

F-MAGMAS

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preliminary version

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You are looking at the text “*F*-magmas” [pdf].

Introduction

A (classical) magma is a multiplication in the simplest sense, a set M together with a map $\mu: M \times M \rightarrow M$.

In Bourbaki (Algebra) such a pair (M, μ) is called a magma with μ the composition law on M [2, Chapter I, §1, 1. Definition 1, p. 1]. The term magma appears also in Bourbaki (Groupes et algèbres de Lie) [1, Chap. II, §2 Algèbres de Lie libres, p. 17] and in Serre (Lie algebras and Lie groups) [6, Chap. IV. Free Lie Algebras, 1. Free magmas, Definition 1.1, p. 18].

If M_X is the free magma on a set X , the map

$$(*) \quad X \amalg (M_X \times M_X) \rightarrow M_X$$

given by inclusion and multiplication is bijective [6, Properties 2), p. 18]. A similar fact holds for multi-magmas as described in [5], see [5, (1.3), p. 6].

The decomposition $M_X = X \amalg M_X^2$ is immediate from the explicit construction of M_X in [6], but also can be directly deduced from the universality of $X \rightarrow M_X$. Namely one may define right away on $X \amalg M_X^2$ the structure of a magma (that is, a multiplication) and the universality of M_X gives a map $M_X \rightarrow X \amalg M_X^2$ yielding the inverse of (*).

The starting point of this text was to formalize this argument (see Lemma (1.6)). We ended up with a very simple generalization of magmas, *F*-magmas. Here F is an endofunctor on a category \mathcal{C} and an *F*-magma is an object M together with a \mathcal{C} -morphism $\mu_M: F(M) \rightarrow M$.

The basic idea to construct the free *F*-magma on an object X of \mathcal{C} is to take the limit of iterations of the functor

$$F_X(M) = X \amalg F(M)$$

see Summary (5.6). The main body of this paper arose from that.

If \mathcal{C} has colimits and F preserves filtered colimits there are universal *F*-magmas and the free *F*-magma on an object. Further, the bijectivity of (*) generalizes to the *F*-decomposition (5.3).

Interestingly, in the case of classical magmas the construction of free magmas is different from that in [6]. The result is the same of course, but the constructions yield different filtrations. See Example (6.2) and also Example (6.3).

The dual notion of an *F*-comagma (a \mathcal{C} -morphism $A \rightarrow F(A)$) appears naturally when constructing universal *F*-magmas. In fact, the basic tool is the inversion of a morphism $\delta: A \rightarrow F(A)$ generalizing the standard case $F = \text{id}_{\mathcal{C}}$ (see 3.2). Besides this I haven’t looked further into *F*-comagmas themselves and more possible interplays with *F*-magmas.

There is an apparent formal similarity of convolution-stable morphisms between comagmas and magmas (see §2 and Theorem (4.1)) to twisting morphisms for differential graded associative (co)algebras [3, Chapter 2, Twisting Morphisms, p. 37]. Again, I haven’t looked into this further.

General provisions

The general framework is a category \mathcal{C} and an endofunctor $F: \mathcal{C} \rightarrow \mathcal{C}$ of \mathcal{C} . Beginning in §3 we assume that direct limits of the form

$$L = \lim_{k \rightarrow \infty} X_k$$

exist in \mathcal{C} and beginning in §4 that F preserves such limits:

$$F(L) = \lim_{k \rightarrow \infty} F(X_k)$$

From §5 on we assume that \mathcal{C} has coproducts $X \amalg Y$.

Recall that an initial object of \mathcal{C} is an object 0 such that $\text{Hom}_{\mathcal{C}}(0, X)$ consists of a single element for each X .

The basic example is the category **Sets** of sets and $F(Z) = Z^2$. Here $X \amalg Y$ is disjoint union and the empty set \emptyset is the unique initial object.

Another example is the category of R -modules for some ring R and $F(Z) = Z^{\otimes 2}$. Here $X \amalg Y = X \oplus Y$ is direct sum and the zero module 0 is an initial object.

§1. Magmas and Comagmas

(1.1) Definition.

An F -magma is a pair (M, μ) consisting of an object M of \mathcal{C} and a \mathcal{C} -morphism

$$\mu: F(M) \rightarrow M$$

An F -comagma is a pair (A, δ) consisting of an object A of \mathcal{C} and a \mathcal{C} -morphism

$$\delta: A \rightarrow F(A)$$

(1.2) **Examples.** In **Sets** let $F(Z) = Z^2$. Then an F -magma is a magma in the classical sense, consisting of a set M and a map $M^2 \rightarrow M$, see [6, p. 18].

