E 20577 F
100. Band Heft 2
ausgegeben am 25.6.1998

DMV

Jahresbericht

der Deutschen Mathematiker-Vereinigung

Herausgegeben von A. Krieg
unter Mitwirkung von
U. Gather, E. Heintze, B. Kawohl,
H. Lange, H. Triebel

B. G. Teubner Stuttgart 1998

Jahresbericht
der Deutschen Mathematiker-Vereinigung

Der „Jahresbericht“ ist das offizielle Veröffentlichungsorgan der Deutschen Mathematiker-Vereinigung, für dessen inhaltliche Gestaltung im Auftrag des Präsidiums der jeweilige Herausgeber zuständig ist. Im „Jahresbericht“ sollen vornehmlich Überblicksartikel über Teilgebiete der reinen und angewandten Mathematik, Nachrufe sowie historische Artikel und Buchbesprechungen veröffentlicht werden.

Manuskripte

Alle für die Schriftleitung des Jahresberichts bestimmten Sendungen (Briefe, Manuskripte, Ankündigung von neuerscheinenden Büchern) sind an Prof. Dr. A. Krieg zu richten. Bücher, von denen eine Besprechung erfolgen soll, werden bei den Verlagen angefordert. Die Autoren werden gebeten, bei der Vorbereitung ihrer Manuskripte die „Hinweise für Autoren“ zu beachten.

Grundsätzlich sollen nur solche Manuskripte eingereicht werden, die nicht gleichzeitig an anderer Stelle zur Veröffentlichung eingereicht oder bereits veröffentlicht worden sind. Mit der Annahme zur Veröffentlichung erwirbt der Verlag das Verlagsrecht zur Vervielfältigung und Verbreitung sowie das Recht der Übersetzung in andere Sprachen.

Bezugsinformationen

Jährlich wird ein Band veröffentlicht, bestehend aus 4 Heften, die vierteljährlich erscheinen. Der Bezug ist nur bandweise möglich.

Der im voraus zahlbare Bezugspreis pro Band beträgt zur Zeit DM 148,– einschließlich Versand. Bestellungen nehmen jede Buchhandlung oder der Verlag entgegen.

Für persönliche Mitglieder der DMV, die den Jahresbericht zu beziehen wünschen, ist der zwischen DMV und Verlag vereinbarte Bezugspreis maßgebend, der im Rahmen des Mitgliedsbeitrags erhoben wird.

Verlag:
B. G. Teubner GmbH, Industriestraße 15, D-70565 Stuttgart
Postfach 80 10 69, D-70510 Stuttgart, Tel. (07 11) 7 89 01-0, Telefax (07 11) 7 89 01-10
e-mail: info@teubner.de
Teubner Home Page http://www.teubner.de

Verantwortlich für den Anzeigenteil: Albrecht Luscher

Printed in Germany – ISSN 0012-0456
Satz: Elsner & Behrens GdbR, D-68723 Oftersheim
Druck: pagina media gmbh, D-69502 Hemsbach

Jahresbericht
der Deutschen Mathematiker-Vereinigung

Inhalt Band 100, Heft 2

1. Abteilung

In den nächsten Heften erscheinende Arbeiten:

A. Bergmann, H. W. Knobloch: Hermann Schmidt 1902–1993
G. Burde, W. Schwarz: Wolfgang Franz zum Gedächtnis
G. Harder: Galoismoduln und Shimura-Varietäten
H. Karcher: Eingebettete Minimalflächen und ihre Riemannschen Flächen
J. Zabczyk: Infinite Dimensional Diffusions in Modelling and Analysis

---

Anschriften der Herausgeber

Prof. Dr. Aloys Krieg, Lehrstuhl A für Mathematik, RWTH Aachen, Templergraben 55, 52056 Aachen

Prof. Dr. Ursula Gather, Fachbereich Statistik, Universität Dortmund, Vogelpothsweg 87, 44221 Dortmund

Prof. Dr. Ernst Heintze, Institut für Mathematik, Universität Augsburg, 86135 Augsburg

Prof. Dr. Bernhard Kawohl, Mathematisches Institut, Universität zu Köln, 50923 Köln

Prof. Dr. Herbert Lange, Mathematisches Institut, Friedrich-Alexander-Universität Erlangen-Nürnberg, Bismarckstraße 1 1/2, 91054 Erlangen

Prof. Dr. Hans Triebel, Mathematisches Institut, Friedrich-Schiller-Universität, Ernst-Abbe-Platz 1–4, 07740 Jena

Bezugshinweis

Früher erschienene Bände (ab Band 68) des „Jahresberichts der Deutschen Mathematiker-Vereinigung" können durch den Buchhandel oder den Verlag bezogen werden.

