NORM VARIETIES AND THE CHAIN LEMMA

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Notes taken by Christian Haesemeyer. Preliminary version.

Recall from the fall term lectures ([4]) that we want to prove the following theorem. Here p is a fixed prime, k a field of characteristic 0, containing the p-th roots of unity.

Theorem 0.1. *Let $n \geq 2$ and $0 \neq \{\underline{a}\} = \{a_1, \dots, a_n\} \in K_n^M(k)/p$. Then*

1) There exists an absolutely irreducible projective p-generic splitting variety for $\{\underline{a}\}$ of dimension $p^{n-1} - 1$.

Assume further that k is p-special. If X is a projective p-generic splitting variety for $\{\underline{a}\}$ of dimension $p^{n-1} - 1$, then

2) X is a ν_{n-1}-variety;

3) each element of $\overline{A}_0(X, \mathcal{K}_1)$ is of the form $[x, \lambda]$, where $x \in X$ is a closed point of degree p and $\lambda \in k(x)^\times$.

In the fall term, the splitting varieties in the theorem were constructed (see [4, ??]) and the necessary properties proved, modulo two results. For the notation and terminology used, see the notes from the fall term lectures. First, the chain lemma.

Theorem 0.2 (Chain Lemma)**.** *Assume k is p-special. Let $\{a_1, \dots, a_n\} \in K_n^M(k)/p$ be a nontrivial symbol. Then there exists a smooth projective cellular variety S/k and a collection of invertible sheaves $J = J_1, J_1', \dots, J_{n-1}, J_{n-1}'$ equipped with nonzero p-forms $\gamma = \gamma_1, \dots, \gamma_{n-1}'$ satisfying the following conditions.*

(1) $\dim S = p(p^{n-1} - 1) = p^n - p$;
(2) $\{a_1, \dots, a_n\} = \{a_1, \dots, a_{n-2}, \gamma_{n-1}, \gamma_{n-1}'\} \in K_n^M(k(S))/p$, *and for each* $2 \leq i \leq n-1$, *we have* $\{a_1, \dots, a_{i-1}, \gamma_i\} = \{a_1, \dots, a_{i-2}, \gamma_{i-1}, \gamma_{i-1}'\} \in K_i^M(k(S))/p$. *In particular,* $\{a_1, \dots, a_n\} = \{\gamma, \gamma_1', \dots, \gamma_{n-1}'\} \in K_n^M(k(S))/p$;
(3) $\gamma \notin \Gamma(S, J)^{\otimes(-p)}$;
(4) *for any* $s \in V(\gamma_i) \cup V(\gamma_i')$, *the field* $k(s)$ *splits* $\{a_1, \dots, a_n\}$;
(5) $I(V(\gamma_i)) + I(V(\gamma_i')) \subset p\mathbb{Z}$ *for all* i;
(6) $\deg(c_1(J)^{\dim S})$ *is relatively prime to* p.

And, the norm principle.

Proposition 0.3 (Norm Principle)**.** *Again, suppose k is p-special and that X is a p-generic slitting variety of dimension $p^{n-1} - 1$ for some symbol $\{\underline{a}\}$. Let $[x, \lambda] \in \overline{A}_0(X, \mathcal{K}_1)$ be such that $[k(x) : k] = p^m$ for $m > 1$. Then there exist $x_i \in X$, $\lambda_i \in k(x_i)^\times$ with $[k(x_i) : k] < [k(x) : k]$ for all i such that $[x, \lambda] = \sum_i [x_i, \lambda_i]$.*

The goal of these lectures is to furnish proofs for these two results.

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1. Lecture 1. January 19, 2005.

This lecture makes a start on proving the chain lemma in weight 2, that is, in the case of central simple algebras, as the constructions can be made very explicit in that case. The general case will be tackled in subsequent lectures.

As a matter of notation, we fix a prime p, sometimes also called l; a field k of characteristic different from p, containing the p-th roots of unity (in fact, we will assume that k is p-special); and a primitive p-th root of unity ζ.

Suppose $a, b \in k^\times$. Then we define a central simple algebra

$$B = A_\zeta(a,b) = k\{X, Y | X^p = a, Y^p = b, YX = \zeta XY\}. \tag{1.1}$$

Observe that $A_\zeta(1,1) \cong M_p(k)$, with generators X_0 a permutation matrix of order p and $Y_0 = \mathrm{diag}(1, \zeta, \zeta^2, \dots, \zeta^{p-1})$.

There is a reduced norm map $\mathrm{Nrd} : B \to k$ (in the split case $B = M_p(k)$, this is just the determinant) with the property that for any maximal commutative subalgebra $E \subset B$, $\mathrm{Nrd}|_E = N_{E/k}$ the reduced norm is just the ordinary norm. We will mainly be interested in the case $E = k[X] \cong k[t]/(t^p - 1)$.

Using the reduced norm, we can define a characteristic polynomial $\mathrm{Nrd}(t \cdot 1 + x) = t^p + T_1(x)t^{p-1} + \cdots + T_p(x)$ for $x \in B$, with $T_p(x) = \mathrm{Nrd}(x)$.

