THE METRIC OF A $(n+2)$-GON IN AFFINE n-SPACE

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Contents

Summary

For $n+2$ points in affine n-space in general position there is a canonical metric (unique up to a similarity factor) such that complementary faces of the $(n+2)$-gon are orthogonal.

We describe this metric in terms of a sum over all $(n-1)$-dimensional faces (see Proposition 2) and discuss some of its properties.

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1. Preliminaries

1.1. **Affine spaces.** An *affine space* over a field F consists of a set A, a vector space V over F and an operation

$$\begin{aligned} A \times V &\to A \\ (a,v) &\mapsto a+v \end{aligned}$$

which makes A into a principal homogeneous V-space.

By a *dilation* of A we understand an automorphism of A whose linear part (in $\mathrm{GL}(V)$) is a scalar multiple of the identity.

Affine spaces can be presented as follows. Let W be a vector space over F and let

$$\varepsilon\colon W \to F$$

be an epimorphism. Then $A = \varepsilon^{-1}(1)$ is an affine space with underlying vector space $V = \varepsilon^{-1}(0)$. The pair (W,ε) is uniquely determined by $(A,V,+)$ up to unique isomorphism.

There are natural exact sequences

$$0 \to \Lambda^{i+1}V \xrightarrow{\iota} \Lambda^{i+1}W \xrightarrow{\varepsilon_i} \Lambda^i V \to 0$$

where ε_i is characterized by

$$\varepsilon_i(a \wedge \omega) = \omega$$

with $a \in A$ and $\omega \in \Lambda^i V$.

1.2. **Symmetric bilinear forms.** Let V be a finite-dimensional vector space and let L be a 1-dimensional vector space. We consider symmetric bilinear maps

$$\Phi\colon V \times V \to L$$

For dual vector spaces we use the notation $V^\vee = \mathrm{Hom}(V,F)$.

The map

$$\begin{aligned} \widehat{\Phi}\colon V &\to \mathrm{Hom}(V,L) = V^\vee \otimes L \\ \widehat{\Phi}(v)(w) &= \Phi(v,w) \end{aligned}$$

is called the *duality* associated to Φ.

Let $n = \dim V$. The *determinant* of Φ is defined as

$$\det(\Phi) = \Lambda^n \widehat{\Phi} \in \mathrm{Hom}\big(\Lambda^n V, \Lambda^n(V^\vee \otimes L)\big) = (\Lambda^n V)^{\otimes -2} \otimes L^{\otimes n}$$

1.3. **The orientation module of a finite set.** Let M be a finite set. The *orientation module* of M is defined as

$$\mathcal{O}_M = \Lambda^{|M|}\mathbf{Z}[M]$$

where $\mathbf{Z}[M]$ is the free abelian group on M. The group $\mathcal{O}_M$ is free of rank 1 and the natural action of the group of permutations of M on $\mathcal{O}_M$ is given by the signum. Clearly $\mathcal{O}_M \otimes \mathcal{O}_M \equiv \mathbf{Z}$.

1.4. **M-gons.** Let A be an affine space (with notations V, $\varepsilon\colon W \to F$ as above) and let M be a finite set. By an *M-gon* in A we understand a map

$$x\colon M \to A$$
$$i \mapsto x_i$$

By a r-gon we understand a M-gon for some set M with $|M| = r$, usually $M = \{1, \dots, r\}$.

For $v \in V$ we denote by $x + v$ the translated M-gon, defined by

$$(x+v)_i = x_i + v$$

with $i \in M$.

For a M-gon $x \in A^M$ and a subset $I \subset M$ we denote by

$$A_I(x) \subset A$$

the affine span of the points x_i with $i \in I$. Its underlying vector space

$$V_I(x) \subset V$$

is generated by the elements $x_i - x_j$ with $i, j \in I$.

We denote by

$$W^M \to \Lambda^M W$$
$$x \mapsto \wedge x$$

be the universal alternating map. Note that

$$\Lambda^M W \equiv \Lambda^{|M|} W \otimes \mathcal{O}_M$$

1.5. $(n+2)$**-gons.** Let n be an integer, let A be an affine space with $\dim A = n$, let M be a set with $|M| = n+2$ and let $x \in A^M$ be a M-gon in A. For $i \in M$ we define the element

$$\theta_i(x) \in \Lambda^n V \otimes \mathcal{O}_M$$

by

$$\theta_i(x) = \varepsilon_n \left(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n+1)}\right) \otimes \left(i \wedge \sigma(1) \wedge \cdots \wedge \sigma(n+1)\right)$$

where

$$\sigma\colon \{1, \dots, n+1\} \to M \setminus \{i\}$$

is any bijection.