In **Sets** let

$$F(Z) = \coprod_{n \geq 2} Z^n$$

Then an F -magma is a multi-magma in the sense of [5].

Let R be a ring and let $F(V) = V^{\otimes 2}$ in the category of R -modules. Then an F -magma is an R -algebra (non-unital, non-associative, non-commutative).¹ Similarly, an F -comagma is an R -coalgebra.

An F -magma is mostly written in the form $M = (M, \mu_M^F) = (M, \mu_M)$ and μ_M^F is called the F -multiplication of M . Similarly for F -comagmas, which appear as $A = (A, \delta_A^F) = (A, \delta_A)$ with δ_A^F called the F -diagonal of A .

A homomorphism of F -magmas is a \mathcal{C} -morphism f such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \mu_M \uparrow & & \uparrow \mu_N \\ F(M) & \xrightarrow{F(f)} & F(N) \end{array}$$

is commutative.

Obviously, composites of F -magma homomorphisms are F -magma homomorphisms.

¹The prefix “non-” stands for “not required”.

We denote by

$$\mathrm{Hom}_F(M, N) = \{ f \in \mathrm{Hom}_{\mathcal{C}}(M, N) \mid f\mu_M = \mu_N F(f) \}$$

the set of F -magma homomorphisms $M \rightarrow N$. Similarly, F -comagma homomorphisms $A \rightarrow B$ are the elements of the set

$$\mathrm{Hom}^F(A, B) = \{ f \in \mathrm{Hom}_{\mathcal{C}}(A, B) \mid \delta_B f = F(f)\delta_A \}$$

The category of F -magmas is denoted by \mathcal{C}_F , the category of F -comagmas by \mathcal{C}^F .

In the following (until §5) the functor F is fixed and we call an F -magma simply a magma. Similarly for comagmas.

If M is a magma, then $F(M)$ is a magma with

$$\mu_{F(M)} = F(\mu_M)$$

And if A is a comagma, so is $F(A) = (F(A), F(\delta_A))$.

Obviously $\mu_M \in \mathrm{Hom}_F(F(M), M)$ and $\delta_A \in \mathrm{Hom}^F(A, F(A))$.

(1.3) Remark. One may call the morphisms μ, δ themselves a (co)magma. Doing this systematically would however yield cryptic statements like $\mu \in \mathrm{Hom}_F(F(\mu), \mu)$ (which has something, on the other hand).

(1.4) Definition.

A magma (M, μ) is called *invertible* if μ is an isomorphism. The *inverse* of an invertible magma (M, μ) is the comagma (M, μ^{-1}) .

A comagma (A, δ) is called *invertible* if δ is an isomorphism. The *inverse* of an invertible comagma (A, δ) is the magma (A, δ^{-1}) .

Clearly, if a magma M is invertible, so is $F(M)$. Similarly for comagmas.

We now restrict to magmas.²

(1.5) Definition. A magma M is called *universal* if for each magma N the set $\mathrm{Hom}_F(M, N)$ has exactly one element. In other words, M is an initial object of the category \mathcal{C}_F .

A universal magma is unique up to unique isomorphism. If 0 is an initial object of \mathcal{C} and $F(0) \simeq 0$, then $(0, F(0) \rightarrow 0)$ is a universal magma.

Here is a key fact:

(1.6) Lemma. *A universal magma is invertible.*

Proof: Let M be a universal magma and let $s: M \rightarrow F(M)$ be the unique magma homomorphism. Then $\mu_M s = \mathrm{id}_M$ by uniqueness. Moreover

$$s\mu_M = \mu_{F(M)}F(s) = F(\mu_M)F(s) = F(\mu_M s) = F(\mathrm{id}_M) = \mathrm{id}_{F(M)} \quad \square$$

(It was this computation which started this paper.)

²There is no need to enter the games of “co-universality” and “co-free”.

§2. Convolutions

Let M be a magma and let A be a comagma.

(2.1) Definition. The self-map

$$\begin{aligned} c_F: \operatorname{Hom}_{\mathcal{C}}(A, M) &\rightarrow \operatorname{Hom}_{\mathcal{C}}(A, M) \\ c_F(f) &= \mu_M F(f) \delta_A \end{aligned}$$

is called *convolution*.

