BASIC NOTES ON ELLIPSES AND KEPLER/NEWTON

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Date: September 20, 2026.

Introduction

The core of the solutions to the Newton differential equation for planet orbits can be formulated as follows:

Theorem. *Let* a, b, $c \in \mathbf{R}$ *be real numbers with*

$$a^2 = b^2 + c^2, \qquad a > 0,\, b \neq 0$$

and let

$$z\colon \mathbf{R} \to \mathbf{C} \setminus \{0\}$$
$$z(s) = c + a\cos s + ib\sin s$$

The map

$$t\colon \mathbf{R} \to \mathbf{R}$$
$$t(s) = as + c\sin s$$

is a diffeomorphism and z *satisfies the differential equation*

$$\frac{d}{dt}\frac{dz}{dt} = -a\frac{z}{|z|^3}$$

One formulation of the Newton/Kepler problem is to a fix positive real number $k > 0$ and ask for solutions $z(\tau)$ to

$$\frac{d}{d\tau}\frac{dz}{d\tau} = -\frac{k}{|z|^2}\frac{z}{|z|}$$

The theorem provides solutions by taking

$$\tau = \sqrt{\frac{a}{k}}t$$

These are in fact the solutions for negative energy and nonzero angular momentum (i.e., for closed planet orbits). See Section 3 for details.

The theorem is proved in Section 1.1 in an ad hoc fashion, without references to geometry or physics. The idea is to display the computations in the most straightforward and simpleminded way. They are easy enough and meant to serve as a basis for forthcoming considerations.

A remarkable fact is that the absolute value $|z|$ of z has the simple form

$$|z(s)| = r(s) = a + c\cos s$$

Section 1.2 considers the effective potential. It has been added for completeness and maybe skipped on a first reading. In Section 1.3 we consider the angle function $\theta(s)$ given by

$$z(s) = r(s)e^{i\theta(s)}$$

and derive the relevant basic properties.

Section 2 contains excursional remarks not needed later.

The application to planet orbits is discussed in Section 3. Here we assume basic knowledge of conic sections and celestial mechanics. Given the standard constants k, L, E (massless, assuming the planet has unit mass), the corresponding ellipse parameters are

$$a = -\frac{k}{2E}, \qquad b = \frac{L}{\sqrt{-2E}}, \qquad c = P + \frac{k}{2E} \qquad (c^2 = a^2 - b^2)$$

where P is the longest distance from the sun.

The text is inspired by Milnor 1983 [2, Appendix 1. Conic sections, p. 359].

It is mainly a collection of explicit computations which grew over time.

Missing are details on the velocity circle described in Milnor [2, 1. The Velocity Circle, p. 353ff]. For now the velocity circle is only briefly mentioned in Section 1.4.

1. The main computations

1.1. **Proof of the theorem.** We denote by $\mathcal{F}$ the ring of real-analytic functions

$$f\colon \mathbf{R} \to \mathbf{R}, \qquad s \mapsto f(s)$$

Let a, b, $c \in \mathbf{R}$ be real numbers with

$$a^2 = b^2 + c^2, \qquad a, b \neq 0$$

and consider the following elements of $\mathcal{F}$:

$$\begin{aligned} x &= c + a\cos s \\ y &= b\sin s \\ r &= a + c\cos s \\ t &= as + c\sin s \end{aligned} \tag{1.1}$$

Obviously

$$\frac{dt}{ds} = r$$

Lemma.

$$r^2 = x^2 + y^2 \tag{1.2}$$

Proof.

$$\begin{aligned} x^2 + y^2 &= (c + a\cos s)^2 + (b\sin s)^2 \\ &= c^2 + 2ac\cos s + a^2\cos^2 s + b^2\sin^2 s \\ &= a^2 + 2ac\cos s - a^2\sin^2 s + c^2 + (a^2 - c^2)\sin^2 s \\ &= a^2 + 2ac\cos s + c^2\cos^2 s = r^2 \end{aligned}$$

□

Let us stress that we do not assume $a > 0$. This assumption is unnecessary and would be rather disruptive.

