The p-adic Uniformization of Shimura Curves

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ABSTRACT: We reprove a theorem of Ceredník on the p-adic uniformization of Shimura curves associated to a quaternion algebra over a totally real number field F at certain bad primes. The uniformization is given over the Shimura field. Our method goes back to Drinfeld in the case $F = \mathbb{Q}$. It gives also a p-adic uniformization of higher dimensional quternionic Shimura varieties.

Introduction

Let us denote the complex manifold $\mathbb{C} \setminus \mathbb{R}$ by X. The group $\mathrm{Gl}_2(\mathbb{R})$ acts via linear fractional transformations from the left on X.

We consider arithmetically defined subgroups $\Gamma \subset \mathrm{Gl}_2(\mathbb{R})$, which are obtained as follows. Let D be a quaternion division algebra over a totally real number field F. We assume that there is a single archimedean place $\alpha : F \to \mathbb{R}$ such that D splits in α:

$$D \otimes_{F,\alpha} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R})$$

At all other archimedian places D is a division algebra.

Let G be the multiplicative group of D considered as an algebraic group over $\mathbb{Q}$. We have a natural decomposition:

$$G_{\mathbb{R}} \cong \prod_{\rho : F \to \mathbb{R}} G_\rho$$

The group $G(\mathbb{R})$ acts on X via the projection to $G_\alpha(\mathbb{R}) \cong \mathrm{Gl}_2(\mathbb{R})$.

For any congruence subgroup $\Gamma \subset G(\mathbb{Q})$ the quotient $\Gamma \backslash X$ is a projective algebraic curve. By Shimura it is canonically defined over a number field. Alternatively one can consider an open compact subgroup $C \subset G(\mathbb{A}_f)$ of the finite adelic points of G. Then the curve

$$\mathrm{Sh}_C = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)/C)$$

has a canonically defined model over the Shimura field $E = \alpha(F)$. This model is called the Shimura curve. Over $\mathbb{C}$ the curve Sh_C is a disjoint union of curves $\Gamma \backslash X$ of the type above. For varying C one obtains a projective system (or tower) of curves over E with a right action of the group $G(\mathbb{A}_f)$.

Let us identify the fields F and E by the isomorphism α. Consider a place $\mathfrak{p}$ of F such that $D_\mathfrak{p}$ is a division algebra. Then the curve Sh_C has bad reduction in $\mathfrak{p}$ even if the group C is maximal in $\mathfrak{p}$ (i.e. the assumptions of corollary 3.2 below hold).

In the case where C is maximal in $\mathfrak{p}$ the Shimura curve has a model $\mathcal{S}h_C$ over $O_{F_\mathfrak{p}}$, which admits a nice p-adic description discovered by Čerednik:

One starts with the formal scheme $\hat{\Omega}^2_{F_\mathfrak{p}}$ over $\mathrm{Spf}\, O_{F_\mathfrak{p}}$ defined by Mumford. It is defined as follows [BC]. For each $O_{F_\mathfrak{p}}$-lattice $M \subset F_\mathfrak{p}^2$ one considers the projective space $\mathbb{P}(M)$ over $\mathrm{Spec}\, O_{F_\mathfrak{p}}$. All these projective spaces are birationally isomorphic. One can take the join of any finite set of them, i.e. the closed graph of the birational correspondence. Going to an inductive limit one obtains a scheme whose p-adic completion is $\hat{\Omega}^2_{F_\mathfrak{p}}$. Each irreducible component of this formal scheme is isomorphic to $\mathbb{P}^1_\kappa$, where κ denotes the residue class field of F. The dual graph of the special fibre is the combinatorial Bruhat-Tits building of $\mathrm{Gl}_2(F_\mathfrak{p})$. It follows from the definition, that the group $\mathrm{PGl}_{F_\mathfrak{p}}$ acts naturally from the left on $\hat{\Omega}^2_{F_\mathfrak{p}}$.

Let $\bar{D}$ be the quaternion algebra obtained from D by changing the local invariants exactly in the places α and $\mathfrak{p}$. One may choose an isomorphism between the restricted products over all finite places $w \neq \mathfrak{p}$ of F:

$$\prod_w{}' \bar{D}^*_w \cong \prod_w{}' D^*_w$$

In particular we get an action of the left hand side on $G(\mathbb{A}_f) = \prod'_{w finite} D^*_w$. Since we assume that $C_\mathfrak{p}$ is the maximal compact open subgroup of $D_\mathfrak{p}$, we have a natural isomorphism:

$$D^*_\mathfrak{p}/C_\mathfrak{p} \cong F^*_\mathfrak{p}/O^*_{F_\mathfrak{p}}$$

The group $\bar{D}_\mathfrak{p}$ acts by the determinant on the right hand side of this isomorphism. Hence altogether we obtain an action of the group $(\bar{D} \otimes \mathbb{A}_f)^*$ on $G(\mathbb{A}_f)/C$ from the left, and hence also an action of the subgroup $\bar{D}^*$.

Choosing an isomorphism $\bar{D}^*_\mathfrak{p} \cong \mathrm{Gl}_2(F_\mathfrak{p})$, we obtain an action of this group on $\hat{\Omega}^2_{F_\mathfrak{p}}$. We let $\bar{D}^*$ act by the embedding $\bar{D} \subset \bar{D}_\mathfrak{p}$.

The following more precise formulation of Čerednik's theorem was proved by Drinfeld [D] in the case $F = \mathbb{Q}$.

Theorem 0.1. *Let $\breve{F}_\mathfrak{p}$ be the completion of the maximal unramified extension of F. There is a $G(\mathbb{A}_f)$-equivariant isomorphism of towers of formal p-adic schemes over $\breve{F}_\mathfrak{p}$:*

$$\bar{D}^* \backslash (\hat{\Omega}^2_{F_\mathfrak{p}} \times_{SpfO_{F_\mathfrak{p}}} SpfO_{\breve{F}_\mathfrak{p}} \times G(\mathbb{A}_f)/C) \cong \mathcal{S}h^\wedge_C \times_{SpfO_{F_\mathfrak{p}}} SpfO_{\breve{F}_\mathfrak{p}}$$

Here $\mathcal{S}h^\wedge_C$ denotes the completion along the special fibre.

Let τ be the Frobenius automorphism relative to $F_\mathfrak{p}$ acting on $\breve{F}_\mathfrak{p}$. Then τ acts on both sides of the isomorphisms above via the factor $SpfO_{\breve{F}_\mathfrak{p}}$. Let $\Pi \in D^*_\mathfrak{p} \subset G(\mathbb{A}_f)$ be a prime element, acting by multiplication on $G(\mathbb{A}_f)/C$. Then the action of $\tau \times \Pi$ on the left hand side of the isomorphism above induces on the right hand side the action of τ.

We show how this result may be deduced from the general non archimedean uniformization theorems for Shimura varieties of PEL-type in [RZ]. By this method one can extend the result easily to quaternionic Shimura varieties of higher dimension (corollary 3.4).

If $C_{\mathfrak{p}}$ is not maximal compact it is not easy to describe a model $\mathcal{S}h_C$ over $\operatorname{Spec} O_{F_{\mathfrak{p}}}$. But one can obtain a uniformization theorem in the rigid analytic setting.

In the theorem above it is natural to consider the formal scheme $\mathcal{N} = (\hat{\Omega}^2_{F_{\mathfrak{p}}} \times_{\operatorname{Spf} O_{F_{\mathfrak{p}}}} \operatorname{Spf} O_{\breve{F}_{\mathfrak{p}}}) \times D^*_{\mathfrak{p}}/C_{\mathfrak{p}}$. It is equipped with an action of $\tau \times \Pi$, which we call the natural Weil descent datum. By Drinfeld the associated rigid analytic space $\mathcal{N}^{rig}$ admits a pro-analytic étale covering $\mathbb{N}_{F_{\mathfrak{p}}}$ with Galois group $O^*_{D_{\mathfrak{p}}}$ acting from the right (see (44) - (47) below).

The space $\mathbb{N}_{F_{\mathfrak{p}}}$ is a rigid analytic space over $\operatorname{Sp} \bar{\breve{F}}_{\mathfrak{p}}$ equipped with a left action of $\bar{D}^*_{\mathfrak{p}}$, a right action of $D^*_{\mathfrak{p}}$, and a Weil descent datum realtive to $\breve{F}_{\mathfrak{p}}/F_{\mathfrak{p}}$.

The space $\mathbb{N}_{F_{\mathfrak{p}}}$ is used to obtain a uniformization for the rigid analytic spaces $(\mathrm{Sh}_C \times_{\operatorname{Spec} F} \operatorname{Spec} F_{\mathfrak{p}})^{rig}$ (see theorem 3.1). The proof is based on the existence of a determinant map with connected fibers:

$$\det : \mathbb{N}_{F_{\mathfrak{p}}} \times_{\operatorname{Sp} \breve{F}_{\mathfrak{p}}} \operatorname{Sp} \bar{\breve{F}}_{\mathfrak{p}} \longrightarrow F^*_{\mathfrak{p}}$$

In fact we show the existence of a determinant map more generally for the pro-analytic covering spaces associated to the formal schemes $\hat{\Omega}^d_{F_{\mathfrak{p}}}$ for any $d \geq 1$, and determine the action of the Galois group on the connected components of the covering spaces (theorem 2.3 below). The case $d = 2$ is dicussed in [C] 4.3. We note that there is an analogue of the determinant map in equal characteristic defined by Genestier [G].

To obtain the uniformization theorems in the generality above we have to redo the uniformization theorem for the unitary group [RZ] 6.50 under slightly more general assumptions. This is the contents of chapter 1. Similiar results have been obtained by Y.Varshavsky, by a totally different method. Chapter 2 is devoted to the determinant map. Chapter 3 contains the p-adic uniformization of Shimura curves.

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0.1 Uniformization for the unitary group

Let K be a CM-field with $F \subset K$ the maximal totally real subfield. We consider a division algebra B over K of rank d^2 with a positive involution $b \longmapsto b', \; b \in B$. Let $W = B$ considered as a B-bimodule. Let

$$\psi : W \times W \longrightarrow \mathbb{Q}$$

be an alternating nondegenerate bilinear form, which satisfies

$$\psi(bw_1, w_2) = \psi(w_1, b'w_2).$$

We fix once for all a rational prime number p. Let us denote by $\mathfrak{p} = \mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_m$ the prime ideals of O_F over p. We assume that they all split in K:

$$\mathfrak{p}_i O_K = \mathfrak{q}_i \overline{\mathfrak{q}}_i, \qquad \mathfrak{q}_i \neq \overline{\mathfrak{q}}_i, \quad i = 0, \ldots, m.$$

The prime ideals $\mathfrak{q}_0$ and $\overline{\mathfrak{q}}_0$ will be also denoted by $\mathfrak{q}$ and $\overline{\mathfrak{q}}$.

We will use the notations $B_{\mathfrak{p}_i} = B \otimes_F F_{\mathfrak{p}_i}$, $B_{\mathfrak{q}_i} = B \otimes_K K_{\mathfrak{q}_i}, W_{\mathfrak{p}_i} = W \otimes_F F_{\mathfrak{p}_i}, W_{\mathfrak{q}_i} = W \otimes_K K_{\mathfrak{q}_i}$ for the completions in the corresponding prime ideals. We assume that $B_{\mathfrak{q}}$ is a division algebra of invariant $1/d$.

Let $O_B \subset B$ be a maximal order, which has the property that $O_B \otimes \mathbb{Z}_p$ is fixed by the involution $b \longmapsto b'$. We write $\Gamma = O_B \otimes \mathbb{Z}_p$ if we view it as a submodule of $W \otimes \mathbb{Q}_p$.

We assume that ψ induces a perfect pairing

$$\Gamma \otimes \Gamma \longrightarrow \mathbb{Z}_p.$$

The form ψ determines uniquely an alternating F-bilinear form

$$\tilde{\psi} : W \times W \longrightarrow F \tag{1}$$

by the equation

$$\psi(fw_1, w_2) = Tr_{F/\mathbb{Q}} \mathrm{diff}^{-1} f \tilde{\psi}(w_1, w_2),$$

Here diff is an element of F, which locally at the primes over p generates the different ideal of $F/\mathbb{Q}_p$. Let $\tilde{G}^\bullet$ be the algebraic group over F, whose group of points with values in a F-algebra R is

$$\tilde{G}^\bullet = \{g \in Gl_{B\otimes_F R}(W \otimes_F R) \mid \tilde{\psi}(gw_1, gw_2) = \tilde{\gamma}(g)\tilde{\psi}(w_1, w_2),$$
$$\tilde{\gamma}(g) \in R^\star, w_1, w_2 \in W \otimes_F R\}$$

Let $b \longmapsto b^\star$ be the involution on B given by

$$\psi(w_1 b, w_2) = \psi(w_1, w_2 b^\star), \quad w_1, w_2 \in W \otimes_F R, b \in B. \tag{2}$$

We may rewrite the definition of $\tilde{G}^\bullet$ as follows:

$$\tilde{G}^\bullet = \{g \in ((B \otimes_F R)^{opp})^\star \mid gg^\star \in R^\star\}$$

The multiplicator is $\tilde{\gamma}(g) = gg^\star$.

Let $G^\bullet = \mathrm{Res}_{F/\mathbb{Q}}\tilde{G}^\bullet$ be the restriction of scalars à la Weil. For any $\mathbb{Q}$-algebra R we have

$$G^\bullet = \{g \in Gl_{B\otimes_{\mathbb{Q}} R}(W \otimes_{\mathbb{Q}} R) \mid \psi(gw_1, gw_2) = \psi(\gamma(g)w_1, w_2)$$
$$\gamma(g) \in (F \otimes_{\mathbb{Q}} R)^\star, w_1, w_2 \in W \otimes_{\mathbb{Q}} R\}$$

On the group of points $\tilde{G}^\bullet(F) = G^\bullet(\mathbb{Q})$ the maps $\tilde{\gamma}$ and γ will induce the same map to F^*.

The module schemes we are going to define will be associated to certain open and compact subgroups $C \subset G^\bullet(\mathbb{A}_f) = \tilde{G}^\bullet(\mathbb{A}_{F,f})$. We will assume that C is of the following type:

$$C = C_p C^p \tag{3}$$

where

$$C^p \subset G^\bullet(\mathbb{A}_f^p)$$

and

$$C_p \subset G^\bullet(\mathbb{Q}_p) = \prod_{i=0}^{m} \tilde{G}^\bullet(F_{\mathfrak{p}_i}).$$

The subgroup C_p should decompose

$$C_p = \prod_{i=0}^{m} C_{\mathfrak{p}_i},\ C_{\mathfrak{p}_i} \subset \tilde{G}^\bullet(F_{\mathfrak{p}_i}),$$

where $C_{\mathfrak{p}_i}$ are subgroups of the following form. Since $\mathfrak{p}_i O_K = \mathfrak{q}_i \overline{\mathfrak{q}}_i$ we have a decomposition $B_{\mathfrak{p}_i} = B_{\mathfrak{q}_i} \times B_{\overline{\mathfrak{q}}_i}$. The involution $*$ interchanges the factors and hence defines an isomorphism

$$* : B_{\overline{\mathfrak{q}}_i} \xrightarrow{\sim} B_{\mathfrak{q}_i}^{opp}.$$

With this identification we may write

$$\tilde{G}^{\bullet}(F_{\mathfrak{p}_i}) = \{(b_1, b_2) \in (B_{\mathfrak{q}_i}^{opp})^{\star} \times B_{\mathfrak{q}_i}^{\star} \mid b_1 b_2 \in F_{\mathfrak{p}_i}^{\star}\} \tag{4}$$

We assume that there is an open and compact subgroup $C_{\mathfrak{q}_i} \subset B_{\mathfrak{q}_i}^{\star}$, such that

$$C_{\mathfrak{p}_i} = \{(c_1, c_2) \in C_{\mathfrak{q}_i} \times C_{\mathfrak{q}_i}^{opp} \mid c_1 c_2 \in F_{\mathfrak{p}_i}^{\star}\} \tag{5}$$

We assume that $C_{\mathfrak{q}} \subset (B_{\mathfrak{q}}^{opp})^{\star}$ is the maximal open compact subgroup of this division algebra.

Let us consider the category of abelian schemes over a base S. An abelian O_K-scheme is a pair (A, ι), where A is an abelian scheme and $\iota : O_K \hookrightarrow \mathrm{End} A$ is an action. An isogeny $\rho : (A, \iota) \longrightarrow (A', \iota')$ is called of order prime to $\mathfrak{p}$, if locally for the Zariski topology on S, there is an element $f \in O_F$, $f \notin \mathfrak{p}$ which annihilates the kernel of ρ. If we invert in the category of O_K-schemes all isogenies of order prime to $\mathfrak{p}$ we obtain a category $\underline{AV}$. Its objects will be called abelian O_K-schemes up to isogeny of order prime to $\mathfrak{p}$.

For an abelian O_K-scheme (A, ι) we have a decomposition of the associated p-divisible group

$$X = \prod_{i=0}^{m} X_{\mathfrak{q}_i} \times X_{\overline{\mathfrak{q}}_i}.$$

Here O_K acts on $X_{\mathfrak{q}_i}$ via the embedding $O_K \longrightarrow O_{K_{\mathfrak{q}_i}}$ etc.. If $A \in \underline{AV}$ it makes still sense to speak of the p-divisible groups $X_{\mathfrak{q}}$ and $X_{\overline{\mathfrak{q}}}$, while $X_{\mathfrak{q}_i}$ for $i \neq 0$ makes only sense up to isogeny. We note that the height of the p-divisible group $X_{\mathfrak{q}_i}$ is $\frac{2 \dim A}{[K:\mathbb{Q}]} \lfloor K_{\mathfrak{q}_i} : \mathbb{Q}_p]$.

Let A_0 be an abelian O_K-scheme and $A \in \underline{AV}$ its class. Then

$$\mathrm{End} A = (\mathrm{End} A_0) \otimes_{O_F} (O_F)_{\mathfrak{p}}.$$

We will denote by $\hat{A}_0$ the dual abelian scheme and by $\hat{A} \in \underline{AV}$ its class. A polarization of A will be a quasiisogeny $\lambda : A_0 \longrightarrow \hat{A}_0$, such that locally on

S for a suitable natural number n the quasiisogeny $n\lambda$ is an isogeny induced by a line bundle ampel relative to S. In fact this condition depends only on the induced morphism $\lambda : A \longrightarrow \hat{A}$ in $\underline{AV}$ and not on the choice of A_0. A set of quasiisogenies $A \longrightarrow \hat{A}$ of the form $F^\star\lambda = \lambda F^\star$, where λ is a polarization, will be called a F-homogenous polarization. Let us denote by Λ the F-vector space $F \cdot \lambda$.

If there is a $\lambda' \in F^\star\lambda$ which induces an isomorphism in $\underline{AV}$ we call Λ a F-homogenous polarization, which is principal in $\mathfrak{p}$.

We consider the subset $\Phi \subset \mathrm{Hom}(K, \overline{\mathbb{Q}_p})$ given by

$$\Phi = \bigsqcup_{i=0}^{m} \mathrm{Hom}(K_{\mathfrak{q}_i}, \overline{\mathbb{Q}}_p). \tag{6}$$

If $\overline{\Phi}$ denotes the conjugate of Φ with respect to the conjugation of K/F we have

$$\Phi \cup \overline{\Phi} = \mathrm{Hom}(K, \overline{\mathbb{Q}}_p).$$

Let us fix an embedding $\alpha : K_{\mathfrak{q}_0} \longrightarrow \overline{\mathbb{Q}}_p$. The image of α will be denoted by E.

We define a B invariant subspace $W_0 \subset W \otimes \overline{\mathbb{Q}}_p$, which is isotropic with respect to ψ. To do this we consider the decomposition

$$W \otimes \overline{\mathbb{Q}}_p = \bigoplus_{\rho : K \longrightarrow \overline{\mathbb{Q}}_p} W \otimes_{K,\rho} \overline{\mathbb{Q}}_p$$

The space W_0 will be a direct sum of $B \otimes_{K,\rho} \overline{\mathbb{Q}}_p$-submodules of $W_{0,\rho} \subset W_\rho = W \otimes_{K,\rho} \overline{\mathbb{Q}}_p$, which satisfy

$$\dim W_{0,\rho} = \begin{cases} 0 & \text{if } \rho \in \Phi, \rho \neq \alpha \\ d & \text{if } \rho = \alpha \\ d^2 - d & \text{if } \rho = \overline{\alpha} \\ d^2 & \text{if } \rho \in \overline{\Phi}, \rho \neq \overline{\alpha} \end{cases} \tag{7}$$

These conditions define the isotropic subspace $W_0 \subset W \otimes \overline{\mathbb{Q}}_p$ up to a symplectic B-linear automorphism of $W \otimes \overline{\mathbb{Q}}_p$.

The polynomial function on B

$$\det_{\overline{\mathbb{Q}}_p}(b|W_0)$$

is defined over E and independent of our choice of W_0.

We are now ready to define the following variant of the moduli problems considered in $\lfloor RZ]$ §6.

Let $\underline{\mathcal{A}}_C$ be the functor on the category of O_E-schemes, whose points $\underline{\mathcal{A}}_C(S)$ with values in an O_E-scheme S are given by isomorphism classes of the following data

1) An object A of $\underline{AV}$ over S with an action of O_B

$$\iota : O_B \longrightarrow \mathrm{End}A.$$

2) A F-homogenous polarization Λ of A, which is principal in $\mathfrak{p}$.

3) For each $i = 1, \dots m$ a generator $\lambda_i \in \Lambda \otimes_F F_{\mathfrak{p}_i} \mod C_{\mathfrak{p}_i} \cap F^{\star}_{\mathfrak{p}_i}$.

4) A class of isomorphisms of $B \otimes \mathbb{A}^p_f$-modules

$$\overline{\eta}^p : V^p(A) \longrightarrow W \otimes \mathbb{A}^p_f \mod C^p$$

such that the Riemann form on $V^p(A)$ given by a polarization $\lambda \in \Lambda$ and ψ on $W \otimes \mathbb{A}^p_f$ are respected up to a constant in $(F \otimes \mathbb{A}^p_f)^{\star}$.

5) For each $i = 1, \dots, m$ a class of $B_{\mathfrak{q}_i}$-module isomorphisms

$$\overline{\eta}_{\mathfrak{q}_i} : V_{\mathfrak{q}_i}(A) \longrightarrow W_{\mathfrak{q}_i} \mod C_{\mathfrak{q}_i}.$$

The following conditions should be satisfied

(i) The Rosati involution on $\mathrm{End}A$ defined by Λ induces on O_B the given involution $b \longmapsto b'$.