A \mathcal{C} -morphism $f: A \rightarrow M$ is called *c-stable* (convolution-stable) if $c_F(f) = f$. We denote by

$$S_F(A, M) = \{ f \in \operatorname{Hom}_{\mathcal{C}}(A, M) \mid c_F(f) = f \}$$

the set of *c-stable* \mathcal{C} -morphisms $A \rightarrow M$.

(2.2) Example. Let $F(V) = V^{\otimes 2}$ in the category of R -modules. Then $c_F(f) = f * f$ is the convolution square of an R -module homomorphism from an R -coalgebra to an R -algebra (see for instance [3, 1.6 Convolution, p. 32]).

Let $k \geq 0$ be a non-negative integer. Then $F^k(M)$ is a magma and $F^k(A)$ is a comagma³ with multiplication resp. diagonal ($a + b = k$)

$$\begin{aligned} \mu_{F^k(M)} &= F^k(\mu_M) = F^a(\mu_{F^b(M)}) \\ \delta_{F^k(A)} &= F^k(\delta_A) = F^a(\delta_{F^b(A)}) \end{aligned}$$

We use the abbreviated notations ($k, h \geq 0$)

$$\begin{aligned} \mu: \operatorname{Hom}_{\mathcal{C}}(F^k(A), F^{h+1}(M)) &\rightarrow \operatorname{Hom}_{\mathcal{C}}(F^k(A), F^h(M)) \\ \mu(f) &= \mu_{F^h(M)} f \\ \delta: \operatorname{Hom}_{\mathcal{C}}(F^{k+1}(A), F^h(M)) &\rightarrow \operatorname{Hom}_{\mathcal{C}}(F^k(A), F^h(M)) \\ \delta(f) &= f \delta_{F^k(A)} \end{aligned}$$

These maps and

$$\begin{aligned} F: \operatorname{Hom}_{\mathcal{C}}(F^k(A), F^h(M)) &\rightarrow \operatorname{Hom}_{\mathcal{C}}(F^{k+1}(A), F^{h+1}(M)) \\ f &\mapsto F(f) \end{aligned}$$

commute whenever the composites are defined. More precisely, on

$$\operatorname{Hom}_{\mathcal{C}}(F^k(A), F^h(M))$$

one has (considering μ, δ, F as indexed families)

$$\begin{aligned} \mu \delta &= \delta \mu & (k, h \geq 1) \\ \mu F &= F \mu & (k \geq 0, h \geq 1) \\ \delta F &= F \delta & (k \geq 1, h \geq 0) \end{aligned}$$

For instance,

$$(\mu F)(f) = \mu_{F^h(M)} F(f) = F(\mu_{F^{h-1}(M)}) F(f) = F(\mu_{F^{h-1}(M)} f) = (F \mu)(f)$$

³Another topic is to consider M, A as F^k -(co)magmas (for example, $\mu_M^{F^2} = \mu_M^F F(\mu_M^F)$). This is not further discussed in this text.

Note that F , μF , δF and the convolution

$$c = \mu\delta F = \delta\mu F$$

are defined on $\text{Hom}_{\mathcal{C}}(F^k(A), F^h(M))$ for $k, h \geq 0$.

In particular, the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(A, M) & \begin{array}{c} \xrightarrow{\mu F} \\ \xleftarrow{\delta} \end{array} & \text{Hom}_{\mathcal{C}}(F(A), M) \\ \begin{array}{c} \uparrow \mu \\ \downarrow \delta F \end{array} & \searrow Fc & \begin{array}{c} \uparrow \mu \\ \downarrow \delta F \end{array} \\ \text{Hom}_{\mathcal{C}}(A, F(M)) & \begin{array}{c} \xrightarrow{\mu F} \\ \xleftarrow{\delta} \end{array} & \text{Hom}_{\mathcal{C}}(F(A), F(M)) \end{array}$$

yields 4 commutative square diagrams (one for each corner) and 2 commutative triangles. On the c -stable subsets (defined by $\mu\delta F = \text{id}$) these induce bijections

$$\begin{array}{ccc} S_F(A, M) & \begin{array}{c} \xleftarrow{\cong} \\ \xrightarrow{\cong} \end{array} & S_F(F(A), M) \\ \begin{array}{c} \uparrow \cong \\ \downarrow \cong \end{array} & \searrow F & \begin{array}{c} \uparrow \cong \\ \downarrow \cong \end{array} \\ S_F(A, F(M)) & \begin{array}{c} \xleftarrow{\cong} \\ \xrightarrow{\cong} \end{array} & S_F(F(A), F(M)) \end{array}$$

There result canonical bijections ($k, h \geq 0$)

$$S_F(A, M) \begin{array}{c} \xleftarrow{\cong} \\ \xrightarrow{\cong} \end{array} S_F(F^k(A), F^h(M))$$