Note that $r = a + c\cos s$ vanishes nowhere since $|a| > |c|$ because of $b \neq 0$. The function

$$r\colon \mathbf{R} \to \mathbf{R} \setminus \{0\}$$

has the same sign as a.

Since $dt/ds = r$ and $\lim_{s\to\pm\infty} t = \pm\infty$ (with a sign switch if $a < 0$), it follows that t is an analytic diffeomorphism $\mathbf{R} \to \mathbf{R}$ and one has

$$\frac{d}{dt} = \frac{1}{r}\frac{d}{ds}$$

Put

$$z = x + iy\colon \mathbf{R} \to \mathbf{C} \setminus \{0\}$$

Then

$$z(0) = c + a = r(0) = \frac{dt}{ds}(0), \qquad \frac{dz}{ds}(0) = ib \tag{1.3}$$

Clearly, $z(s) = Z(e^{is})$ and $r(s) = R(e^{is})$ where

$$\begin{aligned} &Z\colon \mathbf{S}^1 \to \mathbf{C} \setminus \{0\}, \qquad && Z(u + iv) = c + au + ibv \\ &R\colon \mathbf{S}^1 \to \mathbf{R} \setminus \{0\}, \qquad && R(u + iv) = a + cv \end{aligned}$$

with $\mathbf{S}^1 \subset \mathbf{C}$ the unit circle ($u^2 + v^2 = 1$). Moreover, (1.2) shows $|z/r| = 1$. Hence z/r and Z/R map to $\mathbf{S}^1$. For a related diagram see Section 1.3.

The $\times$-product of two complex numbers is defined here as

$$\times\colon \mathbf{C}\times\mathbf{C}\to\mathbf{R}$$
$$z\times w=\Im(\overline{z}w)$$
$$(x+iy)\times(u+iv)=xv-yu$$

Theorem 1.

$$r^3\frac{d}{dt}\frac{dz}{dt}=-az \tag{1.4}$$

$$z\times\frac{dz}{dt}=b \tag{1.5}$$

$$\frac{dz}{dt}\frac{d\overline{z}}{dt}-\frac{2a}{r}=-1 \tag{1.6}$$

Proof. One has

$$\frac{dz}{ds}=-a\sin s+ib\cos s$$

Equation (1.4) follows from

$$\begin{aligned}\frac{d}{dt}\frac{dz}{dt}&=\frac{1}{r}\frac{d}{ds}\left(\frac{1}{r}\frac{dz}{ds}\right)=\frac{1}{r}\left(\frac{1}{r}\frac{d}{ds}\frac{dz}{ds}-\frac{1}{r^2}\frac{dr}{ds}\frac{dz}{ds}\right)\\&=\frac{1}{r^3}\left[(a+c\cos s)(-a\cos s-ib\sin s)-(-c\sin s)(-a\sin s+ib\cos s)\right]\\&=\frac{1}{r^3}\left[-a^2\cos s-ac(\cos^2 s+\sin^2 s)-ib\sin s(a+c\cos s-c\cos s)\right]\\&=-\frac{1}{r^3}\left[a(a\cos s+c)+iab\sin s\right]=-\frac{a}{r^3}z\end{aligned}$$

Moreover, (1.5) follows from

$$\begin{aligned}rz\times\frac{dz}{dt}=z\times\frac{dz}{ds}&=(c+a\cos s+ib\sin s)\times(-a\sin s+ib\cos s)\\&=cb\cos s+ab\cos^2 s+ab\sin^2 s=cb\cos s+ab=rb\end{aligned}$$

and (1.6) from

$$\begin{aligned}r^2\frac{dz}{dt}\frac{d\overline{z}}{dt}=\frac{dz}{ds}\frac{d\overline{z}}{ds}&=a^2\sin^2 s+b^2\cos^2 s\\&=a^2(1-\cos^2 s)+(a^2-c^2)\cos^2 s\\&=a^2-c^2\cos^2 s\\r^2\left(\frac{2a}{r}-1\right)&=r\big(2a-(a+c\cos s)\big)\\&=(a+c\cos s)(a-c\cos s)=a^2-c^2\cos^2 s\end{aligned}$$ □