Definition 1.2. Let B be a central simple algebra of degree p. A *Kummer element* $x \in B$ is an element such that $T_i(x) = 0$ for $i = 1, \dots, p-1$ and $\mathrm{Nrd}(x) \neq 0$. In particular, $x^p = (-1)^{p+1}\mathrm{Nrd}(x)$.

Remark 1.3. In the split case, when $k = \overline{k}$ and thus $B = M_p(k)$, any Kummer element is conjugate to some scalar multiple of X_0 or Y_0.

Definition 1.4. A *Kummer pair* relative ζ in the central simple algebra B of degree p is a pair (x, y) of elements of $B^\times$ such that $xy = \zeta yx$ (equivalently, $yxy^{-1} = \zeta x$).

We list some elementary properties of Kummer pairs.

Lemma 1.5. *If (x, y) is a Kummer pair, then x and y are Kummer elements.*

Proof. As $yxy^{-1} = \zeta x$, the eigenvalues of x (as element of $B_{\overline{k}}$) are of the form $\lambda, \zeta\lambda, \dots, \zeta^{p-1}\lambda$ for some $\lambda \neq 0$, and hence the characteristic polynomial of x is of the form $t^p - \lambda^p$. □

Lemma 1.6 (parametrization lemma)**.** *Let (x, y) be a Kummer pair. Then (x, z) is a Kummer pair if and only if $z \in y \cdot (k[x])^\times$.*

Proof. Because x is semisimple, the centralizer $C_B(x)$ of x in B is exactly $k[x]$. It follows from the definition of a Kummer pair that (x, z) is a Kummer pair if and only if z is invertible and yz^{-1} is in $C_B(x)$. □

Lemma 1.7. *If $k = \overline{k}$, then any Kummer pair (x, y) in $M_p(k)$ is (simultaneously) conjugate to a pair (cX_0, dY_0) for some $c, d \in k^\times$*

Proof. By Lemma 1.5, we can assume that $x = c'X_0$ for some $c' \in k^\times$. Now an application of Lemma 1.6 and Hilbert's Theorem 90 allows to complete the proof. □

The choice of a Kummer pair in a central simple algebra of degree p allows us to realize it in the form of 1.1.

Lemma 1.8. *Let (x,y) be a Kummer pair in B. The assignment $X \mapsto x$, $Y \mapsto y$ defines an isomorphism of algebras*

$$A_\zeta(x^p, y^p) \to B$$

Next, we study varieties of Kummer elements and *Kummer chains.*

Given a central simple algebra B of degree p, we define a variety of Kummer elements $\mathfrak{K} = \mathfrak{K}(B) \subset \mathbb{P}(B) \cong \mathbb{P}^{p^2-1}$ by

$$\mathfrak{K}(A) = \{[x] \in \mathbb{P}_A(B \otimes A) | x \in (B \otimes A)^\times \text{ and } T_i(x) = 0, i = 1, \dots, p-1\}$$

for a (commutative) k-algebra A. Note that $\mathfrak{K}$ is defined by $p-1$ equations inside $\mathbb{P}(B)$ and hence $\dim(\mathfrak{K}) = p^2 - p$.

For a Kummer pair (x,y) in B we define a *Kummer chain* of length n starting from (x,y) to be a sequence $(x_1, \dots, x_n)$ of elements in $B^\times$ such that $x_i x_{i-1} = \zeta^{(-1)^i} x_{i-1} x_i$ for $i = 0, \dots, n$, with $x_0 = y$. That is, (x_i, x_{i-1}) is a Kummer pair relative $\zeta^{(-1)^i}$.

To connect all this to K_2, we make the following observations. Let $(x_1, \dots, x_n)$ be a Kummer chain starting at (x,y), let $x_{-1} = x$ and $x_0 = y$ and define $a_i = x_i^p \in k^\times$ for all i. By Lemma 1.8, we have isomorphisms

$$B \cong A_{\zeta^{(-1)^i}}(a_{i-1}, a_i), \quad i = 0, \dots n.$$

In $K_2(k)/p$, we have the relations $\{a_{-1}, a_0\} = -\{a_0, a_1\} = \dots = \pm\{a_{n-1}, a_n\}$ and $a_i \in a_{i-2} N(k(\sqrt[p]{a_{i-1}}))$. That is, the symbols $\{a_{-1}, a_0\}$ and $\{a_{n-1}, a_n\}$ are connected by a chain of elementary changes.

The *variety of Kummer chains of length n* starting at (x,y) is the locally closed subvariety $\mathfrak{K}_n = \mathfrak{K}_n(B,y) \subset \mathbb{P}(B)^{\times n}$ with points

$$\mathfrak{K}_n(A) = \{([x_1], \dots, [x_n]) | x_i \in (B \otimes A)^\times \text{ and } x_i x_{i-1} = \zeta^{(-1)^i} x_{i-1} x_i, i = 1, \dots, n\}$$

for a (commutative) k-algebra A (here, $x_0 = y \otimes 1$). We observe that $\mathfrak{K}_n(B,y)$ depends on B, ζ and y, but not on x.