§3. Inverting comagmas

3.1. Limits. (Here: filtered colimits with directed index set \mathbf{N}_0 , see [4, Filtered Limits, pp. 211-212])

We assume that \mathcal{C} has direct limits of the form

$$\text{Lim}_{k \rightarrow \infty} \bar{A} = \text{Lim}_{k \rightarrow \infty} (A_k, a_k)$$

for families

$$\bar{A} = (a_k : A_k \rightarrow A_{k+1})_{k \geq 0}$$

With such a limit there come along its canonical morphisms

$$j_k : A_k \rightarrow \text{Lim}_{k \rightarrow \infty} \bar{A}, \quad j_k = j_{k+1} a_k$$

which have the property that for each object B there is the bijection

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(\text{Lim}_{k \rightarrow \infty} \bar{A}, B) &\xrightarrow{\cong} \{ (\varphi_k : A_k \rightarrow B)_{k \geq 0} \mid \varphi_k = \varphi_{k+1} a_k \} \\ \varphi &\mapsto (\varphi_k)_{k \geq 0}, \quad \varphi_k = \varphi j_k, \quad \varphi = \text{Lim}_{k \rightarrow \infty} \varphi_k \end{aligned}$$

(we just described the universal property).

Given the endofunctor F , let

$$\Phi_{\bar{A}} = \text{Lim}_{k \rightarrow \infty} F(j_k) : \text{Lim}_{k \rightarrow \infty} F(\bar{A}) \rightarrow F(\text{Lim}_{k \rightarrow \infty} \bar{A})$$

where

$$F(\bar{A}) = (F(A_k), F(a_k))_{k \geq 0}$$

We say that F commutes with limits $\text{Lim}_{k \rightarrow \infty}$ if $\Phi_{\bar{A}}$ is an isomorphism for each family \bar{A} , cf. [4, Preservation of Limits, p. 116]. (We assume this from §4 on).

3.2. **The limit** $L(A)$. Given a comagma $\delta: A \rightarrow F(A)$, consider the limit

$$L(A, \delta): \quad A \xrightarrow{\delta} F(A) \xrightarrow{F(\delta)} \cdots \rightarrow F^k(A) \xrightarrow{F^k(\delta)} F^{k+1}(A) \rightarrow \cdots$$

We will see that this is again a comagma, invertible if F commutes with the limits.

If $F = \text{id}_{\mathcal{C}}$ is the identity functor, then a comagma is just an endomorphism $\delta \in \text{End}_{\mathcal{C}}(A)$. In this case $L(A, \delta)$ is a standard construction to invert δ (illustrated in Example (3.3) below).

Formally there is the functor

$$\begin{aligned} L: \mathcal{C}^F &\rightarrow \mathcal{C}^F \\ L(A) &= \text{Lim}_{k \rightarrow \infty} \bar{A} \\ L(f) &= \text{Lim}_{k \rightarrow \infty} F^k(f) \\ \delta_{L(A)} &= \Phi_{\bar{A}} L(\delta_A) \end{aligned}$$

where for a comagma A the family of its F -iterates is denoted by

$$\bar{A} = (F^k(A), F^k(\delta_A))_{k \geq 0}$$

Moreover, for a comagma morphism $f \in \text{Hom}^F(A, B)$ one has

$$F^k(f) \in \text{Hom}^F(F^k(A), F^k(B))$$

and their limit

$$L(f): L(A) \rightarrow L(B)$$

is defined via

$$\begin{array}{ccccc} j_k^A: & F^k(A) & \xrightarrow{F^k(\delta_A)} & F^{k+1}(A) & \longrightarrow & L(A) \\ & \downarrow F^k(f) & & \downarrow F^{k+1}(f) & & \downarrow L(f) = \text{Lim}_{k \rightarrow \infty} F^k(f) \\ j_k^B: & F^k(B) & \xrightarrow{F^k(\delta_B)} & F^{k+1}(B) & \longrightarrow & L(B) \end{array}$$

Further, the diagonal of the limit $L(A)$ is the composite

$$\delta_{L(A)}: L(A) \xrightarrow{L(\delta_A)} L(F(A)) \xrightarrow{\Phi_{\bar{A}}} F(L(A))$$

(recall $\delta_A \in \text{Hom}^F(A, F(A))$).