(ii) We have an identity of polynomial functions on O_B :

$$\det_{O_S}(b, \mathrm{Lie}A_0) = \det_{\overline{\mathbb{Q}}_p}(b|W_0)$$

for any abelian variety A_0 in the class A.

We remark that the last condition (ii) is equivalent to the following: The p-divisible group $X_{\mathfrak{q}}$ of A_0 is a special formal $O_{B_{\mathfrak{q}}}$-module in the sense of Drinfeld and for $i = 1, \ldots, m$ the p-divisible groups $X_{\mathfrak{q}_i}$ are étale. Indeed this follows from [RZ] 3.58 and from the existence of a polarization which is principal in $\mathfrak{p}$.

Proposition 0.2. *For sufficiently small congruence subgroups $C \subset G^{\bullet}(\mathbb{A}_f)$ satisfying the conditions above the sheafification $\mathcal{A}_C$ with respect to the étale topology of the functor $\underline{\mathcal{A}}_C$ is representable by a projective scheme over O_E.*

Proof: Let us introduce the notation $\hat{\mathbb{Z}}^p = \varprojlim_{(n,p)=1} \mathbb{Z}/n$ and $\hat{\mathbb{Z}}^p(1) = \varprojlim_{(n,p)=1} \mu_n$, where μ_n is the sheaf of n-the roots of unity. We denote by $\mathbb{A}^{\mathfrak{p}}_{F,f}$ the ring of adeles of F, which have component 0 at infinity and at the prime $\mathfrak{p}$. We define a twist by the roots of unity outside p:

$$\mathbb{A}^{\mathfrak{p}}_{F,f}(1)^p = (\prod_{i=1}^{m} F_{\mathfrak{p}_i}) \times (F \otimes_{\hat{\mathbb{Z}}^p} \hat{\mathbb{Z}}^p(1))$$

Let $(A, \Lambda, \{\overline{\lambda}_i\}, \overline{\eta}^p, \{\overline{\eta}_{\mathfrak{q}_i}\})$ be a point of $\underline{\mathcal{A}}_C(L)$ for an algebraically closed field L. There exists a polarization $\lambda \in \Lambda$, which is principal in $\mathfrak{p}$. To any polarization λ of this type we associate a class

$$cls\, \lambda \in (\mathbb{A}^{\mathfrak{p}}_{F,f}(1)^p)^{\star}/\gamma(C^{\mathfrak{p}}). \tag{8}$$

Here $(\mathbb{A}^{\mathfrak{p}}_{F,f}(1)^p)^{\star}$ denotes the set of isomorphisms of $\mathbb{A}^{\mathfrak{p}}_{F,f}$-modules

$$Isom(\mathbb{A}^{\mathfrak{p}}_{F,f}, (\mathbb{A}^{\mathfrak{p}}_{F,f}(1)^p))$$

To do this we choose an isomorphisms $\eta^p \in \overline{\eta}^p$. Then by the definition of $\underline{\mathcal{A}}_C$ there exists an element $f^p \in (\mathbb{A}^{p}_{F,f} \otimes_{\hat{\mathbb{Z}}^p} \hat{\mathbb{Z}}^p(1))^{\star}$, such that

$$\psi(f^p \eta^p(x), \eta^p(y)) = E^{\lambda}(x, y),\ x, y \in V^p(A),$$

where E^{λ} is the Riemann form associated to λ. For each $i = 1, \ldots, m$ there is an element $f_{\mathfrak{p}_i} \in F^{\star}_{\mathfrak{p}_i}$, such that $\lambda = f_{\mathfrak{p}_i}\lambda_i$. Then the residue class of $(f^p, f_{\mathfrak{p}_i}, \ldots, f_{\mathfrak{q}_m})$ in the right hand side of (8) is by definition $cls\, \gamma$. This definition is independent of the choice of η^p.

Let $F^{\star}_{+,\mathfrak{p}}$ be the multiplicative group of totally positive elements of F, which are units in $\mathfrak{p}$. The residue class of $cls\ \lambda$ in

$$F^{\star}_{+,\mathfrak{p}} \setminus (\mathbb{A}^{\mathfrak{p}}_{F,f}(1)^p)^{\star}/\gamma(C^{\mathfrak{p}})$$

is independent of the choice of $\lambda \in \Lambda$, and will be denoted by $cls\ \Lambda$.

More generally we can define a class

$$cls\,\lambda \in (\mathbb{A}^{\mathfrak{p}}_{F,f}(1)^p)^\star/\gamma(C^{\mathfrak{p}}).$$

for a point of $\underline{\mathcal{A}}_C(S)$ for a connected O_E-scheme S, and a polarization $\lambda \in \Lambda$ which is principal in $\mathfrak{p}$ by taking the class in any geometric point. Since the action of the fundamental group leaves $\overline{\eta}$ invariant (by definition), the class is well-defined. Again cls Λ is defined.

Let us choose an isomorphism $\hat{\mathbb{Z}}^p \longrightarrow \hat{\mathbb{Z}}_p(1)$ over $O_{E^{nr}}$, the maximal unramified extension of O_E The induced isomorphism

$$(\mathbb{A}^{\mathfrak{p}}_{F,f})^\star/\gamma(C^{\mathfrak{p}}) \xrightarrow{\sim} (\mathbb{A}^{\mathfrak{p}}_{F,f}(1)^p)^\star/\gamma(C^{\mathfrak{p}})$$

is defined over a finite unramified extension R of O_E. The prove the proposition we may work over R and ignore the Tate twist by $\hat{\mathbb{Z}}^p(1)$.

Let us choose representatives $\kappa_1, \dots, \kappa_M \in (\mathbb{A}^{\mathfrak{p}}_{F,f})^\star/\gamma(C^{\mathfrak{p}})$ for the finite set $F^\star_{+,\mathfrak{p}} \setminus (\mathbb{A}^{\mathfrak{p}}_{F,f})^\star/\gamma(C^{\mathfrak{p}})$. We define a functor $\tilde{\underline{\mathcal{A}}}_C$ over R by adding to the data defining $\underline{\mathcal{A}}_C$ the following:

6) A polarization $\lambda \in \Lambda$, which is principal in $\mathfrak{p}$, such that $cls\ \lambda = \kappa_j$ for some $j = 1, \dots, m$.

The functor $\tilde{\underline{\mathcal{A}}}_C$ is a moduli problem of polarized abelian varieties with a polarization of bounded degree and hence representable. Indeed a polarized abelian variety may be extracted from the data 1) – 6) as follows. Let $W_{\mathbb{Z}} = O_B \subset W$ and $W^p = W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p$. Then there is a unique abelian variety $A_0 \subset A$ in the isogeny class A, such that $\eta^p \hat{T}^p(A_0)) = W^p$ for any $\eta^p \in \overline{\eta}^p$, $\eta_{\mathfrak{q}_i}(T_{\mathfrak{q}_i}(A_0)) = O_B \otimes_{0_K} O_{K_{\mathfrak{q}_i}}$ for any $\eta_{\mathfrak{q}_i} \in \overline{\eta}_{\mathfrak{q}_i}$, and moreover the elements λ_i induce perfect pairings between the p-divisible groups $X_{\mathfrak{q}_i}$ and $X_{\overline{\mathfrak{q}}_i}$ for $i = 1, \dots, m$.

Then λ defines an quasiisogeny $A_0 \longrightarrow \hat{A}_0$, whose degree is bounded by a constant depending on $\kappa_1 \dots \kappa_M$ and ψ. Moreover we may find a natural number $c = c(\kappa_1, \dots, \kappa_M, \psi)$, such that $c \cdot \lambda$ is induced by an ample line bundle.

Hence the sheafification of the functor $\underline{\tilde{\mathcal{A}}}_C$ is representable by a quasiprojective scheme $\underline{\tilde{\mathcal{A}}}_C$ over Spec$R$. We claim that the morphism

$$\underline{\tilde{\mathcal{A}}}_C \longrightarrow \underline{\mathcal{A}}_{C,R}$$

is an isomorphism for sufficiently small C. The surjectivity of this map is obvious. For the injectivity we assume that there are two points of $\underline{\tilde{\mathcal{A}}}_C(S)$ for a connected scheme S are mapped to the same point of $\underline{\mathcal{A}}_{C,R}(S)$. That means, that we have a point $(A, \Lambda, \{\overline{\lambda}_i\}, \overline{\eta}^p, \{\overline{\eta}_{\mathfrak{q}_i}\})$ of $\underline{\mathcal{A}}_{C,R}(S)$ and two polarizations $\lambda', \lambda'' \in \Lambda$ giving use to points of $\tilde{\mathcal{A}}_C(S)$. Then we must have $cls\lambda' = cls\lambda''$.

Therefore the polarizations λ' and λ'' differ by an element $f \in F^{\star}_{+,\mathfrak{p}} \cap \gamma(C^{\mathfrak{p}})$, which is a subgroup of the group of units of F. By a theorem of Chevalley for sufficiently small $C^{\mathfrak{p}}$ any element of $F^{\star}_{+,\mathfrak{p}} \cap \gamma(C^g)$ is a square in F. Then $f = u^2$ for $u \in F$. But then the multiplication by $u : A \longrightarrow A$ defines an isomorphism between the points of $\underline{\tilde{\mathcal{A}}}_C(S)$ corresponding to λ' and λ''.

Hence we have shown that the sheafification of $\underline{\mathcal{A}}_C$ is representable by a quasiprojective scheme $\mathcal{A}_C$. To finish the proof we verify by the valuative criterion that $\mathcal{A}_C$ is proper.

Let R be a discrete valuation ring with algebraically closed residue field k and field of fractions Q. Let $(A, \Lambda, \lambda_i, \overline{\eta}^p, \overline{\eta}_{\mathfrak{q}_i})$ be a point of $\underline{\mathcal{A}}_C(Q)$ and $A_0 \in A$ an abelian variety. We have to verify that A_0 has potentially good reduction. This is standard by Drinfeld [1]: We may assume that A_0 has semistable reduction. Then the connected component of the special fibre of the Néron modell $\overline{A}_0$ is an extension

$$0 \longrightarrow \mathbb{G}^r_{m,k} \longrightarrow \overline{A}_0 \longrightarrow M \longrightarrow 0$$

where M is an abelian variety. Then we get an action of B on the character group $X^{\star}(\mathbb{G}^r_{m,k}) \otimes \mathbb{Q} \simeq \mathbb{Q}^r$. Since $r \leq \dim A_0 = \frac{1}{2}[B : \mathbb{Q}]$ and B is a division algebra this is only possible for $r = 0$. $\square$

The O_E-schemes $\mathcal{A}_C$ form for varying C of the type described above a projective system with finite transition maps:

$$\mathcal{A}_{C_2} \longrightarrow \mathcal{A}_{C_1} \text{ for } C_2 \subset C_1.$$

We will now define a right action of $G^\bullet(\mathbb{A}_f)$ on the projective system $\{\mathcal{A}_C\}$.

For $g \in G^\bullet(\mathbb{A}_f^p)$ we consider the action in [RZ]:

$$g : \mathcal{A}_C \longrightarrow \mathcal{A}_{g^{-1}Cg},$$

which maps a point $(A, \Lambda, \{\overline{\lambda}_i\}, \overline{\eta}^p, \{\overline{\eta}_{\mathfrak{q}_i}\})$ to $(A, \Lambda, \{\overline{\lambda}_i, g^{-1}\overline{\eta}^p, \{\overline{\eta}_{\mathfrak{q}_i}\})$.

Next we consider an idele $g_{\mathfrak{p}_j} \in \tilde{G}^\bullet(F_{\mathfrak{p}_j})$. According to the decomposition $W_{\mathfrak{p}_j} = W_{\mathfrak{q}_j} \oplus W_{\overline{\mathfrak{q}}_j}$ we will write

$$g_{\mathfrak{p}_j} = (g_{\mathfrak{q}_j}, g_{\overline{\mathfrak{q}}_j}).$$

Then we define a right action

$$g_{\mathfrak{p}_j} : \mathcal{A}_C \longrightarrow \mathcal{A}_{g_{\mathfrak{p}_j}^{-1}Cg_{\mathfrak{p}_j}}$$

by associating to $(A, \Lambda, \{\overline{\lambda}_i\}, \overline{\eta}^p, \{\overline{\eta}_{\mathfrak{q}_i}\})$ the point obtained by changing $\overline{\lambda}_j$ to $\overline{\lambda}_j \cdot \gamma(g_{\mathfrak{p}_j}^{-1})$ and $\overline{\eta}_{\mathfrak{q}_j}$ to $g_{\mathfrak{q}_j}^{-1}\overline{\eta}_{\mathfrak{q}_j}$ while leaving the remaining data unchanged.

Finally we define the action of $\tilde{G}^\bullet(F_{\mathfrak{p}})$. Let $b \in B^\star$ be an element and $A \in AV$ be an object with the action $\iota : O_B \longrightarrow \mathrm{End}A$. Then we define a new action by

$$\iota^b(x) = \iota(b^{-1}xb) \text{ for } x \in O_B. \tag{9}$$

Since b normalizes $O_B \otimes_{O_F} O_{F_\mathfrak{p}}$ we see that $\iota^b(x)$ is an endomorphism of A. We will write A^b for the pair (A, ι^b).

Let Λ be a F-homogenous polarization of A, such that ι respects the involutions, i.e. the Rosati-involution of Λ induces on B the given involution $b \longmapsto b'$. Then ι^b respects the involutions, if $bb' \in F^\star$. Let H be the corresponding algebraic group over $\mathbb{Q}$.

$$H(\mathbb{Q}) = \{b \in B^\star \mid bb' \in F^\star\}$$

Lemma 0.3. *The group $H(\mathbb{Q})$ has the weak approximation property.* $\square$

Proof: Let H' be the derived group. It satisfies the weak approximation property by [PR] Theorem 7.8. The center Z_H of the group H is isomorphic to $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$. The group H is a Z_H-torsor over H'. Since the Galois cohomology $H^1(\kappa(\overline{H}')/\kappa(H'), Z_H)$ vanishes by Hilbert Satz 90 the variety H

is birationally equivalent to $Z_H \times H'$. This implies the lemma by standard facts on weak approximation. $\square$

Let us consider an idele $g_\mathfrak{p} \in \tilde{G}^\bullet(F_\mathfrak{p})$. We denote by $\Gamma_\mathfrak{p} = O_B \otimes_{O_F} O_{F_\mathfrak{p}} \subset W_\mathfrak{p}$ the $O_B \otimes_{O_F} O_{F_\mathfrak{p}}$ lattice. Then $\Gamma_\mathfrak{p} = \Gamma_\mathfrak{q} \oplus \Gamma_{\overline{\mathfrak{q}}}$ and ψ induces a perfect pairing $\Gamma_\mathfrak{q} \times \Gamma_{\overline{\mathfrak{q}}} \longrightarrow \mathbb{Z}_p$. Using this one check easily, that there exists an element $b_\mathfrak{p} \in B_\mathfrak{p}$, such that $b_\mathfrak{p}\Gamma_\mathfrak{p} = g_\mathfrak{p}\Gamma_\mathfrak{p}$ and $b_\mathfrak{p} b'_\mathfrak{p} \in F^\star_\mathfrak{p}$. By the weak approximation property of H we may assume that $b_\mathfrak{p}$ is the image of an element $b \in B$, such that $bb^\star \in F$.

Definition 0.4. *The action of $g_\mathfrak{p}$ on the tower is a morphism for each level*

$$g_\mathfrak{p} : \mathcal{A}_C \longrightarrow \mathcal{A}_C$$

which maps a point $(A, \Lambda, \{\overline{\lambda}_i\}, \overline{\eta}^p, \{\overline{\eta}_{\mathfrak{q}_i}\})$ *to* $(A^b, \Lambda, \{\overline{\lambda}_i bb'\}, \overline{\eta}^p b, \{\overline{\eta}_{\mathfrak{q}_i} b\})$.

This definition is independent of the choice of b. Indeed if b is a unit in $O_B \otimes_{O_F} O_{F_\mathfrak{p}}$ the multiplication by $\iota(b) : A^b \longrightarrow A$ is an isomorphism in the category AV, which defines an isomorphism of the points above.

Hence the definition of the right action of $G^\bullet(\mathbb{A}_f)$ on the projective system $\mathcal{A}_C$ is finished.

We set $\mathcal{A} = \varprojlim \mathcal{A}_C$, which exists as a scheme since the transition maps are affine. It follows that we have an isomorphsim of locally ringed spaces

$$\mathcal{A}/C \xrightarrow{\sim} \mathcal{A}_C.$$

For any open compact subgroup $C \subset G^\bullet(\mathbb{A}_f)$ which is maximal in $\mathfrak{p}$ we define $\mathcal{A}_C = \mathcal{A}/C$. This is a projective scheme over O_E. We show now that the completion of $\mathcal{A}_C$ along the special fibre admits a p-adic uniformization by Drinfeld's $\hat{\Omega}^d$.

Let us denote by κ the residue class field of O_E. It is an $O_{F_\mathfrak{p}}$-algebra via the isomorphism $\alpha : O_{F_\mathfrak{p}} \longrightarrow O_E$. All special formal O_{B_q}-modules over $\overline{\kappa}$ are isogenous. Let Φ be a fixed one. The dual p-divisible group $\hat{\Phi}$ is naturally $O^{opp}_{B_\mathfrak{q}}$-module. The fixed involution $b \longmapsto b'$ induces an isomorphism $O_{B_{\overline{\mathfrak{q}}}} \longrightarrow O^{opp}_{B_\mathfrak{q}}$. We consider the action thus obtained

$$O_{B_\mathfrak{p}} = O_{B_\mathfrak{q}} \times O_{B_{\overline{\mathfrak{q}}}} \longrightarrow \mathrm{End}(\Phi \times \hat{\Phi})$$

Then the natural polarization on $\Phi \times \hat{\Phi}$ induces on $O_{B_\mathfrak{p}}$ the given involution $b \longmapsto b'$. We will use the notation $\mathbb{X} = \Phi \times \hat{\Phi}$.

Let us denote by $\breve{E}$ the completion of the maximal unramified extension E^{nr}. We are going to define a module problem of p-divisible groups on the category of $O_{\breve{E}}$-schemes T, such that p is locally nilpotent on T. We set $\overline{T} = T \times_{\mathrm{Spec}\ \mathbb{Z}_p} \mathrm{Spec}\ \mathbb{F}_p$. Let $\varphi : T \longrightarrow \mathrm{Spec}\ O_{\breve{E}}$ be the structure morphism and $\overline{\varphi} : \overline{T} \longrightarrow \mathrm{Spec}\ O_{\breve{E}}/(p) \longrightarrow \mathrm{Spec}\ \overline{\kappa}$ its reduction.

Let X be a p-divisible group over T with an action of $O_{B_{\mathfrak{p}}}$. According to the decomposition $O_{B_{\mathfrak{p}}} = O_{B_{\mathfrak{q}}} \times O_{B_{\overline{\mathfrak{q}}}}$ we get

$$X = X_1 \times X_2$$

By the isomorphism $O_{B_{\mathfrak{q}}}^{opp} \longrightarrow O_{B_{\mathfrak{q}}}$, we view the dual $\hat{X}_2$ as an $O_{B_{\mathfrak{q}}}$-module. We say that X is of special type if X_1 and $\hat{X}_2$ are special formal $O_{B_{\mathfrak{q}}}$-modules.

Definition 0.5. *Let $\breve{\mathcal{M}}$ be a functor on the category of $O_{\breve{E}}$-schemes T, where p is locally nilpotent, such that a point of $\breve{\mathcal{M}}(T)$ is given by the following set of data up to isomorphism.*

1) *A p-divisible $O_{B_{\mathfrak{p}}}$-module on X of special type on T.*

2) *A quasiisogeny of $O_{B_{\mathfrak{p}}}$-modules over $\overline{T}$*

$$\rho : \overline{\varphi}^{\star}\mathbb{X} \longrightarrow \overline{X}$$

These data are subject to the following condition. The quasiisogeny ρ splits naturally into the direct sum of two isogenies $\rho = \rho_1 \times \rho_2$:

$$\rho_1 : \overline{\varphi}^{\star}\Phi \longrightarrow \overline{X}_1, \quad \rho_2 : \overline{\varphi}^{\star}\hat{\Phi} \longrightarrow \overline{X}_2$$

By rigidity the quasiisogeny $\rho_1\hat{\rho}_2 : \hat{\overline{X}}_2 \longrightarrow \overline{X}_1$ lifts uniquely to a quasi-isogeny of special formal $O_{B_{\mathfrak{q}}}$-modules $\delta : \hat{X}_2 \longrightarrow X_1$. The condition is that locally for the Zariski topology on T there is an element $f \in F_{\mathfrak{p}}^{\star}$, such that $f\delta : \hat{X}_2 \longrightarrow X_1$ is an isomorphism. Two points (X, ρ) and (X', ρ') are called isomorphic, if there is an isomorphism $\alpha : X \longrightarrow X'$ of $O_{B_{\mathfrak{p}}}$-modules, such *that the following diagram is commutative*

$$\begin{array}{ccc} & & \overline{X} \\ & \nearrow & \\ \overline{\varphi}^\star \mathbb{X} & & \downarrow \;\; \overline{\alpha} \\ & \searrow & \\ & & \overline{X'} \end{array}$$

By [RZ] theorem 3.25 the functor $\breve{\mathcal{M}}$ is representable by a formal scheme locally formally of finite type over $\operatorname{Spf} O_{\breve{E}}$. In fact it will follow later (30) and the remark after definition 0.11 that $\breve{\mathcal{M}}$ is a p-adic formal scheme.

Let us denote by $\tau : \text{Spec } O_{\breve{E}} \longrightarrow \text{Spec } O_{\breve{E}}$ the morphism induced by the Frobenius automorphism of the unramified field extension $\breve{E}/E$. For an $O_{\breve{E}}$-scheme T we denote by $T_{[\tau]}$ the $O_{\breve{E}}$-scheme which is obtained by replacing the structure morphism φ by $\tau\varphi$. A formal $O_{B_{\mathfrak{p}}}$-module X of special type on T remains of special type when it is regarded on $T_{[\tau]}$. Let us denote by

$$\text{Frob} : \mathbb{X} \longrightarrow \overline{\tau}^\star \mathbb{X}$$

the Frobenius relative to κ.