Lemma 1.9. *A sequence $(x_1, \dots, x_n)$ is a Kummer chain starting at (x,y) if and only if $x_1 \in k[y]^\times \cdot x$, and $x_i \in k[x_{i-1}]^\times \cdot x_{i-2}$ for $i = 2, \dots, n$ and $x_0 = y$.*

Proof. This follows from Lemma 1.6 and the observation that a pair (z,w) is a Kummer pair relative ζ if and only if (w,z) is a Kummer pair relative ζ^{-1}. □

This lemma implies that the morphism $\mathfrak{K}_n \to \mathfrak{K}_{n-1}$ given by dropping the last component is a torus bundle of rank $p-1$; in particular, we have that $\dim(\mathfrak{K}_n) = n(p-1)$.

Let $h_n : \mathfrak{K}_n \to \mathfrak{K}$ be the projection onto the last component. Then a point $[z] \in \mathfrak{K}(k)$ is in the image of $h_n(k)$ if and only if z can be obtained from (x,y) via a Kummer chain of length n. The morphism $h_p : \mathfrak{K}_p \to \mathfrak{K}$ is a morphism of varieties of the same dimension. We will prove the following result, which will in turn be used to prove the chain lemma (Theorem 0.2) in weight 2 (the cellular variety S in that theorem will be the iterated projective space bundle associated to the iterated torus bundle $\mathfrak{K}_n$).

Theorem 1.10. *The degree of $h_p : \mathfrak{K}_p \to \mathfrak{K}$ is prime to p. More precisely,* $\deg(h_p) \equiv -1 \mod p$.

Proof. Will be given in a future lecture. □

We remark that in special cases, we can be more precise. In general, we do not know the exact value of $\deg(h_p)$, even though the construction of the morphism is very explicit. We do know that for $p = 2$, we have $\deg(h_2) = 1$ and for $p = 3$, we have $\deg(h_3) = 2$ (see [1]). Moreover, if $p = 3$, then h_4 has a rational section (see [3], [2]).

2. Lecture 2. January 26, 2005.

In this lecture, we turn to an approach of constructing the cellular variety of the chain lemma 0.2 that works in all weights. At the end of the lecture, we will relate this to the constructions using Kummer chains explained in the first lecture.

We start by recalling a number of well-known facts about vector bundles and Chern classes; in the process, we introduce some notation.

The Chow ring of a projective bundle. Let X/k be a quasiprojective variety, and $\pi_V : V \to X$ a (geometric) vector bundle. We write $\pi_{\mathbb{P}(V)} : \mathbb{P}(V) \to X$ for the associated projective space bundle (that is, if $V = \mathbf{Spec}(S^*(\mathcal{M}))$ for some locally free $\mathcal{O}_X$-module $\mathcal{M}$, then $\mathbb{P}(V) = \mathbf{Proj}(S^*(\mathcal{M}))$).

The tautological line bundle of $\mathbb{P}(V) \to X$ is the subbundle $\mathbb{L}(V) \subset \pi^*_{\mathbb{P}(V)}V = V \times_X \mathbb{P}(V)$ whose fiber over a point $(x, L) \in \mathbb{P}(V)$ (here $x \in X$ and $L \subset V_x$ is a line in the fiber V_x of V over x) is L. This is the geometric vector bundle with sheaf of sections $\mathcal{O}_V(-1)$. Observe that the natural map (induced by projection) $\phi_V : \mathbb{L}(V) \to V$ is the blow-up of the zero section of V in V.

Now assume that X is smooth, $r = \mathrm{rk}(V)$, and $z = c_1(\mathbb{L}(V)) \in CH^1(\mathbb{P}(V))$ is the first Chern class of the tautological line bundle. A standard computation shows that
$$CH^*(\mathbb{P}(V)) = CH^*(X)[z]/(c_V(z))$$
where $c_V(z) = z^r - c_1(V)z^{r-1} + \cdots + (-1)^r c_r(V)$. The classes $c_i(V) \in CH^i(X)$ are the Chern classes of V. In particular, $CH^*(\mathbb{P}(V))$ is a free module over $CH^*(X)$ with basis $1, z, \ldots, z^{r-1}$. We write $c(V) = 1 + c_1(V) + \cdots + c_r(V)$ for the total Chern class of V.

Example 2.1. Let $H \to X$ be a line bundle, $x = c_1(H) \in CH^1(X)$ and $V = A(H) := \bigoplus_{i=0}^{p-1} H^{\otimes i}$. Then the total Chern class of V is
$$c(V) = \prod_{i=0}^{p-1}(1 + ix) \equiv (1 - x^{p-1}) \mod p.$$
Hence $CH^*(\mathbb{P}(V))/p \cong (CH^*(X)/p)[z]/(z^p - zx^{p-1})$.

Euler class of a vector bundle. Again, let $V \to X$ be a vector bundle of rank r, with X equidimensional (but not necessarily smooth) of dimension d over k. Let $i : X \to V$ be the zero section.

Definition 2.2. The *Euler class* of V is the class $e(V) = i^* i_*([X]) \in CH_{d-r}(X)$. Note that i^* makes sense because the zero section is a regular embedding.

Remark 2.3. When the Chern classes of V make sense (e.g., if X is smooth), then $e(X) = c_r(X)$.