(3.1) Remark. The limits

$$\begin{aligned} L(A) &= \text{Lim}_{k \rightarrow \infty} (F^k(A), F^k(\delta_A)) \\ L(F(A)) &= \text{Lim}_{k \rightarrow \infty} (F^{k+1}(A), F^{k+1}(\delta_A)) \end{aligned}$$

are canonically isomorphic since they differ only by a shift of indices. In fact, $L(\delta_A)$ is that canonical isomorphism $L(A) \rightarrow L(F(A))$ (see the subsequent Lemma (3.2)).

One is tempted to write

$$L(A) = F^\infty(A)$$

leading to $F^\infty F = F^\infty$. In §4 we assume that F commutes with the limits, meaning $FF^\infty = F^\infty F$.

Denote the canonical morphisms of the limits $L(A)$ resp. $L(F(A))$ by

$$\begin{aligned} j_k: F^k(A) &\rightarrow L(A) & j_k &= j_{k+1}F^k(\delta_A) \\ j'_k: F^{k+1}(A) &\rightarrow L(F(A)) & j'_k &= j'_{k+1}F^{k+1}(\delta_A) \end{aligned}$$

(3.2) Lemma. *One has*

$$\begin{aligned} L(\delta_A)j_{k+1} &= j'_k \\ \Phi_{\bar{A}}j'_k &= F(j_k) \\ \delta_{L(A)}j_{k+1} &= F(j_k) \end{aligned}$$

The morphism

$$L(\delta_A): L(A) \rightarrow L(F(A))$$

is an isomorphism.

Proof: The first claim follows from $L(\delta_A)j_k = j'_kF^k(\delta_A)$ (the definition of $L(\delta_A)$). The second claim is the definition of $\Phi_{\bar{A}}$. The third claim is now immediate.

The last claim follows from the universal properties and reindexing $k \leftrightarrow k+1$. More explicitly: The inverse of $L(\delta_A)$ is

$$s = \varinjlim_{k \rightarrow \infty} j_{k+1}: L(F(A)) \rightarrow L(A)$$

(defined since $j_{k+1} = j_{k+2}F^{k+1}(\delta_A)$) as can be seen from the commutative diagram

$$\begin{array}{ccccccc} L(A) & \xrightarrow{L(\delta_A)} & L(F(A)) & \xrightarrow{s} & L(A) & \xrightarrow{L(\delta_A)} & L(F(A)) \\ j_k \uparrow & & j'_k \uparrow & & j_{k+1} \uparrow & & j'_{k+1} \uparrow \\ F^k(A) & \xrightarrow{F^k(\delta_A)} & F^{k+1}(A) & \xlongequal{\quad} & F^{k+1}(A) & \xrightarrow{F^{k+1}(\delta_A)} & F^{k+2}(A) \quad \square \end{array}$$

(3.3) Example. (for the reader's relaxation)

Let F be the identity functor in the category of abelian groups and consider $L = L(\mathbf{Z}, \delta)$ with $\delta: \mathbf{Z} \rightarrow F(\mathbf{Z}) = \mathbf{Z}$ the multiplication by 2. Then

$$\begin{aligned} L &= \varinjlim (\mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{2} \dots) = \mathbf{Z}[\frac{1}{2}] \\ j_k(x) &= \frac{1}{2^k}x \\ \delta_L(x) &= 2x \end{aligned}$$

§4. $L(A)$ as invertible magma

We now assume that F commutes with limits $\varinjlim_{k \rightarrow \infty}$.

Continuing with a fixed comagma A , this means that $\Phi_{\bar{A}}$ is an isomorphism. Thus $L(A)$ is an invertible comagma. Its inverse is the invertible magma $L(A)$ with multiplication

$$\mu_{L(A)} = \delta_{L(A)}^{-1} = L(\delta_A)^{-1}\Phi_{\bar{A}}^{-1}: F(L(A)) \rightarrow L(A)$$

Let again

$$j_k: F^k(A) \rightarrow L(A)$$

denote the canonical morphisms of the limit $L(A)$.

Note that the canonical morphism j_0 is a morphism $A \rightarrow L(A)$.

(4.1) Theorem. *For each magma M , the map*

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(L(A), M) &\rightarrow \text{Hom}_{\mathcal{C}}(A, M) \\ \varphi &\mapsto \varphi j_0 \end{aligned}$$

induces a bijection of subsets

$$\text{Hom}_F(L(A), M) \rightarrow S_F(A, M)$$

Hence $L(A)$ is a universal magma if and only if $S_F(A, M)$ consists of a single element for each magma M .

