On Galois Cohomology, Norm Functions and Cycles

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Galois Cohomology

p a prime
F a field, $\operatorname{char} F \neq p$

$\bar{F}$ a separable closure of F
$G_F = \mathrm{Gal}(\bar{F}/F)$ the absolute Galois group

$$H^n(F, \mathbf{Z}/p) = H^n(G_F, \mathbf{Z}/p)$$

$$H^n(F, \mathbf{Z}/p) = \varinjlim_{L/F} H^n(\mathrm{Gal}(L/F), \mathbf{Z}/p)$$

(Limit over the finite Galois field extensions L of F)

$F^\times = F \setminus \{0\}$ the multiplicative group of F

F contains a primitive p-th root ζ_p of unity
$\mu_p \subset F^\times$ the subgroup generated by ζ_p

$$H^n(F, \mu_p^{\otimes m}) = H^n(F, \mathbf{Z}/p) \otimes \mu_p^{\otimes m}$$

Computation of $H^1(F, \mu_p)$:

$$H^1(F, \mu_p) = \mathrm{Hom}(G_F, \mu_p)$$

Hilbert Satz 90, Kummer theory:

$$F^\times/(F^\times)^p \xrightarrow{\simeq} H^1(F, \mu_p)$$

$$a \to (a) = [F(\sqrt[p]{a})/F]$$

Bloch-Kato conjecture:

For any field F with $\operatorname{char} F \neq p$, the Galois cohomology ring

$$\bigoplus_{n \geq 0} H^n(F, \mu_p^{\otimes n})$$

is generated by $H^1(F, \mu_p)$

The basic relation in $H^2(F,\mu_p^{\otimes 2})$:

$$(a) \cup (1-a) = 0 \qquad (a \in F \setminus \{0,1\})$$

Proof: $E = F(\alpha)$, $\alpha^p = a$

$$\begin{aligned}(a) \cup (1-a) &= (a) \cup (N_{E/F}(1-\alpha)) \\ &= N_{E/F}((a)_E \cup (1-\alpha)) \\ &= N_{E/F}((\alpha^p) \cup (1-\alpha)) = 0\end{aligned}$$

Milnor's K-ring of a field F:

$$\begin{aligned}K_*^{\mathsf{M}} F &= K_0 F \oplus K_1 F \oplus K_2 F \oplus \cdots \\ &= T_{\mathbf{Z}}(F^\times)/\langle a \otimes (1-a), a \in F \setminus \{0,1\}\rangle\end{aligned}$$

$$\begin{aligned}K_0 F &= \mathbf{Z} && \text{(integers)} \\ K_1 F &= F^\times && \text{(multiplicative group)}\end{aligned}$$

Bloch-Kato conjecture:

The ring homomorphism

$$K_*^{\mathsf{M}} F/p \longrightarrow \bigoplus_{n \geq 0} H^n(F,\mu_p^{\otimes n})$$

$$a_1 \otimes \cdots \otimes a_n \to (a_1) \cup \cdots \cup (a_n)$$

is bijective

The elements

$$(a_1) \cup \cdots \cup (a_n) \in H^n(F, \mu_p^{\otimes n})$$

with $a_1, \ldots, a_n \in F^\times$ are called **symbols**

Bloch-Kato conjecture (mod p, weight n):
$H^n(F, \mu_p^{\otimes n})$ is additively generated by symbols

Proofs:

$n = 1$	classical, Hilbert's Satz 90
$p = 2$, $n = 2$	Merkurjev (1982)
$n = 2$	Merkurjev/Suslin (1982)
$p = 2$, $n = 3$	Merkurjev/Suslin, Rost (1986)
$p = 2$	Voevodsky (1996–2002)
$\forall\, p, n$???	Voevodsky/Rost (1997–2007 ?)