The element $\theta_i(x)$ has the standard interpretation of a volume element for the i-th face of x. It is invariant under translations:

$$\theta_i(x+v) = \theta_i(x)$$

This can be easily deduced for instance from

$$x_1 \wedge \cdots \wedge x_r = x_1 \wedge (x_2 - x_1) \wedge \cdots \wedge (x_r - x_1)$$

with $x_i \in A$.

We say that x is *nondegenerate* if $\theta_i(x) \neq 0$ for all $i \in M$.

Lemma 1. *Let $\dim A = n$, let M be a set with $|M| = n+2$ and let $x \in A^M$ be a M-gon in A. Then*

$$\sum_{i \in M} \theta_i(x) x_i = 0$$

in $\Lambda^n V \otimes \mathcal{O}_M \otimes W$.

In particular, by applying ε, one gets

$$\sum_{i\in M}\theta_i(x)=0$$

in $\Lambda^n V\otimes\mathcal{O}_M$.

Proof. Basic multilinear algebra: For $x_i\in W$ the expression

$$\sum_{i=1}^{n+2}(-1)^i(x_1\wedge\cdots\widehat{x_i}\cdots\wedge x_{n+2})x_i$$

is alternating in the x_i. Since $n+2>\dim W$, it vanishes. □

For a subset $I\subset M$ we write

$$\theta_I(x)=\prod_{i\in I}\theta_i(x)\in\big(\Lambda^n V\otimes\mathcal{O}_M\big)^{\otimes|I|}$$

Moreover for $i\in M$ we write

$$\rho_i(x)=\theta_{M\setminus\{i\}}(x)$$

2. The main statements

Let $n=\dim A$, let M be a set with $|M|=n+2$ and let $x\in A^M$ be a M-gon in A.

Notations V, $\varepsilon\colon W\to F$ are as in the previous section.

2.1. **The (dual) metric of a $(n+2)$-gon.** This subsection contains a simple definition of the metric. I found it only after typing the other parts of the text, which turned out to be much more complicated than necessary.

Let

$$L=\Lambda^n V\otimes\mathcal{O}_M$$

Let a, $b\in A$. One considers the tensor

$$\Omega_x=\sum_{i\in M}(x_i-a)\otimes(x_i-b)\otimes\theta_i(x)\in V\otimes V\otimes L$$

Proposition 1. (1) *The element Ω_x does not depend on the choice of a, $b\in A$.*
(2) *The element Ω_x is invariant under switch involution on $V\otimes V$.*
(3) *Let $I\subset M$ and let $f\in V^\vee$. If $f\big(V_I(x)\big)=0$, then*

$$(f\otimes\mathrm{id}_V\otimes\mathrm{id}_L)(\Omega_x)\in V_{M\setminus I}(x)\otimes L$$

Proof. (1) follows from $\sum_i\theta_i(x)x_i=0$ (cf. Lemma 1). (2) is obvious. As for (3), we may assume $I\neq\varnothing$, M. Choose $a\in A_I(x)$ and $b\in A_{M\setminus I}(x)$. Then

$$(f\otimes\mathrm{id}_V\otimes\mathrm{id}_L)(\Omega_x)=\sum_{i\in M\setminus I}f(x_i-a)(x_i-b)\otimes\theta_i(x)$$

□

The tensor Ω_x defines a symmetric duality

$$\widehat{\Omega}_x\colon V^\vee\to V\otimes L$$

Consider its $(n-1)$-fold exterior power

$$\Lambda^{n-1}\widehat{\Omega}_x\colon\Lambda^{n-1}V^\vee\to\Lambda^{n-1}V\otimes L^{\otimes(n-1)}$$

Since $\Lambda^{n-1}V = V^\vee \otimes \Lambda^n V$, it defines a symmetric bilinear form

$$\Phi_x \colon V \times V \to L^{\otimes(n+1)}$$

We call the form Φ_x the *metric of x*.

In the following we give some other descriptions.