The formal scheme $\breve{\mathcal{M}}$ is equipped with a Weil descent datum [RZ]

$$\xi : \breve{\mathcal{M}}(T) \longrightarrow \tau^\star \breve{\mathcal{M}}(T) = \breve{\mathcal{M}}(T_{[\tau]}). \tag{10}$$

It associates to a point $(X, \rho) \in \breve{\mathcal{M}}(T)$ the point $(X', \rho') \in \breve{\mathcal{M}}(T_{[\tau]})$, where $X' = X$ and ρ' is the composite

$$\overline{\varphi}^\star \overline{\tau}^\star \mathbb{X} \quad \overset{\overline{\varphi}^\star(\text{Frob})^{-1}}{\longrightarrow} \quad \overline{\varphi}^\star \mathbb{X} \overset{\rho}{\longrightarrow} X \tag{11}$$

Let us denote by $\mathcal{J}^\bullet(F_{\mathfrak{p}})$ the group of all quasiisogenies of the $O_{B_{\mathfrak{p}}}$-module $\mathbb{X}$, which respect the polarization of $\mathbb{X}$ up to a constant in $F_{\mathfrak{p}}^\star$. If we write $h = (h_1, h_2) : \Phi \times \hat{\Phi} \longrightarrow \Phi \times \hat{\Phi}$ the condition $h \in \mathcal{J}^\bullet(F_{\mathfrak{p}})$ is equivalent to $\hat{h}_2 h_1 \in F_{\mathfrak{p}}^\star$. The homomorphism $h \mapsto \hat{h}_2 h_1$ will be denoted by:

$$\gamma_{\mathcal{J}^\bullet} : \mathcal{J}^\bullet(F_{\mathfrak{p}}) \longrightarrow F_{\mathfrak{p}}^\star \tag{12}$$

There is a natural left action of $\mathcal{J}^\bullet(F_{\mathfrak{p}})$ on $\breve{\mathcal{M}}$:

Definition 0.6. *Let (X, ρ) be a point of $\breve{\mathcal{M}}(T)$ and $g \in \mathcal{J}^\bullet(F_\mathfrak{p})$. Then $h(X, \rho)$ is defined to be $(X, \rho h^{-1})$.*

We next define a right action of the group $\tilde{G}^\bullet(F_\mathfrak{p})$ on $\breve{\mathcal{M}}$. For $g \in \tilde{G}^\bullet(F_\mathfrak{p})$ there is an element $b \in O_{B_\mathfrak{p}}$, such that $g\Gamma = b\Gamma$, where $bb' \in F_\mathfrak{p}^\star$.

Definition 0.7. *The right action of $g \in \tilde{G}^\bullet(F_\mathfrak{p})$ associates to a point (X, ρ) the point $(X^b, \iota(b^{-1})\rho)$, where X^b is defined exactly in the same way as for abelian varieties.*

Next we define the uniformization morphism

$$\Theta : \breve{\mathcal{M}} \times \tilde{G}^\bullet(\mathbb{A}_{F,f}^\mathfrak{p})/C^\mathfrak{p} \longrightarrow \mathcal{A}_C \times_{\mathrm{Spec}\ O_E} \mathrm{Spec}\ O_{\breve{E}}. \tag{13}$$

We note that we have defined a right action of $\tilde{G}^\bullet(\mathbb{A}_{F,f}) = G^\bullet(\mathbb{A}_f)$ on the projective system on the right hand side. We have also a right action on the left hand side of (13) of the group $\tilde{G}^\bullet(\mathbb{A}_{F,f})$. Indeed the action of $\tilde{G}^\bullet(\mathbb{A}_{F,f}^\mathfrak{p})$ is by definition the obvious one, by while $\tilde{G}^\bullet(F_\mathfrak{p})$ acts via $\breve{\mathcal{M}}$ by definition 0.7.

By the required equivariance it is enough to define Θ on $\breve{\mathcal{M}} \times \{1\}$. By the same reason we may assume that the group C is sufficiently small in the set of open compact subgroups of $\tilde{G}^\bullet(\mathbb{A}_{F,f})$ which are maximal in $\mathfrak{p}$. The definition of the morphism Θ will depend on the choice of a point indexed by the letter s: $(A_s, \Lambda_s, \{\overline{\lambda}_{s,i}\}, \overline{\eta}_s^p, \{\overline{\eta}_{s,\mathfrak{q}_i}\}) \in \mathcal{A}_C(\overline{\kappa})$ for some sufficiently small C. We also choose elements $\eta_s^p \in \overline{\eta}_s^p$ $\eta_{s,\mathfrak{q}_i} \in \eta_{s,\overline{\mathfrak{q}}_i}$ and $\lambda_{s,i} \in \overline{\lambda}_{s,i}$. The p-divisible group X_s of A_s will have a decomposition:

$$X_s = \prod_{i=0}^{m} (X_{s,q_i} \times X_{s,\overline{\mathfrak{q}}_i}).$$

Let us denote by $X_{s,\mathfrak{p}} = X_{s,\mathfrak{q}} \times X_{s,\overline{\mathfrak{q}}}$ the $\mathfrak{p}$-component of the p-divisible group of A_s. It does not change $\breve{\mathcal{M}}$ if we assume $\mathbb{X} = X_{s,\mathfrak{p}}$. Given a point $(X, \rho) \in \breve{\mathcal{M}}(T)$ there is up to a unique isomorphism an object $\overline{A}$ in the category AV over $\overline{T}$ whose $\mathfrak{p}$-component of its p-divisible group is $\overline{X}$ together with a quasiisogeny $(A_s)_{\overline{T}} \longrightarrow \overline{A}$ which extends $\rho : \mathbb{X}_T \longrightarrow \overline{X}$. Pushing forward the data $\lambda_{s,i}$, η_s^p, $\eta_{s,\mathfrak{q}_i}$ and Λ_s, we obtain a point $(\overline{A}, \Lambda, \{\overline{\lambda}_i\}, \overline{\eta}^p, \{\overline{\eta}_{\mathfrak{q}_i}\}) \in \mathcal{A}_C(\overline{T})$.

Finally by the criterion of Serre and Tate the given lifting X of $\overline{X}$ defines a unique lifting of that point to a point

$$\Theta((X, \rho) \times 1) \in \mathcal{A}_C(T).$$

One checks immediately that the morphism thus defined is equivariant with respect to the $\tilde{G}^\bullet(F_\mathfrak{p})$ actions given by the definitions 0.6 and 0.7.

Hence Θ is $\tilde{G}^\bullet(\mathbb{A}_{F,f})$-equivariant by definition. Our next aim is to determine the fibres of the morphism Θ. Let us denote by $\varphi \longmapsto \varphi'$ the Rosati involution induced by Λ on the finite dimensional $\mathbb{Q}$-algebra $\mathrm{End}^0_B A_s$. We set

$$\tilde{I}^\bullet(F) = \{\varphi \in \mathrm{End}^0_B A_s \mid \varphi\varphi' \in F^\star\}. \tag{14}$$

We regard $\tilde{I}^\bullet$ as an algebraic group over F. Let $\tilde{\gamma} : \tilde{I}^\bullet \longrightarrow \mathbb{G}_{m,F}$ be the morphism given by $\varphi \longmapsto \varphi\varphi'$. Let us denote by $I^\bullet = \mathrm{Res}_{F/\mathbb{Q}}\tilde{I}^\bullet$ the restriction á la Weil. We are going to define group homomorphisms:

$$\begin{aligned} &I^\bullet(\mathbb{Q}) \longrightarrow \mathcal{J}^\bullet(F_\mathfrak{p}) \\ &I^\bullet(\mathbb{Q}) \longrightarrow \tilde{G}^\bullet(F_{\mathfrak{p}_i}) \qquad i = 1, \dots m \\ &I^\bullet(\mathbb{Q}) \longrightarrow \tilde{G}^\bullet(\mathbb{A}^p_{F,f}) \end{aligned} \tag{15}$$

Since $\mathcal{J}^\bullet(F_\mathfrak{p})$ acts on $\breve{\mathcal{M}}$ from the left definition 0.6 we get an obvious action $I^\bullet(\mathbb{Q})$ on $\breve{\mathcal{M}} \times \tilde{G}^\bullet(\mathbb{A}^\mathfrak{p}_{F,f})/C^\mathfrak{p}$ from the left.

The first morphism of (15) associates to $\varphi \in I^\bullet(\mathbb{Q})$ the induced quasi-isogeny of the $\mathfrak{p}$-component $\mathbb{X}$ of the p-divisible group of A_s.

To define the second type of homomorphism we use the isomorphism

$$\tilde{G}^\bullet(F_{\mathfrak{p}_i}) \xrightarrow{\sim} GL_{B_{\mathfrak{q}_i}}(W \otimes_K K_{\mathfrak{q}_i}) \times F^\star_{\mathfrak{p}_i}$$

which is given by the natural action of $\tilde{G}^\bullet(F_{\mathfrak{p}_i})$ on $W \otimes_K K_{\mathfrak{q}_i}$ on the first factor and by $\tilde{\gamma}$ on the second factor. Hence it is enough to define homomorphisms $\xi_i : I^\bullet(\mathbb{Q}) \longrightarrow Gl_{B_{\mathfrak{q}_i}}(W \otimes_K K_{\mathfrak{q}_i})$ and $I^\bullet(\mathbb{Q}) \longrightarrow F^\star_{\mathfrak{p}_i}$. The last morphism is given by $\tilde{\gamma}$ on $I(\mathbb{Q})$, while ξ_i is defined by the commutative diagram

$$\begin{array}{ccc} V_{\mathfrak{q}_i}(A_s) & \xrightarrow{V_{\mathfrak{q}_i}(\varphi)} & V_{\mathfrak{q}_i}(A_s) \\ {\scriptstyle \eta_{s,\mathfrak{q}_i}}\downarrow & & \downarrow{\scriptstyle \eta_{s,\mathfrak{q}_i}} \\ W \otimes_K K_{\mathfrak{q}_i} & \xrightarrow{\xi_i(\varphi)} & W \otimes_K K_{\mathfrak{q}_i} \end{array}$$

The third morphism of (15) is defined by a similar diagram involving $V^p(A_s)$ and $W \otimes_{\mathbb{Q}} \mathbb{A}^p_f$.

Since the p-divisible groups $X_{s,\mathfrak{q}_i}$ and $X_{s,\bar{\mathfrak{q}}_i}$ for $i = 0 \ldots m$ are isoclinic, it follows from [RZ] 6.29 that the algebra $\bar{B} = End^0_B A_s$ with its Rosati involution $'$ is characterized up to isomorphism by the following properties: $(\bar{B}, ')$ is a central division algebra over K with a positive involution. We have isomorphisms of algebras with an involution:

$$\begin{aligned} (\bar{B} \otimes_F \mathbb{A}^{\mathfrak{p}}_{F,f}, ') &\cong (B^{opp} \otimes_F \mathbb{A}^{\mathfrak{p}}_{F,f}, *) \\ (\bar{B} \otimes_F F_{\mathfrak{p}}, ') &\cong (M_d(F_{\mathfrak{p}}) \times M_d(F_{\mathfrak{p}})^{opp}, \mathrm{switch}) \end{aligned} \tag{16}$$

Here $*$ denotes the involution defined by (2) and switch denotes the involution which interchanges the factors.

Hence $\tilde{I}^\bullet$ is the inner form of $\tilde{G}^\bullet$, such that $\tilde{I}^\bullet(F \otimes \mathbb{R})$ is compact modulo center and such that we have the following isomorphisms:

$$\begin{aligned} \tilde{I}^\bullet(\mathbb{A}^{\mathfrak{p}}_{F,f}) &\cong \tilde{G}^\bullet(\mathbb{A}^{\mathfrak{p}}_{F,f}) \\ \tilde{I}^\bullet(F_{\mathfrak{p}}) &\cong \mathcal{J}^\bullet(F_{\mathfrak{p}}) \end{aligned} \tag{17}$$

It follows as in [RZ] 6.29 and 6.30 that Θ induces an isomorphism of formal schemes

$$\Theta : I^\bullet(\mathbb{Q}) \setminus \breve{\mathcal{M}} \times \tilde{G}^\bullet(\mathbb{A}^{\mathfrak{p}}_{F,f})/C^{\mathfrak{p}} \longrightarrow \hat{\mathcal{A}}_C \times_{\mathrm{Spf}\, O_E} \mathrm{Spf}\, O_{\breve{E}}, \tag{18}$$

where $\hat{\mathcal{A}}_C$ is completion along the special fibre of $\mathcal{A}_C$. Let us now compare the descent data on both sides of (18).

We define $\xi_j \in \tilde{G}^\bullet(F_{\mathfrak{p}_j})$ for $j = 1, \ldots, m$ as follows. The action ξ_j on the first summand of the decomposition $W_{\mathfrak{p}_j} = W_{\mathfrak{q}_j} \oplus W_{\bar{\mathfrak{q}}_j}$ is the identity, and on the second direct summand is multiplication by q, where q is the number of elements in the residue class field κ of E. Let us denote by $\xi^{\mathfrak{p}} \in \tilde{G}^\bullet(\mathbb{A}^{\mathfrak{p}}_{F,f})$ the idele with components ξ_j at the primes $\mathfrak{p}_j, j = 1, \ldots, m$ and 1 elsewhere.

Lemma 0.8. *The canonical descent datum on the right hand side of*

$$\Theta : \breve{\mathcal{M}} \times \tilde{G}^\bullet(\mathbb{A}^{\mathfrak{p}}_{F,f})/C^{\mathfrak{p}} \longrightarrow \mathcal{A}_C \times_{Spec\, O_E} Spec\ O_{\breve{E}}.$$

induces on the left hand side the Weil descent (10) datum on $\breve{\mathcal{M}}$ multiplied with $\xi^{\mathfrak{p}}$.

Proof: Let T be a scheme over $\mathrm{Spf}\, O_{\breve{E}}$ and $\varphi : T \longrightarrow \mathrm{Spf}\, O_{\breve{E}}$ be the structure morphism. To compare the descent data on both sides of (18) we start with a T-valued point $(X, \rho) \times g$ of the left hand side. Here $\rho : \overline{\varphi}^\star \mathbb{X} \longrightarrow X_{\overline{T}}$ is a quasiisogeny. For the comparsion we may assume $g = 1$. Let $(A, \Lambda, \{\overline{\lambda}_i\}, \overline{\eta}^p, \{\overline{\eta}_{\mathfrak{q}_i}\}) \in \mathcal{A}_C(T)$ be the point described in the definition of Θ. The natural descent datum

$$\mathcal{A}_C(T) \longrightarrow \mathcal{A}_C(T_{[\tau]})$$

maps this point to the same abelian variety A with additional structure regarded on $T_{[\tau]}$. The descent datum on $\breve{\mathcal{M}}$ maps ρ to the quasiisogeny

$$\rho_1 = \rho \circ \overline{\varphi}^\star \mathrm{Frob}_{\mathbb{X}}^{-1} : \overline{\varphi}^\star \tau^\star \mathbb{X} \longrightarrow \overline{\varphi}^\star \mathbb{X} \longrightarrow X_{\overline{T}}.$$

The image of (X, ρ_1) by Θ is obtained as follows. We extend ρ_1 to a quasi-isogeny of abelian varieties

$$\overline{\varphi}^\star \overline{\tau}^\star A_s \xrightarrow{\overline{\varphi}^\star(\mathrm{Frob}_{A_s}^{-1})} \overline{\varphi}^\star A_s \longrightarrow A_{\overline{T}}$$

and push forward the data on $\overline{\varphi}^\star \overline{\tau}^\star A_s$ induced by $\Lambda_s, \lambda_{s,i}, \eta_s^p, \eta_{s,\mathfrak{q}_i}$. We note that $\mathrm{Frob}_{A_s}^{-1}$ just induces the identity on the Tate-modules.

$$V_\ell(A_s) = \tau^\star V_\ell(A_s) = V_\ell(\tau^\star A_s) \xrightarrow{\mathrm{Frob}_{A_s}^{-1}} V_\ell(A_s)$$

Hence the push-forwards of η_s^p and η_{s,q_i} are $\overline{\eta}^p$ and $\overline{\eta}_{q_i}$. We claim that the push-forward of an element $\lambda \in \Lambda_s$ by $\mathrm{Frob}_{A_s}^{-1}$ is $q \cdot \lambda$. Indeed this follows from the commutative diagram

$$\begin{array}{ccc}
\overline{\tau}^\star A_s & \xrightarrow{\mathrm{Frob}_{A_s}^{-1}} & A_s \\
{\scriptstyle \overline{\tau}^\star \lambda_i}\downarrow & & \downarrow{\scriptstyle \lambda_i} \\
\overline{\tau}^\star \hat{A}_s & \xrightarrow[\mathrm{Frob}_{\hat{A}_s}^{-1}]{} & \hat{A}_s
\end{array}$$

and the equality

$$\widehat{\mathrm{Frob}_{A_s}^{-1}} = \mathrm{Frob}_{\hat{A}_s} \cdot q^{-1}.$$

The assertion of the lemma is now obvious.

To the group $G^\bullet$ and the module (W, ψ) there is associated a Shimura variety. We set $\mathcal{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. Let

$$h^\bullet : \mathcal{S} \longrightarrow G^\bullet_{\mathbb{R}} \tag{19}$$

be a morphism of algebraic groups such that $h^\bullet$ defines on $W_{\mathbb{R}}$ a Hodge structure of type $(1,0), (0,1)$, and such that $\psi(w_1, h^\bullet(\sqrt{-1})w_2), w_1, w_2 \in W_{\mathbb{R}}$ is symmetric and positive definite. Then $h^\bullet$ is determined by these properties up to conjugacy. Let $\mathrm{Sh}_{(G^\bullet, h^\bullet), C}$ the corresponding Shimura variety, i.e. tower of projective algebraic varieties indexed by $C \subset G^\bullet(\mathbb{A}_f)$. As above we restrict our attention to those C, which satisfy the conditions under (5). Then $\mathrm{Sh}_{(G^\bullet, h^\bullet), C}$ is for sufficiently small C a fine module scheme for the étale sheafification of the following functor. Let $E(h^\bullet)$ be the Shimura field.

Then a point of the functor over a $E(h^\bullet)$-scheme S is given by the same data and conditions as a point of $\underline{\mathcal{A}}_C$ except that the data 3) and 5) are replaced by a single datum 3' :

3') For each $i = 1, \ldots, m$ a class of $B_{\mathfrak{p}_i}$-module isomorphisms

$$\overline{\eta}_{\mathfrak{p}_i} : V_{\mathfrak{p}_i}(A) \longrightarrow W_{\mathfrak{p}_i} \mod C_{\mathfrak{p}_i}$$

which respects the bilinear forms on both sides given by Λ respectively ψ up to a constant in $F^\star_{\mathfrak{p}_i}$.

We fix once for all a diagram:

$$\mathbb{C} \longleftarrow \bar{\mathbb{Q}} \xrightarrow{\nu} \bar{\mathbb{Q}}_p \tag{20}$$

According to this diagram the Hodge structure $W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}$ given by the morphism $h^\bullet$ defines a corresponding decomposition:

$$W_{\bar{\mathbb{Q}}_p} = W^{1,0} \oplus W^{0,1} \tag{21}$$

We require that $W^{1,0}$ satisfies the conditions of (7) on the space W_0.

The condition may be reformulated as follows. The trace of an element $b \in B$ acting on $W^{1,0}$ is of the following form

$$Tr_{\mathbb{C}}(b|W^{1,0}) = \sum_{\rho: K \to \mathbb{C}} r_\rho(Tr^0 b), \tag{22}$$

where r_ρ are natural numbers, such that $r_\rho + r_{\bar{\rho}} = d$.

Then the condition (7) is equivalent to:

$$r_\rho = \begin{cases} 0 & \text{if } \rho \in \Phi,\ \rho \neq \alpha \\ 1 & \text{if } \rho = \alpha \\ d-1 & \text{if } \rho = \bar{\alpha} \\ d & \text{if } \rho \in \bar{\Phi}, \rho \neq \bar{\alpha} \end{cases} \tag{23}$$

The diagram (20) determines a p-adic place of the Shimura field $E(h^\bullet)$. One checks easily, that under the conditions (23):

$$E(h^\bullet)_\nu = \alpha(K_{\mathfrak{q}}) = E.$$

We compare now $\mathrm{Sh}_{(G^\bullet,h^\bullet),C}$ and $\mathcal{A}_C$ as varieties over E. Let E_{p^∞}/E be the field extension obtained by adjoining all p^n-th roots of units for all $n \geq 0$. An element $\sigma \in \mathrm{Gal}(E_{p^\infty}/E)$ operates on the E_{p^∞}-valued point of $\mathbb{Q}_p(1)$ by multiplication with an element $u_\sigma \in \mathbb{Z}_p^\star$. It may be helpful to note that the descent datum on the constant scheme $\mathbb{Q}_p$ relative to E_{p^∞}/E giving rise to $\mathbb{Q}_p(1)$ is multiplication by u_σ^{-1}.

Lemma 0.9. *There is an isomorphism*

$$Sh_{(G^\bullet,h^\bullet),C} \times_{Spec\, E} Spec\, E_{p^\infty} \xrightarrow{\sim} \mathcal{A}_C \times_{Spec\, O_E} Spec\, E_{p^\infty},$$

such that the action of $id_{\mathcal{A}_C} \times \sigma$ on the right hand side gives on the left hand side the action by $\tilde{u}_\sigma \times \sigma$, where $\tilde{u_\sigma} \in K^\star(\mathbb{A}_f)$ is the image of u_σ by the diagonal embedding

$$\mathbb{Q}_p^\star \longrightarrow \prod_{i=1}^{m} K_{\mathfrak{q}_i}^\star \subset K^\star(\mathbb{A}_f)$$

Proof: We choose an isomorphism $\mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}_p(1)$ over $\mathrm{Spec}\, E_{p^\infty}$ and denote by $\zeta \in \mathbb{Z}_p(1)$ the image of 1.