Now assume $d = r$ and X is proper. In that case $e(V) \in CH_0(X)$ and we can use the degree map $\deg : CH_0(X) \to \mathbb{Z}$ to obtain an integer $\deg(e(V))$, called the degree of V.

Example 2.4. For a line bundle $L \to X$ we call $\delta(L) = \deg(L^{\oplus d}) \in \mathbb{Z}$ the *degree* of L; observe that $\delta(L) = \deg(c_1(L)^d)$.

Example 2.5. If V/k is a vector space of dimension n, then the degree of the tautological line bundle over $\mathbb{P}(V)$ is $\delta(\mathbb{L}(V)) = (-1)^{n-1}$.

Construction of Varieties in the Chain Lemma. The goal is to construct certain iterated projective bundles together with line bundles and differential forms on them as needed in the chain lemma 0.2. Recall from Example 2.1 that for a line bundle H, $A(H)$ denotes the vector bundle $\bigoplus_{i=0}^{p-1} H^{\otimes i}$.

We start with a variety X_0 and line bundles H_{-1}, H_0 on X_0 (for example, we could choose $X_0 = \mathrm{Spec}(k)$).

From this initial data, we construct inductively varieties X_r and line bundles H_r on X_r, for $r \geq 1$, together with morphisms $f_r : X_r \to X_{r-1}$; all f_r will be projective space bundles of relative dimension $p-1$.

Namely, we define

$$f_r = \pi_{\mathbb{P}(A(H_{r-1}))} : X_r := \mathbb{P}(A(H_{r-1})) \to X_{r-1}.$$

Setting $L_r := \mathbb{L}(A(H_{r-1}))$ we define

$$H_r := (f_{r-1} \circ f_r)^*(H_{r-2}) \otimes_{\mathcal{O}_{X_r}} L_r.$$

(For $r = 1$, let $f_0 = \mathrm{id}_{X_0}$.)

Let us consider the easiest example of this construction, when $X_0 = \mathrm{Spec}(k)$.

In this case, H_{-1}, H_0 are obviously trivial, and $\dim X_r = r(p-1)$. The next result computes $\delta H_r \mod p$; it is the basis of all our future computations of degrees.

Proposition 2.6. *Suppose $1 \leq r \leq p$. Then $\delta(H_r) \equiv 1 \mod p$.*

Proof. The proof has two parts: First, we compute an explicit presentation of the Chow ring $CH^*(X_r)/p$; then we use this explicit computation and some elementary algebra to calculate $\delta(H_r)$.

So, let $x_i = c_1(H_i) \in CH^1(X_i)/p$ and $z_i = c_1(L_i) \in CH^1(X_i)/p$. From Example 2.1 recall that

$$CH^*(X_r)/p = (CH^*(X_{r-1})/p)[z_r]/(z_r^p - z_r x_{r-1}^{p-1})$$

whence

$$CH^*(X_r)/p \cong \mathbb{F}_p[x_1, \ldots, x_r, z_1, \ldots, z_r]/I$$

where

$$I = (z_i^p - z_i x_{i-1}^{p-1}, x_i - x_{i-2} - z_i | i = 0, 1, \ldots r)$$

(here we set $x_{-1} = x_0 = 0$ to simplify notation).

So, as $\mathbb{F}_p$ vector space, $CH^*(X_r)/p$ has a basis consisting of the monomials $z_1^{n_1} \cdots z_r^{n_r}$ for $0 \leq n_i \leq p-1$. In particular, $CH_0(X_r)/p$ is generated by one element $u = z_1^{p-1} \cdots z_r^{p-1}$ and $x_r^{r(p-1)} \equiv du \mod p$ for some $d \in \mathbb{F}_p$.

We can compute $\deg(u)$ by induction. For $r = 1$, Example 2.5 shows that $\deg((z_1)^{p-1}) = (-1)^{p-1}$. The projection formula implies that

$$\deg(u) = \deg(z_r^{p-1} \cdot (z_1^{p-1} \cdots z_{r-1}^{p-1})) = \deg((f_r)_*(z_r^{p-1}) \cdot (z_1^{p-1} \cdots z_{r-1}^{p-1}))$$

(note we abuse notation here, writing z_i for the classes $z_i \in CH^1(X_i)$, $f_{i+1}^*(z_i) \in CH^1(X_{i+1})$ etc.). Applying Example 2.5 one more time, we conclude that $(f_r)_*(z_r^{p-1}) = (-1)^{p-1}[X_{r-1}]$; by induction, $\deg(u) = (-1)^{(p-1)r} \equiv 1 \mod p$.