Proof: For a \mathcal{C} -morphism $\varphi: L(A) \rightarrow M$ the corresponding sequence

$$\varphi_k = \varphi j_k: F^k(A) \rightarrow M$$

satisfies

$$(4.2) \quad \varphi_k = \varphi_{k+1} F^k(\delta_A)$$

Clearly, (4.2) recovers $\varphi = \text{Lim}_{k \rightarrow \infty} \varphi_k$.

If φ is a magma homomorphism, the diagrams

$$\begin{array}{ccccc} \varphi_{k+1}: & F^{k+1}(A) & \xrightarrow{j_{k+1}} & L(A) & \xrightarrow{\varphi} & M \\ & \parallel & & \uparrow \mu_{L(A)} & & \uparrow \mu_M \\ F(\varphi_k): & F^{k+1}(A) & \xrightarrow{F(j_k)} & F(L(A)) & \xrightarrow{F(\varphi)} & F(M) \end{array}$$

are commutative since $j_{k+1} = \mu_{L(A)} F(j_k)$ by Lemma (3.2). Thus

$$(4.3) \quad \varphi_{k+1} = \mu_M F(\varphi_k)$$

Conversely, since $\mu_{L(A)}$ is an isomorphism, (4.3) implies that φ is a magma homomorphism.

Now use §2. Conditions (4.2) and (4.3) read as

$$\begin{aligned} \varphi_k &= \delta(\varphi_{k+1}) \\ \varphi_{k+1} &= (\mu F)(\varphi_k) \end{aligned}$$

Thus $\varphi_k \in S_F(F^k(A), M)$, in particular $\varphi_0 \in S_F(A, M)$. It follows that $\varphi_0 \in S_F(A, M)$ and $\varphi = \text{Lim}_{k \rightarrow \infty} \varphi_k \in \text{Hom}_F(L(A), M)$ determine each other. \square

(4.4) Example. A constant functor is a functor with constant value on objects and sending a morphism to the identity.

Let $F(Z) = Y$ be a constant functor. Then an F -magma is a pair $(M, Y \rightarrow M)$ and (Y, id_Y) is universal.

In this case $L(A) = (Y, \text{id}_Y)$ for each comagma A . Indeed, $F^k(A) = Y$, $F^k(\delta_A) = \text{id}_Y$ for $k \geq 1$. Moreover, $S_F(A, M)$ consists of $\mu_M \delta_A$.

An initial object 0 of \mathcal{C} is a comagma with δ_0 the unique morphism $0 \rightarrow F(0)$.

(4.5) Corollary. *Let 0 be an initial object of \mathcal{C} . Then $L(0)$ is a universal magma.*

Proof: For a magma M , the unique element of $\text{Hom}_{\mathcal{C}}(0, M)$ is clearly c -stable and therefore the only element of $S_F(0, M)$. The claim follows from Theorem (4.1). \square

§5. Free magmas

We now assume that \mathcal{C} has coproducts $A \amalg B$.

Given the endofunctor F and an object X of \mathcal{C} , define the endofunctor F_X of \mathcal{C} by

$$\begin{aligned} F_X(M) &= X \amalg F(M) \\ F_X(f) &= \text{id}_X \amalg F(f) \end{aligned}$$

In other words, F_X is the coproduct of the constant functor with value X and F . Or, if Ψ_X is the endofunctor

$$\begin{aligned} \Psi_X(Y) &= X \amalg Y \\ \Psi_X(f) &= \text{id}_X \amalg f \end{aligned}$$

then F_X is the composite

$$F_X = \Psi_X \circ F$$

It follows that F_X commutes with limits $\text{Lim}_{k \rightarrow \infty}$ since F and Ψ_X do.

We write $L = L(F, ?)$ for the functor L from §3 to indicate the dependence on F . In particular, there is the functor

$$L(F_X, ?): \mathcal{C}^{F_X} \rightarrow \mathcal{C}^{F_X}$$

and for an F_X -comagma A there is the invertible F_X -magma (cf. §4)

$$L(F_X, A) = \text{Lim}_{k \rightarrow \infty} (F_X^k(A), F_X^k(\delta_A))$$

An F_X -magma M consists of an F -magma M and a \mathcal{C} -morphism $\lambda_M: X \rightarrow M$ with μ_M^F, λ_M the components of the F_X -multiplication:

$$\mu_M^{F_X} = (\lambda_M, \mu_M^F): X \amalg F(M) \rightarrow M$$

(5.1) Proposition. *Let A be an object of \mathcal{C} and let $\theta: A \rightarrow X$ be a morphism in \mathcal{C} . Consider A as F_X -comagma with F_X -diagonal*

$$\delta_\theta = i_X \theta: A \rightarrow F_X(A) = X \amalg F(A)$$

where i_X is the injection of X .