Let $\varphi : T \longrightarrow \mathrm{Spec}\, E_{p^\infty}$ be a scheme. We define a morphism

$$\xi : \mathrm{Sh}_{(G^\bullet,h^\bullet),C}(T) \longrightarrow \mathcal{A}_C(T)$$

Let $(A, \Lambda, \overline{\eta}^p, \overline{\eta}_{\mathfrak{p}_i}) \in \mathrm{Sh}_{(G^\bullet,h^\bullet),C}(T)$ be a point. The image $(A, \Lambda, \{\overline{\lambda}_i\}, \overline{\eta}^p, \{\overline{\eta}_{\mathfrak{q}_i}\})$ by α is defined as follows. Let $\eta_{\mathfrak{p}_i} \in \overline{\eta}_{\mathfrak{p}_i}$ be an isomorphism. It decomposes as a direct sum

$$\eta_{\mathfrak{p}_i} = \eta_{\mathfrak{q}_i} \oplus \eta_{\bar{\mathfrak{q}}_i} : V_{\mathfrak{q}_i}(A) \oplus V_{\bar{\mathfrak{q}}_i}(A) \longrightarrow W_{\mathfrak{q}_i} \oplus W_{\bar{\mathfrak{q}}_i}.$$

This already defines $\eta_{\mathfrak{q}_i}$. To finish the definition of ξ we need to say what $\overline{\lambda}_i$ is. We give it by the equation

$$E^{\lambda_i}(x, y) = \varphi^\star(\zeta)\psi(\eta_{\mathfrak{q}_i}(x), \eta_{\bar{\mathfrak{q}}_i}(y)),\ x \in V_{\mathfrak{q}_i}(A), y \in V_{\bar{\mathfrak{q}}_i}(A).$$

Here E^{λ_i} denotes the Riemann form associated an element $\lambda_i \in \Lambda \otimes_F F_{\mathfrak{p}_i}$. Let $\sigma \in \mathrm{Gal}(E_{p^\infty}/E)$ be an element of the Galois group. We regard $(A, \Lambda, \overline{\eta}^p, \overline{\eta}_{\mathfrak{p}_i})$ as a point of $\mathrm{Sh}_{(G^\bullet,h^\bullet),C}(T_{[\sigma]})$. Its image by ξ has the form $(A, \Lambda, \{\overline{\lambda}_i'\}, \overline{\eta}^p, \{\overline{\eta}_{\mathfrak{q}_i}\})$.

The classes $\overline{\lambda}_i'$ are given by

$$\begin{aligned} E^{\lambda_i'}(x, y) &= \varphi^\star\sigma^\star(\zeta)\psi(\eta_{\mathfrak{q}_i}(x), \eta_{\bar{\mathfrak{q}}_i}(y)) \\ &= u_\sigma\varphi^\star(\zeta)\psi(\eta_{\mathfrak{q}_i}(x), \eta_{\bar{\mathfrak{q}}_i}(y)) \end{aligned}$$

This implies $\lambda_i' = \lambda_i u_\sigma$. By definition of the action of $K^\star(\mathbb{A}_f)$ on $\mathcal{A}_C$ this implies that we have a commutative digram

$$\begin{array}{ccc} \mathrm{Sh}_{(G^\bullet,h^\bullet),C}(T) & \longrightarrow & \mathcal{A}_C(T) \\ {\scriptstyle can}\downarrow & & \downarrow{\scriptstyle can\cdot\tilde{u}_\sigma^{-1}} \\ \mathrm{Sh}_{(G^\bullet,h^\bullet),C}(T_{[\sigma]}) & \longrightarrow & \mathcal{A}_C(T_{[\sigma]}) \end{array}$$

The lemma follows. □

We may now state the main theorem on the uniformization of the Shimura variety $\mathrm{Sh}_{(G^\bullet,h^\bullet)}$. Let $\mathbb{Q}_p^{ab}$ be the maximal abelian extension of $\mathbb{Q}_p$. We set $\tilde{E} = E\mathbb{Q}_p^{ab} = E_{p^\infty}E^{nr}$. Let $a \in \mathbb{Q}_p^\star$ and σ be its image by the Artin reciprocity map $\mathbb{Q}_p^\star \longrightarrow \mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$. We denote by $\tilde{a}$ the image of a by the diagonal map

$$\mathbb{Q}_p^\star \longrightarrow \prod_{i=1}^m K_{\bar{\mathfrak{q}}_i}^\star \subset K^\star(\mathbb{A}_f). \tag{24}$$

We will now assmue that a is in the inverse image of $\mathrm{Gal}(\tilde{E}/E) \subset \mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$. Then we define a Galois twist $\mathrm{Sh}^t_{(G^\bullet,h^\bullet),C}$ of $\mathrm{Sh}_{(G^\bullet,h^\bullet),C}$ over E by the condition that there is an isomorphism over $\tilde{E}$

$$\mathrm{Sh}_{(G^\bullet,h^\bullet),C} \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \tilde{E} \xrightarrow{\sim} \mathrm{Sh}^t_{(G^\bullet,h^\bullet),C} \times_{\mathrm{Spec} E} \mathrm{Spec} \tilde{E},$$

such that for any $\tilde{E}$-scheme T the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Sh}_{(G^\bullet,h^\bullet),C}(T) & \xrightarrow{\sim} & \mathrm{Sh}^t_{(G^\bullet,h^\bullet),C}(T) \\ {\scriptstyle can\cdot\tilde{a}^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle can} \\ \mathrm{Sh}_{(G^\bullet,h^\bullet),C}(T_{[\sigma]}) & \xrightarrow{\sim} & \mathrm{Sh}^t_{(G^\bullet,h^\bullet),C}(T_{[\sigma]}). \end{array}$$

Here can is the descent datum which comes from the E-structure on both sides.

We may interpret $\mathrm{Sh}^t_{(G^\bullet,h^\bullet)}$ as a Shimura variety: Let us denote by $Z^\bullet = Res_{K/\mathbb{Q}}\mathbb{G}_{m,K}$ the center of the group $G^\bullet$.

Over $\mathbb{Q}_p$ we have a decomposition:

$$Z^\bullet_{\mathbb{Q}_p} = \prod_{i=0}^{m} Res_{K_{\mathfrak{q}_i}/\mathbb{Q}_p}\mathbb{G}_{m,K_{\mathfrak{q}_i}} \times \prod_{i=0}^{m} Res_{K_{\bar{\mathfrak{q}}_i}/\mathbb{Q}_p}\mathbb{G}_{m,K_{\bar{\mathfrak{q}}_i}} \tag{25}$$

Let

$$\mu^{\mathfrak{p}}_K : \mathbb{G}_{m,\mathbb{Q}_p} \longrightarrow Z^\bullet_{\mathbb{Q}_p}$$

be the morphism, which is trivial on the first $m+2$ factors, and which is given by (24) on the last m factors. According to the diagram (20) we may view $\mu^{\mathfrak{p}}_K$ as a morphism $\mathbb{G}_{m,\mathbb{C}} \to Z^\bullet_{\mathbb{Q}_p}$. Let $h^{\mathfrak{p}}_K : \mathcal{S} \to Z^\bullet_{\mathbb{R}}$ be the morphism with first component $\mu^{\mathfrak{p}}_K$. Then $\mathrm{Sh}^t_{(G^\bullet,h^\bullet)}$ coincides with the Shimura variety $\mathrm{Sh}_{(G^\bullet,h^\bullet(h^{\mathfrak{p}}_K)^{-1})}$. Let us also introduce the morphism

$$\mu_K : \mathbb{G}_{m,\mathbb{Q}_p} \longrightarrow Z^\bullet_{\mathbb{Q}_p},$$

which is trivial on the first $m+1$ factors of (25) and is given on the last factors by the canonical adjunction morphism

$$\mathbb{G}_{m,\mathbb{Q}_p} \longrightarrow Res_{K_{\bar{\mathfrak{q}}_i}/\mathbb{Q}_p}\mathbb{G}_{m,K_{\bar{\mathfrak{q}}_i}}$$

There is a morphism $h_K : \mathcal{S} \to Z^\bullet_{\mathbb{R}}$ associated to μ_K, exactly as above for $\mu^{\mathfrak{p}}_K$.

Proposition 0.10. *There exists a tower of projective schemes $\mathcal{S}h^t_C$ over O_E, where C runs through the open and compact subgroups of $G^{\bullet}(\mathbb{A}_f)$, which satisfy the conditions after (3). The tower is equipped with a $G^{\bullet}(\mathbb{A}_f)$-action from the right. The general fibre of this tower is the tower $Sh_{(G^{\bullet},h^{\bullet}(h^{\mathfrak{p}}_K)^{-1}),C}$ with its natural $G^{\bullet}(\mathbb{A}_f)$-action. There is a $G^{\bullet}(\mathbb{A}_f)$-equivariant isomorphism of formal schemes over $SpfO_{\breve{E}}$*

$$\Theta : I^{\bullet}(\mathbb{Q}) \setminus \breve{\mathcal{M}} \times \tilde{G}^{\bullet}(\mathbb{A}^{\mathfrak{p}}_{F,f})/C^{\mathfrak{p}} \longrightarrow \hat{\mathcal{S}h}^t_C \times_{SpfO_E} SpfO_{\breve{E}}. \qquad (26)$$

Here $\hat{\mathcal{S}h}^t_C$ denotes the completion of $\mathcal{S}h^t_C$ along the special fibre. The left hand side of (26) is equipped with a Weil-descent datum via $\breve{\mathcal{M}}$ (see (10)). This is mapped by Θ to the canonical Weil descent datum on the right hand side.

Proof: This is a consequence of the lemma (0.8), the lemma (0.9), and the explicit computation of the Artin reciprocity law for $\mathbb{Q}_p$ ([CF] Chap. VI, Thm. 3.2). $\square$

We will reformulate the last proposition in terms of the formal scheme $\hat{\Omega}^d_E$ defined by Deligne ([RZ] 3.71).

Let $\breve{\mathcal{N}}$ be the formal scheme over $\operatorname{Spf} O_{\breve{E}}$, which classifies quasiisogenies of the special formal $O_{B_{\mathfrak{q}}}$-module Φ over the $O_{F_{\mathfrak{p}}}$-algebra $\bar{\kappa}$. We recall the definition from [RZ] 3.59 and 3.21, using the notation of our definition (0.5).

Let T be a scheme over $\operatorname{Spf} O_{\breve{E}}$, where p is locally nilpotent.

Definition 0.11. *An element of $\breve{\mathcal{N}}(T)$ is given by the following data:*

1) *A special formal $O_{B_{\mathfrak{q}}}$-module Y over T, with respect to $T \to SpfO_{\breve{E}} \xrightarrow{\alpha} SpfO_{F_{\mathfrak{p}}}$.*

2) *A quasiisogeny*

$$\rho_1 : \Phi_{\bar{T}} \longrightarrow Y_{\bar{T}}.$$

This is a p-adic formal scheme as shown in [RZ] 3.63. It is equipped with a Weil-descent datum relative to $\breve{E}/E$

$$\xi_1 : \breve{\mathcal{N}}(T) \longrightarrow \tau^{\star}\breve{\mathcal{N}}(T),$$

which is defined exactly in the same way as the Weil-descent datum for $\breve{\mathcal{M}}$ after definition (0.5). Let us denote by $G_{\mathfrak{p}}$ the algebraic group over $F_{\mathfrak{p}}$ given by:

$$G_{\mathfrak{p}}(F_{\mathfrak{p}}) = Aut_{B_{\mathfrak{q}}} W_{\mathfrak{q}} \cong (B^{opp}_{\mathfrak{q}})^{\star}. \qquad (27)$$

Let us denote by $\mathcal{J}(F_{\mathfrak{p}})$ the group of quasiisognies of the special formal $B_{\mathfrak{q}}$-module Φ. There is an isomorphism $\mathcal{J}(F_{\mathfrak{p}}) \cong Gl_d(F_{\mathfrak{p}})$,which we will fix ([RZ] 3.71).

We define a left action of $\mathcal{J}(F_{\mathfrak{p}})$ and a right action of $G_{\mathfrak{p}}(F_{\mathfrak{p}})$ on $\breve{\mathcal{N}}$, as follows:

$$h(Y, \rho_1) = (Y, \rho_1 h^{-1}), \quad h \in \mathcal{J}(F_{\mathfrak{p}}) \tag{28}$$
$$(Y, \rho_1)g = (Y^b, \iota(b^{-1})\rho_1), \quad g \in G_{\mathfrak{p}}(F_{\mathfrak{p}}) \tag{29}$$

In the last definition $b \in B_{\mathfrak{q}}^{\star}$ is any element satisfying $g\Gamma_{\mathfrak{q}} = b\Gamma_{\mathfrak{q}}$.

We have a natural isomorphism of formal schemes:

$$\begin{aligned} \breve{\mathcal{M}} &\longrightarrow \breve{\mathcal{N}} \times (F_{\mathfrak{p}}^{\star}/O_{F_{\mathfrak{p}}}^{\star}) \\ (X, \rho) &\mapsto (X_1, \rho_1) \times f \end{aligned} \tag{30}$$

Here $(X, \rho), (X_1, \rho_1)$, and f have the same meaning as in definition 0.5 .

By $F_{\mathfrak{p}}^{\star}/O_{F_{\mathfrak{p}}}^{\star}$ we denote the constant formal scheme over $\operatorname{Spf} O_{\breve{E}}$. We define a Weil descent datum relative to $\breve{E}/E$ on this formal scheme, such that

$$F_{\mathfrak{p}}^{\star}/O_{F_{\mathfrak{p}}}^{\star} \longrightarrow \tau^{\star}(F_{\mathfrak{p}}^{\star}/O_{F_{\mathfrak{p}}}^{\star}) = F_{\mathfrak{p}}^{\star}/O_{F_{\mathfrak{p}}}^{\star}$$

is the multiplication by $q = \operatorname{card} \kappa$. If we take this Weil descent datum into account we write $F_{\mathfrak{p}}^{\star}/O_{F_{\mathfrak{p}}}^{\star}(1)$.

Proposition 0.12. *The map (30) defines a morphism compatible with the Weil descent data :*

$$\breve{\mathcal{M}} \longrightarrow \breve{\mathcal{N}} \times F_{\mathfrak{p}}^{\star}/O_{F_{\mathfrak{p}}}^{\star}(1) \tag{31}$$

Proof: This follows because the push forward of the canonical polarization $\tau^{\star}(\lambda_0)$ on $\tau^{\star}\mathbb{X}$ by $\mathrm{Frob}^{-1} : \tau^{\star}\mathbb{X} \longrightarrow \mathbb{X}$ gives the polarization $q\lambda_0$. □

We define isomorphisms

$$\tilde{G}^{\bullet}(F_{\mathfrak{p}}) \longrightarrow G_{\mathfrak{p}}(F_{\mathfrak{p}}) \times F_{\mathfrak{p}}^{\star} \tag{32}$$
$$\mathcal{J}^{\bullet}(F_{\mathfrak{p}}) \longrightarrow \mathcal{J}(F_{\mathfrak{p}}) \times F_{\mathfrak{p}}^{\star}, \tag{33}$$

such that the morphism (31) becomes equivariant with respect to the various actions defined above. The morphism $\tilde{G}^{\bullet}(F_{\mathfrak{p}}) \to G_{\mathfrak{p}}(F_{\mathfrak{p}})$ is the restriction with respect to the inclusion $W_{\mathfrak{q}} \to W_{\mathfrak{q}} \oplus W_{\bar{\mathfrak{q}}} = W_{\mathfrak{p}}$. The map $\tilde{G}^{\bullet}(F_{\mathfrak{p}}) \to F_{\mathfrak{p}}^{\star}$ is the multiplicator $\tilde{\gamma}$. The morphism $\mathcal{J}^{\bullet}(F_{\mathfrak{p}}) \to \mathcal{J}(F_{\mathfrak{p}})$ is the restriction with respect to the inclusion $\Phi \to \Phi \times \hat{\Phi}$. Finally $\gamma_{\mathcal{J}^{\bullet}} : \mathcal{J}^{\bullet}(F_{\mathfrak{p}}) \to F_{\mathfrak{p}}^{\star}$ is the multiplicator (12).Therefore we may rewrite the uniformization morphism (26) as follows

$$\Theta : I^{\bullet}(\mathbb{Q}) \setminus \breve{\mathcal{N}} \times F_{\mathfrak{p}}^{\star}/O_{F_{\mathfrak{p}}}^{\star}(1) \times \tilde{G}^{\bullet}(\mathbb{A}_{F,f}^{\mathfrak{p}}) \longrightarrow \hat{\mathcal{S}h}_{C}^{t} \times_{\operatorname{Spf} O_E} \operatorname{Spf} O_{\breve{E}}$$

Taking into account that $\hat{\mathcal{S}h}_{C}^{t}$ is a model over $\operatorname{Spf} O_E$ of $\mathrm{Sh}_{(G^{\bullet}, h^{\bullet}(h_K)^{-1}),C}$, we may reformulate the proposition 0.10 as follows:

Theorem 0.13. *Let $C \subset G^{\bullet}(\mathbb{A}_f)$ be an open compact subgroup which is maximal in idealp. Then there exists a model $\mathcal{S}h_{(G^{\bullet},h^{\bullet},(h_K)^{-1}),C}$ over $SpecO_E$ of $Sh_{(G^{\bullet},h^{\bullet}(h_K)^{-1}),C}$ and a $G^{\bullet}(\mathbb{A}_f)$-equivariant morphism of formal schemes:*

$$I^{\bullet}(\mathbb{Q}) \setminus \breve{\mathcal{N}} \times F_{\mathfrak{p}}^{\star}/O_{F_{\mathfrak{p}}}^{\star} \times \tilde{G}^{\bullet}(\mathbb{A}_{F,f}^{\mathfrak{p}})/C \xrightarrow{\sim} \mathcal{S}h_{(G^{\bullet}h^{\bullet}(h_K)^{-1}),C} \times_{SpfO_E} SpfO_{\breve{E}}, \quad (34)$$

which is compatible with the canonical Weil descent data on both sides relative to $\breve{E}/E$. The canonical Weil descent datum on the left hand side is here the descent datum induced from the given Weil descent datum on the factor $\breve{\mathcal{N}}$.
□

Using [RZ] 3.72 it is easy to rewrite the theorem above in terms of the formal scheme $\hat{\Omega}_E^d$. Let us denote by Π a prime element of the division algebra $B_{\mathfrak{q}}$. Then we have an isomorphism of formal schemes:

$$\begin{aligned} \breve{\mathcal{N}} &\longrightarrow (\hat{\Omega}_E^d \times_{\operatorname{Spf} O_E} \operatorname{Spf} O_{\breve{E}}) \times \mathbb{Z} \\ (Y, \rho_1) &\longmapsto \quad (Y^{\Pi^n}, \Pi^{-n})\rho_1 \quad \times -n \end{aligned} \quad (35)$$

Here $\mathbb{Z}$ denotes the constant formal scheme over $\operatorname{Spf} O_{\breve{E}}$. The integer n is the relative height of ρ_1 with respect to the action of $O_{B_{\mathfrak{q}}}$. The relative height is the multiple of the usual height, which satisfies that the isogeny induced by Π on a special formal $O_{B_{\mathfrak{q}}}$-module has relative height 1 (see [RZ] 3.53).

If we equip $\mathbb{Z}$ with the Weil descent datum relative to $\operatorname{Spf} O_{\breve{E}}/\operatorname{Spf} O_E$ given by

$$\begin{aligned} &\mathbb{Z} \longrightarrow \tau^{\star}\mathbb{Z} = \mathbb{Z}, \\ &n \mapsto n+1 \end{aligned}$$

and $\hat{\Omega}^d_E \times_{\mathrm{Spf}\, O_E} \mathrm{Spf}\, O_{\breve{E}}$ with the natural Weil descent datum the morphism (35) becomes compatible with the Weil descent data.

The action of $G_{\mathfrak{p}}(F_{\mathfrak{p}})$ on $\breve{\mathcal{N}}$ induces on the right hand side of (35) the trivial action on the first two factors and the addition with $\mathrm{ord}_{F_{\mathfrak{p}}} \mathrm{Nm}^0 g$ on the factor $\mathbb{Z}$. Here $g \in G_{\mathfrak{p}}(F_{\mathfrak{p}}) \cong (B^{opp}_{\mathfrak{q}})^{\star}$, and Nm^0 is the reduced norm from $B^{opp}_{\mathfrak{q}}$ to $F_{\mathfrak{p}}$. One may choose the isomorphism $\mathcal{J}(F_{\mathfrak{p}}) \cong Gl_2(F_{\mathfrak{p}})$ in such a way that the action of $h \in Gl_2(F_{\mathfrak{p}})$ induced on the right hand side of (35) is the natural action of h on the first factor $\hat{\Omega}^d_E$, is trivially on the second factor and is addition with $\mathrm{ord}_{F_{\mathfrak{p}}} \det h$ on the factor $\mathbb{Z}$.

We have surjective homomorphisms:

$$
\begin{aligned}
\tilde{G}^{\bullet}(F_{\mathfrak{p}}) &\longrightarrow \mathbb{Z} \times F^{\star}_{\mathfrak{p}}/O^{\star}_{F_{\mathfrak{p}}} \\
g = (g_1, g_2) &\mapsto (\mathrm{ord}_{F_{\mathfrak{p}}} \mathrm{Nm}^0 g_1, \tilde{\lambda}(g))
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}^{\bullet}(F_{\mathfrak{p}}) &\longrightarrow \mathbb{Z} \times F^{\star}_{\mathfrak{p}}/O^{\star}_{F_{\mathfrak{p}}} \\
h = (h_1, h_2) &\mapsto (\mathrm{ord}_{F_{\mathfrak{p}}} \det h_1, \lambda_{\mathcal{J}^{\bullet}}(h))
\end{aligned}
$$

The kernel of the first of these maps is the maximal compact subgroup $C_{\mathfrak{p}} \subset \tilde{G}^{\bullet}(F_{\mathfrak{p}})$. Hence we obtain a homomorphism

$$
\mathcal{J}^{\bullet}(F_{\mathfrak{p}}) \longrightarrow \tilde{G}^{\bullet}(F_{\mathfrak{p}})/C_{\mathfrak{p}} \tag{36}
$$

Taking this into account we may rewrite the uniformization isomorphism (34) as a $G^{\bullet}(\mathbb{A}_f)$-equivariant isomorphism of towers of formal schemes over $\breve{E}$:

$$
\begin{aligned}
\Theta : I^{\bullet}(\mathbb{Q}) \setminus (\hat{\Omega}^d_E \times_{\mathrm{Spf}\, O_E} \mathrm{Spf}\, O_{\breve{E}}) \times G^{\bullet}(\mathbb{A}_f)/C \xrightarrow{\sim} & \\
\mathcal{S}h_{(G^{\bullet}, h^{\bullet}(h_K)^{-1}), C} \times_{\mathrm{Spf}\, O_E} \mathrm{Spf}\, O_{\breve{E}}
\end{aligned} \tag{37}
$$

The group $G^{\bullet}(\mathbb{A}_f)$ acts on the right hand side of that isomorphism only over the factor $G^{\bullet}(\mathbb{A}_f)$. The action of $I^{\bullet}(\mathbb{Q})$ defining the quotient on the left hand side is via the morphisms (15). Here the action of the group $\tilde{G}^{\bullet}(\mathbb{A}^{\mathfrak{p}}_{F,f})$ from the left is the obvious one, while the left action of $\mathcal{J}^{\bullet}(F_{\mathfrak{p}})$ on the factor $\hat{\Omega}^d_E$ is via the obvious morphism $\mathcal{J}^{\bullet}(F_{\mathfrak{p}}) \to \mathcal{J}(F_{\mathfrak{p}})$ defined above (33) and the action of $\mathcal{J}^{\bullet}(F_{\mathfrak{p}})$ on $\tilde{G}^{\bullet}(F_{\mathfrak{p}})/C_{\mathfrak{p}}$ is given by the morphism (36).