Nachdruck der Bände 1 bis 40 liefert: Johnson Reprint Corp., 111 Fifth Avenue, New York, N.Y. 10003

Nachdruck der Bände 41 bis 67 liefert: Swets & Zeitlinger, Heereweg 347b, POB 810, NL-2160 SZ Lisse/Holland

Jber. d. Dt. Math.-Verein. © 1998 B. G. Teubner Stuttgart
100 (1998) 45–208
Mathematics Subject Classification: 53 A, B, C & 58 A, C, D, E, F, G

Riemannian geometry during the second half of the twentieth century

M. Berger, Bures-Sur-Yvette

Contents

−1 Introduction

From the 50's to the 70's the task of describing the evolution, the techniques and the results of Riemannian Geometry would have been easy enough. But since the 70's Riemannian Geometry has experienced such a dramatic increase that this task has now become almost impossible. This is not only for reasons of volume, but also because of the difficulties of forming an ordering. In fact today's results, techniques and examples are so strongly interrelated that we have been forced to take several steps.

First: not to be exhaustive. Secondly we have adopted quite an artificial classification. As a consequence this text is not really a complete survey. On the contrary, we will refer to existing partial surveys whenever we know of their existence. Finally we will mention briefly at the end some topics which are important but are not treated in this text explicitly or even in general, due to the lack of space. We still hope that we will be clear and fair enough. And also that our text is, at the present time, reasonably up-to-date.

There is also the problem of defining *"Riemannian Geometry"*. For obvious reasons of dimension and competence we restrict it quite strongly to *what's really happening downstairs on the Riemannian manifold itself, in particular as a metric space.* In particular we are going to ignore completely most of the topics of *differential geometry,* even *Riemannian bundles.* This is a field which has to this day undergone tremendous developments, including among others: differential topology, fiber bundles and connections, singularities and transversality, contact manifolds, symplectic geometry, minimal surfaces and generalizations of them, Yang-Mills (gauge) theory, twistors, foliations, submanifolds of $\mathbf{R}^d$, conformal geometry. For example, despite their great importance among various Riemannian bundles, Yang-Mills fields are not considered, at least here, as Riemannian Geometry but of course they are an important part of differential geometry. We will of course mention them, though only very briefly in TOP. 6 . The tangent bundle, in particular the unit tangent bundle, will be used, but typically in connection with the geodesic flow. Some people may find our "definition " of Riemannian Geometry too narrow. They might be right, as it reflects the (biased) "elementary geometry" temper of the present author as well as his liking for results which are simple to state. Because of their basic use in Riemannian Geometry and the way they are constructed, we will explain exterior differential forms and spinors in a little more detail in TOP. 6.A and B. One conclusion is the desire that some form of "Handbook of differential geometry" will appear quite soon; for the moment one can consult the three volumes of (Greene & Yau, 1993).

We also took one more gamble: we aim to give an historical overview, but, at the same time, describe the state of affairs as it is today. This will not take too much space since we will use an "author-date" reference system, which will hence give credits automatically. Concerning references we will be far from exhaustive, since we are not presenting a complete survey and also because our bibliography would have become unreasonably extensive. In most cases what we will do is to give a reference which is good with respect to date and will, at the same time, enable the reader to go backwards using one or more bibliographies by induction. This will be especially helpful for the topics which are only mentioned briefly.

manifold (unless otherwise stated) =
Riemannian manifold = a (the) "Riemannian metric"
= metric

most manifolds will be COMPACT
and never with a boundary (closed)
and if not compact then always complete

Acknowledments are very important for the present text. First, to the editors of a book to appear on mathematics during the second half of the century, who asked me to write such a text for the differential geometry part. I soon started writing and very quickly found myself embarked upon a project whose length would exceed by far (as the reader can now see) the twenty page allowance I was offered. Therefore I decided to submit it to the Jahresbericht of the DMV.

