of the numbers s_d of candidates for generic splitting varieties seems to be more subtle. Here the only successful technique I am aware of is to use a theorem of Conner and Floyd.

ref The principle is as follows. Suppose p is odd. In this case the numbers s_d are Pontryagin numbers and oriented bordism invariants. Let Z be an oriented differentiable manifold of (real) dimension $2d$ with $d = p^m - 1$ and suppose that there is a fixed point free G-action with $G = (\mathbb{Z}/p)^m$. Then the Conner-Floyd theorem tells that Z lies in a certain ideal of the oriented cobordism ring, and by the structure of this ideal it follows that

$$s_d(Z) = 0 \bmod p^2.$$

Now suppose we have two manifolds X, Y of dimension $2d$ with G-actions having only finitely many fixed points, and suppose that the sets of fixed point together with the G-actions on the tangent spaces are isomorphic (in this case we say that X and Y are G-fixed point equivalent). Then one can form the multifold connected sum of X and $-Y$, along their fixed points, and obtains a manifold Z with a fixed point free G-action. It follows that

$$s_d(X) = s_d(Y) \bmod p^2.$$

Now in many cases of candidates X of generic splitting varieties it is possible to find an appropriate G-action and a G-variety Y such that X and Y have only finitely many fixed points, are G-fixed point equivalent, and such that the number $s_d(Y)$ is comparatively easy to compute. Then we know also the number $\widetilde{s}(X)$. The varieties Y we consider here usually don't split the symbols.

The settled cases.

Theorem. *Let n be a natural number and let p be a prime and suppose that n, p are subject to one of the following conditions*:

1. *$n = 2$ and p arbitrary.*
2. *$p = 2$ and $n \geq 2$.*
3. *$p = 3$ and $n = 3$, or 4.*

Moreover let k be a field with $\operatorname{char} k \neq p$ containing the p-th roots of unity and let $a_1, \ldots, a_n \in k^*$.

Then there exists a smooth projective variety X over k satisfying (V_1)–(V_5).

In the case $n = 2$ one takes here the Brauer-Severi variety corresponding to the symbol. The only subtle condition is here (V_5), which has been proved in [6].

In the case $p = 2$ one takes here the projective quadric with form $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \perp \langle -a_n\rangle$ corresponding to the symbol. Again the only subtle condition is here (V_5), which has been proved in [12].

The essentially new cases are $p = 3$ and $n = 3$, 4. Here we used exceptional Jordan algebras and their relation with 3-symbols mod 3 (see [13]) and some cubic tricks. The varieties X considered are desingularizations of the varieties

$$\{\, [x, t] \in \mathbb{P}(J \oplus k) \mid N_J(x) = bt^3 \,\}$$

where J is either a central simple algebra of degree 3 or an exceptional Jordan algebra. The varieties Y mentioned above used for the computations of the characteristic numbers are (diagonal) cubic hypersurfaces. Details will appear in [11]. ref

The program of this paper. The most difficult one among the conditions (V_3)–(V_5) seemed to be for some time (V_5). Indeed for $n = 2$ or $p = 2$ conditions (V_3), (V_4) are almost trivial to check for the mentioned varieties. Also for $p = 3$ and $n = 3, 4$, condition (V_3) is easy to check and (V_4) can be surely settled without using the Conner-Floyd theorem. Namely the varieties in question have a lot of additional structure, e.g., big automorphism groups of type $\mathrm{A}_2 + \mathrm{A}_2$ or E_6. The Conner-Floyd theorem was used here more for convenience.

In this paper we propose some arguments in order to attack (V_4) and (V_5). The basic idea of the approach is to reduce the problems for symbols of arbitrary length to symbols of length 2, that is to algebras.

If the program is successful, then one could conclude the following:

- Suppose that X has a point of degree p, satisfies (V_2), and $(\mathrm{V}_3)'$, and that X is a p-generic splitting variety of $\{a_1, \ldots, a_n\} \bmod p$.