Remark 1. More general arguments use only (1.4) to show that the left hand sides of (1.5), (1.6) have vanishing derivatives, i.e., they are constant. Then (1.5), (1.6) follow by evaluating at $s=0$. In brief, with $a>0$ so that $r=|z|$,

$$\begin{aligned}(z\times\dot z)^{\cdot}&=\dot z\times\dot z+z\times\ddot z=0+0\\(\dot z\dot{\overline{z}})^{\cdot}&=\ddot z\dot{\overline{z}}+\dot z\ddot{\overline{z}}=2\Re(\ddot z\dot{\overline{z}})=-2a\Re(z\dot{\overline{z}})|z|^{-3}\\\left(\frac{1}{(z\overline{z})^{1/2}}\right)^{\cdot}&=-\frac{1}{2}\frac{\dot z\overline{z}+z\dot{\overline{z}}}{(z\overline{z})^{3/2}}=-\Re(\dot z\overline{z})|z|^{-3}\end{aligned}$$

Lemma.

$$ar - cx = b^2 \tag{1.7}$$

Proof. $a(a + c\cos s) - c(c + a\cos s) = a^2 - c^2 = b^2$ □

1.2. **Effective potential.** Let

$$U\colon a\mathbf{R}_{>0} \to \mathbf{R}, \qquad U(\rho) = \frac{b^2}{2\rho^2} - \frac{a}{\rho}$$

The analytic function $\rho \mapsto U(\rho)$ is a massless variant of the effective potential, see Section 3.5. It and its derivative may be composed with $r\colon \mathbf{R} \to a\mathbf{R}_{>0}$ resulting in elements of $\mathcal{F}$ (here the assumption $a > 0$ wouldn't be inconvenient). One has (using (1.7))

$$\begin{aligned} U(\rho)\Big|_{\rho=r} &= \frac{b^2}{2r^2} - \frac{a}{r} = -\frac{ar+cx}{2r^2} \\ \frac{dU(\rho)}{d\rho}\bigg|_{\rho=r} &= -\frac{b^2}{r^3} + \frac{a}{r^2} = c\frac{x}{r^3} \end{aligned} \tag{1.8}$$

The effective potential yields a second order differential equation "involving only r":

Proposition 1.

$$\frac{d}{dt}\frac{dr}{dt} = -\frac{dU(\rho)}{d\rho}\bigg|_{\rho=r} \tag{1.9}$$

Proof. One has

$$\frac{dr}{ds} = -c\sin s$$

and (1.9) follows from (1.8) and

$$\begin{aligned} \frac{d}{dt}\frac{dr}{dt} &= \frac{1}{r}\frac{d}{ds}\left(\frac{1}{r}\frac{dr}{ds}\right) = \frac{1}{r}\left(\frac{1}{r}\frac{d}{ds}\frac{dr}{ds} - \frac{1}{r^2}\frac{dr}{ds}\frac{dr}{ds}\right) = \frac{1}{r^3}\left(r\frac{d}{ds}\frac{dr}{ds} - \left(\frac{dr}{ds}\right)^2\right) \\ &= \frac{1}{r^3}\big[(a + c\cos s)(-c\cos s) - c^2\sin^2 s\big] = \frac{-c}{r^3}\big[a\cos s + c\big] = \frac{-cx}{r^3} \end{aligned}$$ □