I was able to write such a text with enthusiasm thanks to the Rome University La Sapienza, the Indian Institute of Technology at Powai-Bombay, the University of Pennsylvania and the Zürich Polytechnicum who all invited me to give lectures, Roma in 1992, Bombay in 1993, Penn in the fall of 94 and Zürich in the Wintersemester 95–96. Most important is the fact that these four departments permitted me to give lectures entitled "Topics in Riemannian Geometry" in which I covered a lot of material but with almost no proofs, simply sketching ideas and ingredients. I would like to thank those four mathematics departments very much for having accepted my lectures which did not fit into any classical framework.

Last but not least, a preliminary version was sent out to fifty colleagues. I got many answers, all of them important. I will now give a list of names and for obvious reasons I will do this without trying to state the importance of the individuals' help. Those who helped me a lot, and whom I even bothered many, many times, will recognize themselves. All of them gave help which was very valuable: Stephanie Alexander, Michael Anderson, Ivan Babenko, Victor Bangert, Pierre Bérard, Lionel Bérard Bergery, Gérard Besson, Armand Borel, Jean-Benoît Bost, Jean-Pierre Bourguignon, Robert Bryant, Dan Burghelea, Peter Buser, Jeff Cheeger, Shiing-shen Chern, Tobias Colding, Alain Connes, Yves Colin de Verdière, Thibault Damour, Jost Eschenburg, Kenji Fukaya, Jacques Gasqui de Saint-Joachim, Paul Gauduchon, Robert Greene, Mikhael Gromov, Nancy Hingston, Dominique Hulin, Mikhail Katz, Ruth Kellerhals, Bruce Kleiner, Horst Knörrer, Jean-Louis Koszul, Jacques Lafontaine, Rémi Langevin, Blaine Lawson, André Lichnerowicz, Paul Malliavin, Wolfgang Meyer, René Michel, Robert Osserman, Pierre Pansu, Peter Petersen, Hans-Bert Rademacher, Takashi Sakai, Katsuhiro Shiohama, Shanta Shrinivasan, Alain-Sol Sznitman, Iskander Taimanov, Domingo Toledo, Lieven Vanhecke, Takao Yamaguchi, Wolfgang Ziller.

I do hope that, whatever its imperfections are, the present text will be of help to Riemannian geometers of every age and might even be pleasant to read in part or in totality for non-experts.

Finally I heartfully thank the DMV, the editors of the Jahresbericht, in particular Ernst Heintze, for having made very special efforts to publish the present text which does not fit into any classical category. Special thanks are due to Karin Seeger for the difficult and lengthy job of transforming my inaccurate English into the present form, as well as for converting my oldfashioned Word 5 into LaTeX.

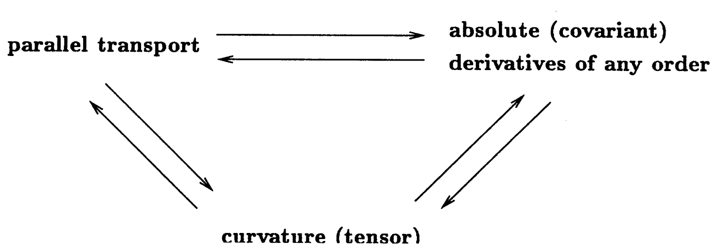
0 Riemannian Geometry up to 1950

A. Gauss, Riemann, Christoffel and Levi-Civita

In 1827 Gauss was the first, for ordinary surfaces in $\mathbf{R}^3$, to decouple the intrinsic metric from the surfaces' embedding. In fact he had arrived at that result long before 1827, see (Dombrowski, 1979). More precisely he proved that the total curvature, namely the product of the two principal curvatures, could be computed using only the intrinsic metric ("theorema egregium").

In 1854 Riemann defined surfaces with a "Riemannian metric" in a completely abstract way, making them free of any embedding or immersion. Moreover he did so in arbitrary dimension and for all manifolds, even if his notion of a manifold was somewhat imprecise. See volume II of (Spivak, 1970) for a perfect analysis of Riemann's founding paper. With the now solidly established notion of a differentiable manifold one can define a general Riemannian manifold (M, g) as a differentiable manifold M endowed with a Riemannian metric g, i.e. one is given, at every point m of M, some positive definite quadratic form $g(m)$ on the tangent space $T_m M$ to M at m. Moreover, one requires the mapping $m \to g(m)$ to be differentiable. Riemann defined the metric $d(p, q)$ as the infimum of the length of all the curves joining p and q, the length making sense precisely because of g. He also defined the curvature tensor, as well as the sectional curvature. We will come back later to this very hard object, which will be tackled in the digression at the end of COMMENTS on TOPIC I.