This means we can calculate $\delta(H_r) = \deg(x_r^{r(p-1)}) \equiv d \mod p$. We will prove by induction on r that $d = 1$. This is obvious if $r = 1$. Suppose $1 < r \leq p$. Recall that $x_r = z_r + x_{r-2}$. So

$$x_r^{r(p-1)} = (z_r + x_{r-2})^{p(r-1)+(p-r)} \equiv (z_r^p + x_{r-2}^p)^{r-1} \cdot (z_r + x_{r-2})^{p-r} \mod p$$

(because $p - r \geq 0$). Employing the relation $z_r^p \equiv z_r x_{r-1}^{p-1} \mod p$, we get that

$$x_r^{r(p-1)} \equiv (z_r x_{r-1}^{p-1} + x_{r-2}^p)^{r-1} (z_r + x_{r-2})^{p-r} \equiv z_r^{p-1} x_{r-1}^{(r-1)(p-1)} + T \mod p$$

where T is not divisible by z_r^{p-1} and hence $T \equiv 0 \mod p$ for dimension reasons. By inductive hypothesis,

$$x_{r-1}^{r(p-1)} = z_r^{p-1} x_{r-1}^{(r-1)(p-1)} = z_r^{p-1} \cdot (z_1^{p-1} \cdots z_{r-1}^{p-1}) = u$$

in $CH^*(X_r)/p$, as claimed. $\square$

Connection with Kummer Chains. Recall from the first lecture that given a central simple algebra B over k of degree p and a Kummer pair (x, y) in B we constructed quasiprojective varieties $\mathfrak{K}_r$, for $0 \leq r$, together with maps $g_{r+1} : \mathfrak{K}_{r+1} \to \mathfrak{K}_r$ that are torus bundles of rank $p-1$; we also constructed another variety $\mathfrak{K}$ and a morphism $h_p : \mathfrak{K}_p \to \mathfrak{K}$ of varieties of the same dimensions, and formulated a theorem asserting that the degree of this map was invertible $\mod p$ (cf. Theorem 1.10). To prove this assertion, we will apply the degree formula [4, ??] to a rational map $X_p \to \overline{\mathfrak{K}} \subset \mathbb{P}(B)$ and the line bundles H_p and $\mathbb{L}(B)|\overline{\mathfrak{K}}$ and show that $\mathfrak{K}_p$ has an open embedding into X_p.

Proposition 2.7. *Let B be a central simple algebra over k of degree p and (x, y) a Kummer pair in B. For all $r \geq 0$, there is natural open embedding $i_r : \mathfrak{K}_r = \mathfrak{K}_r(B, y) \subset X_r$ such that the diagram*

$$\begin{array}{ccc} \mathfrak{K}_{r+1} & \xrightarrow{g_{r+1}} & \mathfrak{K}_r \\ {\scriptstyle i_{r+1}}\big\downarrow & & {\scriptstyle i_r}\big\downarrow \\ X_{r+1} & \xrightarrow{f_{r+1}} & X_r \end{array}$$

commutes.

Proof. Instead of a rigorous proof, we give more of an explanation. To make any sense of what follows, view all appearing k-vector spaces, k-alegbras and Zariski open subsets of them as k-vector space schemes, k-algebra schemes and open subschemes. For example, if E/k was a quadratic field extension, then $E^\times$ and $E - \{0\}$ would have the same k-rational points, but they are not isomorphic as schemes: the E-valued points of $E^\times$ (viewed as scheme) are $(E \otimes_k E)^\times = E^\times \times E^\times$, while the E-valued points of $E - \{0\}$ are $((E \otimes_k E) - \{0\}) = (E \times E) - \{(0, 0)\}$.

First, we define i_1. Since $X_0 = \mathrm{Spec}(k)$, X_1 is just a projective space, which we identify with $\mathbb{P}(k[y] \cdot x)$. The embedding $i_1 : \mathfrak{K}_1 \to X_1$ is now induced by the embedding $k[y]^\times \cdot x \to (k[y] - \{0\}) \cdot x$. With this identification of X_1, the fiber of $H_1 = L_1$ over a point $[x_1] \in \mathbb{P}(k[y] \cdot x)$ is exactly the line $k \cdot x_1$. Therefore the fiber of $A(H_1)$ over $[x_1]$ can be naturally identified with $k[x_1] \cdot y$. In this way, we obtain a natural open embedding $i_2 : \mathfrak{K}_2 \to X_2$, over X_1; fiberwise, it is induced by the open embedding $k[x_1]^\times \cdot y \to (k[x_1] - \{0\}) \cdot y$. We proceed inductively in the same way to obtain the embeddings $i_3, i_4, \ldots$. On A-valued points, for a commutative k-algebra A, we obtain an identification of $X_r(A)$ with sequences $(([x_1], [x_2], \ldots, [x_r])$ such that $x_i \in (A[x_{i-1}] - \{0\}) \cdot x_{i-2} \subset B \otimes_k A$ for $1 \le i \le r$, where we write $x_{-1} = x$ and $x_0 = y$; with this identification, $i_r(A)$ becomes the obvious embedding of the subset $\mathfrak{K}_r(A)$ of $X_r(A)$.

Note however that X_r does not depend on the choice of (B, y) (even though the description of $X_r(A)$ given above does), while the open subscheme $\mathfrak{K}_r$ does. □

Recall that for a proper k-scheme Y we define $I(Y)$ as the ideal in $\mathbb{Z}$ generated by the image of the degree map $\deg : CH_0(Y) \to \mathbb{Z}$.

Proposition 2.8. *Let $0 \le r \le p$ and suppose the central simple algebra B is a skew field over k. Then $I(X_r - i_r(\mathfrak{K}_r)) \subset p\mathbb{Z}$.*

Proof. Let F/k be a finite field extension. Given the identifications indicated in the proof of Proposition 2.7, a point in $X_r(F)$ can be given as a sequence $([x_1], [x_2], \dots, [x_r])$ such that $x_i \in (F[x_{i-1}] - \{0\}) \cdot x_{i-2} \subset B \otimes_k F$ for $1 \le i \le r$, where we write $x_{-1} = x$ and $x_0 = y$.