Then for each F_X -magma M the set $S_{F_X}(A, M)$ consists of the single element $\lambda_M \theta$. In particular, $L(F_X, A)$ is a universal F_X -magma.

Proof: For a \mathcal{C} -morphism $f: A \rightarrow M$ the convolution is

$$c_{F_X}(f) = \mu_M^{F_X} F_X(f) \delta_\theta = \lambda_M \theta$$

as can be seen from the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{c_{F_X}(f)} & M \\ \theta \swarrow & \delta_\theta \downarrow & \uparrow \mu_M^{F_X} \\ X & \xrightarrow{i_X} & X \amalg F(A) \xrightarrow{F_X(f)} X \amalg F(M) \\ & & \text{= id}_X \amalg F(f) \end{array} \quad \mu_M^{F_X} = (\lambda_M, \mu_M^F)$$

The last claim follows from Theorem (4.1). □

We consider X itself as F_X -comagma via the injection

$$i_X: X \rightarrow F_X(X) = X \amalg F(X)$$

Then $L(F_X, X)$ is a universal F_X -magma (take $\theta = \text{id}_X$ in Proposition (5.1)).

(5.2) Definition. The *free F -magma on X* is the universal F_X -magma

$$M(F, X) = L(F_X, X)$$

Hence (abbreviating $M_X = M(F, X)$) in \mathcal{C} the triple

$$(M_X, \mu_{M_X}^F, \lambda_{M_X})$$

is universal among triples

$$(M, F(M) \rightarrow M, X \rightarrow M)$$

Since M_X is invertible (by Lemma (1.6) or by construction) it follows that

$$(5.3) \quad (\lambda_{M_X}, \mu_{M_X}^F) : X \amalg F(M_X) \xrightarrow{\simeq} M_X$$

is an isomorphism. We call (5.3) the *F -decomposition* of the free F -magma on X .

(5.4) Remark. Universal magmas are unique up to unique isomorphisms. Thus $L(F_X, A)$ of Proposition (5.1) does not depend on θ and A . The isomorphism

$$L(F_X, A) \rightarrow L(F_X, X)$$

is induced by the F_X -comagma homomorphism θ , its inverse

$$L(F_X, X) \rightarrow L(F_X, F_X(A)) \simeq L(F_X, A)$$

by the injection $X \rightarrow F_X(A) = X \amalg F(A)$ and $L(F_X, \delta_\theta)^{-1}$. (Informally speaking, in the limit A gets “eaten” by X via θ .)

(5.5) Remark. If there exists an initial object 0 of \mathcal{C} , then $L(F_X, 0)$ is universal by Corollary (4.5) and one may define the free F -magma on X right away as

$$M_X = L(F_X, 0)$$

without reference to Proposition (5.1). If additionally $F(0) \simeq 0$, then

$$F_X(0) = X \amalg F(0) = X$$

is the F_X -comagma $X = (X, i_X)$ and the limits $L(F_X, 0)$, $L(F_X, X)$ differ just by an index shift.

In particular, in **Sets** one may define the free F -magma on a set X as

$$M_X = L(F_X, \emptyset)$$

Similarly for the category of R -modules.

(5.6) Summary. The free F -magma on X is the limit of terms

$$\begin{aligned} M_1 &= X \\ M_2 &= X \amalg F(X) \\ M_3 &= X \amalg F(M_2) \\ M_4 &= X \amalg F(M_3) \\ &\dots \end{aligned}$$

with the transitions given by the identity on X and the F -transform of the preceding transition morphism.

§6. Examples

(6.1) **Example.** If $F = \text{id}_{\mathcal{C}}$ is the identity functor, then

$$F_X^k(X) = \prod_{h=1}^{k+1} X = X \times \{1, \dots, k+1\}$$

and the free F -magma $M_X = M(\text{id}_{\mathcal{C}}, X)$ on X is

$$M_X = X \times \mathbf{N} = X \amalg X \amalg X \amalg \dots$$

with $\lambda_{M_X} = \text{id}_X \times \{1\}$ the injection of the first term and $\mu_{M_X} = \text{id}_X \times \{+1\}$ the shift to the right.