Finally the Weil descent datum on the right hand side gives on the left hand side the Weil descent datum induced from $\hat{\Omega}^d_E$ times the action with the idele $(\Pi, \Pi^{-1}) \in \tilde{G}^\bullet(F_\mathfrak{p}) \subset G^\bullet(\mathbb{A}_f)$ (with the notation of (4)).

There is a rigid analytic pro-space $\mathbb{M}$ associated to the functor $\breve{\mathcal{M}}$, which was introduced in [RZ] (5.34). The pro-space $\mathbb{M}$ is defined over $\breve{E}$. It is equipped with an action of $\tilde{G}^\bullet(F_\mathfrak{p})$ from the right and an action of $\mathcal{J}^\bullet(F_\mathfrak{p})$ from the left.

We set:

$$\tilde{G}^\bullet(O_{F_\mathfrak{p}}) = \{g \in \tilde{G}^\bullet(F_\mathfrak{p}) | g\Gamma_\mathfrak{p} = \Gamma_\mathfrak{p}\}$$

By definition 0.7 this group acts trivially on $\breve{\mathcal{M}}$.

Let $\breve{\mathcal{M}}^{rig}$ be the rigid analytic space associated to $\breve{\mathcal{M}}$. There is an equivariant étale covering map

$$\mathbb{M} \longrightarrow \breve{\mathcal{M}}^{rig}, \tag{38}$$

with respect to the actions of $\tilde{G}^\bullet(F_\mathfrak{p})$ and $\mathcal{J}^\bullet(F_\mathfrak{p})$.

Moreover there is a Weil descent datum on $\mathbb{M}$ relative to $\breve{E}/E$, which is compatible with the Weil descent datum on $\breve{\mathcal{M}}^{rig}$ with respect to the map (38).

We will work with compact open subgroups $C_\mathfrak{p} \subset \tilde{G}^\bullet(F_\mathfrak{p})$ of the form (5), but we do no longer assume that $C_\mathfrak{q} \subset (B^{opp}_\mathfrak{q})^\star$ is maximal. The pro-space $\mathbb{M}$ is a projective limit of rigid analytic spaces $\mathbb{M}_{C_\mathfrak{p}} = \mathbb{M}/C_\mathfrak{p}$.

To define the various actions on $\mathbb{M}$ we make the following remark. Let $\mathcal{Z}$ be a formal p-adic scheme over $\operatorname{Spf} O_{\breve{E}}$. We denote by $\mathcal{Z}^{rig}$ the associated rigid analytic space in the sense of Raynaud. We work in the category whose objects are p-adic formal schemes $\mathcal{Z}$ over $\breve{\mathcal{M}}$ and whose morphisms are morphisms of rigid analytic spaces over $\breve{\mathcal{M}}^{rig}$. Then the spaces $\mathbb{M}_{C_\mathfrak{p}}$ may be regarded as representing objects for the following functors. A point of $\mathbb{M}_{C_\mathfrak{p}}(\mathcal{Z}^{rig})$ is a pair (X, ρ) over $\mathcal{Z}$ as in the definition 0.5 and a class of isomorphisms $\bar{\eta}_\mathfrak{p} : T_p(X^{rig}) \to \Gamma_\mathfrak{p}$ modulo $C_\mathfrak{p}$. We require, that an element $\eta_\mathfrak{p} \in \bar{\eta}_\mathfrak{p}$ respects the symplectic structures in the following sense: The natural polarization on $\mathbb{X} = \Phi \times \hat{\Phi}$ induces a quasipolarization λ on X. We note that λ induces a quasiisogeny $X_1 \to \hat{X}_2$, which was denoted by δ^{-1} in the definition 0.5. Let

$$E^\lambda : T_p(X^{rig}) \times T_p(X^{rig}) \to \mathbb{Q}_p(1)$$

be the corresponding Riemann form. Then we require the existence of an element $f \in (F_{\mathfrak{p}} \otimes \mathbb{Q}_p(1))^{\star}$ such that:

$$E^{\lambda}(t_1, t_2) = \psi(f\eta_{\mathfrak{p}}(t_1), \eta_{\mathfrak{p}}(t_2)), \quad t_1, t_2 \in T_p(X^{rig}) \tag{39}$$

The class of f in $(F_{\mathfrak{p}} \otimes \mathbb{Q}_p(1))^{\star}/\gamma(C_{\mathfrak{p}})$, does not depend on the choice of $\eta_{\mathfrak{p}} \in \bar{\eta}_{\mathfrak{p}}$. We may also work with the $F_{\mathfrak{p}}$-bilinear form associated to E^{λ} and ψ (compare (1)). Then the equation (39) becomes:

$$\tilde{E}^{\lambda}(t_1, t_2) = f\tilde{\psi}(\eta_{\mathfrak{p}}(t_1), \eta_{\mathfrak{p}}(t_2))$$

The spaces $\mathbb{M}$ are used for the rigid analytic uniformization of $\mathrm{Sh}_{G^{\bullet},C}$ for open compact aubgroups $C \subset G^{\bullet}(\mathbb{A}_f)$, which are not necessarily maximal in $\mathfrak{p}$. For this we introduce a functor $\underline{\mathbf{A}}_C$ on the category of schemes over E, which in the case $C_{\mathfrak{p}}$ maximal, is the general fibre of the functor $\mathcal{A}_C$. The functor $\underline{\mathbf{A}}_C$ is defined by adding to the definition of $\mathcal{A}_C$ the following datum, which makes only sense in characteristic 0.

7) A class of $B_{\mathfrak{p}}$-module isomorphisms

$$\bar{\eta}_{\mathfrak{p}} : V_{\mathfrak{p}}(A) \longrightarrow W_{\mathfrak{p}} \quad \mathrm{modulo}\, C_{\mathfrak{p}} \tag{40}$$

Here $V_{\mathfrak{p}}(A)$ denotes the $\mathfrak{p}$-part of the rational Tate module. We require the following two conditions are satisfied: Firstly the Riemann form on $V_{\mathfrak{p}}(A)$ induced by $\lambda \in \Lambda$ and the form ψ are respected by $\bar{\eta}_{\mathfrak{p}}$ up to a constant in $F_{\mathfrak{p}}^{\star}$. Secondly the Tate module $T_{\mathfrak{p}}(A)$ is mapped by any $\eta_{\mathfrak{p}}$ in $\bar{\eta}_{\mathfrak{p}}$ to $\Gamma_{\mathfrak{p}}$.

Again the sheafification of the functor $\underline{\mathbf{A}}_C$ is representable by a projective scheme $\mathbf{A}_C$.

Let us denote by $C_{\mathfrak{p},m} \subset \tilde{G}^{\bullet}(F_{\mathfrak{p}})$ the maximal open compact subgroup. Let $C_m \subset \tilde{G}^{\bullet}(\mathbb{A}_f)$ be the subgroup obtained from C by changing the $\mathfrak{p}$-part $C_{\mathfrak{p}}$ to $C_{\mathfrak{p},m}$.

We remark that for $C = C_m$ our second requirement in the definition of $\mathbf{A}_C$ already determines the datum $\bar{\eta}_{\mathfrak{p}}$ uniquely. Hence in this case the schemes $\mathcal{A}_{C,E}$ and $\mathbf{A}_C$ agree. In general there is an étale covering morphism $\mathbf{A}_C \rightarrow \mathbf{A}_{C_m}$, which classifies isomorphisms $\bar{\eta}_{\mathfrak{p}}$ as above for the universal abelian scheme on $\mathbf{A}_{C_m}$. From this it follows that the uniformization isomorphism (18) gives rise to a uniformization isomorphism of rigid analytic spaces:

$$I^{\bullet}(\mathbb{Q}) \setminus \mathbb{M} \times \tilde{G}^{\bullet}(\mathbb{A}^{\mathfrak{p}}_{F,f})/C \longrightarrow \mathbf{A}_C^{rig} \times_{\mathrm{Sp}\, E} \mathrm{Sp}\, \breve{E} \tag{41}$$

The Weil descent datum (10) on $\breve{\mathcal{M}}^{rig}$ relative to $\breve{E}/E$ extends naturally to the following Weil descent datum on $\mathbb{M}$:

$$\begin{aligned} \mathbb{M}(\mathcal{Z}^{rig}) &\longrightarrow \mathbb{M}(\mathcal{Z}^{rig}_{[\tau]}) \\ (X, \rho, \bar{\eta}_{\mathfrak{p}}) &\mapsto (X, \rho \mathrm{Frob}_{\mathbb{X}}^{-1}, \bar{\eta}_{\mathfrak{p}}) \end{aligned} \tag{42}$$

The assertion concerning the Weil descent data of the lemma 0.8 remains true for the morphism (41). Hence we obtain by twisting the morphism (41) in the same manner as in proposition 0.10 :

Proposition 0.14. *There is a $G^{\bullet}(\mathbb{A}_f)$-equivariant isomorphism of rigid analytic spaces over $Sp\,\breve{E}$*

$$I^{\bullet}(\mathbb{Q}) \setminus \mathbb{M} \times \tilde{G}^{\bullet}(\mathbb{A}^{\mathfrak{p}}_{F,f})/C \longrightarrow Sh^{rig}_{(G^{\bullet}, h^{\bullet}(h^{\mathfrak{p}}_{K})^{-1}), C} \times_{Sp\,E} Sp\,\breve{E}, \tag{43}$$

which is compatible with the Weil descent data given on both sides. The right action of $G^{\bullet}(\mathbb{A}_f)$ on the right hand side of (43) for varying C is induced on the left hand side the obvious right action of $\tilde{G}^{\bullet}(\mathbb{A}^{\mathfrak{p}}_{F,f})$ and the right action of $\tilde{G}^{\bullet}(F_{\mathfrak{p}})$ via the factor $\mathbb{M}$. The quotient by $I^{\bullet}(\mathbb{Q})$ is defined exactly in the same way as after (15).

We now refomulate this result in terms of the rigid pro-analytic covering space $\mathbb{N} \to \breve{\mathcal{N}}^{rig}$, which is defined in the same manner as $\mathbb{M}$ (see [RZ] 5.34 for the general case). The space $\mathbb{N}$ is equipped with a right action of $G_{\mathfrak{p}}(F_{\mathfrak{p}})$, a left action of $\mathcal{J}(F_{\mathfrak{p}})$, and a Weil descent datum relative to $\breve{E}/E$.

Let $\mathcal{Z}$ be a formal p-adic scheme over $\mathrm{Spf}\, O_{\breve{E}}$. Let $\mathcal{Z} \to \breve{\mathcal{N}}^{rig}$ be a morphism given by a pair (Y, ρ_1) as in definition 0.11. For an open compact subgroup $C_{\mathfrak{q}} \subset G_{\mathfrak{p}}(F_{\mathfrak{p}}) = (B^{opp}_{\mathfrak{q}})^{\star}$ a morphism $\mathcal{Z}^{rig} \to \mathbb{N}_{C_{\mathfrak{q}}}$ is given by a class of $O_{B_{\mathfrak{q}}}$-module isomorphisms

$$\bar{\eta}_1 : T_p(Y) \longrightarrow \Gamma_{\mathfrak{q}} \,\mathrm{modulo}\, C_{\mathfrak{q}} \tag{44}$$

The actions of $G_{\mathfrak{p}}(F_{\mathfrak{p}})$, $\mathcal{J}(F_{\mathfrak{p}})$, and the Weil descent datum are given in the same notations as (28) and (29) as follows:

$$
\begin{aligned}
h(Y,\rho_1,\bar\eta_1) &= (Y,\rho_1 h^{-1},\bar\eta_1) && (45)\\
(Y,\rho_1,\bar\eta_1)g &= (Y^b,\iota(b^{-1})\rho_1, g^{-1}\bar\eta_1 T_p(\iota(b))) && (46)\\
\mathbb{N}(\mathcal{Z}^{rig}) &\longrightarrow \mathbb{N}(\mathcal{Z}^{rig}_{[\tau]}) && (47)\\
(Y,\rho_1,\bar\eta_1) &\mapsto (Y,\rho_1 \mathrm{Frob}_\Phi^{-1}\bar\eta_1)
\end{aligned}
$$

Consider a congruence open compact subgroup of $C_\mathfrak{p} \subset \tilde{G}^\bullet(F_\mathfrak{p}) \subset (B_\mathfrak{q}^{opp})^\star \times B_\mathfrak{q}^\star$ of the form $C_\mathfrak{p} = C_\mathfrak{q} \times C_\mathfrak{q}^{opp}$. By the formula (39) we obtain a morphism:

$$
\begin{aligned}
\mathbb{M}_{C_\mathfrak{p}} &\longrightarrow \mathbb{N}_{C_\mathfrak{q}} \times (F_\mathfrak{p}^\star(1)/\gamma(C_\mathfrak{p}))_{\breve{E}} && (48)\\
(X,\rho,\bar\eta) &\mapsto (X_1,\rho_1,\bar\eta_\mathfrak{q}) \times f
\end{aligned}
$$

Here the notations X_1,ρ_1 are from the definition 0.5. The rigid space $F_\mathfrak{p}^\star(1)/\gamma(C_\mathfrak{p})$ is a space over E, which over E_{p^∞} becomes isomorphic to the constant space $F_\mathfrak{p}^\star/\gamma(C_\mathfrak{p})$ equipped with the descent datum given before lemma 0.9:

$$
\begin{aligned}
F_\mathfrak{p}^\star/\gamma(C_\mathfrak{p})(T) &\longrightarrow F_\mathfrak{p}^\star/\gamma(C_\mathfrak{p})(T_{[\sigma]}) && (49)\\
f &\mapsto u_\sigma^{-1} f
\end{aligned}
$$

Let us denote by $F_\mathfrak{p}^\star(rec_{\mathbb{Q}_p})$ the constant pro-scheme $F_\mathfrak{p}^\star = \text{``}\lim_{\leftarrow}\text{''} F_\mathfrak{p}^\star/U_\mathfrak{p}^n$ over $\bar{\mathbb{Q}}_p$ with the following Weil descent datum. Let $rec_{\mathbb{Q}_p} : W_{\mathbb{Q}_p} \to \mathbb{Q}_p^\star$ be the reciprocity law of class field theory. Then for $\epsilon \in W_{\mathbb{Q}_p}$ the Weil descent datum is given by:

$$
\begin{aligned}
F_\mathfrak{p}^\star(T) &\longrightarrow F_\mathfrak{p}^\star(T_{[\epsilon]})\\
f &\mapsto rec_{\mathbb{Q}_p}(\epsilon) f
\end{aligned}
$$

We may regard $F_\mathfrak{p}^\star(rec_{\mathbb{Q}_p})$ as a pro-scheme over $\breve{\mathbb{Q}}_p$ with a Weil descent datum relative to $\breve{\mathbb{Q}}_p/\mathbb{Q}_p$, but we may not consider it as a scheme over $\mathbb{Q}_p$. Let us denote by $F_\mathfrak{p}(rec_{\mathbb{Q}_p})_E$ the pro-scheme over $\breve{E}$ obtained by base change from $\breve{\mathbb{Q}}_p$ to $\breve{E}$ together with the induced Weil descent datum relative to $\breve{E}/E$.

Proposition 0.15. *The morphism (48) defines an isomorphism of rigid pro-analytic spaces*

$$\mathbb{M} \longrightarrow \mathbb{N} \times F_{\mathfrak{p}}^{\star}(rec_{\mathbb{Q}_p})_E, \tag{50}$$

which is compatible with the Weil descent data on both sides. The map (50) is equivariant with respect to the morphisms (32) and (33).

Proof: We have to show, that the morphism (48) is compatible with the Weil descent data on both sides relative to $\breve{E}/E$. Hence it is enough to look for the Frobenius element in $\mathrm{Gal}(\breve{E}/E)$. The effect of the Weil descent datum at the Frobenius element on $\mathbb{M}$ is by definition, that it changes ρ to $\rho Frob_{\mathbb{X}}^{-1}$ (see (42)). If λ on X is the push-forward by ρ of the canonical polarization on $\mathbb{X}$, then the push-forward by $\rho \circ Frob_{\mathbb{X}}^{-1}$ is qf. $\square$

We obtain a rigid analytic version of theorem 0.13.

Theorem 0.16. *For any compact open subgroup $C \subset G^{\bullet}(\mathbb{A}_f)$ there is an isomorphism of rigid analytic spaces over $Sp\,\breve{E}$:*

$$I^{\bullet}(\mathbb{Q}) \setminus \mathbb{N} \times F_{\mathfrak{p}}^{\star} \times \tilde{G}^{\bullet}(\mathbb{A}_{F,f}^{\mathfrak{p}})/C \xrightarrow{\sim} Sh^{rig}_{(G^{\bullet},h^{\bullet}(h_K)^{-1}),C} \times_{Sp\,E} Sp\,\breve{E}, \tag{51}$$

which is $G^{\bullet}(\mathbb{A}_f)$-equivariant, and compatible with the Weil descent data on both sides. The right action of $\tilde{G}^{\bullet}(\mathbb{A}_{F,f}^{\mathfrak{p}})$ on the left hand side is the obvious one, while the right action of $\tilde{G}^{\bullet}(F_{\mathfrak{p}})$ is the action on $\mathbb{N} \times F_{\mathfrak{p}}^{\star}$ given by by the homomorphism (32). The action of $I^{\bullet}(\mathbb{Q})$ defining the quotient on the left hand side is given by the homomorphisms (15), and (33).

Proof: One inserts (50) in the isomorphism (43) and twists the result by the reciprocity law belonging to $h_K^{-1}h_K^{\mathfrak{p}}$.

It is obvious that our proof gives the following generalization of the last theorem. Consider the situation F, K, B, ψ at the beginning of this chapter. Let Φ be the CM-type defined by (6). Let $P = \{\mathfrak{p}_0, \ldots \mathfrak{p}_s\}$ for some rational number $s \leq m$. We fix embeddings $\alpha_0 : F_{\mathfrak{p}_0} \to \bar{\mathbb{Q}}_p, \ldots \alpha_s : F_{\mathfrak{p}_s} \to \bar{\mathbb{Q}}_p$. Let us assume that $B_{\mathfrak{q}_i}$ is a division algebra of invariant $1/d$, for $i = 0, \ldots, s$. Let $h^{\bullet} : \mathcal{S} \to G^{\bullet}(\mathbb{R})$ be the morphism defined up to conjugacy by (19). We require that the numbers r_ρ defined by the Hodge structure $h^{\bullet}$ (22) are as follows:

$$r_\rho = \begin{cases} 0 & \text{if } \rho \in \Phi,\ \rho \notin \{\alpha_0, \ldots, \alpha_s\} \\ 1 & \text{if } \rho \in \{\alpha_0, \ldots, \alpha_s\} \\ d-1 & \text{if } \rho \in \{\bar{\alpha_0}, \ldots, \bar{\alpha_s}\} \\ d & \text{if } \rho \in \bar{\Phi},\ \rho \notin \{\bar{\alpha_0}, \ldots, \bar{\alpha_s}\} \end{cases}$$

Let us denote by $E_i = \alpha_i(K_{\mathfrak{q}_i})$ Let $\mathcal{N}_i$ be the formal scheme over $\operatorname{Spf} O_{\breve{E}_i}$, which classifies special formal $O_{B_{\mathfrak{q}_i}}$ with respect to α_i for $i = 0, \ldots, s$ (cf. definition 0.11). Let $\mathbb{N}_i \to \mathcal{N}_i^{rig}$ be the rigid pro-analytic covering space over $\operatorname{Sp} \breve{E}_i$. We denote by E the compositum of the fields E_i in $\bar{\mathbb{Q}}_p$. Let $\mathcal{N}_{i,E} = \mathcal{N}_i \times_{\operatorname{Sp} \breve{E}_i} \operatorname{Sp} \breve{E}$ be the space obtained by base change. It inherits from $\mathcal{N}_i$ a Weil descent datum relative to $\breve{E}/E$. With these assumptions we have:

Corollary 0.17. *For any compact open subgroup $C \subset G^\bullet(\mathbb{A}_f)$ there is an isomorphism of rigid analytic spaces over $Sp\,\breve{E}$:*

$$I^\bullet(\mathbb{Q}) \setminus \prod_{i=0}^{i=s} (\mathbb{N}_{i,E} \times F^\star_{\mathfrak{p}_i}) \times \tilde{G}^\bullet(\mathbb{A}^P_{F,f})/C \xrightarrow{\sim} Sh^{rig}_{(G^\bullet, h^\bullet(h_K)^{-1}),C} \times_{Sp\,E} Sp\,\breve{E}, \quad (52)$$

which is $G^\bullet(\mathbb{A}_f)$-equivariant, and compatible with the Weil descent data on both sides.

Here $\tilde{I}^\bullet$ is the inner form of $\tilde{G}^\bullet$, such that $\tilde{I}^\bullet(F \otimes \mathbb{R})$ is compact modulo center and such that we have the following isomorphisms:

$$\begin{aligned} \tilde{I}^\bullet(\mathbb{A}^P_{F,f}) &\cong \tilde{G}^\bullet(\mathbb{A}^P_{F,f}) \\ \tilde{I}^\bullet(F_{\mathfrak{p}_i}) &\cong \mathcal{J}^\bullet_i(F_{\mathfrak{p}_i}), \quad \text{for } i = 0, \ldots, s \end{aligned}$$

0.2 The connected components of the rigid analytic covering spaces

We will obtain the connected components of $\mathbb{M}$ from the uniformization theorem and the knowledge of the connected components of a Shimura variety associated to the group $G^\bullet$. In contrast to chapter 1 we will denote an open compact subgroup of $G^\bullet(\mathbb{A}_f)$ by $C^\bullet$.