On the other hand, such a point will be in $\mathfrak{K}_r(F)$ provided $x_i \in (F[x_{i-1}]^\times) \cdot x_{i-2}$ for $1 \le i \le r$. Observe that x_j is a Kummer element, for $-1 \le j \le r$, by Lemma 1.5; because B is a skew field and of degree p over k, this implies that $F[x_j]$ is a field if $\gcd([F : k], p) = 1$. In that case, $F[x_j]^\times = F[x_j] - \{0\}$. Hence, $\mathfrak{K}_r(F) = X_r(F)$ or $p | [F : k]$ (note, however, that $\mathfrak{K}_r(E) \ne X_r(E)$ over any field E splitting B). This finishes the proof. □

Recall that $\mathfrak{K} = \mathfrak{K}(B)$ was the variety of Kummer elements of B. We are interested in the morphism $h_p : \mathfrak{K}_p \to \mathfrak{K}$ given by projection onto the last component. To apply some version of the degree formula, we extend this to a morphism of projective varieties $X_p \to X$, defining $X(A) = \{[x] \in \mathbb{P}_A(B \otimes A) | T_i(x) = 0, i = 1, \dots, p-1\}$ for a commutative k-algebra A (recall that $T_i(x)$ denotes coefficients of the characteristic polynomial of x). Clearly, we have an open embedding $\mathfrak{K} \to X$, and h_p extends to a morphism $\tilde{h}_p : X_p \to X$. Note also that by an argument as in Proposition 2.8 we see that $I(X - \mathfrak{K}) \subset p\mathbb{Z}$ if B is a skew field. Let L be the restriction of the tautological line bundle $\mathbb{L}(B)$ of $\mathbb{P}(B)$ to X; the fiber of L over $[x]$ is $k \cdot x \subset B$.

Lemma 2.9. *There is an isomorphism of line bundles on $\mathfrak{K}_p$, $h_p^*(L|\mathfrak{K}) \cong H_p|\mathfrak{K}_p$.*

Now we can use the degree formula for the morphism h_p with completion $\tilde{h}_p$ and the line bundles L and H_p, applying Proposition 2.8 to conclude:

Proposition 2.10. *Assume that B is a skew field. Then $\delta(H_p) \equiv \deg(h_p)\delta(L) \bmod p$.*

In turn, this implies Theorem 1.10 in case B is a skew field, in light of the computations of $\delta(L)$ in Example 2.5 and of $\delta(H_p)$ in Proposition 2.6. To obtain Theorem 1.10 in the general case, it suffices to observe that $\mathfrak{K}$ and $\mathfrak{K}_p$ are absolutely irreducible so that the degree of h_p does not change when scalars are extended; hence the case where B is a skew field implies the case where k is algebraically closed (so B is split), and that in turn implies the assertion for arbitrary B.

3. Lecture 3. February 2, 2005.

This lecture consists of two parts. First, we complete the proof of the chain lemma in weight 2 by introducing the necessary forms into the picture. In the second part, we give some heuristic ideas concerning the chain lemma in higher weights.

Notation. As before, $V \to X$ denotes a (geometric) vector bundle. We write Θ for the trivial line bundle over $\mathrm{Spec}(k)$. A *p-form* or *form of degree p* $\gamma : V \to \Theta$ is a fiberwise linear map $\mathrm{Sym}_p(V) \to Theta$ along $f : X \to \mathrm{Spec}(k)$. (Equivalently, this is a section of the bundle $\underline{\mathrm{Hom}}(\mathrm{Sym}_p(V), f^*\Theta) = \mathrm{Sym}_p(V)^*$.) Here $\mathrm{Sym}_p(V) = S_p(V) = \Gamma^p(V) = (V^{\otimes p})^{\Sigma_p}$ is the bundle of p-th divided powers of V.

Given a line bunlde $L \to X$ and a p-form γ on L, we define a bundle of $\mathcal{O}_X$-algebras $A_\gamma = A_\gamma(L)$ together with a p-form N_γ as follows.

$$A_\gamma = T_{\mathcal{O}_X}(L)/ < x^{\otimes p} - \gamma(x) >$$

(here $T_{\mathcal{O}_X}(L)$ is the tensor algebra bundle of L) and N_γ is the fiberwise norm on A_γ. Note that the natural morphism $\mathbf{Spec}(A_\gamma) \to X$ is flat of degree p, as A_γ is a bundle of degree p algebras.