For a triple $(M, \mu: M \rightarrow M, \lambda: X \rightarrow M)$ the corresponding morphism $\varphi: M_X \rightarrow M$ is given by

$$\varphi|_{X \times \{k\}} = \mu^{k-1} \lambda$$

The F -decomposition (5.3) is the isomorphism

$$X \amalg (X \times \mathbf{N}) \xrightarrow{\cong} X \times \mathbf{N}$$

induced from the bijection

$$\mathbf{N}_0 \xrightarrow{+1} \mathbf{N}$$

(6.2) **Example.** We consider the case of classical magmas. So let $\mathcal{C} = \mathbf{Sets}$ and $F(Z) = Z^2$ (this stands of course for $F(Z) = Z \times Z$, $F(f) = f \times f$).

Then the free magma $M(F, X)$ on X is the limit of

$$\begin{aligned} M_1 &= X \\ M_2 &= X \amalg X^2 \\ M_3 &= X \amalg (X \amalg X^2)^2 \\ &= X \amalg X^2 \amalg (X \times X^2) \amalg (X^2 \times X) \amalg (X^2 \times X^2) \\ M_4 &= X \amalg (X \amalg (X \amalg X^2)^2)^2 \\ &\dots \end{aligned}$$

This limit is actually a union with $M_{k+1} \setminus M_k$ consisting of the parenthesized expressions with maximal depth of nested paren pairs equal to k (here $X^2 = (X \times X)$ counts for 1 pair).

In contrast, in [6, p. 18] (also in [2, Chapter I, §7.1, p. 81], [1, p. 17]) the free magma on X (denoted as M_X) is defined as follows:

$$\begin{aligned} X_1 &= X \\ X_n &= \prod_{p+q=n} X_p \times X_q \quad (n \geq 2) \\ M_X &= \prod_{n \geq 1} X_n \end{aligned}$$

This description corresponds to the decomposition by length with first terms

$$\begin{aligned}
X_1 &= X \\
X_2 &= X^2 \\
X_3 &= (X \times X^2) \amalg (X^2 \times X) \\
X_4 &= (X \times (X \times X^2)) \amalg (X \times (X^2 \times X)) \\
&\quad \amalg (X^2 \times X^2) \\
&\quad \amalg ((X \times X^2) \times X) \amalg ((X^2 \times X) \times X)
\end{aligned}$$

The filtration by length is more natural and convenient for classical magmas. However the notion of length does not generalize to arbitrary F .

The F -decomposition (5.3) is

$$M_X = X \amalg M_X^2$$

described as $M_X \setminus X = M_X \times M_X$ in [6, Properties 2), p.18], [2, p.81], [1, p.17].

(6.3) Example. Similar remarks apply to multi-magmas (see (1.2)). Here the F -decomposition (5.3) is

$$M_X = X \amalg \coprod_{n \geq 2} M_X^n$$

In [5] it is called arity-decomposition ([5, (1.3), p.6]) and an indispensable tool for inductive definitions and proofs.

(6.4) Example. More generally, let P be a set of ordered finite nonempty sets and consider in **Sets** the endofunctor

$$F(Z) = \coprod_{I \in P} Z^I$$

This setup includes Examples (6.1) (for $\mathcal{C} = \mathbf{Sets}$), (6.2), (6.3).

The general construction of $M_X = M(F, X)$ (Definition (5.2)) establishes the existence of free F -magmas right away without much ado about the details of P .

One way to construct the free F -magma M_X directly is to consider parenthetical expressions with nested “ I -paren pairs”

$$(\alpha_1 \cdots \alpha_{|I|})_I$$

starting from $\alpha_i \in X$.

Alternatively, M_X can be identified with the set of isomorphism classes of finite rooted planar trees with labels as follows: Each leaf of the tree (a vertex of valency 1, excluding the root) is marked with an element of X . Further, for each inner node (a vertex with valency ≥ 2) the ordered set of incoming edges (coming from leaves) is identified with some $I \in P$ (so the valency of the node is $|I| + 1$).

If $|I| = 1$ for some $I \in P$, then the subsets of M_X consisting of the trees with a given number of leaves (the length in the preceding examples) are not finite already for $|X| = 1$ since any number of nodes of valency 2 is possible.

In the particular case

$$F(Z) = Z$$

(the case $P = \{\{*\}\}$) the element

$$(x, k) \in M_X = X \times \mathbf{N}$$

(see Example (6.1)) is represented in terms of parenthetical expressions by

$$(\cdots((x))\cdots)$$

with $k - 1$ paren pairs and in terms of trees by

$$x \bullet \longrightarrow \circ \longrightarrow \circ \cdots \cdots \circ \longrightarrow \bullet \text{ root}$$

with $k - 1$ inner nodes.

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