Thereafter, Christoffel succeeded partially to understand what was going on, namely the, at that time still obscure, notion of connection. This was around 1865, see (Klingenberg & Pinl, 1981). Then at the turn and the beginning of the century it was known that Riemannian manifolds can be associated with a *golden triangle*, which is the primary tool in Riemannian Geometry. This result was due to Levi-Civita and Ricci: (Levi-Civita & Ricci, 1901) and (Levi-Civita, 1917). The triangle is made up of the three vertices



Let us be more precise. The aim is to be able to develop a differential calculus of any order on a Riemannian manifold. In affine spaces the tangent spaces at various points can all be identified with a single vector space by translations and that is good enough. But this is not possible a priori in a smooth manifold since there is no way of comparing tangent spaces at different points. The discovery was that such a comparison is possible in a unique way in infinitesimal terms if one demands that the Euclidean structures of the tangent spaces are preserved and that, moreover, the second differential of a function is symmetric. The precise notion of such a *connection* was seen in various ways, mostly in terms of coordinates. Elie Cartan used mobile frames. The notion of connection was introduced in various settings by Elie Cartan, and Ehresmann put it into the most general framework in (Ehresmann, 1950), for more on the history of connection see Note 2 in Volume I of (Kobayashi & Nomizu, 1963–1969). One can also use, in the tangent bundle to the tangent bundle, the notion of a horizontal subspace. Since Koszul the best way of writing the canonical connection (as well as connections in various bundles with ad hoc modifications) has been to define it as an operation $D_X Y$ on vector fields which is linear in X and a derivation on Y. Namely, if f is an arbitrary function, then $D_{fX} Y = f D_X Y$ but $D_X(f.Y) = X(f).Y + f.D_X Y$. Koszul introduced this concept in 1951, it was used soon in (Nomizu, 1954) and finally presented in a systematic expository way in (Koszul, 1960). Preservation of the metric is insured by $X\langle Y,Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$ for all X, Y, Z and the symmetry required for the second differential is insured if and only if the connection is torsion-free, i.e. $D_X Y - D_Y X = [X, Y]$ for every pair of vector fields.

The connection can be integrated along any curve (but not on a surface in general) because it is then a linear first order differential linear equation. The result is the Levi-Civita parallelism and the tangent spaces along the curve can then be identified with a single one of them: one can now take derivatives of any order along a curve. It is important to note the fact that the parallel transport along a loop is in general different from the identity. This lack of "holonomy" is the starting point for TOP. 3.

Let us now comment on the arrows between the three vertices of the golden triangle. Note first that the metric can be derived from it because shortest paths (called *segments*) between points are supported by geodesics. The geodesics can be defined as the self-paralleled transported curves γ, i.e. $D_{\gamma'}\gamma' = 0$ for their speed vector γ'. The curvature tensor is given by the derivative of the parallel transport along infinitesimal parallelograms and in writing it is nothing but the defect of commutation in $D_X \circ D_Y$, namely $R(X, Y) = D_Y \circ D_X - D_X \circ D_Y$. So the curvature tensor is an antisymmetric differential form with values in the endomorphisms of the tangent spaces.

Now the absolute (covariant) derivative $D_v S$ of a form (or more generally of any tensor S) is obtained by taking the ordinary derivative at 0 of the values taken by the form on a set of vectors which are transported parallel along any curve whose initial speed vector is v .