Remark 2. Here is a more general way to prove (1.9) using only (1.4) and (1.5). We assume again $a > 0$ so that $r = |z|$. Let $\alpha = \dot z\overline{z}$. The claim follows from

$$\begin{aligned} \dot r &= \Big((z\overline{z})^{1/2}\Big)^{\cdot} = \frac{\dot z\overline{z} + z\dot{\overline{z}}}{2(z\overline{z})^{1/2}} = \frac{\Re(\alpha)}{r} \\ r^2\dot z\dot{\overline{z}} &= \alpha\overline{\alpha} = \Re(\alpha)^2 + \Im(\alpha)^2 = (r\dot r)^2 + (z \times \dot z)^2 = (r\dot r)^2 + b^2 \\ \ddot r &= -\frac{\Re(\alpha)}{r^2}\dot r + \frac{\Re(\ddot z\overline{z}) + \Re(\dot z\dot{\overline{z}})}{r} = -\frac{\dot r^2}{r} - \frac{a\Re(z\overline{z})}{r^4} + \frac{\dot z\dot{\overline{z}}}{r} = \frac{b^2}{r^3} - \frac{a}{r^2} \end{aligned}$$

From the second equation we record

$$\dot z\dot{\overline{z}} = \dot r^2 + \frac{b^2}{r^2} \tag{1.10}$$

This is (1.14) below, proved there using polar coordinates.

1.3. **The angle.** Define $\theta \in \mathcal{F}$ as the lift of $z/r \colon \mathbf{R} \to \mathbf{S}^1$ to the standard universal cover $\mathbf{R} \to \mathbf{S}^1$,

$$\begin{tikzcd} \mathbf{R} \arrow[r, "\theta"] \arrow[d, "s \mapsto e^{is}"'] \arrow[dr, "\frac{z}{r}"] & \mathbf{R} \arrow[d, "\varphi \mapsto e^{i\varphi}"] \\ \mathbf{S}^1 \arrow[r, "\frac{Z}{R}"] & \mathbf{S}^1 \end{tikzcd}$$

with $\theta(0) = 0$ (note that $(z/r)(0) = (c+a)/(a+c) = 1$).

Thus

$$z = re^{i\theta}, \qquad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{1.11}$$

Lemma.

$$r(a - c\cos\theta) = b^2 \tag{1.12}$$

Proof. Follows from $r(a - c\cos\theta) = ra - cx$ and (1.7). □

For a system (1.11) one has generally

$$rdr = xdx + ydy, \qquad r^2 d\theta = xdy - ydx$$

as can be seen from

$$\begin{aligned} \overline{z}dz &= (x - iy)(dx + idy) = (xdx + ydy) + i(xdy - ydx) \\ \overline{z}dz &= \overline{z}d(re^{i\theta}) = \overline{z}(dr + ird\theta)e^{i\theta} = r(dr + ird\theta) \end{aligned}$$

Proposition 2.

$$r^2 \frac{d\theta}{dt} = b \tag{1.13}$$

$$\frac{dz}{dt}\frac{d\overline{z}}{dt} = \left(\frac{dr}{dt}\right)^2 + \frac{b^2}{r^2} \tag{1.14}$$

Proof. (1.13) is a reformulation of (1.5). This follows from

$$z \times dz = \Im(\overline{z}dz) = r^2 d\theta$$

Moreover, (1.14) is evident from

$$dz = (dr + ird\theta)e^{i\theta}$$

and (1.13). □

1.4. **Velocity circle.** As for the velocity circle (cf. Milnor [2, 1. The Velocity Circle, p. 353ff]):

Lemma.

$$\frac{d}{d\theta}\frac{dz}{dt} = -\frac{a}{b}e^{i\theta}, \qquad \frac{dz}{dt} = \frac{a}{b}ie^{i\theta} - i\frac{c}{b}, \qquad \left|\frac{dz}{dt} + i\frac{c}{b}\right|^2 = \frac{a^2}{b^2}$$

Proof. TODO.

2. Remarks

The following comments will not be used in the next section.

2.1. **Diffeomorphisms.** Equations (1.13) or (1.5) show that

$$Z/R\colon \mathbf{S}^1 \to \mathbf{S}^1$$

and its lift to the universal cover

$$\theta\colon \mathbf{R} \to \mathbf{R}$$

are analytic diffeomorphisms.

2.2. **Power series.** To prove Theorem 1 and other computations, it would suffice to look at the Taylor expansions at 0, i.e., one may pass from $\mathcal{F}$ to the formal power series ring $\mathbf{R}((s))$.