Let us introduce the torus $\tilde{T}^\bullet$:

$$\tilde{T}^\bullet = \{(k,f) \in Res_{K/F}\mathbb{G}_{m,K} \times \mathbb{G}_{m,F} | k\bar{k} = f^d\}$$

We do not indicate the dependence on the degree d^2 of the divison algebra B/K in the notation of $\tilde{T}^\bullet$. We have a surjective homomorphism:

$$\tilde{\vartheta}^\bullet : \tilde{G}^\bullet \longrightarrow \tilde{T}^\bullet \tag{53}$$

It maps $g \in \tilde{G}^\bullet$ to $(\mathrm{Nm}^0 g, \tilde{\gamma}(g)) \in \tilde{T}^\bullet$. We denote here by Nm^0 the reduced norm $End_B W \longrightarrow K$. The kernel of the map $\tilde{\vartheta}^\bullet$ is the derived group $\tilde{G}^{\bullet,der}$. We also consider the Weil restriction of (53):

$$\vartheta^\bullet : G^\bullet \longrightarrow T^\bullet \tag{54}$$

Let $(h_1, h_2) : \Phi \times \hat{\Phi} \to \Phi \times \hat{\Phi}$ be an element in $\mathcal{J}^\bullet(F_\mathfrak{p})$. We define a morphism:

$$\begin{aligned} \vartheta^\bullet_{\mathcal{J}} : \mathcal{J}^\bullet(F_\mathfrak{p}) &\longrightarrow \tilde{T}^\bullet(F_\mathfrak{p}) \subset K_\mathfrak{q} \times K_{\bar{\mathfrak{q}}} \times F_\mathfrak{p} \\ (h_1, h_2) &\mapsto (\det(h_1) \times \det(h_2), \hat{h_2} h_1) \end{aligned} \tag{55}$$

Here det denotes the determinant on the matrix algebra $\mathrm{End}_{B_\mathfrak{q}}\Phi$ respectively $\mathrm{End}_{B_\mathfrak{q}^{opp}}\hat{\Phi}$.

We will denote by $\bar{\breve{E}}$ the algebraic closure of $\breve{E}$. If we speak about a rigid analytic space over $\bar{\breve{E}}$ we mean that it is naturally defined over some finite complete extension of $\breve{E}$.

Proposition 0.18. *Let $C^\bullet_\mathfrak{p} \subset \tilde{G}^\bullet(F_\mathfrak{p})$ be an open compact normal subgroup. The right action of $\tilde{G}^\bullet(F_\mathfrak{p})$ and the left action of $\mathcal{J}^\bullet(F_\mathfrak{p})$ on $\mathbb{M}_{C^\bullet_\mathfrak{p}}$ provide actions on the geometric connected components $\pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}})$. There is an equivariant isomorphism*

$$\det\nolimits_{\mathbb{M}} : \pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}}) \longrightarrow \tilde{T}^\bullet(F_\mathfrak{p})/\vartheta^\bullet(C^\bullet_\mathfrak{p}) \tag{56}$$

with respect to the morphisms

$$\vartheta^\bullet : \tilde{G}^\bullet(F_\mathfrak{p}) \longrightarrow \tilde{T}^\bullet(F_\mathfrak{p}), \qquad \vartheta^\bullet_{\mathcal{J}} : \mathcal{J}(F_\mathfrak{p}) \longrightarrow \tilde{T}^\bullet(F_\mathfrak{p}).$$

The isomorphisms (56) are functorial in $C^\bullet_\mathfrak{p}$.

If we view $\pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}})$ as a constant rigid analytic space over $\bar{\breve{E}}$, it is provided with a $W(\overline{E}/E)$-descent datum. Hence on the right hand side of (56) we get a similar descent datum. It is described as follows: Consider the following "reciprocity" map

$$\begin{array}{lllcl} Rez : W(\overline{E}/E) & \longrightarrow E^* \longrightarrow \tilde{T}^\bullet(F_\mathfrak{p}) \subseteq K^*_\mathfrak{q} & \times \quad K^*_{\bar{\mathfrak{q}}} & \times & F^*_\mathfrak{p}, \\ \sigma & \mapsto e \longmapsto \quad \alpha^{-1}(e) & \times\, \alpha^{-1}(e^{-1})(Nm_{E/\mathbb{Q}_p}e)^d & \times & Nm_{E/\mathbb{Q}_p}e \end{array}$$

where the first arrow is the reciprocity law of local class field theory. Then the descent datum on the right hand side of (56) is given by multiplication with $Rez(\sigma) \in \tilde{T}^\bullet(F_\mathfrak{p})$:

$$\tilde{T}^\bullet(F_\mathfrak{p})/\vartheta^\bullet(C^\bullet_\mathfrak{p}) \overset{Rez(\sigma)}{\longrightarrow} \tilde{T}^\bullet(F_\mathfrak{p})/\vartheta^\bullet(C^\bullet_\mathfrak{p}) \cong (\tilde{T}^\bullet(F_\mathfrak{p})/\vartheta^\bullet(C^\bullet_\mathfrak{p}))^\sigma$$

Here the last identification is the descent datum on the constant rigid analytic space $\tilde{T}^\bullet(F_\mathfrak{p})/\vartheta^\bullet(C^\bullet_\mathfrak{p})$ over E.

Proof: We do not know a local decription of the map (56), but we should note here that in the theory of Drinfeld modules there is an analog of the map (56) which admits a purely local description (Genestier [G]). For this proof we denote by $S_{C^\bullet}$ the rigid analytic space $\mathrm{Sh}^{rig}_{(G^\bullet,h^\bullet(h^\mathfrak{p})^{-1}),C^\bullet} \times_{\mathrm{Sp}\,E} \mathrm{Sp}\,\bar{\breve{E}}$ for an open compact subgroup $C^\bullet \subset G^\bullet(\mathbb{A}_f)$. Since we work over $\bar{\breve{E}}$, we will denote during the proof by $\mathbb{M}$, what is $\mathbb{M} \times_{\mathrm{Sp}\,\breve{E}} \mathrm{Sp}\,\bar{\breve{E}}$ in our usual notation.

The uniformization isomorphism (43) then reads

$$\Theta : I^\bullet(\mathbb{Q}) \setminus \mathbb{M} \times \tilde{G}^\bullet(\mathbb{A}^\mathfrak{p}_{F,f})/C^\bullet \overset{\sim}{\longrightarrow} S_{C^\bullet}.$$

Taking the connected components we get an isomorphism

$$I^\bullet(\mathbb{Q}) \setminus \pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}}) \times \tilde{G}^\bullet(\mathbb{A}^\mathfrak{p}_{F,f})/C^{\bullet,\mathfrak{p}} \overset{\sim}{\longrightarrow} \pi_0(S_{C^\bullet}). \tag{57}$$

Let us denote by $T^\bullet(\mathbb{Q})_+$ the elements of $T^\bullet(\mathbb{Q})$, which lie in the connected of $1 \in T^\bullet(\mathbb{R})$. More explicitly $T^\bullet(\mathbb{Q})_+$ is the subgroup of

$$T^\bullet(\mathbb{Q}) = \{(k,f) \in K^* \times F^* \mid k\bar{k} = f^d\},$$

which consists of elements (k,f), such that f is totally positive. If d odd we obtain $T^\bullet(\mathbb{Q})_+ = T^\bullet(\mathbb{Q})$.

By [De] 2.7 we have a $G^\bullet(\mathbb{A}_f)$-equivariant isomorphism

$$\pi_0(S_{C^\bullet}) \simeq T^\bullet(\mathbb{Q})_+ \setminus T^\bullet(\mathbb{A}_f)/\vartheta^\bullet(C^\bullet),$$

where the action of $G^\bullet(\mathbb{A}_f)$ on the right hand side for varying $C^\bullet$ is via the homomorphism

$$\vartheta^\bullet : G^\bullet(\mathbb{A}_f) \twoheadrightarrow T^\bullet(\mathbb{A}_f).$$

In particular the derived group $\tilde{G}^{\bullet,der}(\mathbb{A}^{\mathfrak{p}}_{F,f})$ acts trivially. It follows that (57) induces a $G^\bullet(\mathbb{A}_f)$-equivariant isomorphism

$$I^\bullet(\mathbb{Q}) \setminus \pi_0(\mathbb{M}_{C^\bullet_{\mathfrak{p}}}) \times \tilde{T}^\bullet(\mathbb{A}^{\mathfrak{p}}_{F,f})/\vartheta^\bullet(C^{\bullet,\mathfrak{p}}) \xrightarrow{\sim} T^\bullet(\mathbb{Q})_+ \setminus T^\bullet(\mathbb{A}_f)/\vartheta^\bullet(C^\bullet) \quad (58)$$

We note that $\tilde{I}^\bullet$ is an inner form of $\tilde{G}^\bullet$ and that $I^\bullet(\mathbb{R})$ is compact modulo center. Hence we have a morphism $\tilde{\vartheta}_{I^\bullet} : \tilde{I}^\bullet \longrightarrow \tilde{T}^\bullet$. Let $\Delta = Hom(X^*_{F_{\mathfrak{p}}}(\tilde{T}^\bullet), \mathbb{Z})$. Then we have maps [RZ] 3.52

$$\begin{aligned} \tilde{G}^\bullet(F_{\mathfrak{p}}) &\longrightarrow \Delta \\ \tilde{I}^\bullet(F_{\mathfrak{p}}) &\longrightarrow \Delta \end{aligned} \quad (59)$$

It follows from [RZ] 6.17 that in the case where $C^\bullet_{\mathfrak{p}} = C^\bullet_{m,\mathfrak{p}}$ is the maximal compact subgroup of $\tilde{G}^\bullet(F_{\mathfrak{p}})$ we have

$$\pi_0(\mathbb{M}_{C^\bullet_{m,\mathfrak{p}}}) = \pi_0(\mathcal{M}^{rig}) \xrightarrow{\sim} \Delta,$$

such that the last isomorphism is equivariant with respect to maps (59)

We note that the actions of $\tilde{G}^\bullet(F_{\mathfrak{p}})$ and $\tilde{I}^\bullet(F_{\mathfrak{p}}) = \mathcal{J}^\bullet(F_{\mathfrak{p}})$ on $\pi_0(\mathbb{M}_{C^\bullet_{\mathfrak{p}}})$ are continous. This follows because the fibres of $\pi_0(\mathbb{M}_{C^\bullet_{\mathfrak{p}}}) \longrightarrow \pi_0(\mathbb{M}_{C^\bullet_{m,\mathfrak{p}}})$ are finite and because an action of a p-adic Lie group on a finite set is always continous. Indeed for any number N the N-th powers of elements form an open subset.

The kernel of the map $\tilde{I}^\bullet(F_{\mathfrak{p}}) \longrightarrow \tilde{T}^\bullet(F_{\mathfrak{p}})$ is isomorphic to the special linear group $Sl_d(F_{\mathfrak{p}})$. Since it is generated by N-the powers for any N, the action of $\tilde{I}^\bullet(F_{\mathfrak{p}})$ on $\pi_0(\mathbb{M}_{C^\bullet_{\mathfrak{p}}})$ factors through $\tilde{T}^\bullet(F_{\mathfrak{p}})$. Hence the action of $I^\bullet(\mathbb{Q})$ used on the left hand side of (58) factors through $\vartheta_{I^\bullet} : I^\bullet(\mathbb{Q}) \longrightarrow T^\bullet(\mathbb{Q})$. Since the image of the last map is $T^\bullet(\mathbb{Q})_+$ we may rewrite (58) as follows

$$T^\bullet(\mathbb{Q})_+ \setminus \pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}}) \times \tilde{T}^\bullet(\mathbb{A}^\mathfrak{p}_{F,f})/\vartheta^\bullet(C^{\bullet,\mathfrak{p}}) \xrightarrow{\sim} T^\bullet(\mathbb{Q})_+ \setminus T^\bullet(\mathbb{A}_f)/\vartheta^\bullet(C^\bullet) \quad (60)$$

This map induces a $\tilde{G}^\bullet(F_\mathfrak{p})$-equivariant isomorphism

$$(T^\bullet(\mathbb{Q})_+ \cap \vartheta^\bullet(C^{\bullet,\mathfrak{p}})) \setminus \pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}}) \longrightarrow (T^\bullet(\mathbb{Q})_+ \cap \vartheta^\bullet(C^{\bullet,\mathfrak{p}})) \setminus \tilde{T}^\bullet(F_\mathfrak{p})/\vartheta^\bullet(C^\bullet_\mathfrak{p}) \quad (61)$$

as follows.

Consider the map from $\pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}})$ to the left hand side of (60), which sends $x \in \pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}})$ to the class of $x \times 1$, where $1 \in \tilde{T}(\mathbb{A}^\mathfrak{p}_{F,f})/\vartheta^\bullet(C^{\bullet,\mathfrak{p}})$. This map induces an embedding of the left hand side of (61) to the left hand side of (60). We claim that there is an element $h \in \tilde{T}^\bullet(\mathbb{A}^\mathfrak{p}_{F,f})/\vartheta^\bullet(C^{\bullet,\mathfrak{p}})$ and a set-theoretic map $\tilde{\Delta} : \pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}}) \longrightarrow \tilde{T}^\bullet(F_\mathfrak{p})$, such that the left hand side of (60) with (60), maps x to the residue class of $\tilde{\Delta}(x) \times h$.

To see this note that the group $\tilde{G}^\bullet(F_\mathfrak{p})$ acts transitively on $\pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}})$. Indeed, we know this if $C^\bullet_\mathfrak{p}$ is the maximal compact subgroup $C^\bullet_{m,\mathfrak{p}}$. The transitivity follows, because $C^\bullet_{m,\mathfrak{p}}$ acts transitively on the fibres of the map $\pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}}) \longrightarrow \pi_0(\mathbb{M}_{C^\bullet_{m,\mathfrak{p}}})$.

Let us choose a point $x \in \pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}})$. Its image in the right hand side of (60) is the class of an element $\tilde{\Delta}(x_0) \times h$. Any other element $x \in \pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}})$ may be written in the form $x_0 g_x,\ g_x \in \tilde{G}^\bullet(F_\mathfrak{p})$. By the equivariance of the morphism (60) we see that x is mapped to the class of $\tilde{\Delta}(x_0)\vartheta^\bullet(g_x) \times h$. Hence we have the map $\Delta(x) = \tilde{\Delta}(x_0)\vartheta^\bullet(g_x) \times h$ we were looking for.

If we embed the right hand side of (61) to the right hand side of (60) by $t \in \tilde{T}^\bullet(F_\mathfrak{p})$ goes to the class of $t \times h$, we see that the map (60) induces a $\tilde{G}^\bullet(F_\mathfrak{p})$-equivariant isomorphism (61). We remark that the morphism (61) depends on the choice of h and is only uniquely determined up to translation by an element $u \in T^\bullet(\mathbb{Q})_+$ on the right hand side of (61).

We have seen that the left action of $\mathcal{J}^\bullet(F_\mathfrak{p}) = \tilde{I}^\bullet(F_\mathfrak{p})$ on $\mathbb{M}_{C^\bullet_\mathfrak{p}}$ induces an action of $\tilde{T}^\bullet(F_\mathfrak{p})$ on $\pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}})$. Hence we get an action of $\tilde{T}^\bullet(F_\mathfrak{p})$ on the left sides of (60) resp. (61), which we call for the moment the left action. On the other hand we have the right action of $\tilde{G}^\bullet(F_\mathfrak{p})$ on the left hand sides of (60) respectively (61). These actions clearly factor through $\tilde{\vartheta}^\bullet : \tilde{G}^\bullet(F_\mathfrak{p}) \longrightarrow \tilde{T}^\bullet(F_\mathfrak{p})$. Let us show that these both actions agree. It suffices to do this for the left hand side of (60). Let $x \in \pi_0(\mathbb{M}_{C^\bullet_\mathfrak{p}}),\quad u \in \tilde{T}^\bullet(\mathbb{A}^\mathfrak{p}_{F,f})$ be elements and denote the class of $x \times u$ in the left hand side of (60) by (x, u). Let $\xi_{\mathfrak{p}} t_{\mathfrak{p}}^{-1} \in \tilde{T}^{\bullet}(F_{\mathfrak{p}})$ acts trivially on $\pi_0(\mathbb{M}_{C_{\mathfrak{p}}^{\bullet}})$ with respect to the left action. Then we obtain the coincidence of the actions:

$$\begin{aligned} (x, u)\xi_{\mathfrak{p}} &= (x, u)(t^{\mathfrak{p}})^{-1} = (x, u(t^{\mathfrak{p}})^{-1}) \\ &= (x, (t^{\mathfrak{p}})^{-1}u) = (t_{\mathfrak{p}}x, u) = (\xi_p x, u) \end{aligned}$$

The first equation may be checked on the right hand side of (60), where it follows from $\xi_{\mathfrak{p}} t_{\mathfrak{p}}^{-1} \in \vartheta^{\bullet}(C_{\mathfrak{p}}^{\bullet})$. The other equations are clear.

Making $C^{\bullet,\mathfrak{p}}$ small and keeping $C_{\mathfrak{p}}^{\bullet}$ fixed, we see in particular, that the action of $\tilde{G}^{\bullet}(F_{\mathfrak{p}})$ on $\pi_0(\mathbb{M}_{C_{\mathfrak{p}}^{\bullet}})$ factors through $\tilde{G}^{\bullet}(F_{\mathfrak{p}}) \longrightarrow \tilde{T}^{\bullet}(F_{\mathfrak{p}})$, and that the right and left actions of $\tilde{T}^{\bullet}(F_{\mathfrak{p}})$ on $\pi_0(\mathbb{M}_{C_{\mathfrak{p}}^{\bullet}})$ agree.

We conclude the proof by showing that the $\tilde{T}^{\bullet}(F_{\mathfrak{p}})$-equivariant map (61) is induced by a $\tilde{T}^{\bullet}(F_{\mathfrak{p}})$-equivariant isomorphism
$\pi_0(\mathbb{M}_{C_{\mathfrak{p}}^{\bullet}}) \longrightarrow \tilde{T}^{\bullet}(F_{\mathfrak{p}})/\vartheta^{\bullet}(C_{\mathfrak{p}}^{\bullet})$.

We already know this for the maximal open compact subgroup $C_{m,\mathfrak{p}}^{\bullet} \subset \tilde{G}^{\bullet}(F_{\mathfrak{p}})$. Therefore we consider the fibre $P_{C_{\mathfrak{p}}^{\bullet}}$ of $\pi_0(\mathbb{M}_{C_{\mathfrak{p}}^{\bullet}}) \longrightarrow \pi_0(\mathbb{M}_{C_{m,\mathfrak{p}}^{\bullet}})$ over a fixed point of $\pi_0(\mathbb{M}_{C_{m,\mathfrak{p}}})$.

Let us use the abbreviations $\overline{C}_{\mathfrak{p}}^{\bullet} = \vartheta^{\bullet}(C_{\mathfrak{p}}^{\bullet})$ etc.. Looking at the fibres of the map from the morphism (61) to the corresponding morphism for $C_{\mathfrak{p}}^{\bullet} = C_{m,\mathfrak{p}}^{\bullet}$, we obtain a $\overline{C}_{m,\mathfrak{p}}^{\bullet}$-equivariant isomorphism

$$(T^{\bullet}(\mathbb{Q})_{+} \cap (\overline{C}^{\bullet,\mathfrak{p}} \times \overline{C}_{m,\mathfrak{p}}^{\bullet})) \setminus P_{C_{\mathfrak{p}}^{\bullet}} \xrightarrow{\sim} (T^{\bullet}(\mathbb{Q})_{+} \cap (\overline{C}^{\bullet,\mathfrak{p}} \times \overline{C}_{m,\mathfrak{p}}^{\bullet})) \setminus \overline{C}_{m,\mathfrak{p}}^{\bullet}/\overline{C}_{\mathfrak{p}}^{\bullet}$$

Let $U \subset T^{\bullet}(\mathbb{Q})$ be the group of units. Then for big numbers N the group U^N acts trivially on the finite sets $P_{C_{\mathfrak{p}}^{\bullet}}$ and $\overline{C}_{m,\mathfrak{p}}^{\bullet}/\overline{C}_{\mathfrak{p}}^{\bullet}$. But a theorem of Chevalley tells us, that for sufficiently small open compact subgroups $\overline{C}^{\bullet,\mathfrak{p}} \subset \tilde{T}^{\bullet}(\mathbb{A}_{F,f}^{\mathfrak{p}})$, we have

$$T_{\bullet}(\mathbb{Q})_{+} \cap (\overline{C}^{\bullet,\mathfrak{p}} \times \overline{C}_{m,\mathfrak{p}}^{\bullet}) \subset U^N.$$

Hence we obtain a $\overline{C}_{m,\mathfrak{p}}^{\bullet}$-equivariant isomorphism

$$P_{C_{\mathfrak{p}}^{\bullet}} \longrightarrow \overline{C}_{m,\mathfrak{p}}^{\bullet}/\overline{C}_{\mathfrak{p}}^{\bullet}$$

by choosing $C^{\bullet,\mathfrak{p}}$ small enough. It follows that the action of $\tilde{T}^{\bullet}(F_{\mathfrak{p}})$ on $\pi_0(\mathbb{M}_{C_{\mathfrak{p}}^{\bullet}})$ provides the desired isomorphism:

$$\pi_0(\mathbb{M}_{C_\mathfrak{p}^\bullet}) \xrightarrow{\sim} \tilde{T}^\bullet(F_\mathfrak{p})/\vartheta^\bullet(C_\mathfrak{p}^\bullet).$$

Passing to the proscheme we get an equivariant isomorphism

$$\pi_0(\mathbb{M}) \longrightarrow \tilde{T}^\bullet(F_\mathfrak{p})$$

Next we compute what the $W(\overline{E}/E)$-descent datum on $\pi_0(\mathbb{M})$ does on $\tilde{T}^\bullet(F_\mathfrak{p})$. The $W(\overline{E}/E)$-descent datum on $\pi_0(S_{C^\bullet})$ is given by the following rule.

To the morphisms $h^\bullet$, $h^\mathfrak{p} : \mathcal{S}_\mathbb{R} \longrightarrow G^\bullet_\mathbb{R}$ there are associated one parameter groups $\mu^\bullet$ and $\mu^\mathfrak{p} : \mathbb{G}_{m,\mathbb{C}} \longrightarrow G^\bullet_\mathbb{C}$.

Consider the morphism $\vartheta^\bullet \circ (\mu^\bullet(\mu^\mathfrak{p})^{-1}) : \mathbb{G}_{m,\mathbb{C}} \longrightarrow \tilde{T}^\bullet_\mathbb{C}$. It is defined over E, according to the chosen embedding $\overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}_p$. Hence we obtain a morphism

$$Rez : W(\overline{E}/E) \longrightarrow E^* \stackrel{\vartheta^\bullet(\mu^\bullet(\mu^\mathfrak{p})^{-1})}{\longrightarrow} T^\bullet(E) \stackrel{\mathrm{Nm}_{E/\mathbb{Q}_p}}{\longrightarrow} T^\bullet(\mathbb{Q}_p)$$

According to [De] the descent datum induced from $\pi_0(S_{C^\bullet})$ by the isomorphism

$$\pi_0(S_{C^\bullet}) \xrightarrow{\sim} T^\bullet(\mathbb{Q})_+ \setminus T^\bullet(\mathbb{A}_f)/\vartheta^\bullet(C^\bullet)$$

on the constant scheme on the right hand side is multiplication by
$Rez\,(\sigma), \quad \sigma \in W(\overline{E}/E)$.