2.2. **The metric of a $(n+2)$-gon.** We consider the following symmetric bilinear map on $\Lambda^2 W$ with values in an appropriate 1-dimensional vector space:

$$\varphi_x \colon \Lambda^2 W \times \Lambda^2 W \to \big(\Lambda^n V \otimes \mathcal{O}_M\big)^{\otimes(n+1)}$$
$$\varphi_x(\alpha,\beta) = \sum_{\substack{I\subset M\\ |I|=n-1}} \theta_I(x)\varepsilon_n\big(\alpha\wedge(\wedge\, x|_I)\big)\varepsilon_n\big(\beta\wedge(\wedge\, x|_I)\big)$$

This is to be read as follows. The product $\alpha\wedge(\wedge\, x|_I)$ is an element of

$$\Lambda^2 W \wedge \Lambda^I W = \Lambda^2 W \wedge \Lambda^{|I|} W \otimes \mathcal{O}_I = \Lambda^{n+1} W \otimes \mathcal{O}_I$$

Thus $\varepsilon_n\big(\alpha\wedge(\wedge\, x|_I)\big)$ is an element of $\Lambda^n V\otimes\mathcal{O}_I$ and since $\mathcal{O}_I\otimes\mathcal{O}_I\equiv\mathbf{Z}$ we have

$$\varepsilon_n\big(\alpha\wedge(\wedge\, x|_I)\big)\varepsilon_n\big(\beta\wedge(\wedge\, x|_I)\big) \in \big(\Lambda^n V\big)^{\otimes 2} = \big(\Lambda^n V\otimes\mathcal{O}_M\big)^{\otimes 2}$$

Lemma 2. *If $\alpha\in\Lambda^2 V$ or $\beta\in\Lambda^2 V$, then $\varphi_x(\alpha,\beta)=0$.*

Proof. Suppose $\alpha\in\Lambda^2 V$. Fix $a\in A$.

For $z\in A^{n-1}$ one has

$$\begin{aligned}\varepsilon_n\big(\alpha\wedge(\wedge\, z)\big) &= \varepsilon_n\big(\alpha\wedge z_1\wedge\cdots\wedge z_{n-1}\big)\\ &= \varepsilon_n\big(\alpha\wedge z_1\wedge(z_2-z_1)\wedge\cdots\wedge(z_{n-1}-z_1)\big)\\ &= \varepsilon_n\big(\alpha\wedge a\wedge(z_2-z_1)\wedge\cdots\wedge(z_{n-1}-z_1)\big)\\ &= \sum_{i=1}^{n-1}(-1)^{i+1}\varepsilon_n\big(\alpha\wedge a\wedge z_1\wedge\cdots\widehat{z_i}\cdots\wedge z_{n-1}\big)\end{aligned}$$

Moreover

$$\varepsilon_n\big(\beta\wedge(\wedge\, z)\big) = (-1)^{i+1}\varepsilon_n\big(\beta\wedge z_i\wedge z_1\wedge\cdots\widehat{z_i}\cdots\wedge z_{n-1}\big)$$

Hence

$$\varepsilon_n\big(\alpha\wedge(\wedge\, z)\big)\varepsilon_n\big(\beta\wedge(\wedge\, z)\big) =$$
$$\sum_{i=1}^{n-1}\varepsilon_n\big(\alpha\wedge a\wedge z_1\wedge\cdots\widehat{z_i}\cdots\wedge z_{n-1}\big)\varepsilon_n\big(\beta\wedge z_i\wedge z_1\wedge\cdots\widehat{z_i}\cdots\wedge z_{n-1}\big)$$

This shows that

$$\varepsilon_n\big(\alpha\wedge(\wedge\, x|_I)\big)\varepsilon_n\big(\beta\wedge(\wedge\, x|_I)\big) =$$
$$\sum_{\substack{i\in I\\ K=I\setminus\{i\}}}\varepsilon_n\big(\alpha\wedge a\wedge(\wedge\, x|_K)\big)\varepsilon_n\big(\beta\wedge x_i\wedge(\wedge\, x|_K)\big)$$

We get

$$\begin{aligned}\varphi_x(\alpha,\beta) &= \sum_{\substack{I\subset M\\ |I|=n-1}} \theta_I(x)\varepsilon_n\big(\alpha\wedge(\wedge x|_I)\big)\varepsilon_n\big(\beta\wedge(\wedge x|_I)\big)\\ &= \sum_{\substack{K\subset M\\ |K|=n-2}}\sum_{i\in M\setminus K} \theta_I(x)\varepsilon_n\big(\alpha\wedge a\wedge(\wedge x|_K)\big)\varepsilon_n\big(\beta\wedge x_i\wedge(\wedge x|_K)\big)\end{aligned}$$