  Then (V_4) and the following condition $(\mathrm{V}_5)'$ holds for X.

$(\mathrm{V}_5)'$ After base change to any p-special field one has: For every $w \in A(X)$ there exist a point $x \in X_{(0)}$ of degree p and $\alpha \in \kappa(x)^*$ such that w is represented by

$$[\alpha] \in \coprod_{x \in X_{(0)}} K_1\kappa(x).$$

We mention that if X has a point of degree p, is not difficult to show that $(\mathrm{V}_3)'$ + $(\mathrm{V}_5)' \Rightarrow (\mathrm{V}_5)$.

If things work out, it is likely that Voevodsky's conditions can be settled for $n \leq 3$ and $p = 3$, $n \leq 5$. In these cases one can deal with condition (V_3) using the fact that one has generic splitting splitting varieties which are forms of the algebraic group $\mathrm{SL}_1(p)$, or one can use some cubic phenomena.

In order to be not misleading, we would like to point out that to settle condition (V_3) in general, it still seems important to have available generic splitting fields (their existence is known for $n \leq 3$), not only p-generic splitting fields.

At the moment there are two major gaps in our program.

The first is that we need a G-equivariant resolution of singularities (in characteristic 0). Namely our candidates X will be not smooth, but proper and with a G-action having only finitely many fixed points in the smooth locus. Moreover there will be always G-varieties Y which are G-fixed point equivalent to X and whose number s_d is not too difficult to compute. Having G-equivariant resolution of singularities (away from the fixed points) available, we could assume that X is smooth and the birational invariant $\widetilde{s}(X)$ could be computed.

The second gap is hopefully only due to my minor knowledge of the topological Morava K-theories. We need a generalization of the degree formula which involves morphisms to products of independent norm varieties. In the case $(n,p) = (2,2)$, this generalization is basically the following:

- Let $C_1, \ldots, C_r$ be conics such that their classes in $\mathrm{Br}(k)$ are linearly independent. Let $Y = C_1 \times \cdots \times C_r$, let X be a smooth proper variety of dimension r and let $f\colon X \to Y$ be a morphism. Then

$$\mathrm{td}(X) = \deg f \bmod 2.$$

Here td is the Todd number.

ref This fact can be proven using Riemann-Roch.

Multiplicative functions. Before describing our program in more detail, we discuss an exemplary application of the degree formula to problems related with symbols.

In an ideal (but boring) world one could associate to a symbol $\{a_1, \ldots, a_n\} \bmod p$ a kind of algebra A of dimension p^n and a form $N_A\colon A \to k$ of degree p such that $\{a_1, \ldots, a_n, b\} = 0$ if and only if b is a value of N_A (this should hold over all fields F/k and for $b \in F^*$). The set of values $\neq 0$ of such a form N_A would necessarily form a group and would be stable under transfer from finite extensions.

The existence of such forms N_A for all n and p seems to much to ask for. However one can hope that there exist rational functions N_A on some varieties A which are very near to a rational variety (by this we mean e.g., that A has many rational points, $\mathrm{CH}_0(A) = \mathbb{Z}$, etc.).

Indeed, if there exists for a symbol $\{a_1, \ldots, a_n\} \bmod p$ a generic splitting variety X of dimension p^{n-1} then one could try to take for A the canonical bundle

$A \to S^p(X)$ of commutative degree p algebras (=transfer of the structure sheaf of X^p) and for N_A the norm map of this algebra bundle. This approach leads hopefully in the end to multiplicative functions as desired. However, the singularities of $S^p(X)$ cause some troubles, and at the moment one does not know how to deal with this approach.

Later we discuss the chain lemma construction of Kummer structures. The Kummer structures are supposed to give parametrizations of the Kummer subfields of the (non-existent) algebras A over p-special fields.

