A FORMULA FOR THE EULER-PONCELET POINT

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Generally speaking, the Euler-Poncelet point EP of a quadrangle is the common point of the Feuerbach circles of the four subtriangles. (The Feuerbach or 9-point circle of a triangle is the circumcircle of the median triangle formed by the midpoints of the sides.)

If the quadrangle is an orthocentric system (a triangle completed by the intersection of its altitudes), then all Feuerbach circles coincide and EP is not defined.

If 2 points of the quadrangle coincide, then 2 Feuerbach circles are not defined and 2 Feuerbach circles coincide.

If the 4 points lie on a straight line, then the Feuerbach circles coincide with that line (or are not defined, if you wish).

Otherwise EP is defined.

If 3 of the 4 points lie on a straight line, then EP is the orthogonal projection of the 4th point to that line.

Let

$$z = (z_0, z_1, z_2, z_3) \in \mathbf{C}^4$$

As usual, complex conjugates are denoted by

$$\overline{z} = (\overline{z}_0, \overline{z}_1, \overline{z}_2, \overline{z}_3)$$

Consider the expression

$$EP(z) = \frac{1}{2} \cdot \frac{\displaystyle\sum_{i=0}^{3} \frac{z_i^2}{\prod_{j\neq i}(\overline{z}_i - \overline{z}_j)}}{\displaystyle\sum_{i=0}^{3} \frac{z_i}{\prod_{j\neq i}(\overline{z}_i - \overline{z}_j)}} \tag{1}$$

Proposition. *$EP(z)$ is the Euler-Poncelet point of the quadrangle z.*

In this text we will not prove this claim. However we will show that $EP(z)$ is a “quadrangle center”. This means that $EP(z)$ is invariant under permutations of the z_i (which is obvious) and that $EP(z)$ is equivariant with respect to similarities of the Euclidean plane (including reflections):

Lemma. *Let $a, b \in \mathbf{C}$, $a \neq 0$. Then*

$$EP(az_0 + b, az_1 + b, az_2 + b, az_3 + b) = a\,EP(z_0, z_1, z_2, z_3) + b$$

Moreover

$$EP(\overline{z}_0, \overline{z}_1, \overline{z}_2, \overline{z}_3) = \overline{EP(z_0, z_1, z_2, z_3)}$$

Proof. The first statement for $b = 0$ and the second statement are obvious.

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Let $e=(1,1,1,1)$. It remains to show

$$EP(z+e)=EP(z)+1 \tag{2}$$

We consider first the denominator

$$A(z)=\sum_{i=0}^{3}\frac{z_i}{\prod_{j\neq i}(\overline{z}_i-\overline{z}_j)}$$

and show that it is invariant under translations:

$$A(z+e)=A(z)$$

This amounts to

$$\sum_{i=0}^{3}\frac{1}{\prod_{j\neq i}(\overline{z}_i-\overline{z}_j)}=0 \tag{3}$$

The latter is a general identity in variables $x_i=\overline{z}_i$. It may be verified may an explicit computation. It follows also from the residue theorem applied to the differential form

$$\omega=\frac{dt}{(t-x_0)(t-x_1)(t-x_2)(t-x_3)}$$

Indeed, the residues of ω at $t=x_i$ are the terms appearing in (3). There are no further poles including $t=\infty$.

For the numerator

$$B(z)=\sum_{i=0}^{3}\frac{z_i^2}{\prod_{j\neq i}(\overline{z}_i-\overline{z}_j)}$$

one finds

$$B(z+e)=B(z)+2A(z)+0$$

using again (3).

Now (2) is immediate. □

One may rewrite formula (1) as follows. For variables $x=(x_0,x_1,x_2,x_3)$ let

$$D_0(x)=(x_1-x_2)(x_2-x_3)(x_3-x_1)$$

and for $i=1,2,3$ let $D_i(x)$ be obtained from $(-1)^iD_0(x)$ by replacing x_i with x_0. In other words,

$$D_i(x)\prod_{j\neq i}(x_j-x_i)=\delta(x),\qquad \delta(x)=\prod_{k<\ell}(x_k-x_\ell)$$

Clearing denominators in (1) by multiplying with $\delta(\overline{z})$ yields

$$EP(z)=\frac{1}{2}\cdot\frac{\sum_{i=0}^{3}z_i^2D_i(\overline{z})}{\sum_{i=0}^{3}z_iD_i(\overline{z})}$$

Appendix

We conclude with a general discussion of "points" and "centers".