For $i\in K$ one has $x_i\wedge(\wedge x|_K)=0$. Hence we may extend the range of i to all of M and get

$$\varphi_x(\alpha,\beta) = \sum_{\substack{K\subset M\\ |K|=n-2}}\sum_{i\in M} \theta_K(x)\theta_i(x)\varepsilon_n\big(\alpha\wedge a\wedge(\wedge x|_K)\big)\varepsilon_n\big(\beta\wedge x_i\wedge(\wedge x|_K)\big)$$

This vanishes because of $\sum_i \theta_i(x)x_i=0$ (cf. Lemma 1). □

By Lemma 2, the form φ_x is essentially a form on $\Lambda^2W/\Lambda^2V\simeq V$. We describe this as follows:

Proposition 2. *Let a, $b\in A$. The form*

$$\Phi_x\colon V\times V\to \big(\Lambda^nV\otimes\mathcal{O}_M\big)^{\otimes(n+1)}$$

$$\Phi_x(v,w) = \sum_{\substack{I\subset M\\ |I|=n-1}} \theta_I(x)\varepsilon_n\big(v\wedge a\wedge(\wedge x|_I)\big)\varepsilon_n\big(w\wedge b\wedge(\wedge x|_I)\big)$$

does not depend on the choices of a and b.

We call the form Φ_x the *metric of x*.

Lemma 3. *Let i, j, k, $\ell\in M$ be distinct elements. Then*

(1) $$\Phi_x(x_i-x_j,x_k-x_\ell)=0$$

(2) $$\Phi_x(x_i-x_j,x_i-x_k)=-\rho_i(x)$$

(3) $$\Phi_x(x_i-x_j,x_i-x_j)=-\rho_i(x)-\rho_j(x)$$

Proof. It is easy to see that (1) and (3) follow from (2).

As for (2) we choose $a=b=x_i$ in the definition of Φ_x. Then

$$\Phi_x(x_i-x_j,x_i-x_k) = \sum_{\substack{I\subset M\\ |I|=n-1}} \theta_I(x)\varepsilon_n\big(-x_j\wedge x_i\wedge(\wedge x|_I)\big)\varepsilon_n\big(-x_k\wedge x_i\wedge(\wedge x|_I)\big)$$

Every summand vanishes except for $I=M\setminus\{i,j,k\}$ and for this term the last two factors amount to $-\theta_j(x)\theta_k(x)$. This shows (2). □

Remark 1. One may check (3) also as follows. We have with $a=b=x_j$

$$\Phi_x(x_i-x_j,x_i-x_j) = \sum_{\substack{I\subset M\\ |I|=n-1}} \theta_I(x)\varepsilon_n\big(x_i\wedge x_j\wedge(\wedge x|_I)\big)^2$$

Here every summand vanishes except for $I = M \setminus \{i,j,h\}$ with $h \neq i,j$. Hence

$$\begin{aligned}\Phi_x(x_i - x_j, x_i - x_j) &= \sum_{\substack{h \in M\setminus\{i,j\} \\ I = M\setminus\{i,j,h\}}} \theta_I(x)\varepsilon_n\big(x_i \wedge x_j \wedge (\wedge x|_I)\big)^2 \\ &= \sum_{\substack{h \in M\setminus\{i,j\} \\ I = M\setminus\{i,j,h\}}} \theta_I(x)\theta_h(x)^2 \\ &= \theta_{M\setminus\{i,j\}}(x) \sum_{h \in M\setminus\{i,j\}} \theta_h(x) \\ &= \theta_{M\setminus\{i,j\}}(x)(-\theta_i(x) - \theta_j(x))\end{aligned}$$

Here we have used $\sum_i \theta_i(x) = 0$ (cf. Lemma 1). Claim (3) is now immediate.

Remark 2. Condition (1) in Lemma 3 is equivalent to

$$V_I(x) \perp_{\Phi_x} V_{M\setminus I}(x)$$

for all subsets $I \subset M$.

Suppose x is nondegenerate. Then

$$V = V_I(x) + V_{M\setminus I}(x)$$

for all subsets $I \subset M$. Moreover the form Φ_x is determined by (1) in Lemma 3 up to multiplication by a scalar.

