CHEAT SHEET FOR PLANET ORBITS

Given real numbers $k > 0$ and P, Q with

$$L = PQ \neq 0, \qquad E = \frac{1}{2}Q^2 - \frac{k}{P} < 0$$

the solution to the Newton differential equation

$$\frac{d}{d\tau}\frac{dz}{d\tau} = -\frac{k}{|z|^2}\frac{z}{|z|}, \qquad z(0) = P, \qquad \frac{dz}{d\tau}(0) = iQ$$

is

$$z(s) = c + a\cos s + ib\sin s, \qquad \tau(s) = \sqrt{\frac{a}{k}}(as + c\sin s)$$

where

$$a = -\frac{k}{2E}, \qquad b = \frac{L}{\sqrt{-2E}}, \qquad c = P + \frac{k}{2E} \qquad (c^2 = a^2 - b^2)$$

$P = c+a$ is the longest or shortest distance to the sun, resulting in $c \geq 0$ resp. $c \leq 0$. One has

$$|z(s)| = a + c\cos s$$

and the polar ellipse equation of the orbit is

$$|z| = \frac{L^2/k}{1 + (1 + 2PE/k)\cos\theta} = \frac{b^2}{a - c\cos\theta} \qquad (z = |z|e^{i\theta})$$

The eccentricity and the semi-latus rectum of the ellipse are

$$\varepsilon = |1 + 2PE/k| = |c|/a, \qquad \Lambda = L^2/k = b^2/a$$

The constants of motion

$$L = z \times \frac{dz}{d\tau} = |z|^2\frac{d\theta}{d\tau} = \sqrt{\frac{k}{a}}b, \qquad E = \frac{1}{2}\left|\frac{dz}{d\tau}\right|^2 - \frac{k}{|z|} = -\frac{k}{2a}$$

satisfy

$$2EL^2 + k^2 \geq 0$$

The period time is

$$T = \frac{2\pi k}{(-2E)^{3/2}} = 2\pi\sqrt{\frac{a^3}{k}}$$