Hence one may formulate and verify the statements at first in a purely algebraic way, without referring to convergence etc.

Obviously

$$\theta = \arcsin\left(\frac{y}{r}\right) = \arcsin\left(\frac{b\sin s}{a + c\cos s}\right) \qquad (\text{at } s = 0)$$

2.3. **Holomorphic extensions.** The function

$$Z\colon \mathbf{S}^1 \to \mathbf{C} \setminus \{0\}$$

(with $Z(e^{is}) = z(s)$) extends to a complex analytic function as follows:

$$Z\colon \mathbf{C} \setminus \{0\} \to \mathbf{C}$$
$$\begin{aligned} 2Z(w) &= 2c + a\left(w + \frac{1}{w}\right) + b\left(w - \frac{1}{w}\right) \\ &= 2c + (a+b)w + (a-b)\frac{1}{w} \end{aligned}$$

If $c = 0$, then

$$Z(w) = aw^{\pm 1}$$

(In this case the ellipse is a circle.)

If $c \neq 0$, then

$$Z(w) = \frac{a+b}{2w}(w - w_0)^2$$

where

$$w_0 = -\frac{a-b}{c} = -\frac{c}{a+b}$$

Note that

$$w_0^2 = \frac{a-b}{a+b}, \qquad w_0 + \frac{1}{w_0} = -2\frac{a}{c}$$

Likewise, the function

$$R\colon \mathbf{S}^1 \to \mathbf{R} \setminus \{0\}$$

extends to

$$R\colon \mathbf{C} \setminus \{0\} \to \mathbf{C}$$

$$\begin{aligned} 2R(w) &= 2a + c\left(w + \frac{1}{w}\right) \\ R(w) &= a && (c = 0) \\ R(w) &= \frac{c}{2w}(w - w_0)(w - w_0^{-1}) && (c \neq 0) \end{aligned}$$

Thus

$$Z/R\colon \mathbf{S}^1 \to \mathbf{S}^1$$

is the linear fractional

$$\begin{aligned} Z/R(w) &= w^{\pm 1} && (c = 0) \\ Z/R(w) &= \frac{-w_0^{-1} w + 1}{w - w_0^{-1}} && (c \neq 0) \end{aligned}$$

Remark 3. The linear fractional puts us into the world of PGL(2). This ought to be useful.

2.4. **Squares of central ellipses are planet orbits.** With $w = \lambda^2$ one gets the symmetric forms

$$2Z(\lambda^2) = (a+b)\left(\lambda + \frac{c}{a+b}\frac{1}{\lambda}\right)^2 = (a-b)\left(\frac{1}{\lambda} + \frac{c}{a-b}\lambda\right)^2$$

(note that $a \pm b \neq 0$ in at least one case).

The fact that

$$\frac{2Z(\lambda^2)}{a+b}, \qquad \frac{2Z(\lambda^2)}{a-b}$$

are the squares of

$$\lambda + \frac{c}{a+b}\frac{1}{\lambda}, \qquad \frac{1}{\lambda} + \frac{c}{a-b}\lambda$$

respectively, is employed in Arnold 1990 (1989) [1, Appendix 1. Proof that orbits are elliptic, p. 95]. (For $|\lambda| = 1$, the latter functions parameterize central ellipses, their squares are ellipses with the origin a focus point.)

3. Newton and Kepler

3.1. **Ellipses are solutions to the Newton equation.** We assume $a > 0$ (now this assumption is convenient). Then $r > 0$ and (1.2) shows

$$r = |z|$$

The function $z\colon \mathbf{R} \to \mathbf{C} \setminus \{0\}$ is a solution to the Newton equation (cf. (1.4))

$$\frac{d}{dt}\frac{dz}{dt} = -\frac{az}{|z|^3}$$

with initial conditions (cf. (1.3))

$$z(0) = c + a$$

$$\frac{dz}{dt}(0) = i\frac{b}{a+c} = i\frac{a-c}{b} = \pm i\sqrt{\frac{a-c}{a+c}}$$

The constants of motion are (cf. (1.5) and (1.6))

$$L = z \times \frac{dz}{dt} = b, \qquad E = \frac{1}{2}\left|\frac{dz}{dt}\right|^2 - \frac{a}{|z|} = -\frac{1}{2}$$

The first Kepler law says that a planet orbit is an ellipse with a focus point at the location of the sun. This is clear for z from $c^2 = a^2 - b^2$ and the definitions of x, y in (1.1) or from (1.12) (and standard definitions/descriptions for ellipses).

