ICM 1998, Berlin, Aug. 18–27

Abstracts of Plenary and Invited Lectures

Section:

6. Topology

1991 MS Classification:

Bob Oliver, Département de Mathématiques Université Paris-Nord, Villetaneuse, France

Vector bundles over classifying spaces

A vector bundle over a space X can be thought of as a collection of (complex) vector spaces, one for each point in X, which are combined in one topological space. For any X, let $\mathrm{Vect}(X)$ denote the set of isomorphism classes of vector bundles over X, made into a commutative monoid by the operation of direct sum. Define $\mathbb{K}(X)$ to be the Grothendieck group of $\mathrm{Vect}(X)$; i. e., the abelian group of all formal differences $(x - x')$ for $x, x' \in \mathrm{Vect}(X)$ (with the obvious relations). When X is compact, then $K(X) \stackrel{\mathrm{def}}{=} \mathbb{K}(X)$, the K-theory of X, defines a cohomology theory; and in particular has nice properties such as forming exact sequences and Bott periodicity which make it (relatively) easy to calculate these groups.

When X is non-compact, and in particular when $\dim(X) = \infty$, then this is no longer the case, and the groups $\mathbb{K}(X)$ no longer define a cohomology theory. So instead, one must define $K(X)$ via classifying spaces. In general, for any topological group G, Milnor constructed a classifying space BG for G, with the property that for any space X, bundles over X with "structure group" G are in one-to-one correspondence with the set $[X, BG]$ of homotopy classes of maps $X \to BG$. In particular, if U denotes the infinite unitary group, then one defines $K(X) = [X, \mathbb{Z} \times BU]$ for arbitrary X, and this coincides with the above definition when X is compact. There is a natural homomorphism $\beta_X : \mathbb{K}(X) \to K(X)$ for each X, which is an isomorphism when X is compact, but not in general. In fact, the geometrically defined functor $\mathbb{K}(-)$ can behave very differently from $K(-)$: it is not a cohomology theory, and is in general much more difficult to compute than $K(-)$.

If G is a compact Lie group and BG is its classifying space, then $\mathbb{K}(BG)$ and $K(BG)$ can both be studied by comparison with the representation ring $R(G)$ of G. Let

$$\alpha_G : R(G) \longrightarrow \mathbb{K}(BG)$$

be the homomorphism defined by sending a complex G-representation V to its Borel construction $(EG \times_G V)$, regarded as a vector bundle over BG. Alternatively, if one thinks of a representation (with hermitian product) as a homomorphism $\rho : G \to U(n)$, then α_G sends ρ to the vector bundle classified by $B\rho : BG \to BU(n)$. The *completion theorem* of Atiyah and Segal says that

$$\beta_{BG} \circ \alpha_G : R(G) \longrightarrow K(BG)$$

extends to an isomorphism $\widehat{\alpha}_G : R(G)\widehat{\ } \xrightarrow{\cong} K(BG)$, where $R(G)\widehat{\ } = \varprojlim_n (R(G)/I^n)$ is completion with respect to the augmentation ideal $I = \mathrm{Ker}[R(G) \to \mathbb{Z}]$.

The new result which I want to describe, proven in joint work with Stefan Jackowski, is a description of the Grothendieck group $\mathbb{K}(BG)$ itself, also in terms of representations. The starting point for doing this was to consider the case where G is p-toral; i. e., an extension of a torus by a finite p-group. By theorems of Dwyer–Zabrodsky (when G is a finite p-group) and of Notbohm (when G is p-toral), $\alpha_G : R(G) \to \mathbb{K}(BG)$ is an isomorphism in this case. Hence, for an arbitrary compact Lie group G, there are homomorphisms

$$\mathbb{K}(BG) \xrightarrow{\text{restr}} \mathbb{K}(BP) \xrightarrow[\cong]{\alpha_P^{-1}} R(P),$$

for each p-toral subgroup $P \subseteq G$; and these combine to define a homomorphism

$$r_G : \mathrm{Vect}(BG) \longrightarrow R_{\mathcal{P}}(G) \overset{\text{def}}{=} \varprojlim_P R(P).$$

Here, the inverse limit is taken over all p-toral subgroups of G (for all primes p) with respect to inclusion and conjugation of subgroups.

Our main theorem is that r_G is an isomorphism; i. e., that $\mathbb{K}(BG) \cong R_{\mathcal{P}}(G)$, for any compact Lie group G. This is proven by first studying $\mathrm{Vect}_n(BG) \cong [BG, BU(n)]$ for each n, using techniques developed in joint work with Jim McClure, based on a decomposition of BG as a homotopy direct limit of spaces BP for p-toral $P \subseteq G$. These sets $\mathrm{Vect}_n(BG)$ are in themselves quite complicated in general (they contain non-vanishing higher derived functors of inverse limits, and can even be uncountable); but all of the "difficult" parts of $\mathrm{Vect}(BG)$ vanish when one takes the Grothendieck construction.

Among the consequences of this theorem are that β_{BG} is always injective, that α_G is surjective if G is finite or connected; and that $\mathrm{Coker}(\alpha_G)$ always has finite exponent. In fact, $\mathbb{K}(BG) \cong R(G)$ (α_G is an isomorphism) whenever if $\pi_0(G)$ (the group of connected components of G) has the property that all of its elements have prime power order.

As one example, let G be a finite group, and let r_p (for any prime p) denote the number of conjugacy classes of elements of p-power order. Then $\mathbb{K}(BG) \cong \mathbb{Z} \times \prod_p (\mathbb{Z})^{r_p - 1}$, while in contrast $K(BG) \cong \mathbb{Z} \times \prod_p (\widehat{\mathbb{Z}}_p)^{r_p - 1}$.