The morphism $\mu^\bullet$ may be described as follows. Let $\mathbb{G}_{m,\mathbb{C}}$ act on $W \otimes_\mathbb{Q} \mathbb{C}$ via $\mu^\bullet$. Then the corresponding weight decomposition contains only the weights 0 and 1.

$$W \otimes_\mathbb{Q} \mathbb{C} = W_0 \oplus W_1.$$

The space W_0 should satisfy the condition (7). We make this a little more explicit. Consider the decompsition

$$W \otimes_\mathbb{Q} \mathbb{C} = \bigoplus_{\varphi : K \longrightarrow \mathbb{C}} W \otimes_{K,\varphi} \mathbb{C}.$$

The form ψ defines perfect pairings for each φ

$$\psi : W \otimes_{K,\varphi} \mathbb{C} \times W \otimes_{K,\overline{\varphi}} \mathbb{C} \longrightarrow \mathbb{C}$$

We extend the involution $\mathbb{C}$-linear to $B \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\varphi: K \longrightarrow \mathbb{C}} B \otimes_{K,\varphi} \mathbb{C}$. Then the involution induces isomorphisms

$$B \otimes_{K,\varphi} \mathbb{C} \longrightarrow (B \otimes_{K,\overline{\varphi}} \mathbb{C})^{opp} \tag{62}$$

We may choose isomorphisms $B \otimes_{K,\varphi} \mathbb{C} \simeq M_d(\mathbb{C})$ in such a way, that the isomorphisms (62) are the transposition of a matrix. Then $W \otimes_{K,\varphi} \mathbb{C}$ becomes an $M_d(\mathbb{C})$-module and my be written in the form

$$W_\varphi = W \otimes_{K,\varphi} \mathbb{C} \simeq \mathbb{C}^d \otimes_{\mathbb{C}} U_\varphi.$$

The U_φ are d-dimensional $\mathbb{C}$-vectorspaces. By the Morita equivalence we obtain from ψ non-degenrate pairings

$$\beta_\varphi : U_\varphi \times U_{\overline{\varphi}} \longrightarrow \mathbb{C}.$$

For the given embedding $\alpha : K \longrightarrow \mathbb{C}$ we choose any decomposition

$$U_\alpha = U'_\alpha \oplus U''_\alpha$$

such that U'_α has dimension 1 and U''_α has dimension $d-1$. Then the spaces W^0 and W^1 defined as follows

$$\begin{aligned} W^0 =& \mathbb{C}^d \otimes U''_\alpha \oplus \mathbb{C}^d \otimes (U''_\alpha)^\perp \oplus (\bigoplus_{\varphi \in \Phi \setminus \alpha} W_\varphi) \\ W^1 =& \mathbb{C}^d \otimes U'_\alpha \oplus \mathbb{C}^d \otimes (U'_\alpha)^\perp \oplus (\bigoplus_{\varphi \in \overline{\Phi} \setminus \overline{\alpha}} W_\varphi) \end{aligned}$$

An element $g \in G^\bullet_{\mathbb{C}}$ induces an endomorphism $g_\varphi : U_\varphi \longrightarrow U_\varphi$ for each $\varphi : K \longrightarrow \mathbb{C}$. Then

$$\mathrm{Nm}^0 g = \prod \det g_\varphi \in K \otimes \mathbb{C} \cong \prod_{\varphi: K \longrightarrow \mathbb{C}} \mathbb{C}_\varphi$$

Here we use the notation $\mathbb{C}_\varphi = K \otimes_{K,\varphi} \mathbb{C}$.

Then the morphism $\vartheta^\bullet \mu^\bullet : \mathbb{G}_{m,\mathbb{C}} \longrightarrow (K \otimes \mathbb{C})^* \times (F \otimes \mathbb{C})^*$ is given as follows. The projection of $\vartheta^\bullet \mu^\bullet(z),\ z \in \mathbb{C}^*$ on the factor $(F \otimes \mathbb{C})^*$ is $1 \otimes z$ while the projection $\vartheta^\bullet \mu^\bullet(z)_\varphi$ to the factor $\mathbb{C}^*_\varphi$ of $(K \otimes \mathbb{C})^*$ is:

$$\vartheta^\bullet\mu^\bullet(z)_\varphi = \begin{cases} z & \text{if } \varphi = \alpha \\ z^{d-1} & \text{if } \varphi = \overline{\alpha} \\ 1 & \text{if } \varphi \in \Phi \setminus \alpha \\ z^d & \text{if } \varphi \in \overline{\Phi} \setminus \overline{\alpha} \end{cases}$$

Over the field $E = \alpha(K_\mathfrak{q})$ the map $\vartheta^\bullet\mu^\bullet$ is given as follows. The algebra $K_\mathfrak{q} \otimes_{\mathbb{Q}_p} E$ is a direct sum of all composite of the field K and E. One of the composite is $\alpha \otimes id : K_\mathfrak{q} \otimes_{\mathbb{Q}_p} E \longrightarrow E$. Let $\varepsilon : E \longrightarrow K_\mathfrak{q} \otimes_{\mathbb{Q}_p} E$ the section of $\alpha \otimes id$ given by that direct sum decomposition. In the same way the composition $\overline{\alpha} \otimes id : K_{\overline{\mathfrak{q}}} \otimes_{\mathbb{Q}_p} E \longrightarrow E$ defines a section $\overline{\varepsilon} : E \longrightarrow K_{\overline{\mathfrak{q}}} \otimes_{\mathbb{Q}_p} E$.

The map

$$\vartheta^\bullet\mu^\bullet : \mathbb{G}_{m,E} \longrightarrow T_E^\bullet \subset (Res_{K/\mathbb{Q}}\mathbb{G}_{m,K})_E \times (Res_{F/\mathbb{Q}}\mathbb{G}_{m,F})_E$$

is given on the second factor by

$$\begin{aligned} \mathbb{G}_{m,E} &\longrightarrow (Res_{F/\mathbb{Q}}\mathbb{G}_{m,F})_E = (F \otimes E)^*, \\ e &\longmapsto 1 \otimes e \end{aligned}$$

and is given on the first factor by

$$\begin{array}{ccccccccc} \mathbb{G}_{m,E} \longrightarrow (K_\mathfrak{q} \otimes_{\mathbb{Q}_p} E)^* & \times & (K_{\overline{\mathfrak{q}}} \otimes_{\mathbb{Q}_p} E)^* & \times & \prod_{i=1}^m (K_{\mathfrak{q}_i} \otimes_{\mathbb{Q}_p} E)^* & \times & \prod_{i=1}^m (K_{\overline{\mathfrak{q}_i}} \otimes_{\mathbb{Q}_p} E)^* \\ e \longmapsto \varepsilon(e) & \times & \overline{\varepsilon}(e^{-1})(1 \otimes e)^d & \times & 1 & \times & (1 \otimes e)^d \end{array}$$

The corresponding reciprocity map is

$$\begin{array}{ccccccccc} E^* \longrightarrow T^\bullet(\mathbb{Q}_p) \subset K_\mathfrak{q}^* & \times & K_{\overline{\mathfrak{q}}}^* & \times & \prod_{i=1}^m K_{\mathfrak{q}_i}^* & \times & \prod_{i=1}^m K_{\overline{\mathfrak{q}_i}}^* & \times & (F \otimes \mathbb{Q}_p)^*, \\ e \longmapsto \alpha_0^{-1}(e) & \times & \overline{\alpha}_0^{-1}(e^{-1})(\mathrm{Nm}\, e)^d & \times & 1 & \times & (\mathrm{Nm}\, e)^d & \times & \mathrm{Nm}\, e \end{array}$$

where $\mathrm{Nm}\, e$ denotes $\mathrm{Nm}_{E/\mathbb{Q}_p} e$.

It is clear that the map $\vartheta^\bullet\mu_K^\mathfrak{p} : \mathbb{G}_m \longrightarrow T^\bullet$ is defined over $\mathbb{Q}_p$ and that the corresponding reciprocity map is

$$
\begin{array}{ccccccccccccc}
\mathbb{Q}_p & \longrightarrow & K_{\mathfrak{q}}^* & \times & K_{\overline{\mathfrak{q}}}^* & \times & \prod_{i=1}^{m} K_{\mathfrak{q}_i}^* & \times & \prod_{i=1}^{m} K_{\overline{\mathfrak{q}_i}}^* & \times & F_{\mathfrak{p}}^* & \times & \prod_{i=1}^{m} F_{\mathfrak{p}_i}^* \\
z & \longmapsto & 1 & \times & 1 & \times & 1 & \times & z^d & \times & 1 & \times & z
\end{array}
$$

Finally we obtain that the reciprocity map associated to $\vartheta^\bullet \circ (\mu^\bullet(\mu_K^{\mathfrak{p}})^{-1})$ is given by the map

$$
\begin{array}{rlccccc}
E^* \to & \tilde{T}^\bullet(F_{\mathfrak{p}}) \subset K_{\mathfrak{q}}^* & \times & K_{\overline{\mathfrak{q}}}^* & \times & F_{\mathfrak{p}}^* \\
e \longmapsto & \alpha^{-1}(e) \times \overline{\alpha}^{-1}(e^{-1})(\mathrm{Nm}\, e)^d \times \mathrm{Nm}\, e
\end{array} \tag{63}
$$

followed by the inclusion $\tilde{T}^\bullet(F_{\mathfrak{p}}) \longrightarrow T^\bullet(\mathbb{Q}_p)$.

That the descent datum induced from $\pi_0(\mathbb{M}_{C_{\mathfrak{p}}^\bullet})$ on the right hand side of (53) is as desired follows from the fact that the descent datum on $\pi_0(S_{C^\bullet})$ is given by (63) and from the isomorphsim (60). This completes the proof of the proposition.

The proposition gives us the following refinement of the uniformization for $S_{C^\bullet}$:

Corollary 0.19. *The map induced by* $\det_{\mathbb{M}}$ *and* $\vartheta^\bullet$

$$
I^\bullet(\mathbb{Q}) \setminus \mathbb{M} \times \tilde{G}(\mathbb{A}_{F,f}^{\mathfrak{p}})/C^\bullet \longrightarrow T^\bullet(\mathbb{Q})_+ \setminus T^\bullet(\mathbb{A}_f)/\vartheta^\bullet(C^\bullet)
$$

is up to translation by an element $t \in T^\bullet(\mathbb{A}_f)$ induced via (43) by the map given by [De]

$$
Sh_{(G^\bullet, h^\bullet(h_K^{\mathfrak{p}})^{-1}), C^\bullet} \longrightarrow T^\bullet(\mathbb{Q})_+ \setminus T^\bullet(\mathbb{A}_f)/\vartheta^\bullet(C^\bullet).
$$

Theorem 0.20. *Let $D_{\mathfrak{p}}/F_{\mathfrak{p}}$ be a division algebra of invariant $1/d$ over a local field $F_{\mathfrak{p}}$. Let $\tilde{G}_{\mathfrak{p}}$ be the multiplicative group of $D_{\mathfrak{p}}$ considered as algebraic group over $F_{\mathfrak{p}}$ and $\tilde{\mathcal{J}} = Gl_{d,F_{\mathfrak{p}}}$. Let $\vartheta : \tilde{G}_{\mathfrak{p}} \longrightarrow \mathbb{G}_{m,F_{\mathfrak{p}}}$ be the reduced norm and* $\det : Gl_{d,F_{\mathfrak{p}}} \longrightarrow \mathbb{G}_{m,F_{\mathfrak{p}}}$ *be the determinant.*

Choose an embedding $\alpha : F_{\mathfrak{p}} \longrightarrow \overline{\mathbb{Q}}_p$ and set $E = \alpha(O_{F_{\mathfrak{p}}})$ Let $\breve{\mathcal{N}}$ be the formal scheme over $O_{\breve{E}}$ which classifies special formal $O_{D_{\mathfrak{p}}}$-modules. Let $\mathbb{N} \longrightarrow \breve{\mathcal{N}}^{rig}$ be the associated rigid pro-analytic covering space over $\breve{E}$. $\mathbb{N}$ is equipped with a left action by $\tilde{\mathcal{J}}(F_{\mathfrak{p}})$ and a right action of $\tilde{G}_{\mathfrak{p}}(F_{\mathfrak{p}})$.

There is an equivariant map over $\breve{\bar{E}}$

$$\det_{\mathbb{N}} : \mathbb{N} \longrightarrow F_{\mathfrak{p}}^{*} \tag{64}$$

with respect to the morphisms ϑ *and* det. *This map is compatible with the Weil-descent data, if we equip the right hand side of (64) with the* $W(\overline{E}/E)$-*descent datum given by*

$$W(\overline{E}/E) \longrightarrow E^{*} \xleftarrow{\alpha} F_{\mathfrak{p}}^{*}.$$

For any open compact-normal subgroup $C_{\mathfrak{p}} \subset \tilde{G}_{\mathfrak{p}}(F_{\mathfrak{p}})$ *the map (64) induces an isomorphism*

$$\pi_0(\mathbb{N}_{C^{\mathfrak{p}}}) \simeq F_{\mathfrak{p}}^{*}/\vartheta(C_{\mathfrak{p}}).$$

Proof: The map

$$\tilde{T}^{\bullet}(F_{\mathfrak{p}}) \subset K_{\mathfrak{q}}^{*} \times K_{\bar{\mathfrak{q}}}^{*} \times F_{\mathfrak{p}}^{*} \overset{projection}{\longrightarrow} K_{\mathfrak{q}}^{*} \times F_{\mathfrak{p}}^{*}$$

defines an isomorphism $\tilde{T}^{\bullet}(F_{\mathfrak{p}}) \xrightarrow{\sim} K_{\mathfrak{q}}^{*} \times F_{\mathfrak{p}}^{*}$

From the proposition (0.15) we get a diagram over $\breve{\bar{E}}$.

$$\begin{array}{ccc} \mathbb{M} & \xrightarrow{\sim} & \mathbb{N} \times F_{\mathfrak{p}}^{*} \\ \big\downarrow & & \big\downarrow \\ \tilde{T}^{\bullet}(F_{\mathfrak{p}}) & \xrightarrow{\sim} & K_{\mathfrak{q}}^{*} \times F_{\mathfrak{p}}^{*} \end{array}$$

The right vertical map is defined by the commutativity of the diagram. It is compatible with the projections to $F_{\mathfrak{p}}^{*}$. The left vertical map is given by the proposition 0.18. It follows that the diagram above is equivariant with respect to the following diagram

$$\begin{array}{ccc} \tilde{G}^{\bullet}(F_{\mathfrak{p}}) & \xrightarrow{\sim} & \tilde{G}(F_{\mathfrak{p}}) \times F_{\mathfrak{p}}^{*} \\ \big\downarrow & & \big\downarrow \\ \tilde{T}^{\bullet}(F_{\mathfrak{p}}) & \xrightarrow{\sim} & K_{\mathfrak{q}}^{*} \times F_{\mathfrak{p}}^{*} \end{array}$$

The proposition follows immediately from the proposition 0.18. □

For the map (64) we will also use the notation

$$\det_{\mathbb{N}} : \mathbb{N} \longrightarrow F_{\mathfrak{p}}^{*}(rec_{\alpha}) \tag{65}$$

to indicate the Weil descent datum on the right hand side, which is respected by this map.

0.3 The uniformization of Shimura curves

Let D be a quaternion division algebra over the totally real number field F. Let $\alpha : F \longrightarrow \mathbb{R}$ be a place of F such that

$$D \otimes_{F,\alpha} \mathbb{R} \simeq M_2(\mathbb{R})$$

We assume that D is ramified at all other infinite places of F.

Let us denote by $\tilde{G}$ the multiplicative group D^* considered as algebraic group over F and let $G = Res_{F/\mathbb{Q}}\tilde{G}$ be the Weil restiction. The action of $\mathbb{C}^*$ on $\mathbb{C} = \mathbb{R}^2$ defines a group homomorphism

$$h : \mathcal{S} \longrightarrow Gl_2(\mathbb{R}) \simeq \tilde{G} \otimes_{F,\alpha} \mathbb{R} \subset G_{\mathbb{R}}.$$

Let $\mathrm{Sh}_G = \mathrm{Sh}_{(G,h)}$ be the associated Shimura variety.

By the diagram (20) α defines an embedding $\alpha : F \longrightarrow \overline{\mathbb{Q}}_p$ an hence a prime ideal $\mathfrak{p}$ of O_F. Let us assume that $D_\mathfrak{p}$ is a division algebra. Let $\overline{D}$ a quaternion algebra over F, such that $\overline{D}_\mathfrak{p} \simeq M_2(F_\mathfrak{p})$, $\overline{D} \otimes_{F,\alpha} \mathbb{R}$ is a division algebra, and such for all places w of F, which are different from $\mathfrak{p}$ and α, the algebras $\overline{D}_w$ and D_w are isomorphic.

For an open and compact subgroup $C \subset G(\mathbb{A}_f)$ the Shimura varieties $\mathrm{Sh}_{G,C}$ are projective and defined over the Shimura field $E(h) = \alpha(F)$. Let $E = E(h)_\nu = \alpha(F_\mathfrak{p})$ the localization at the place ν given by (20). We denote by $Sh^{rig}_{G,C}$ the rigid analytic space over E associated to the algebraic variety $\mathrm{Sh}_{G,C} \times_{\mathrm{Spec}\, E(h)} \mathrm{Spec}\, E$.

The uniformization theorem describes the tower of rigid analytic spaces $\mathrm{Sh}^{rig}_{G,C}$ over E together with the $G(\mathbb{A}_f)$-action as follows.

The group $\overline{D}^*$ acts from the left on $\mathbb{N}$ by the isomorphisms $\overline{D}^*_\mathfrak{p} \cong Gl_2(F_\mathfrak{p}) \cong \mathcal{J}(F_\mathfrak{p})$, and from the left on $\tilde{G}(\mathbb{A}^{\mathfrak{p}}_{F,f})$ by the isomorphism
$(\overline{D} \otimes_F \mathbb{A}^{\mathfrak{p}}_{F,f})^* \simeq (D \otimes_F \mathbb{A}^{\mathfrak{p}}_{F,f})^* = \tilde{G}(\mathbb{A}^{\mathfrak{p}}_{F,f})$.

The group $G(\mathbb{A}_f)$ acts from the right on $\mathbb{N} \times \tilde{G}(\mathbb{A}^{\mathfrak{p}}_{F,f})$, where the action on the first factor is by the projection $G(\mathbb{A}_f) \longrightarrow G(F_\mathfrak{p}) = G_\mathfrak{p}(F_\mathfrak{p})$ and the action on the second factor is right translation. This action is compatible with the Weil descent datum given by (47).

Theorem 0.21. *There is a $G(\mathbb{A}_f)$-equivariant isomorphism of towers of rigid analytic spaces over $\breve{E}$*

$$\overline{D}^* \setminus \mathbb{N} \times \tilde{G}(\mathbb{A}^{\mathfrak{p}}_{F,f})/C \xrightarrow{\sim} Sh^{rig}_{G,C} \times_{SpE} Sp\breve{E} \tag{66}$$

If we equip the left hand side with the Weil-descent-datum coming from the factor $\mathbb{N}$, the isomorphism (66) becomes compatible with the Weil descent data on both sides.

Let

$$\tilde{\vartheta} : \tilde{G} \longrightarrow \mathbb{G}_{m,F} \tag{67}$$

be the map induced by the reduced norm of D. Together with the determinant map

$$\det_{\mathbb{N}} : \mathbb{N} \longrightarrow F^*_{\mathfrak{p}}(rec_\alpha),$$

we obtain a map of rigid analytic spaces over $\breve{E}$ compatible with the Weil descent data

$$\overline{D}^* \setminus \mathbb{N} \times \tilde{G}(\mathbb{A}^{\mathfrak{p}}_{F,f})/C \longrightarrow (F^*_+ \setminus (F \otimes \mathbb{A}_f)^*(rec_\alpha)/\vartheta(C), \tag{68}$$

and which is equivariant with respect to the map induced by $\tilde{\vartheta}$:

$$\vartheta : G(\mathbb{A}_f) \longrightarrow (F \otimes \mathbb{A}_f)^*$$

The geometric fibres of the map (68) are connected rigid analytic spaces.

If we assume that $C = C^{\mathfrak{p}}\, C_{\mathfrak{p}}$, with $C^{\mathfrak{p}} \subset \tilde{G}(\mathbb{A}^{\mathfrak{p}}_{F,f})$, and that $C_{\mathfrak{p}} \subset \tilde{G}(F_{\mathfrak{p}})$ is maximal compact, we may formulate the theorem in terms of formal schemes.

Corollary 0.22. *There is a model $\mathcal{S}h_{G,C}$ of the tower $Sh_{G,C}$ over O_E with $G(\mathbb{A}_f)$-action, such that there is a $G(\mathbb{A}_f)$-equivariant isomorphism of formal schemes*

$$\overline{D}^* \setminus \breve{\mathcal{N}}^{rig} \times (\tilde{G}(\mathbb{A}^{\mathfrak{p}}_{F,f})/C^{\mathfrak{p}}) \xrightarrow{\sim} \mathcal{S}h_{G,C} \tag{69}$$

We prove the theorem by embedding $\mathrm{Sh}_{G,C}$ into a Shimura variety associated to a unitary group of the type considered before.

Let $B = D^{opp} \otimes_F K$, and let $\star$ be the involution on B, which is the tensor product of the main involution on D^{opp} and the conjugation on K. There is a naturally defined algebraic group $G^\bullet$ over $\mathbb{Q}$, such that

$$G^\bullet(\mathbb{Q}) = \{b \in (B^{opp})^* \mid bb^* \in F^*\}.$$

We introduce the notations $Z = Res_{F/\mathbb{Q}}\, \mathbb{G}_{m,F},\ Z^\bullet = Res_{K/\mathbb{Q}}\, \mathbb{G}_{m,K}$. It is easly seen that the group $G^\bullet$ sits in an exact sequence.

$$\begin{aligned} 1 \longrightarrow Z \longrightarrow & G \times Z^\bullet \longrightarrow G^\bullet \longrightarrow 1, \\ f \longmapsto & f \times f^{-1} \\ & d \times k \longmapsto d \otimes k \end{aligned} \tag{70}$$

where we have indicated, what the maps do on the $\mathbb{Q}$-valued points $f \in F^* = Z(\mathbb{Q}),\ d \in D^* = G(\mathbb{Q}),\ k \in K^\bullet = Z^\bullet(\mathbb{Q})$. The notation Z resp. $Z^\bullet$ means that we have identify these groups with the centers of G resp. $G^\bullet$.

