The focus form

Let V be a (locally) free R-module of rank 2. Then $V = V^\vee \otimes \Lambda^2 V$ where $V^\vee$ is the dual.

Denote by S^2V the second symmetric power and by $S_2V \subset V \otimes V$ the module of symmetric 2-tensors. Clearly $\operatorname{rank} S^2V = \operatorname{rank} S_2V = 3$.

Consider the bilinear product

$$\times\colon S_2V \times S_2V \xrightarrow{\wedge} \Lambda^2 S_2V = (S_2V)^\vee \otimes \Lambda^3 S_2V = S^2V \otimes \Lambda^2 V$$

and let

$$\det\colon S_2V \to (\Lambda^2 V)^{\otimes 2}$$

be the determinant.

Given h, $p \in S_2V$, put

$$q = h \times p \in S^2V \otimes \Lambda^2 V$$

$$f = \frac{\det(p)}{\det(B_q)}\, B_q \times h \in S^2 V$$

where

$$B\colon S^2V \to S_2V$$

is the symmetrization. Note that f is of degree 1 in p and of degree 0 in h.

The "focus scheme" ($q = 0$, $f = 1$) is finite of degree 4.

Example:

$$h = x^2 + y^2, \qquad p = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \qquad q = \frac{a^2 - b^2}{a^2b^2}\, xy \otimes (x \wedge y), \qquad f = \frac{x^2 - y^2}{a^2 - b^2}$$

The focus scheme consists of $(\pm\sqrt{a^2-b^2}, 0)$, $(0, \pm i\sqrt{a^2-b^2})$.

Example: For $R = \mathbf{C}$ and $(a,b) = (5,4)$ one gets the equations

$$\frac{x^2}{25} + \frac{y^2}{16} = 1, \qquad xy = 0, \qquad \frac{x^2 - y^2}{9} = 1$$

for the ellipse, its axes and the "focus hyperbola".

The image shows the configuration in the (real) (x,y)-plane and in the (x, iy)-plane. In the latter, the ellipse equation yields an hyperbola and the focus hyperbola becomes a circle.