One can go even further and try to define multiplicative functions ad hoc as follows. Let $a_1, \ldots, a_n \in k^*$ and define rational functions Φ_n in p^n variables inductively by:

$$\Phi_0(t) = t^p,$$

$$\Phi_n(x_0, x_1, \ldots, x_{p-1}) = \Phi_{n-1}(x_0) \prod_{i=1}^{p-1} \bigl(1 - a_n \Phi_{n-1}(x_i)\bigr)^{(-1)^i}.$$

It is easy to see that $\{a_1, \ldots, a_n, b\} = 0$ for any value $b \neq 0, \infty$ of Φ_n.

For the functions Φ_n I expect:

- Let $b \in k^*$ with
$$\{a_1, \ldots, a_n, b\} = 0 \text{ in } K^{\mathrm{M}}_{n+1} k/p.$$
Then there exist a finite extension F/k of degree prime to p such that b is over F a value of the form Φ_n.

The reasoning for this is the following. Let

$$X = \{\Phi_n(x) = b\}.$$

On the variety X one can define a certain G-action ($G = \mu_p^n$) for which one can apply the above mentioned method to compute the invariant $\widetilde{s}(X)$ (using equivariant resolution of singularities). It seems very probable that $\widetilde{s}(X) \neq 0$ (if $\{a_1, \ldots, a_n, b\} \neq 0$). If we assume the existence of a generic splitting variety of $\{a_1, \ldots, a_n, b\}$ of dimension $\dim X = p^{n-1} - 1$, then, by the degree formula, there would exist a generically finite map $X \to Y$ of degree prime p.

The definition of the functions Φ_n is suggested by looking at small values for n and p.

For $n = 1$ it is an elementary exercise to show that any element

$$\lambda = \alpha_0 + t\alpha_1 + \cdots + t^{p-1}\alpha_{p-1} \in k[t]/(t^p - a_1)$$

can be written in the form

$$\lambda = \beta_0 \prod_{i=1}^{p-1} (1 + t\beta_i)^{(-1)^i}$$

over an extension of degree

$$u = \left[\left(\frac{p-1}{2}\right)!\right]^2.$$

One can do a similar game for $n = 2$.

For $p = 2$ it is not difficult to see that the function Φ_n is rationally equivalent to the Pfister form $\langle\!\langle a_1, \ldots, a_n \rangle\!\rangle$.

For $p = 3$ the functions Φ_1, Φ_2, Φ_3 are rationally equivalent to the norm forms of a Kummer extension, of a central simple algebra, and of an exceptional Jordan algebra, respectively. One can show that for Φ_4 the set of values forms a group. Details concerning $p=3$ will appear in [11].

One may ask for a bound $\epsilon_{n,p}$ such that the extension F needed to represent b by Φ_n can be always chosen with $[F:k]$ dividing $\epsilon_{n,p}$. Playing around assuming the existence of the "ideal algebras A" one is led to guess

$$\epsilon_{n+1,p} = u^{p^n}\epsilon_{n,p}^p$$

which gives

$$\epsilon_{n,p} = \left[\left(\frac{p-1}{2}\right)!\right]^{2np^{(n-1)}}$$

Here p is odd and if $n=1$ one should assume that the field k is infinite. Note that $\epsilon_{n,3}=1$.

Concerning the choice of functions like Φ_n (disregarding the bounds $\epsilon_{n,p}$) one has a lot of freedom. For example one may replace the exponents $(-1)^i$ by any integer prime to p, for Φ_1 and Φ_2 one can choose the norm forms of Kummer extensions resp. cyclic algebras, etc.

Multiplicativity and the norm principle. In order to settle (V_5) one may assume that the ground field is p-special (if X has a point of degree p and $(\mathrm{V}_3)'$ holds). Then one can replace (V_5) by the following two statements (M) and (N).

Let $X' \subset X_{(0)}$ denote the set of points of degree p.