The Chain Lemma in Weight 2. Recall that we constructed in the previous lecture, starting with a variety X_0 and line bundles H_{-1} and H_0 on X_0 a sequence of projective bundles

$$X_p \xrightarrow{f_p} X_{p-1} \xrightarrow{f_{p-1}} \cdots \xrightarrow{f_1} X_0$$

together with line bundles L_r and H_r on X_r. We will inductively define p-forms ϵ_r on L_r and γ_r on H_r, starting from given forms γ_{-1} and γ_0. Assume $\gamma_{-1}, \dots, \gamma_{r-1}$ are already defined. Recall that $L_r = \mathbb{L}(A(H_{r-1}))$ and $H_r = (f_{r-1} \circ f_r)^*(H_{r-2}) \otimes L_r$. Observe that $A(H_{r-1}) \cong A_{\gamma_{r-1}}(H_{r-1})$ as vector bundle, and on the latter we have the p-form $N_{\gamma_{r-1}}$. Now we define ϵ_r to be the composition

$$\mathrm{Sym}_p(\mathbb{L}(A(H_{r-1}))) \to \mathrm{Sym}_p(A(H_{r-1})) \xrightarrow{N_{\gamma_{r-1}}} \Theta$$

where the first map is induced by the canonical inclusion $\mathbb{L}(A(H_{r-1})) \to f_r^*(A(H_{r-1}))$.

In turn we define $\gamma_r = (f_{r-1} \circ f_r)^*(\gamma_{r-2}) \otimes \epsilon_r)$ (note there is a natural isomorphism $\mathrm{Sym}_p(L \otimes H) \cong \mathrm{Sym}_p(L) \otimes \mathrm{Sym}_p(H)$ for line bundles L and H, so that the tensor product of p-forms ϵ and γ on L and H, respectively, defines qa p-form $\epsilon \otimes \gamma$ on $L \otimes H$).

Now we go back to the example of this construction where $X_0 = \mathrm{Spec}(k)$ and $H_{-1} = H_0 = k$ are just trivial line bundles. A p-form on k is determined, up to isomorphism, by an element $a \in k^\times/(k^\times)^p$. Choose $a, b \in k^\times$ to define the forms $\gamma_{-1}(t) = at^p$ and $\gamma_0(t) = bt^p$ and write $u = \{a, b\} \in K_2(k)/p$ for the resulting symbol.

As explained in above, we obtain from this initial data a sequence $(X_r, L_r, H_r, \epsilon_r, \gamma_r)$ of (smooth projective cellular) varieties, line bundles, and p-forms. The next proposition will finish the proof of the chain lemma for weight 2. For a scheme Z, a point Q on some Z-scheme $f : S \to Z$ and a vector bundle V on Z we write $V|Q$ for the $k(Q)$ vector space $f^*(V)_Q \otimes_{\mathcal{O}_{S,Q}} k(Q)$.

Proposition 3.1. *Let $P \in X_r$ be a point with residue field $E = k(P)$; choose generators $x_i \in H_i|P$ and $y_i \in L_i|P$ for the one-dimensional E vector spaces $H_i|P$ and $L_i|P$, for $-1 \le i \le r$ in such a way that $x_i = x_{i-2} \otimes y_i$.*

1. *If for some $1 \le i \le r$ we have $\gamma_i|P = 0$ (than is, $\gamma_i(x_i) = 0$), then $\{a, b\}_E = 0 \in K_2(E)/p$.*
2. *If $\gamma_i|P \neq 0$ for $-1 \le i \le r$, then*

$$\{a, b\}_E = (-1)^r \{\gamma_{r-1}(x_{r-1}), \gamma_r(x_r)\} \in K_2(E)/p.$$

Proof. By induction on r. Both parts are obvious if $r = 0$. To prove the first part, we may assume that $\gamma_i|P \neq 0$ for $1 \leq i \leq r-1$, but $\gamma_r|P = 0$. We have $x_r = x_{r-2} \otimes y_r$ and by the definition of γ_r, we conclude

$$0 = \gamma_r(x_r) = \gamma_{r-2}(x_{r-2}) N_{\gamma_{r-1}}(y_r)$$

whence $N_{\gamma_{r-1}}(y_r) = 0$. Now $N_{\gamma_{r-1}}$ is, by definition, simply the norm of the Kummer extension $E[\sqrt[p]{\gamma_{r-1}(x_{r-1})}]/E$ and $y_r \neq 0$; we conclude that this Kummer extension is split, that is, $\gamma_{r-1}(x_{r-1})$ is a p-th power in E. Consequently, $\{\gamma_{r-2}(x_{r-2}), \gamma_{r-1}(x_{r-1})\}_E = 0 \in K_2(E)/p$, but this symbol is, by induction, equal to $\pm\{a, b\}_E$ in $K_2(E)/p$. That finishes the proof of the first assertion.

For the second claim, we can assume by induction that

$$\{a, b\}_E = \pm\{\gamma_{r-2}(x_{r-2}), \gamma_{r-1}(x_{r-1})\}_E.$$

Now $\gamma_r(x_r) = \gamma_{r-2}(x_{r-2}) N_{\gamma_{r-1}}(y_{r-1})$. But $\{\gamma_{r-1}(x_{r-1}), N_{\gamma_{r-1}}(y_{r-1})\} = 0$; indeed, for any field k containing the p-th roots of unity, any element $a \in k^\times$ and any $y \in K_a = k[\sqrt[p]{a}]$, the symbol $\{a, N_{K_a/k}(y)\} = 0 \in K_2(k)/p$. We conclude that