Further down we will write the exact sequence (70) in the form

$$G \times^Z Z^\bullet = G^\bullet \tag{71}$$

We identify the group $G^\bullet$ with one of the groups considered at the beginning of chapter 1. For this we have to define a suitable alternating $\mathbb{Q}$-bilinear form ψ on $W = B$ that satisfies the relation

$$\psi(w_1, w_2 b) = \psi(w_1 b^*, w_2) \tag{72}$$

We choose K and the CM-type Φ as in chapter 1. There is a natural isomorphism

$$K \otimes_{\mathbb{Q}} \mathbb{C} \simeq \prod_{\varphi : K \longrightarrow \mathbb{C}} \mathbb{C}_\varphi,$$

where $\mathbb{C}_\varphi$ is an exemplar of $\mathbb{C}$ for every embedding $\varphi : K \longrightarrow \mathbb{C}$. Consider the morphism $\mu_K^\alpha : \mathbb{G}_{m,\mathbb{C}} \longrightarrow Z^\bullet_{\mathbb{C}}$, which is given on the $\mathbb{C}$-valued points as follows

$$\begin{aligned} \mathbb{C}^* \to Z^\bullet(\mathbb{C}) = & \prod_{\varphi : K \longrightarrow \mathbb{C}} \mathbb{C}^*_\varphi \\ z \longmapsto & \prod z^{\epsilon(\varphi)}, \end{aligned}$$

where the exponents $\epsilon(\varphi)$ are given by

$$\epsilon(\varphi) = \begin{cases} 0, & \text{if } \varphi \in \Phi \cup \overline{\alpha} \\ 1, & \text{if } \varphi \in \Phi \setminus \overline{\alpha} \end{cases}$$

Here $\alpha : K \longrightarrow \mathbb{C}$ is the unique extension, which belongs to Φ.

We denote by $h_K^\alpha : \mathcal{S} \longrightarrow Z_{\mathbb{R}}^\bullet$ the morphism with first component μ_K^α. We set

$$h^\bullet = h \circ h_K^\alpha$$

One proves [De] that there is a form ψ on W as above (72), such that the $\mathbb{R}$-bilinear form: $\psi_{\mathbb{R}}(w_1, h^\bullet(\sqrt{-1})w_2),\ w_1, w_2 \in W \otimes \mathbb{R}$ is symmetric and positive definite. Then ψ induces a positive involution $b \longmapsto b'$ on B and we are exactly in the situation of chapter 1:

$$G^\bullet(\mathbb{Q}) = \{g \in Gl_B W \mid \psi(gw_1, gw_2) = \psi(\gamma^\bullet(g)w_1, w_2)\}.$$

The Shimura varieties $\mathrm{Sh}_{(G^\bullet, h^\bullet h_K^{-1})}$ and $\mathrm{Sh}_{(G^\bullet, h)}$ where h denotes here the map $h : \mathcal{S} \xrightarrow{h \times 1} G \times Z^\bullet \longrightarrow G^\bullet$, differ by a Galois twist associated to the central cocharacter $\mu_{K,\alpha} : \mathbb{C}^* \xrightarrow{id} \mathbb{C}_{\overline{\alpha}}^* \subset Z^\bullet(\mathbb{C})$.

Hence by the uniformization isomorphism (51), we obtain after twisting the descent data, a $G^\bullet(\mathbb{A}_f)$-equivariant isomorphism

$$I^\bullet(\mathbb{Q}) \setminus \mathbb{N} \times F_{\mathfrak{p}}^*(rec_\alpha) \times \tilde{G}^\bullet(\mathbb{A}_{F,f}^{\mathfrak{p}})/C^\bullet \xrightarrow{\sim} \mathrm{Sh}_{(G^\bullet,h),C^\bullet}^{rig} \times_{\mathrm{Sp}\, E} \mathrm{Sp}\, \breve{E}, \tag{73}$$

which is compatible with the Weil descent data.

Next we choose $C^\bullet$ in dependence of C as follows. We set $C_Z = Z(\mathbb{A}_f) \cap C$. Then for a sufficiently small subgroup $C'_K \subset Z^\bullet(\mathbb{A}_f)$ we have

$$\begin{aligned} C'_K \cap \mathbb{Z}(\mathbb{A}_f) &\subset C_Z \\ C'_K Z^\bullet(\mathbb{Q}) \cap Z(\mathbb{A}_f) &\subset C_Z Z(\mathbb{Q}). \end{aligned} \tag{74}$$

We set $C_{Z^\bullet} = C'_K C_Z$, which is another open compact subgroup of $Z^\bullet(\mathbb{A}_f)$ satisfying (74). Then in the notation (71) we set

$$C^\bullet = C \times^{C_Z} C_{Z^\bullet} \tag{75}$$

Then (compare [De]) $\mathrm{Sh}_{G,C} \subset \mathrm{Sh}_{(G^\bullet,h),C^\bullet}$ is an open and closed subvariety. Considering the maps to the connected components [De] 2.7 we obtains a cartesian diagram

$$\begin{array}{ccc} \mathrm{Sh}_{G,C} & \xrightarrow{\varkappa} & T(\mathbb{Q})_+ \setminus T(\mathbb{A}_f)/\vartheta(C) \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{Sh}_{(G^\bullet,h),C} & \xrightarrow{\varkappa^\bullet} & T^\bullet(\mathbb{Q})_+ \setminus T^\bullet(\mathbb{A}_f)/\vartheta(C^\bullet) \end{array} \tag{76}$$

The letter T denotes the torus $Res_{F/\mathbb{Q}}\,\mathbb{G}_{m,F}$ considered as factor group of G by (67):

$$\vartheta : G \longrightarrow T.$$

Then $\varkappa$ is equivariant with respect to $G(\mathbb{A}_f) \longrightarrow T(\mathbb{A}_f)$ and has geometrically connected fibres.

We have a commutative diagram

$$\begin{array}{ccc} G \times Z^\bullet & \xrightarrow{\vartheta\times id} & T \times Z^\bullet \\ \Big\downarrow & & \Big\downarrow \\ G^\bullet & \xrightarrow{\vartheta^\bullet} & T^\bullet \end{array}$$

If we view $T^\bullet$ as a subtorus of $Z \times Z^\bullet$

$$T^\bullet(\mathbb{Q}) = \{(f,k) \in F^* \times K^* \mid k\overline{k} = f^2\},$$

the right vertical map is given by the formula

$$(t,z) \in T(\mathbb{Q}) \times Z^\bullet(\mathbb{Q}) \longmapsto (tz\overline{z}, tz^2) \in F^* \times K^*.$$

Now we use this to rewrite the left hand side of (73). Firstly we find:

$$\tilde{G}^\bullet(\mathbb{A}^{\mathfrak{p}}_{F,f})/C^{\bullet,\mathfrak{p}} = (\tilde{G}(\mathbb{A}^{\mathfrak{p}}_{F,f})/C^{\mathfrak{p}}) \times^{Z(\mathbb{A}^{\mathfrak{p}}_{F,f})} (Z^\bullet(\mathbb{A}^{\mathfrak{p}}_{F,f})/C_{Z^\bullet})$$

To rewrite the other factor we note that there is an isomorphism

$$\begin{aligned} \mathbb{N} \times^{F^*_{\mathfrak{p}}} K^*_{\mathfrak{p}} &\xrightarrow{\sim} \mathbb{N} \times F^*_{\mathfrak{p}}(rec_\alpha) \\ n \times (k_1,k_2) &\longmapsto nk_1 \times \det\nolimits_{\mathbb{N}}(n)k_1k_2 \end{aligned}$$

The action of $\tilde{G}^\bullet(F_\mathfrak{p})$ on the right hand side given by (32) becomes on the left hand side the natural action of

$$\tilde{G}(F_\mathfrak{p}) \times^{F_\mathfrak{p}^*} K_\mathfrak{p}^* \cong \tilde{G}^\bullet(F_\mathfrak{p}),$$

i.e. $\tilde{G}(F_\mathfrak{p})$ acts as defined from the right on $\mathbb{N}$ and $K_\mathfrak{p}^*$ acts by multiplication on itself.

Hence we may rewrite the unifromization isomorphism (73) as follows

$$I^\bullet(\mathbb{Q}) \setminus (\mathbb{N} \times \tilde{G}(\mathbb{A}_{F,f}^\mathfrak{p})/C) \times^{Z(\mathbb{A}_f)} (Z^\bullet(\mathbb{A}_f)/C_{Z^\bullet}) \xrightarrow{\sim} \mathrm{Sh}_{(G^\bullet,h),C^\bullet} \times_{\mathrm{Sp}\, E} \mathrm{Sp}\, \breve{E}. \quad (77)$$

The map

$$\det_\mathbb{N} \times \vartheta : \mathbb{N} \times \tilde{G}(\mathbb{A}_{F,f}^\mathfrak{p}) \longrightarrow T(\mathbb{A}_f)(rec_{\alpha_0})$$

induces a map $\varkappa_1^\bullet$ from the left hand side of (77) to

$$\pi_0(\mathrm{Sh}_{(G^\bullet,h),C^\bullet}) = T^\bullet(\mathbb{Q})_+ \setminus T^\bullet(\mathbb{A}_f)(rec_\alpha)/C^\bullet = T(\mathbb{Q})_+ \backslash T(\mathbb{A}_f)(rec_\alpha)/C \times^{Z(\mathbb{A}_f)} Z^\bullet(\mathbb{Q}) \setminus Z^\bullet(\mathbb{A}_f)/C_{Z^\bullet},$$

which is compatible with the Weil descent data and equivariant with respect to $G \times Z^\bullet \xrightarrow{\vartheta \times id} T \times Z^\bullet$. Hence $\varkappa_1^\bullet$ coincides with the map $\varkappa^\bullet$ of (76) up to translation with an element of $(T \times Z^\bullet)(\mathbb{A}_f)$:

$$\varkappa_1^\bullet(t \times k) = \varkappa^\bullet$$

By (76) $Sh_{G,C}^{rig}$ is the preimage of $T(\mathbb{Q})_+ \setminus T(\mathbb{A}_f)(rec_\alpha)/C \subset \pi_0(Sh_{(G^\bullet,h),C^\bullet})$ by $\varkappa^\bullet$. The coincides with the preimage of $\varkappa_1^\bullet(1 \times k)$. Since the preimage of $\varkappa_1^\bullet$ and $\varkappa_1^\bullet(1 \times k)$ are isomorphic by the action of the Hecke operator $k \in Z^\bullet(\mathbb{A}_f)$, we see that $Sh_{G,C}^{rig}$ is isomorphic as a tower with the $G(\mathbb{A}_f)$-action to the inverse image of the map $\varkappa_1^\bullet$.

To obtain the final result, we make the action of $I^\bullet(\mathbb{Q})$ more explicit.

Lemma 0.23. *There is an isomorphism*

$$I^\bullet(\mathbb{Q}) \cong \overline{D}^* \times^{F^*} K^*.$$

The action of $I^\bullet(\mathbb{Q})$ on $(\mathbb{N}\times\tilde{G}(\mathbb{A}^{\mathfrak{p}}_{F,f}))\times^{Z(\mathbb{A}_f)}Z^\bullet(\mathbb{A}_f)$ that defines the left hand side of (73) is given by the following action of $\overline{D}^* \times^{F^*} K^*$: The group $\overline{D}^*$ acts from the left on $\mathbb{N}$ via an isomorphism $\overline{D}^*_{\mathfrak{p}} \simeq Gl_2(F_{\mathfrak{p}})$, and on $\tilde{G}(\mathbb{A}^{\mathfrak{p}}_{F,f})$ from the left by an isomorphism $(\overline{D}^* \otimes_F \mathbb{A}^{\mathfrak{p}}_{F,f})^* \simeq \tilde{G}(\mathbb{A}^{\mathfrak{p}}_{F,f})$, and trivially on the factor $Z^\bullet(\mathbb{A}_f)$. The group K^* acts obviously via the factor $Z^\bullet(\mathbb{A}_f)$.

Proof: To prove the assertion it is enough to work over $\breve{\bar{E}}$. It follows, that we may work with the moduli problem $\mathcal{A}_C$ and $\mathcal{A}_C^{rig}$. The group $I(\mathbb{Q}) = \tilde{I}(F)$ was defined in terms of a $\overline{\kappa}$-valued point $(A_s, \Lambda_s\{\overline{\lambda}_{s,i}\}, \overline{\eta}_s^{-1}, \eta_{s,\mathfrak{q}_i})$. Let $L = \mathrm{End}^0_B A_s$ and denote by $\ell \longmapsto \ell'$ the Rosati involution defined by Λ. The prove the first assertion of the lemma, it is enough to show that there is K-algebra isomorphism $L \simeq \overline{D} \otimes_F K$, which sends the Rosati involution on L to the tensor product of the main involution on $\overline{D}$ and the conjegation on K. We claim that this assertion is local with respect to the number field F. We know that the existence of a K-algebra isomorphism $L \simeq \overline{D} \otimes_F K$ (which does not necessarily respect the involutions) is a local question. If we assume the existence of such an isomorphism, then the given positive involution on $\overline{D} \otimes_F K$ induces a positive involution $\ell \longmapsto \ell^*$ on L. Hence we have to show that two involutions on L are isomorphic of they are locally isomorphic. We easily see by the theorem of Skolem-Noether, that there exists an element $x \in L$, $x = x^*$, such that $\ell' = x\ell^*x^{-1}$. Moreover x is unique up to multiplication by an element of F^*. The two involutions are isomorphic, of there exists an $f \in F^*$, such that the equation $f \cdot x = uu^*$ has a solution $u \in L$. We see that the set of solutions is a right torsor under the group

$$H = \{g \in L \mid gg^* \in F\}$$

But by the isomorphism $L \simeq \overline{D} \otimes K$ the group H fits into an exact sequence

$$1 \longrightarrow F^* \longrightarrow \overline{D}^* \times K^* \longrightarrow H \longrightarrow 1,$$

which shows that H satifies the Hasse principle. Therefore the question of the existence of an involution preserving isomorphism is indeed local.

Let us now check, that $(L,')$ and $(\overline{D}\otimes_F K, *)$ are locally isomorphic. At the infinite places this is clear, since any two positive involutions on $M_2(\mathbb{C})$ are isomorphic. Let us consider a finite prime which does not lie over p. Then $\overline{\eta}^p_s$ provides us with a B-module isomorphism $V_\ell(A_s) \cong W \otimes \mathbb{Q}_\ell$, which takes the Riemann form on $V_\ell(A_s)$ to ψ up to a constant in $(F \otimes \mathbb{Q}_\ell)^*$. Hence we find an involution preserving map

$$L \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{End}_{B\otimes\mathbb{Q}_\ell} W \otimes \mathbb{Q}_\ell, \tag{78}$$

where the involution on the right hand side is induced by ψ. By [RZ] lemma 6.28 this induces the desired involution preserving isomorphism

$$L \otimes \mathbb{Q}_\ell \longrightarrow B^{opp} \otimes \mathbb{Q}_\ell = D \otimes K \otimes \mathbb{Q}_\ell.$$

For the primes $\mathfrak{p}_i, i = 1, \dots m$ we obtain in the same way morphisms

$$L \otimes_K K_{\mathfrak{q}_i} \longrightarrow \mathrm{End}_{B_{\mathfrak{q}_i}} W \otimes_K K_{\mathfrak{q}_i}$$

The involution $\ell \longmapsto \ell'$ defines an isomorphism

$$L \otimes_K K_{\overline{\mathfrak{q}}_i} \simeq (L \otimes_K K_{\mathfrak{q}_i})^{opp}$$

and the form ψ an isomorphism

$$\mathrm{End}_{B_{\overline{\mathfrak{q}}_i}} W \otimes_K K_{\overline{\mathfrak{q}}_i} \simeq (\mathrm{End}_{B_{\mathfrak{q}_i}}(W \otimes_K K_{\mathfrak{q}_i}))^{opp}.$$

One obtains an injection

$$L \otimes_F F_{\mathfrak{p}_i} \longrightarrow \mathrm{End}_{B_{\mathfrak{p}_i}}(W \otimes F_{\mathfrak{p}_i}) \tag{79}$$

which preserves the involutions. This is an isomorphism since the dimensions of both algebras by the case $\ell \neq p$ are the same.

Finally we consider the prime $\mathfrak{p}$. Then we obtain a $K_{\mathfrak{p}}$-algebra isomorphism preserving the involutions.

$$\begin{aligned} L \otimes_F F_{\mathfrak{p}} \longrightarrow & \mathrm{End}_{B_{\mathfrak{p}} \times B_{\overline{\mathfrak{q}}}} \Phi \times \hat{\Phi} \simeq \overline{D}_{\mathfrak{p}} \times \overline{D}_{\mathfrak{p}}^{opp} \\ & \simeq \overline{D}_{\mathfrak{p}} \times \overline{D}_{\mathfrak{p}} \simeq \overline{D}_{\mathfrak{p}} \otimes_{F_{\mathfrak{p}}} K_{\mathfrak{p}}. \end{aligned} \tag{80}$$

Here the isomorphism $\overline{D}_{\mathfrak{p}}^{opp} \simeq \overline{D}_{\mathfrak{p}}$ is the main involution. Hence we obtain globally an involution preserving isomorphism:

$$(L, {}') \cong (\bar{D} \otimes_F K, *) \tag{81}$$

To see that the actions are as stated in the lemma let us consider the prime $\mathfrak{p}$. The action of the group $I^\bullet(\mathbb{Q}) = \overline{D}^* \times^{F^*} K^* \subset L^*$ on $\mathbb{N} \times^{F_{\mathfrak{p}}^*} K_{\mathfrak{p}}^*$ is given by the isomorphism (80). This isomorphism may be written inserting (81):

$$\overline{D}^* \times^{F^*} K^* \longrightarrow \overline{D}_{\mathfrak{p}}^* \times^{F_{\mathfrak{p}}} K_{\mathfrak{p}}^* \subset \overline{D}_{\mathfrak{p}} \otimes_{F_{\mathfrak{p}}} K_{\mathfrak{p}} \tag{82}$$

Since any automorphism of $\overline{D}_{\mathfrak{p}} \otimes_{F_{\mathfrak{p}}} K_{\mathfrak{p}}$ which preserves the involution is the conjugation by an element of $\overline{D}_{\mathfrak{p}}^*$ we conclude that the map (82) is induced by a map of the form $\overline{D} \overset{localization}{\longrightarrow} \overline{D}_{\mathfrak{p}}^* \overset{conjugation}{\longrightarrow} \overline{D}_{\mathfrak{p}}^*$. Hence gives the predicted action on $\mathbb{N} \times^{F_{\mathfrak{p}}} K_{\mathfrak{p}}^*$.

At the other place ν of F it is still true that any automorphism of $\overline{D} \otimes_F K_\nu$ which preserves the polarization is induced by conjugation with an element from $\overline{D}_\nu$. Hence we can agree for the place ν in the same way using the maps (78) respectively (79).

We are now able to prove the uniformization theorem for Shimura curves. The lemma allows us to rewrite the left hand side of (77) as follows

$$(\overline{D}^* \setminus \mathbb{N} \times \tilde{G}(\mathbb{A}_{F,f}^{\mathfrak{p}})/C) \times^{Z(\mathbb{A}_f)} (Z^\bullet(\mathbb{Q}) \setminus Z^\bullet(\mathbb{A}_f)/C_{Z^\bullet})$$

Since the inverse image of $\varkappa_1^\bullet$ is $Sh_{G,C}^{rig}$ we obtain the theorem.

Corresponding to corollary 0.17 we have an obvious generalization of theorem 0.21.

Let D/F be a quaternion division algebra over a totally real number field and let G be its multiplicative group. Let $P = \{\mathfrak{p}_0, \ldots, \mathfrak{p}_s\}$ be a set of ideals of F over the rational prime p. Assume we are given embeddings $\alpha_i : F_{\mathfrak{p}_i} \to \bar{\mathbb{Q}}_p$. By the diagram (20) we obtain embeddings $\alpha_i : F \to \mathbb{R}$.

We make the assumption that

$$D \otimes_{F,\alpha_i} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R}), \quad \text{for } i = 0, \ldots, s,$$

and that D is a division algebra at all other real primes. Let $\bar{D}$ be the quaternion algebra over F obtained from D by twisting exactly in the real places $\alpha_0, \ldots, \alpha_s$ and at the places $\mathfrak{p}_0, \ldots, \mathfrak{p}_s$.

Let $E_i = \alpha_i(F_{\mathfrak{p}_i})$ for $i = 0, \ldots, s$ and let E be the compositum of the fields E_i in $\bar{\mathbb{Q}}_p$.

Consider the morphism

$$h : \mathcal{S} \longrightarrow \prod_{i=0}^{s} \mathrm{Gl}_2(\mathbb{R}) \cong \prod_{i=0}^{s} \tilde{G} \otimes_{F,\alpha_i} \mathbb{R} \subset G_{\mathbb{R}}$$

defined by the natural action of $\mathbb{C}^*$ on $\mathbb{C}^r = \mathbb{R}^{2r}$. The corresponding Shimura variety Sh_G is defined over E.

Let $\mathcal{N}_i$ the formal scheme classifying special formal $O_{D_{\mathfrak{p}_i}}$-modules (definition 0.11). Let $\mathbb{N}_i \to \mathcal{N}^{rig}$ the pro-analytic covering space over $\mathrm{Sp}\,\breve{E}_i$. Let $\mathcal{N}_{i,E} = \mathcal{N}_i \times_{\mathrm{Sp}\,\breve{E}_i} \mathrm{Sp}\,\breve{E}$ be the space obtained by base change. It inherits from $\mathcal{N}_i$ a Weil descent datum relative to $\breve{E}/E$.

Corollary 0.24. *There is a $G(\mathbb{A}_f)$-equivariant isomorphism of towers of rigid analytic spaces over $\breve{E}$ for C running through the compact open subgroups of $G(\mathbb{A}_f)$:*

$$\bar{D}^* \setminus \prod_{i=0}^{s} \mathbb{N}_{i,E} \times \tilde{G}(\mathbb{A}_{F,f}^P)/C \xrightarrow{\sim} Sh_{G,C}^{rig} \times_{SpE} Sp\breve{E} \tag{83}$$

If we equip the left hand side with the Weil descent datum coming from the factors $\mathbb{N}_{i,E}$, the isomorphism (66) becomes compatible with the Weil descent data on both sides.

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