(M) ("Multiplicativity") For $x,\ y \in X'$ and $\alpha \in \kappa(x)^*$, $\beta \in \kappa(y)^*$, there exist $z \in X'$ and $\gamma \in \kappa(z)^*$ such that

$$[\alpha] + [\beta] = [\gamma] \in A(X)$$

(N) ("Norm principle for degree p extensions") For a field F/k of degree p, a point $x \in X'_F$, and $\alpha \in \kappa(x)^*$, there exist points $x_1,\ \ldots,\ x_r \in X'$ and elements $\alpha_i \in (\kappa(x_i)\otimes F)^*$, $i=1,\ \ldots,\ r$ such that

$$N_{X_F}([\alpha]) = \prod_{i=1}^{r} N_{X_F}([\alpha_i]).$$

The names of these conditions have their origin in the corresponding properties of the reduced norms of algebras and for Pfister forms: their set of values form a group, and they are stable under transfer.

If X has a point of degree p, it is not difficult to check $(\mathrm{V}_3)' + (\mathrm{M}) + (\mathrm{N}) \Rightarrow (\mathrm{V}_5)$.

In the following we try to sketch our approach to settle (M) and (N) for the case $n=3$. For the varieties X we choose any smooth proper variety birationally isomorphic to

$$\{\,[x,t] \in \mathbb{P}(A\oplus k) \mid N_A(x) = a_3t^p\,\}$$

where $A = A(a_1,a_2)$ is the degree p algebra with class (a_1,a_2) and N_A is its reduced norm. The variety X is a generic splitting variety of $\{a_1,a_2,a_3\}$, see [5]. Moreover one can show that $(\mathrm{V}_3)'$ holds for X.

The "chain lemma approach". Condition (M) holds for Brauer-Severi varieties, see [6]. Therefore, in order to settle (M) in general, it would be enough to show that given x, y as in (M), there exist a Brauer-Severi variety Y and a morphism $Y \to X$ such that x, y are rationally equivalent to points which lift to Y.

For this it would suffice to find $b_1,\ b_2,\ b_3 \in k^*$ such that

(1) $\{a_1, a_2, a_3\} = \{b_1, b_2, b_3\}$.
(2) $\kappa(x)$ and $\kappa(y)$ split $\{b_1, b_2\}$.

Then we would take for Y the Severi-Brauer variety for $A(b_1, b_2)$ and the existence of a morphism $Y \to X$ would be granted by the fact that X is a generic splitting variety.

The only way I know to produce elements b_i with (1) (besides for $p = 2, 3$) is to use some elementary changes in the entries of the symbols. By an elementary change we mean here a permutation of the entries or replacing a_3 by a norm from $A(a_1, a_2)$. In the case $p = 2$ one knows that all triples (b_1, b_2, b_3) with (1) can be obtained in such a way ("Chain equivalence of Pfister forms"). For $p > 2$ there was until recently not much known about a chain lemma for symbols even for $n = 2$. In order to define the $H^3(\mathbb{Z}/3)$-invariant for F_4, J-P. Serre proposed some years ago to settle the chain lemma for exceptional Jordan algebras (i.e., if $J(a_1, a_2, a_3) \simeq J(b_1, b_2, b_3)$ then (b_1, b_2, b_3) can be obtained from (a_1, a_2, a_3) by elementary changes). This was eventually done by Petersson and Racine [9], at least up to replacing the ground field by a quadratic extension. The analysis of the Petersson-Racine arguments led to our approach to (M) to be discussed in this paper.

The basic idea is the following.

One tries to show the following "chain lemma for splitting fields of symbols":

(3) There exist a natural number m with the following property:
Let $F = k(\sqrt[p]{b})$ be a splitting field of $\{a_1, a_2, a_3\}$. Then after a sequence of m elementary changes of the type

$$a_3 \mapsto a_3 N_{A(a_1,a_2)}(x) \quad \text{and} \quad a_2 \mapsto a_2 N_{A(a_1,a_3)}(x)$$

(so keeping the first entry a_1 fixed) one may arrange that F splits $\{a_1, a_2\}$.