$$\{\gamma_{r-2}(x_{r-2}), \gamma_{r-1}(x_{r-1})\}_E \equiv -\{\gamma_{r-1}(x_{r-1}), \gamma_r(x_r)\}_E \mod p;$$

this concludes the proof of the second assertion. □

We now proved the chain lemma Theorem 0.2 in weight 2. Indeed, writing $\{a, b\}$ for the symbol $\{a_1, a_2\}$ in 0.2, we have $S = X_p$, $J = J_1 = H_p$, $J_1' = f_p^*(H_{p-1})$ and γ_1 (in 0.2) is the form γ_p of 3.1, while γ_1' in 0.2 is the γ_{p-1} of 3.1. Now Theorem 0.2, (1) is immediate from the construction of $S = X_p$; 0.2, (2) is Proposition 3.1, (2); 0.2, (3), is just the fact that $\gamma = \gamma_p$ is not constant; 0.2, (4) is 3.1, (1); 0.2, (5) is a consequence of Proposition 2.8; and 0.2, (6) is Proposition 2.6.

Preview: Higher Weights. `Warning: The rest of this lecture is only poorly understood by the person writing these notes.`

In weight two, splitting varieties have an explicit description (as Severi-Brauer varieties); also for the prime $p = 2$, the splitting varieties in all weights can be described using Pfister quadrics. We indicate an explicit construction of (rational) forms that may play the role of Pfister forms for odd primes, and the role of the reduced norm form in wieghts bigger than 2.

Let $a_1, \ldots, a_n \in k^\times$ be a sequence of units in k; wirte $u_r = \{a_1, \ldots, a_r\} \in K_r^M(k)/p.$. Inductively, we define rational functions $\Phi_r = \Phi_{a_1,\ldots,a_r}$ in p^r variables, as follows.

We set $\Phi_0(t) = t^p$. Assuming $\Phi_0, \ldots, \Phi_{r-1}$ have been defined, let

$$\Psi_r(X_{ij}|1 \leq i \leq p-1 \text{ and } 1 \leq j \leq p^{r-1}) = \prod_{i=1}^{p-1} (1 - a_r \Phi_{r-1}(X_{ij}))$$

and

$$\Phi_r(X_{ij}|0 \leq i \leq p-1 \text{ and } 1 \leq j \leq p^{r-1}) = \Phi_{r-1}(X_{0j}) \Psi_r(X_{ij}|1 \leq i \leq p-1).$$

The next lemma is a generalization of the fact, mentioned before, that the symbol $\{a, b\}$ vanishes $\mod p$ if b is a norm from the Kummer extension defined by a - this was the basic fact used to prove the relations between symbols in Proposition 3.1.

Lemma 3.2. *Let b be a nonzero value of Φ_n. Then $u_n \cdot \{b\} = 0 \in K_{n+1}^M(k)/p$.*

Proof. Use induction and the Steinberg relations. □

For the following remarks, let $P\tilde{h}i_0 = \Phi_0$,

$$\tilde{\Psi}_r(X_{ij}|1 \leq i \leq p-1 \text{ and } 1 \leq j \leq p^{r-1}) = \prod_{i=1}^{p-1}(1 - a_r\tilde{\Phi}_{r-1}(X_{ij}))^{(-1)^i}$$

and

$$\tilde{\Phi}_r(X_{ij}|0 \leq i \leq p-1 \text{ and } 1 \leq j \leq p^{r-1}) = \tilde{\Phi}_{r-1}(X_{0j})\tilde{\Psi}_r(X_{ij}|1 \leq i \leq p-1).$$

Note that Lemma 3.2 holds for $\tilde{\Phi}_n$ as well as for Φ_n.

Remark 3.3. If $p = 2$, then the form $\tilde{\Phi}_n$ is birationally equivalent to the Pfister form $\ll a_1, \ldots, a_n \gg$.

If $p = 3$, then $\tilde{\Phi}_1$ is birationally equivalent to the norm $N_{k[\sqrt[3]{a}]/k}$, $\tilde{\Phi}_2$ is birationally equivalent to the reduced norm Nrd of the algbra $A_\zeta(a_1, a_2)$ and $\tilde{\Phi}_3$ is equivalent to the norm form of the exceptional Jordan algebra $J(a_1, a_2, a_3)$. Here two rational functions are birationally equivalent if they can be transformed into one another by an automorphism (over the base field k) of the field of rational functions.

Moreover, we have the following result.

Lemma 3.4. *If $p = 3$, then the set of nonzero values of $\tilde{\Phi}_4$ is a subgroup of $k^\times$.*

We also have a result that is analogous to Proposition 3.1, with the norm forms replaced by Φ_n.

Proposition 3.5. *There is a smooth proper cellular variety X of dimension $p^n - 1$, a line bundle $L \to X$ and a p-form γ on L such that:*

(1) $\deg(L) \not\equiv 0 \mod p$.
(2) $x \in V(\gamma) \Rightarrow (u_n)_{k(x)} \equiv 0 \mod p$.
(3) *γ is birationally equivalent to Φ_n* `(in some sense yet to be understood precisely by the note-taker -` presumably, γ and Φ_n both define elements of the function field of L (note this is a rational function field as X is cellular) and one can be transformed to the other by an automorphism of that field.)

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