Remark 3. It is clear (for nondegenerate x) that if two lines of the M-gon are parallel, then its metric is isotropic.

2.3. **Second presentation of Φ_x.** Fix $h \in M$ and let $N = M \setminus \{h\}$. Then $|N| = n+1$. We assume that the family $(x_i)_{i\in N}$ is a basis for W.

Then there exists a symmetric bilinear map

$$\Psi_h \colon W \times W \to \big(\Lambda^n V \otimes \mathcal{O}_M\big)^{\otimes(n+1)}$$

with

$$\begin{aligned}\Psi_h(x_i, x_j) &= 0 \\ \Psi_h(x_i, x_i) &= -\rho_i(x)\end{aligned}$$

for $i,\, j \in N$, $i \neq j$.

Lemma 4. *One has*

$$\Psi_h(x_i, x_h) = \rho_h(x)$$

for $i \in M$.

Proof. Indeed,

$$\begin{aligned}\Psi_h(x_i, x_h) &= \Psi_h\big(x_i, -\theta_h(x)^{-1}\sum_{j\in N}\theta_j(x)x_j\big) \\ &= -\theta_h(x)^{-1}\theta_i(x)\big(-\rho_i(x)\big) = \rho_h(x)\end{aligned}$$

and

$$\begin{aligned}\Psi_h(x_h, x_h) &= \Psi_h\bigl(x_h, -\theta_h(x)^{-1}\sum_{j\in N}\theta_j(x)x_j\bigr)\\ &= -\theta_h(x)^{-1}\rho_h(x)\sum_{j\in N}\theta_j(x) = \rho_h(x)\end{aligned}$$

□

Corollary. *The form Φ_x is the restriction of Ψ_h to V. Moreover*

$$\Psi_h(V, x_h) = 0$$

and

$$W = V \oplus x_h F \tag{4}$$

is an orthogonal decomposition with respect to Ψ_h.

2.4. **Third presentation of Φ_x.** Let $|M| = n+2$ and let

$$a \in (F^\times)^M$$

be a M-family of invertible elements in F with

$$\sum_{i\in M} a_i = 0$$

Let $U = F[M]$ be the vector space with basis e_i, $i \in M$. Let

$$\Psi_a \colon U \times U \to F$$

be the symmetric bilinear form with

$$\Psi_a(e_i, e_j) = 0$$
$$\Psi_a(e_i, e_i) = a_i^{-1}$$

for i, $j \in M$, $i \neq j$.

The vector

$$z = \sum_{i\in M} a_i e_i \in U$$

is isotropic. Let us denote by $[z] \subset U$ the subspace generated by z. Note that $\Psi_a(e_i, z) = 1$ for $i \in M$. Hence for i, $j \in M$ one has $e_i - e_j \in [z]^\perp$. Since $z \neq 0$, these elements generate $[z]^\perp$.

Now put

$$W_a = U/[z]$$
$$V_a = [z]^\perp/[z]$$

Further let

$$\varepsilon\colon W_a \to F$$
$$\varepsilon\bigl(u + [z])\bigr) = \Psi_a(u, z)$$

and

$$A_a = \varepsilon^{-1}(1)$$

Then A_a is a n-dimensional affine space with underlying vector space V_a. Moreover,

$$x\colon M \to A_a$$
$$x_i = e_i + [z]$$

defines a M-gon x in A_a.

Let

$$\Phi_a\colon V_a \times V_a \to F$$
$$\Phi_a(u+[z], u'+[z]) = \Psi_a(u,u')$$

be the canonical form associated with Ψ_a and the isotropic vector z.

There is a canonical identification

$$\Lambda^n V_a \otimes \mathcal{O}_M \equiv [z] \otimes U/[z]^\perp \otimes \Lambda^n V_a \otimes \mathcal{O}_M \equiv \Lambda^{n+2} U \otimes \mathcal{O}_M \equiv F$$

given by

$$\overline{\alpha} \otimes (\wedge\sigma) \mapsto z \otimes e_i \otimes \overline{\alpha} \otimes (\wedge\sigma) \mapsto (z \wedge e_i \wedge \alpha) \otimes (\wedge\sigma), \quad (\wedge\sigma)\otimes(\wedge\sigma) \mapsto 1$$

with $\alpha \in \Lambda^n\big([z]^\perp\big)$ and $\sigma\colon \{1,\dots,n+2\} \to M$ a bijection.