Suppose (3) holds. Let

$$F = k(\sqrt[p]{b}),\ F' = k(\sqrt[p]{b'})$$

be two splitting fields of $\{a_1, a_2, a_3\}$. After some elementary changes we could assume that F splits $\{a_1, a_2\}$. Then there exists $c \in k^*$ with $\{a_1, a_2\} = \{b, c\}$, so that we may assume $b = a_1$. Applying (3) now to F', we see that there exist a subsymbol $\{b, d\}$ split by F and by F'. This way (2) would be settled.

The number m of necessary elementary changes is of course not important for the last argument. However it plays an important role in our picture. If we do m elementary changes (alternating in the second and third entry and normalizing the parameters x by say $\mathrm{trace}(x) = 1$) we end up with a triple $\big(a_1, a_2'(r), a_3'(r)\big)$ with functions $a_2'(r)$, $a_3'(r)$ on a rational variety R_m of dimension $m(p^2 - 1)$. If (3) is true, then R_m would give a parameter space for all Kummer splitting fields of degree p, by associating to $r \in R$ the (finite-dimensional) family of Kummer subfields of $A\big(a_1, a_2'(r)\big)$ ($a_2'(r)$ is the last changed entry).

It is obvious from the construction that the function $a_2'(r)$ on R_m parametrizes some Kummer splitting fields of degree p. Since X is generic, we have a rational map

$$f_m\colon R_m \mapsto X^p/(\mathbb{Z}/p)$$

with $\mathbb{Z}/p$ acting by cyclic permutation. A key observation of the "chain lemma approach" is that for $m = p$ we get

$$\dim R_p = p(p^2-1) = \dim X^p/(\mathbb{Z}/p).$$

We are led to compute the degree of $f = f_p$.

Our general conjecture is that the degree of f is prime to p. This would give (3) over p-special fields, which is all what we need.

Variants of the map f can be constructed for all n and p. Concerning explicit computations of the degree, we have looked only at the cases $(n,p) = (2,2)$ (here $\deg f = 1$) and $(n,p) = (2,3)$ (here $\deg f = 2$). I am sure that $\deg f$ can be computed explicitely for $p = 2$ and arbitrary n (probably $\deg f = 1$) and $(n,p) = (3,3)$ (probably $\deg f = 2$) and hopefully for $n = 2$ and arbitrary p (perhaps $\deg f = (p-1)!$?). The Petersson-Racine arguments show that for $(n,p) = (2,3)$ and $(n,p) = (3,3)$ the map f_4 has a rational section. In particular (3) is true in these cases for $m = 4$.

In this paper we show that $\deg f$ is prime to p for $n \leq 3$, assuming equivariant resolution of singularities and the degree formula for varieties of dimension $p^n - 1$. The method is to construct from f a morphism

$$\widehat{f}\colon \widehat{R} \to Z$$

of splitting varieties of the symbol $\{a_1, a_2, a_3, t\}$ over $k(t)$ and to compute the number $\widetilde{s}(\widehat{R})$ using the fixed point method mentioned above.

So even for $n = 2$, where (3) amounts to the chain equivalence for classical algebras, we use at the moment algebraic cobordism.

Kummer structures. We have tried to axiomatize the families R of Kummer splitting fields, which led us to the notion of "Kummer structures". A (split) Kummer structure of degree p and weight n consists of a smooth proper variety R of dimension $\dim R = p^n - p$ together with a line bundle L on R and form δ of degree p on L. Moreover there is a G-action ($G = \mu_p^n$) on (R, L, δ) with some nice properties. It is part of the axioms, that if we twist (R, L, δ) with

$$\mathfrak{a} = \big((a_1), \ldots, (a_n)\big) \in H^1_{\text{ét}}(k, G) = k^*/(k^*)^p \times \cdots \times k^*/(k^*)^p.$$

to $(R_\mathfrak{a}, L_\mathfrak{a}, \delta_\mathfrak{a})$, then for $r \in R$ the Kummer extensions $\kappa(r)(\sqrt[p]{\delta_\mathfrak{a}})$ split the symbol $\{a_1, \ldots, a_n\}$. From a Kummer structure (R, L, δ) of weight n one may easily construct splitting varieties X of symbols of length $n+1$ of the correct dimension $d = p^n - 1$. The axioms of Kummer structures guarantee (modulo equivariant resolution of singularities) that X has always a good number s_d.