The expression in (1) is evidently invariant under transformations

$$z_i \to z_i, \quad \overline{z}_i \to a\overline{z}_i + b \tag{4}$$

of the $\overline{z}_i$ alone.

This is a general feature.

Let E be the Euclidean plane and let us define an "n-gon point" as a function

$$f \colon E^n \to E$$
$$(P_1, \ldots, P_n) \mapsto f(P_1, \ldots, P_n)$$

which is equivariant under the group G of proper similitudes of E:

$$f\big(g(P_1), \ldots, g(P_n)\big) = g\big(f(P_1, \ldots, P_n)\big) \qquad (g \in G)$$

If f is invariant under all permutations of the P_i, it is called an "n-gon center".

For instance, the feet of the altitudes in a triangle are triangle points (but not triangle centers).

We assume that f is a rational map (defined on an appropriate open subset $U \subset E^n$). If we identify $E = \mathbf{R}^2$ and write

$$P_r = (x_r, y_r)$$

this means that f is a pair $f = (f_1, f_2)$ of rational functions

$$f_1, f_2 \in \mathbf{R}(x_1, y_1, \ldots, x_n, y_n)$$

If we identify $E = \mathbf{C}$ and write

$$f = f_1 + if_2, \quad z_r = x_r + iy_r, \quad \overline{z}_r = x_r - iy_r$$

we get an element

$$f \in \mathbf{C}(z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n)$$

in the function field of $2n$ independent variables $z_r, \overline{z}_r$ over $\mathbf{C}$.

The rationality condition is a bit too strong and is made here for simplicity. For example, the incenter of a triangle yields an element f in a biquadratic extension of the field $\mathbf{C}(z_1, \overline{z}_1, z_2, \overline{z}_2, z_3, \overline{z}_3)$ and is not rational (but real-analytic).

Consider then the complex valued function

$$F(P_1, \ldots, P_n) = \frac{f(P_1, \ldots, P_n) - P_1}{P_2 - P_1}$$

Since f is equivariant under G it follows that F is invariant under G. Since G acts generically free on E^2 with dense orbit, it follows that F is of the form

$$F = \widetilde{F}(\eta_3, \ldots, \eta_n), \qquad \eta_r = \frac{P_r - P_1}{P_2 - P_1}$$

Thus f is of the form

$$f(P_1, \ldots, P_n) = P_1 + (P_2 - P_1)\widetilde{F}(\eta_3, \ldots, \eta_n)$$

Considered as element in $\mathbf{C}(z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n)$ this means that

$$f \in \mathbf{C}(z_1, \ldots, z_n, \lambda_3, \ldots, \lambda_n), \qquad \lambda_r = \frac{\overline{z}_r - \overline{z}_1}{\overline{z}_2 - \overline{z}_1}$$

It follows that indeed f is invariant under the transformations (4).

In the example $EP(z)$ it easy to get the corresponding element

$$EP(z) \in \mathbf{C}(z_1, z_2, z_3, z_4, \lambda_3, \lambda_4)$$

One just writes

$$\overline{z}_i - \overline{z}_j = (\overline{z}_i - \overline{z}_1) - (\overline{z}_j - \overline{z}_1)$$

and divides by $(\overline{z}_2 - \overline{z}_1)$.

In the case $n = 3$ the situation is particularly nice. The orthocenter H of the triangle z_1, z_2, z_3 is

$$H = \frac{\lambda(z_2 - z_1)(z_2 - z_3 + z_1) + (z_3 - z_1)(z_2 - z_3 - z_1)}{\lambda(z_2 - z_1) - (z_3 - z_1)}$$

with

$$\lambda = \frac{\overline{z}_3 - \overline{z}_1}{\overline{z}_2 - \overline{z}_1}$$

Since H is a linear fractional in λ (over $\mathbf{C}(z_1, z_2, z_3)$), one can express λ in terms of H. It follows that every rational triangle center is actually an element of

$$\mathbf{C}(z_1, z_2, z_3, H)$$

For more details and examples see my text "The holomorphic extension of triangle functions".

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