$H^2(F, \mu_p)$ **and the Brauer group**

$\mathrm{Br}(F)$ = group of similarity classes of central simple algebras over F

$\mathrm{Br}(F)$ = set of isomorphism classes of skew fields with center F (finite F-dimension)

Cyclic algebras: $\zeta_p \in F$, $a, b \in F^\times$

$$A(a,b) = \langle\, X, Y \mid X^p = a,\ Y^p = b,\ YX = \zeta_p XY \,\rangle$$

There is a natural isomorphism

$$H^2(F, \mu_p) \xrightarrow{\simeq} {}_p\mathrm{Br}(F)$$

${}_p\mathrm{Br}(F) = p$-torsion subgroup of $\mathrm{Br}(F)$

If $\mu_p \subset F$, symbols correspond to cyclic algebras:

$$H^2(F, \mu_p^{\otimes 2}) \xrightarrow{\simeq} {}_p\mathrm{Br}(F)$$

$$(a) \cup (b) \to [A(a,b)]$$

The H^3-invariant for semisimple algebraic groups G

(Rost, Serre; 1993)

$H^1(F, G)$ = isomorphism classes of principal homogeneous G-spaces over F

The H^3-invariant is a collection of maps

$$\Theta\colon H^1(F, G) \longrightarrow H^3(F, Q_G \otimes \mu_{N(G)}^{\otimes 2})$$

functorial in F and G

Q_G = Weyl invariant quadratic forms on the root lattice

$Q_G = \mathbf{Z}$ for simple G

Example: $G = G_2$ ($\operatorname{char} F \neq 2$):

$H^1(F, G_2) =$ isomorphism classes of **octonion algebras** over F
The nontoral subgroup

$$(\mathbf{Z}/2)^3 \xrightarrow{j} G_2$$

yields

$$H^1(F, \mathbf{Z}/2)^3 \xrightarrow{j} H^1(F, G_2) \xrightarrow{\Theta} H^3(F, \mathbf{Z}/2)$$

$$((a),(b),(c)) \to [O(a,b,c)] \to (a) \cup (b) \cup (c)$$

Example: $G = F_4$ ($\operatorname{char} F \neq 3, \mu_3 \subset F$):

$H^1(F, F_4) =$ isomorphism classes of **exceptional Jordan algebras** over F
The nontoral subgroup

$$(\mathbf{Z}/3)^3 \xrightarrow{j} F_4$$

yields

$$H^1(F, \mu_3)^3 \xrightarrow{j} H^1(F, F_4) \xrightarrow{\Theta} H^3(F, \mathbf{Z}/3)$$

$$((a),(b),(c)) \to [J(a,b,c)] \to (a) \cup (b) \cup (c)$$

Multiplicative Norm Functions

Given a symbol

$$u = (a_1) \cup \cdots \cup (a_n) \in H^n(F, \mu_p^{\otimes n})$$

Need some sort of multiplicative function Φ in p^n variables generalizing the classical examples:

Example: $n = 2$: Φ is the reduced norm form

$$\Phi = \mathrm{Nrd}\colon A(a_1, a_2) \to F$$

of the cyclic algebra corresponding to u

Example: $n = 3$, $p = 2$: Φ is the norm form of the octonion algebra $O(a_1, a_2, a_3)$

Example: $n = 3$, $p = 3$: Φ is the norm form of the exceptional Jordan algebra $J(a_1, a_2, a_3)$

Example: $p = 2$: Φ is the Pfister quadratic form $\Phi = \langle\langle a_1, \ldots, a_n \rangle\rangle$

Recall from (complex) cobordism:

$s_d(X) \in \mathbf{Z}$ is Milnor's characteristic number

If $d = \dim X = p^m - 1$, then $s_d(X) \in p\mathbf{Z}$

Using algebraic cobordism and degree formulas one shows:

Theorem: If $u \neq 0$, there exists a rational function

$$\Phi\colon A \longrightarrow \mathbf{A}^1$$

on some variety A such that:

- $(u)_{F(A)} \cup (\Phi) = 0$ in $H^{n+1}(F(A), \mu_p^{\otimes(n+1)})$
- $\dim A = p^n$
- For any smooth compactification X of the generic fiber of Φ one has
$$\frac{s_d(X)}{p} \neq 0 \mod p$$

(A, Φ) is unique "up to extensions of degree prime to p" (at least for $n = 2$ or $p = 2$)

Multiplicativity of Φ: Ideally this means

$$\Phi(\mu(x,y)) = \Phi(x)\Phi(y)$$

for some (bilinear, rational?) map

$$\mu\colon A \times A \longrightarrow A$$

Look for a correspondence

$$\mu\colon A \times A \xleftarrow{\ f\ } W \xrightarrow{\ g\ } A$$

with $(\deg f, p) = 1$

This involves:

- Existence of *generic* splitting varieties of symbols (Voevodsky, see next pages)
- Algebraic cobordism (Morel/Levine)
- Parameterization of the "subfields" of the "algebra A with norm Φ"—motivated by chain lemma for exceptional Jordan algebras (Serre, Petersson/Racine 1995)

Construction of certain Cycles

$u = (a_1) \cup \cdots \cup (a_n) \in H^n(F, \mu_p^{\otimes n})$ a symbol

X a splitting variety of u: $u_{F(X)} = 0$

Using Bloch-Kato conjecture in weight $n-1$, get an element

$$\eta_u \in \mathrm{CH}^b(X^2) \qquad b = \frac{p^{n-1}-1}{p-1}$$

in the Chow group of b-codimensional cycles

Example: $n = 2$: $X =$ Severi-Brauer variety of cyclic algebra $A(a_1, a_2)$

$$(\eta_u)^{p-1} = \text{Diagonal}(X) + \text{decomp. elements}$$

Example: $p = 2$: $X =$ Quadric with quadratic form $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \perp \langle -a_n \rangle$

$$\eta_u = \text{“Rost projector”} + \text{decomp. elements}$$

$H_{\mathcal{M}}^{r,s}$: motivic cohomology (Suslin, Voevodsky)

$\mathcal{X}$ = simplicial scheme : $X \leftleftarrows X^2 \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} X^3 \cdots$

β = Bockstein
Q_i Steenrod/Milnor operations (Voevodsky)

The map j is an isomorphism assuming the Bloch-Kato conjecture in weight $n-1$

Construction of η_u:

$$\begin{array}{c}
u \in \ker[H^n(F, \mu_p^{\otimes(n-1)}) \longrightarrow H^n(F(X), \mu_p^{\otimes(n-1)})] \\
\simeq \Big\uparrow j \\
H_{\mathcal{M}}^{n,n-1}(\mathcal{X}, \mathbf{Z}/p) \\
\Big\downarrow \beta \circ Q_1 \circ \cdots \circ Q_{n-2} \\
H_{\mathcal{M}}^{2b+1,b}(\mathcal{X}, \mathbf{Z}) \\
\Big\downarrow \text{proj} \\
\text{Homology of } [\mathrm{CH}^b(X) \to \mathrm{CH}^b(X^2) \to \mathrm{CH}^b(X^3)]
\end{array}$$

Problem: Find some variety X such that:

(1) $(u)_{F(X)} = 0$ in $H^n(F(X), \mu_p^{\otimes n})$

(2) $d = \dim X = p^{n-1} - 1$

(3) The integer

$$c(X) = (\pi_1)_*(\eta_u^{p-1}) \in \mathrm{CH}^0(X) = \mathbf{Z}$$

is nonzero mod p

Then X would be a generic splitting variety (up to extensions of degree prime to p)

Theorem: There exists X with (1), (2) and

$$\frac{s_d(X)}{p} \neq 0 \mod p$$

Voevodsky announced essentially that

$$c(X) = \frac{s_d(X)}{p} \mod p$$