This way the construction of norm varieties is reduced to the construction of Kummer structures. For this we use the chain lemma approach indicated above.

On the norm principle. We now discuss the basic idea to approach condition (N). We keep the notations used in the formulation of (N).

Given α, we have to solve for some r the equation

$$N_{X_F}([\alpha]) = \prod_{i=1}^{r} N_{X_F}([\alpha_i])$$

in the α_i.

Suppose that conversely the α_i are given and we have to solve the equation in α. Then we would be done by the multiplicativity property (M).

We write

$$F = k[t]/(t^p - c)$$

and $\gamma = \sqrt[p]{c}$.

We make the following Ansatz:

- The elements $\alpha_i \in \kappa(x_i) \otimes F$ are of the form

$$\begin{aligned} \alpha_1 &= \beta_1, \\ \alpha_i &= 1 + \gamma\beta_i, \quad i \geq 2. \end{aligned}$$

  with $\beta_i \in \kappa(x_i)$.

Let $A \to S^p(X)$ be the canonical bundle of commutative degree p algebras and let N_A be its norm map. Then our equation reads as

$$N_{A_F}(\alpha) = N_A(\beta_1) \prod_{i=2}^{r} N_{A_F}(1 + \gamma\beta_i), \quad \alpha \in A_F,\, \beta_i \in A.$$

Let $\lambda \in F$ be a generic element. We consider the two varieties

$$\begin{aligned} Z &= R_{F/k}\Big(\{\, \alpha \in A_F \mid N_{A_F}([\alpha]) = \lambda \,\}\Big), \\ T_r &= \{\, (\beta_1, \dots, \beta_r) \in A^r \mid N_A(\beta_1) \prod_{i=2}^{r} N_{A_F}([1 + \gamma\beta_i]) = \lambda \,\}. \end{aligned}$$

Here $R_{F/k}$ denotes the restriction of varieties.

As mentioned before, the multiplicativity property (M) gives a morphism

$$g_r : T_r \to Z.$$

(Actually, the preceding arguments would give (M) only over p-special fields, so g_r would exist only over a covering $\widetilde{T}_r \to T_r$ of degree prime to p, but we neglect this problem at this place.)

A calculation of the dimensions gives

$$\begin{aligned} \dim Z &= p(p^n - 1), \\ \dim T_r &= rp^n - p \end{aligned}$$

So for $r = p$ the dimensions are equal and we are led to ask for the degree of $g = g_p$. If $\deg g$ is prime to p we would have settled (N) over p-special fields.

If one looks at the cases $n = 1, 2$ and uses for N_A the norm form of a Kummer extension resp. a cyclic algebra, one finds indeed $(\deg g, p) = 1$ (for the natural choice of g).

To compute $\deg g$ we may pass to any field extension, in particular we can make base change $k \to k' = F$. Then $F = k' \times \cdots \times k'$ is split and the element λ becomes a tuple of p generic elements $t_1, \dots, t_p$ of k'. Moreover Z becomes

$$Z' = X_1 \times \cdots \times X_p$$

where X_i is a norm variety for the symbol $\{a_1, \dots, a_n, t_i\}$.

Now having available all the tools considered before to settle (M) and furthermore an appropriate generalization of the degree formula applicable to maps to Z', one should be able to conclude $(\deg g, p) = 1$ from the computation of a certain characteristic number of T_p.

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