The second Kepler law holds by (1.13), since the area A has differential

$$dA = \frac{1}{2}r^2 d\theta$$

3.2. **Scaling the time.** Choose some positive constant $k \in \mathbf{R}$ and put

$$\tau = \sqrt{\frac{a}{k}}t = \sqrt{\frac{a}{k}}(as + c\sin s)$$

Then z is a solution to

$$\frac{d}{d\tau}\frac{dz}{d\tau} = -\frac{kz}{|z|^3}$$

with initial conditions

$$z(0) = P = c + a$$

$$\frac{dz}{d\tau}(0) = iQ = i\sqrt{\frac{k}{a}}\frac{b}{a+c}$$

The constants of motion (defined as above but with respect to τ) are

$$L = PQ = \sqrt{\frac{k}{a}}b, \qquad E = \frac{1}{2}Q^2 - \frac{k}{P} = -\frac{k}{2a}$$

The period time T with respect to τ is $T = \tau|_{s=2\pi}$ since the period time with respect to s is 2π. Hence

$$T = 2\pi\sqrt{\frac{a^3}{k}} = \frac{2\pi k}{(-2E)^{3/2}}$$

in accordance with the third Kepler law (cf. Milnor [2, (6), p. 356]).

3.3. **Solutions for given orthogonal initial position and velocity.** The orthogonality of position $z(0) = P$ and velocity $dz/d\tau(0) = iQ$ means that P is an orbit point with extremal distance to the sun (since $\langle z, z\rangle^{\cdot} = 2\langle z, \dot{z}\rangle$).

The following description of the solutions is as in Milnor [2, p. 360] but with the scale factor $\alpha = \sqrt{k/a}$ for the parameter s eliminated. More precisely, Milnor's function $t(s)$ equals $\tau(\alpha s)$ with c replaced by $-c$.

Given $k > 0$ and P, Q with

$$L = PQ \neq 0, \qquad E = \frac{1}{2}Q^2 - \frac{k}{P} < 0$$

the solution to the Newton equation

$$\frac{d}{d\tau}\frac{dz}{d\tau} = -\frac{k}{|z|^2}\frac{z}{|z|}, \qquad z(0) = P, \qquad \frac{dz}{d\tau}(0) = iQ$$

is

$$z(s) = c + a\cos s + ib\sin s, \qquad \tau(s) = \sqrt{\frac{a}{k}}(as + c\sin s)$$

where

$$a = -\frac{k}{2E}, \qquad b = \frac{L}{\sqrt{-2E}}, \qquad c = P + \frac{k}{2E}$$

The previous sections apply since a, $b \neq 0$ (obviously) and $a^2 = b^2 + c^2$, as can be seen from

$$E(c^2 - a^2) = EP^2 + Pk = \frac{P^2Q^2}{2} = \frac{L^2}{2} = -Eb^2$$

Note also that $E < 0$ implies $P > 0$ and $a > 0$.

The value $P = c + a$ is the longest or shortest distance to the sun, resulting in $c \geq 0$ resp. $c \leq 0$. One has

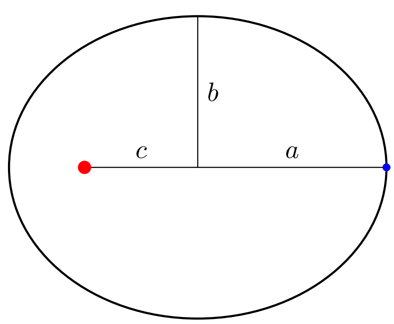
$$|z(s)| = a + c\cos s$$

and the polar ellipse equation of the orbit is (cf. (1.12))

$$|z| = \frac{b^2}{a - c\cos\theta} = \frac{L^2/k}{1 + (1 + 2PE/k)\cos\theta} \qquad (z = |z|e^{i\theta})$$