With respect to this identification, one has

$$\theta_i(x) = a_i$$

and

$$(-a_1 \cdots a_{n+2})^{-1}\Phi_x \perp \mathcal{H} = \Psi_a \tag{5}$$

where $\mathcal{H}$ is a hyperbolic plane.

It is easy to see that every nondegenerate M-gon x in a n-dimensional affine space appears in this way from some

$$a \in (F^\times)^M$$

with

$$\sum_{i\in M} a_i = 0$$

Remark 4. For nondegenerate x this gives a very simple way to define Φ_x. The first definition of Φ_x via a sum over all $(n-1)$-dimensional faces works smoothly for all x and has its own appeal anyway. I don't know an urgent reason to consider the description of Φ_x via the form Ψ_h, $h \in M$—I used it at first to compute the determinant of Φ_x.

Remark 5. Suppose $\operatorname{char} F \neq 2$. Then a n-dimensional quadratic form Φ appears as Φ_x for some x if and only if

$$\Phi \perp \langle 1,-1\rangle \simeq -a_1\cdots a_{n+2}\langle a_1,\dots,a_{n+2}\rangle$$

for some $a_i \in F^\times$ with $\sum_{i=1}^{n+2} a_i = 0$. From this one sees that every similarity class of a n-dimensional quadratic form appears as the similarity class of the metric of a $(n+2)$-gon.

Remark 6. One may also consider twisted forms of $(n+2)$-gons. The setup would be to consider an etale algebra H of rank $n+2$ and a point $x\colon \operatorname{Spec} H \to A$. For nondegenerate x the quadratic form Φ_x would be of the form

$$\Phi_x \perp \langle 1,-1\rangle \simeq -N_{H/F}(a)T_{H/F}(\langle a\rangle)$$

for some $a \in H^\times$ with $T_{H/F}(a) = 0$.

2.5. **The determinant of Φ_x.** Here is the computation:

Lemma 5.
$$\det(\Phi_x) = \big(-\theta_M(x)\big)^{(n-1)} \in (\Lambda^n V)^{\otimes(n+2)(n-1)}$$

In particular we see that Φ_x is nondegenerate if and only if x is nondegenerate.

Proof. Since we have to check a polynomial identity in x, we may assume that x is nondegenerate. Then one may use the description of Φ_x in (5). But one may also use the orthogonal decomposition (4) which shows
$$\det(\Psi_h) = \det(\Phi_x)\Psi_h(x_h, x_h)$$
One has
$$\det(\Psi_h) = \theta_h(x)^{-2}\prod_{i\in N}\big(-\rho_i(x)\big) = \big(-\theta_M(x)\big)^{n-1}\rho_h(x)$$
and we are done by $\Psi_h(x_h, x_h) = \rho_h(x)$ (cf. Lemma 4). □

2.6. **The dual $(n+2)$-gon.** We assume that x is nondegenerate. Fix $a \in A$ and a basis element λ for $\big(\Lambda^n V \otimes \mathcal{O}_M\big)^{\otimes(n+1)}$. Then we get a M-gon
$$y\colon M \to V^\vee$$
$$y_i = \widehat{\Phi}_x(x_i - a)/\lambda$$
in the dual space $V^\vee$. We call it a *dual* M-gon of x. Dual M-gons of x are determined by x up to dilations (translations and scalar multiplications). They are characterized by
$$\langle y_i - y_j, x_k - x_\ell\rangle = 0$$
where i, j, k, $\ell \in M$ are distinct elements and where $\langle\ ,\ \rangle$ denotes the natural pairing $V^\vee \times V \to F$.

Dual M-gons determine the metric up to multiplication by a scalar.

2.7. **The involution of a $(n+2)$-gon.**

Proposition 3. *Let $n = \dim A$, let M be a set with $|M| = n+2$ and let $x \in A^M$ be a M-gon in A. Suppose that x is nondegenerate. Then there exists a unique involution τ_x of orthogonal type on* $\mathrm{End}(V)$ *such that*
$$\tau_x\big(\mathrm{Hom}\big(V, V_I(x)\big)\big) = \mathrm{Hom}\big(V/V_{M\setminus I}(x), V\big)$$
for each subset $I \subset M$.

It is clear that $\tau_x = \tau_y$ if x, y differ only by a dilation of A.

