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2. Algebra

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Tilting Theory and Quasitilted algebras

Tilting theory is a central topic in the representation theory of artin algebras, with origins in work of Bernstein - Gelfand - Ponomarev in the early seventies. There has been extensive interaction with various research directions in representation theory, as well as in other branches of algebra. In this lecture we shall survey the historical development of tilting theory, and in particular we discuss quasitilted algebras, a recent outgrowth of tilting theory.

Let Λ be (for simplicity) a finite dimensional algebra over a field k, and let T be a finitely-generated (left) Λ-module. In general we cannot say much about the connection between the module theories for Λ and for the associated endomorphism algebra $\Gamma := \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. We would like to impose conditions on T which are strong enough so that information can be transferred from Λ to Γ, and at the same time weak enough so that Γ need not be too closely related to Λ. A tilting module provides some balance between these two requirements.

An object T in the category of finitely generated Λ-modules $\mathrm{mod}\,\Lambda$ is a (1-)tilting module if T is of projective dimension at most one, $\mathrm{Ext}^1_\Lambda(T,T) = 0$, and there is an exact sequence of Λ-modules $0 \to \Lambda \to T_0 \to T_1 \to 0$, where T_0 and T_1 are summands of a finite direct sum of copies of T. Associated with a (1-)tilting module T is a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathrm{mod}\,\Lambda$, where $\mathcal{T} = \{C \mid \mathrm{Ext}^1_\Lambda(T,C) = 0\}$, and there are equivalences of $\mathcal{T}$ and $\mathcal{F}$ with subcategories of $\mathrm{mod}\,\Gamma$. Especially important is the case when Λ is hereditary, in which case $\mathrm{mod}\,\Lambda$ has been thoroughly investigated. The corresponding algebras Γ are then called *tilted* algebras. They have played an important role in the representation theory of algebras since many general questions can be reduced to problems about tilted algebras, which are well understood because of their close connection with hereditary algebras. This level of generality was reached in work of Brenner - Butler and Happel - Ringel in the late seventies. Their results generalized the module theoretic interpretation, given in joint work with Auslander and Platzeck, of the reflection functors of Bernstein - Gelfand - Ponomarev.

We discuss three main lines of further developments, where the first two go via a generalization to (n-)tilting modules (where the projective dimension of T is n). One direction is concerned with the correspondence, established in work with Auslander, between (n-)tilting modules and a certain type of subcategories of $\operatorname{mod}\Lambda$, given by

$$T \mapsto \mathcal{T} = \{C \mid \operatorname{Ext}^1_\Lambda(T, C) = 0 \text{ for all } i > 0\}$$

Here there are applications to algebraic groups, via work of Ringel on quasi-hereditary algebras, and connections with dualizing modules in commutative ring theory. The second line of development goes via Happel's discovery of the connection with derived categories. In the third direction we developed, with Happel and Smalø, a theory of tilting with respect to torsion pairs which are not necessarily induced by a (1-)tilting module. Here we were naturally led to deal with abelian categories rather than just module categories.

As a by-product of our general theory we introduced a new class of algebras, the *quasitilted* algebras, as the endomorphism algebras of what we call tilting objects in a hereditary (i. e., $\operatorname{Ext}^2(-,-) = 0$) k-category $\mathcal{H}$ (with finite dimensional homomorphism and extension spaces). Even though it is not clear what the categories $\mathcal{H}$ may be, quasitilted algebras can be described as the algebras of global dimension at most two such that each indecomposable module is of projective or injective dimension at most one. Two important classes of quasitilted algebras are the tilted algebras and the canonical algebras of Ringel. The categories $\mathcal{H} := \operatorname{mod} H$, where H is a finite dimensional hereditary algebra, give rise to the former class, and the categories $\mathcal{H}$ of coherent sheaves on weighted projective lines, introduced by Geigle - Lenzing, give rise to the latter class. It is an important problem to classify all possible categories $\mathcal{H}$. It is conjectured that the two types of $\mathcal{H}$ mentioned above exhaust all possibilities up to derived equivalence. In work with Happel we have shown that this is true if $\mathcal{H}$ is connected and contains either a directing object X (that is, X does not lie on a cycle of non-zero non-isomorphisms between indecomposable modules) or a non-zero object of finite length (in the case k is algebraically closed).