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8. Analysis

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Developments from Nonharmonic Fourier Series

This lecture gives a survey of a subject which has been developed from classical work on nonharmonic Fourier series. In short, we present results on sampling and interpolation for a fairly wide class of spaces of holomorphic functions. It is shown how independent ideas originating with Paley and Wiener (1934), Duffin and Schaeffer (1952), and Beurling (1959) combine with modern techniques like L^2 estimates for $\overline{\partial}$ and Muckenhoupt weights into a quite general theory. We consider some important examples, through which we convey principle ideas of the subject.

The general setting is as follows. We are given a weighted L^p space of holomorphic functions defined on some domain Ω in the complex plane. We denote this space by $\mathcal{B}$ and assume that the functional of point evaluation $f \mapsto f(z)$ is bounded for each $z \in \Omega$. The norm of this functional is called the majorant of $\mathcal{B}$, and it is denoted by $M(z)$. If $\mathcal{B}$ is a Hilbert space, then $M(z) = \sqrt{K(z,z)}$, where $K(z,\zeta)$ is the reproducing kernel of the space. We say that a sequence of distinct points $\Lambda = \{\lambda_k\}$ in Ω is a set of sampling for $\mathcal{B}$ if $\|f\|_{\mathcal{B}} \simeq \|\{f(\lambda_k)/M(\lambda_k)\}\|_{\ell^p}$ for $f \in \mathcal{B}$, meaning that the ratio between the two norms is bounded from above and below by positive constants, independently of f. We say that Λ is an interpolating sequence for $\mathcal{B}$ if the interpolation problem $f(\lambda_k) = a_k$ has a solution $f \in \mathcal{B}$ whenever $\{a_k/M(\lambda_k)\} \in \ell^p$. Finally, we say that Λ is a complete interpolating sequence for $\mathcal{B}$ if it is both sampling and interpolating. Our aim is to give characterizations of such sequences.

Set $\Phi(z) = \log M(z)$, which is seen to be a subharmonic function. A crucial role is then played by its Laplacian $\Delta\Phi$, which induces density conditions for sampling and interpolating sequences. This statement will be made precise through a number of examples. Sharp density conditions of this kind were first stated by Beurling (1959), and we call such results Beurling-type density theorems.

We consider first situations in which complete interpolating sequences exist. The prime example is the Paley-Wiener space L^p_π $(1 < p < \infty)$, which consists of all entire functions of exponential type at most π, whose restrictions to the real line are in $L^p(\mathbb{R})$. The case $p = 2$ corresponds to the problem first studied by Paley and Wiener, namely that of describing unconditional bases of exponential systems for $L^2(-\pi,\pi)$. A characterization of complete interpolating sequences for L^2_π is given in work by B. S. Pavlov (1979), N. K.

Nikolskii (1980), and A. M. Minkin (1992). The techniques of these authors are linked to the Hilbert space geometry of L^2; a simpler approach, yielding also a characterization in the general case $1 < p < \infty$ as well as for weighted spaces as described below, has later been given by Yu. I. Lyubarskii and the author (1997). In the latter work, the description of complete interpolating sequences is turned into a question about boundedness of the Hilbert transform in weighted L^p spaces of functions and sequences.

A remarkable extension of Fourier analysis and in particular the theory of Paley-Wiener spaces is given by L. de Branges in his book "Hilbert spaces of entire functions" (1968). For all de Branges-type spaces there exist complete interpolating sequences, in fact, there exist orthonormal bases of normalized reproducing kernels. In general, however, it is difficult to describe all complete interpolating sequences for such spaces.

We consider weighted spaces of de Branges-type, and put a certain regularity condition on the weight. To begin with, let $\mathcal{H}$ be a Hilbert space of entire functions whose norm is given by

$$\|f\|_w^2 = \int_{-\infty}^{\infty} |f(x)|^2 w(x)dx,$$

where w is a positive weight function. Assume further that $\mathcal{H}$ is closed under the operations $f(z) \mapsto \overline{f(\overline{z})}$ and $f(z) \mapsto f(z)(z-\overline{\zeta})/(z-\zeta)$, provided $f(\zeta) = 0$. It turns out that a natural regularity condition to be put on the weight is $w(x)M^2(x) \simeq 1$. If this holds, we say that w is a majorant weight. Then the following theorem holds: *The positive function w is a majorant weight if and only if w has the form*

$$\log w(x) = u(x) + g(x) - \int_{-\infty}^{\infty} (\log|1 - x/t| + (1 - \chi_{[-1,1]}(t))m(t)dt,$$

where $u \in L^\infty$, g is a real entire function, and $m(x) \simeq 1$. (The function g is an inessential part of the weight, because a replacement of w by we^{-g} corresponds to multiplying all functions in $\mathcal{H}$ by a factor $e^{g/2}$.) If $g(x) = ax$ for some real a, we say that $\mathcal{H}$ is a weighted Paley-Wiener space. It is seen that adding a positive constant b to m does not change the weight. The numbers a and b (the latter plays a role as "type") yield a natural parametrization of all weighted Paley-Wiener spaces associated with the weight w. Examples are provided, and we discuss some basic properties of weighted Paley-Wiener spaces. We show how characterizations of sampling, interpolating, and complete interpolating sequences extend to such spaces. Here the density function m plays the leading role, as it is in essence the Laplacian of $\log M$ along the real line.

Finally, we discuss the situation for spaces of entire functions which satisfy $\Delta\Phi(z) \simeq 1$ (weighted Fock spaces), and spaces of holomorphic functions in the unit disk which satisfy $\Delta\Phi(z) \simeq (1-|z|^2)^{-2}$ (weighted Bergman spaces). It is shown that for such spaces there are no complete interpolating sequences, a fact which permits complete descriptions of sampling and interpolation in terms of densities only.

The talk is based on material from the following references, as well as some unpublished joint work with Yurii Lyubarskii and Joaquim Ortega-Cerdà:

B. Berndtsson and J. Ortega-Cerdà, Interpolation and sampling in Hilbert spaces of analytic functions, *J. Reine Angew. Math.* **464** (1995), 109–120.

Yu. I. Lyubarskii and K. Seip, Complete interpolating sequences for Paley-Wiener spaces and Muckenhoupt's (A_p) condition, *Rev. Mat. Iberoamericana* **13** (1997), 361–376.

J. Ortega-Cerdà and K. Seip, Beurling-type density theorems for weighted L^p spaces of entire functions, *J. Analyse Math.* , to appear.

K. Seip, Beurling type density theorems in the unit disk, *Invent. Math.* **113** (1993), 21–39.

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