The eccentricity and the semi-latus rectum of the ellipse are

$$\varepsilon = |c|/a = |1 + 2PE/k|, \qquad \Lambda = b^2/a = L^2/k$$



The ellipse with $a : b : c = 5 : 4 : 3$

3.4. **Solutions for given non-zero angular momentum and negative energy.**

Lemma. *For real numbers k, L, E with $L \neq 0$ there exist real numbers P, Q with*

$$L = PQ, \qquad E = \frac{1}{2}Q^2 - \frac{k}{P}$$

if and only if

$$k^2 + 2EL^2 \geq 0 \tag{3.1}$$

Proof. The quadratic equation

$$\frac{1}{2}Q^2 - \frac{k}{L}Q - E = 0 \tag{3.2}$$

has a solution $Q \in \mathbf{R}$ if and only if its discriminant

$$\Delta = \left(\frac{k}{L}\right)^2 - 4\frac{1}{2}(-E) = \left(\frac{k}{L}\right)^2 + 2E$$

is non-negative: $\Delta \geq 0$. $\square$

It follows that for given $k > 0$, $L \neq 0$ and E subject to

$$-\frac{k^2}{2L^2} \leq E < 0$$

the ellipses described above in terms of P, Q are corresponding solutions to the Newton differential equation.

In terms of a, b, condition (3.1) resolves to $a^2 - b^2 \geq 0$.

Equality in (3.1) means that the orbit is a circle of radius

$$r = \frac{L^2}{k}$$

Otherwise the two solutions of the quadratic equation (3.2) are the velocities at the positions with longest resp. shortest distance from the sun.

3.5. **Effective potential.** Let generally $z = z(\tau)$ be a solution to

$$\frac{d}{d\tau}\frac{dz}{d\tau} = -\frac{kz}{r^3}$$

with $r = |z|$.

Then

$$L = z \times \frac{dz}{d\tau}$$

$$E = \frac{1}{2}\left|\frac{dz}{d\tau}\right|^2 - \frac{k}{r}$$

are constants of motion (for a proof see Remark 1 after Theorem 1).

We assume $L \neq 0$ (this excludes a straight fall into the sun).

Moreover let

$$U(r) = \frac{L^2}{2r^2} - \frac{k}{r} \qquad (r > 0)$$

be the effective potential. Then (for a proof see Remark 2 after Proposition 1)

$$\frac{d}{d\tau}\frac{dr}{d\tau} = -\frac{dU(r)}{dr}$$

(This fact has been added for completeness, but is not used in the following.)

Decomposing the kinetic part of the energy (cf. decomposition (1.10), (1.14)) yields

$$E = \frac{|\dot{z}|^2}{2} - \frac{k}{r} = \frac{\dot{r}^2}{2} + \frac{L^2}{2r^2} - \frac{k}{r} = \frac{\dot{r}^2}{2} + U(r)$$

Hence

$$E \geq U(r) \tag{3.3}$$

The function $U(r)$ takes its minimum $\min U$ at $r = r_0$ where

$$r_0 = \frac{L^2}{k}, \qquad \min U = -\frac{k^2}{2L^2}$$

This shows that condition (3.1) holds in general.

Inequality (3.3) together with $\lim_{r\to 0} U(r) = +\infty$ shows also that, for given L, E, the distance r cannot approach 0 (the planet stays away from the sun).

At a minimum one has $0 = \langle z, z\rangle^{\cdot} = 2\langle z, \dot{z}\rangle$ so that the velocity Q is orthogonal to the position P as seen from the sun.

For $E < 0$ the corresponding orbit is the described ellipse with initial values P, Q, in the obvious coordinate system.

In this case the distance r has also a maximum. This is not the case for $E \geq 0$ where the orbits are parabolas ($E = 0$) or hyperbolas ($E > 0$).

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