Proof. τ_x is the involution associated with the symmetric bilinear form Φ_x, i. e.,
$$\tau_x\big(\Phi_x(v) \otimes v'\big) = \Phi_x(v') \otimes v$$
□

Remark 7. Suppose $\mathrm{char}\, F = 2$, that n is even and that $\theta_i(x) = \theta_j(x) \neq 0$ for all i, $j \in M$. Then
$$\sum_{i\in M} x_i = 0$$
One finds that there is a unique *alternating* bilinear form $\Omega\colon V \times V \to F$ with
$$\Omega(x_i - x_j, x_i - x_k) = 1$$

for distinct elements i, j, $k \in M$.

This is the only case where an involution τ on $\mathrm{End}(V)$ with

$$\tau_x\big(\mathrm{Hom}\big(V, V_I(x)\big)\big) = \mathrm{Hom}\big(V/V_{M\setminus I}(x), V\big)$$

for each subset $I \subset M$ is possibly symplectic.

3. The case of a plane quadrangle

We now look at the case $\dim A = 2$ and $|M| = 4$.

In this case $V = V^\vee \otimes \Lambda^2 V$ and the duality $\widehat{\Phi}_x$ becomes a map

$$\widehat{\Phi}\colon V \to V^\vee \otimes (\Lambda^2 V \otimes \mathcal{O}_M)^{\otimes 3} = V \otimes (\Lambda^2 V)^{\otimes 2} \otimes \mathcal{O}_M$$

3.1. **On the lines of a plane quadrangle.** We assume that x is nondegenerate. Then $\widehat{\Phi}$ is an isomorphism and induces an involution

$$\sigma_x\colon \mathbf{P}(V) \to \mathbf{P}(V)$$

on the projective space of lines in V. It has the following interpretation: Let $M = \{i, j, k, h\}$. Then

$$\sigma_x([x_i - x_j]) = [x_k - x_h]$$

One may phrase this by saying that "the 6 lines of a plane quadrangle stand in involution". The converse is also true: If 6 points in $\mathbf{P}^1$ stand in involution, they are given by the lines of a quadrangle. (Note: An involution of $\mathbf{P}^1$ is determined by two pairs of points.)

3.2. **On the dual quadrangle.** Since $V = V^\vee \otimes \Lambda^2 V$ we may also speak about dual M-gons in V. They can be described as follows:

Given four general points x_1, x_2, x_3, x_4 in a 2-dimensional affine space, there exists another sequence of four points y_1, y_2, y_3, y_4 such that the line $x_i - x_j$ is parallel to the line $y_k - y_h$ for any permutation $ijkh$ of 1234. The y-tuple is uniquely determined by the x-tuple up to translation and scalar multiplication.

3.3. **Selfdual quadrangles?** It turns out that a nondegenerate quadrangle is never dual to itself (in characteristic different from 2).

What about nondegenerate quadrangles which become dual to itself after a permutation? It turns out that then there exists one side $x_i - x_j$ which is parallel to its opposite side $x_k - x_h$ and the permutation is $(ij)(kh)$. More specifically, let $A = V$, let v, $w \in V$ be linearly independent and let $c \in F^\times$. Then the quadrangle $(0, v, w, w + cv)$ is dual to $(v, 0, w + cv, w)$.

3.4. **The determinant.** Let x_1, x_2, x_3, x_4 be a nondegenerate plane quadrangle and let $a_i \in F^\times$ with

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$$

Then

$$\det(\Phi_x) = -a_1a_2a_3a_4$$

up to multiplication by a square.

Consider the case of a parallelogram. (This amounts to $a_1 = a_2 = -a_3 = -a_4$.) In this case the metric is hyperbolic; the two isotropic lines are given by the pairs of parallel sides.

Suppose that $F = \mathbf{R}$ (real numbers). The Φ_x is definite if and only if one of the points x_i lies inside the triangle formed by the other points x_j, x_k, x_ℓ.

3.5. **Orthocentric quadrangles.** Let us consider the case $F = \mathbf{R}$. Let x be a nondegenerate plane quadrangle and suppose that the metric Φ_x is definite. Then we have an Euclidean structure on A. With respect to this Euclidean structure, the quadrangle x is orthocentric, i. e., each point x_i is the orthocenter of the opposite triangle x_j, x_k, x_ℓ. A dual quadrangle is obtained from x by a rotation of $90°$.

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