THE BASIC CORRESPONDENCE OF A SPLITTING VARIETY

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Summary of the talk at the 2007 Abel Symposium, Oslo, Norway

We discussed some details of a construction used in the proof of the generalized Milnor conjecture. A general overview had been given by Sasha Merkurjev in the morning. Let me just repeat here the statement of the conjecture:

Consider the norm residue homomorphism

$$h_{(n,p)}\colon K_n^M k/p \to H^n_{\text{et}}(k,\mu_p^{\otimes n})$$
$$\{a_1,\dots,a_n\} \mapsto (a_1)\cup\cdots\cup(a_n)$$

from Milnor's K-groups to Galois cohomology. The *generalized Milnor conjecture* (aka Milnor-Bloch-Kato conjecture, aka ...) states the bijectivity of this map for any prime p, any n, and any field k with $\operatorname{char} k \neq p$.

In the talk we concentrated on the "basic correspondence" of a splitting variety. The general reference for this is [3]. For norm varieties and their relation to characteristic numbers and cobordism see [2].

In [3] we introduced also the more abstract notion of a "special correspondence" on a variety (with the basic correspondences on norm varieties as only known examples so far). Varieties with special correspondences have been considered further recently in [9].

The following text is an extended version of [4].

The basic correspondence of a splitting variety of a symbol u (see below) is obtained by the following diagram, which is essentially due to Voevodsky:

$$u \in \ker\bigl[H^n_{\text{et}}(k,\mu_p^{\otimes(n-1)}) \longrightarrow H^n_{\text{et}}(k(X),\mu_p^{\otimes(n-1)})\bigr]$$
$$\simeq \Big\uparrow j$$
$$H^{n,n-1}_{\mathcal M}(\mathcal X,\mathbf Z/p)$$
$$\Big\downarrow \beta\circ Q_1\circ\cdots\circ Q_{n-2}$$
$$\mu \in H^{2b+1,b}_{\mathcal M}(\mathcal X,\mathbf Z)$$
$$\Big\downarrow \text{proj}$$
$$\text{homology of } \bigl[\,\mathrm{CH}^b(X)\to\mathrm{CH}^b(X^2)\to\mathrm{CH}^b(X^3)\bigr]$$

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Here

$$u = (a_1) \cup \cdots \cup (a_n) \in H^n_{\text{et}}(k, \mu_p^{\otimes n})$$

is a symbol (we assume $\mu_p \subset k$ and fix a generator of μ_p) and X is a smooth variety over k over which the symbol is split, i.e., $u_{k(X)} = 0$.

Furthermore, $\mathcal{X}$ is the simplicial scheme

$$\mathcal{X} : X \leftleftarrows X^2 \Lleftarrow X^3 \cdots$$

The map j relating motivic cohomology of $\mathcal{X}$ to Galois cohomology is an isomorphism if one assumes the generalized Milnor conjecture in weight $n-1$. For this one uses results from [6].

Then one applies the Milnor operations Q_i in motivic cohomology (these can be expressed in terms of the motivic Steenrod operations similarly as in topology) and the Bockstein homomorphism β.

One obtains the class

$$\mu \in H^{2b+1,b}_{\mathcal{M}}(\mathcal{X}, \mathbf{Z}), \qquad b = \frac{p^{n-1}-1}{p-1}$$

which plays an essential role in Voevodsky's work on the generalized Milnor conjecture, cf. [7]. If X is a norm variety for the symbol u, Voevodsky uses the class μ to show that X is a generic (up to extensions of degree prime to p) splitting variety for u and to split off from X a certain motive, the so-called generalized Rost motive. (For $p = 2$ genericity and the construction of the motive can be obtained in a much more elementary way using quadratic forms.) All this is essential for the final proof of the conjecture (involving, as for $p = 2$, Margolis homology and the so-called "injectivity", settled in [1], see also [5]).

An important step in handling μ is to verify a certain nontriviality condition. Some ingredients for this part of Voevodsky's work have not been written up in details yet, but it seems that they will appear soon, cf. [8].

Last year I was able to derive genericity and the construction of the motive from μ in a more ad hoc fashion, cf. [3]. One considers the standard spectral sequence for the simplicial scheme $\mathcal{X}$ which leads to the map proj as indicated in the diagram. Then one picks a representative

$$\rho \in \mathrm{CH}^b(X^2)$$

of $\mathrm{proj}(\mu)$. I call any such element a *basic correspondence of the norm variety X of u*. Working with ρ, the necessary nontriviality condition reads as

$$c(\rho) \neq 0 \tag{1}$$

where

$$c(\rho) \in \mathbf{Z}/p\mathbf{Z}$$

is a certain integer mod p (see [3, Section 5, p. 13] for the definition). I could verify condition (1) "by hand", so to speak, namely by investigating the specific examples of norm varieties I had constructed earlier in [1].

Once one knows (1), it is surprisingly easy to prove p-genericity of the norm variety using the functoriality of the definition of μ and ρ. The argument, essentially due to Voevodsky, was discussed in the lecture. It is described in [3, Section 6].

The construction of the motive can be done in the general setting of special correspondences, see [3, Section 7]. For $p = 2$ things become particularly easy, see [3, Section 7.3].

For an illustration, let me describe the basic correspondence in the case $n = 2$. In this case $b = 1$. For X we take a Severi-Brauer variety (of dimension $p - 1$). Thus ρ is an element in the Picard group of X^2:

$$\rho \in \mathrm{CH}^1(X^2) = \mathrm{Pic}(X^2)$$

If we pass to the algebraic closure $\bar{k}$ of k, then

$$X_{\bar{k}} = \mathbf{P}^{p-1}_{\bar{k}}$$

and one finds

$$\rho_{\bar{k}} = \pi_0^*[\mathcal{O}(1)] - \pi_1^*[\mathcal{O}(1)] \mod p\,\mathrm{Pic}(X^2_{\bar{k}})$$

where

$$\pi_0, \pi_1 \colon X \times X \to X$$

are the projections.

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