The third arrow concerns the relation between curvature and covariant derivatives of tensors. In a differentiable manifold the differential of a function is well defined, but the second differential no longer is because the chain rule formula is not valid for the second derivatives when one looks at coordinate changes. The Riemannian canonical Levi-Civita connection yields precisely such an invariant second derivative, called the *Hessian* of the function: it is a bilinear symmetric differential form (this is the commutativity of the second derivatives for Riemannian manifolds). And one can keep going on with derivatives (covariant) of any order. However, then typically the third differential of a function is no longer symmetric, but exactly the defect of this symmetry is given by the curvature tensor. The same holds true for the second derivative of 1-forms, etc. This is fundamental in many places below and can be seen as the revenge of the Riemannian geometer. The explicit formulas for those defects are called *Ricci commutation formulas*, see their first use in D below. In Koszul's notation Ricci formulas are straightforward.

The *curvature tensor*, known to Riemann, is introduced above, as a tensor of type (3,1), namely the exterior 2-form $R(X, Y)$ with values in the endomorphisms of the tangent spaces. But alternatively (using the duality given by the metric) it can be defined as a differential form of degree four (quadrilinear) $R(x, y, z, t) = \langle R(x, y)z, t\rangle$, enjoying moreover some symmetries. We will see below in various places that such an object, even from the point of view of linear algebra alone, remains quite mysterious today, see the digression in 1. COMMENTS on I).

B. Van Mangoldt, Hadamard, Elie Cartan and Heinz Hopf

In (van Mangoldt, 1881) it was already known that negative curvature and simple connectedness for surfaces embedded in $\mathbf{R}^3$ are enough to insure uniqueness of geodesics joining two points, implying that, moreover, any piece of a geodesic is a shortest path (*segment*). In (Hadamard, 1898) the study of negative curvature surfaces was brought into a very general setting, in particular the study of their periodic geodesics and geodesic flow. But this is still a long way from the modern statement: any complete abstract Riemannian manifold of negative curvature (in fact nonpositive and of arbitrary dimension) is the quotient of its universal covering, which is diffeomorphic to $\mathbf{R}^d$, by a discrete group of isometries. This is needed in particular for studying space forms of negative curvature, see II and I.B.4. For the early history of geodesics, see (Nabonnand, 1995).

Missing, as well-defined notions, were abstract manifolds and the completeness of Riemannian manifolds (in any dimension). Hopf was interested in soundly establishing the meaning of completeness (with a good notion of manifolds) in relation to the prolongability of geodesics. This is done in (Hopf & Rinow, 1931), surprisingly enough only for dimension 2. It was remarked only in (Myers, 1935a) that the Hopf-Rinow proof applies to any dimension without any non-trivial change. It should be noted that an important boost to Riemannian Geometry came from the arrival of General Relativity.

In the second edition (Cartan, 1946–1951) of his fundamental book (Cartan, 1928a), Elie Cartan was interested in the technical tools (and ignored systematically what precisely constitutes a manifold). He developed a dual version of the second order differential equation, today called *Jacobi fields* (known to Gauss for surfaces). A Jacobi field is, by definition, the transverse derivative of a one-parameter family of geodesics. In vector writing (and using parallel transport to work with only "ordinary derivatives") the equation for Jacobi fields reads $Y'' + R(Y) = 0$ where R is the $R(\gamma', .)\gamma'$ part of the curvature tensor given by the speed γ' of the geodesic γ, along wich one is working. For the curvature tensor, a frightening object, see various places below. Cartan's tool was "repère mobile" but this is equivalent. He obtained the fact that the curvature tensor and the parallel transport together determine the metric (see the end of TOP. 4 for the global Ambrose problem). Note that one here needs to know the geodesics through a point and not only the curvature, see the digression in I.C.1. From his "philosophy" (Chapter X of (Cartan, 1946–1951)) Cartan deduced that a real analytic Riemannian manifold was determined locally by the values, at any given point, of the curvature tensor and of all its covariant derivatives. In fact Cartan had knowledge of that philosophy long before, when he was studying the symmetric spaces in the mid 20's (II.C), since in that case the curvature tensor has a vanishing covariant derivative and then the metric is known as soon as the curvature tensor is known at a single point. As a corollary he showed that nonpositive curvature implies (simple connectedness is required) an inequality which says that triangles are in this case "larger than or equal to the corresponding Euclidean ones":

$$a^2 \geq b^2 + c^2 - 2bc \cos \alpha$$

In (Cartan, 1929b), Section 16, this inequality is used to prove the existence and uniqueness of the center of mass in those manifolds (see I.C.1). From this he deduced that compact maximal subgroups of semi-simple Lie groups are always conjugate because a symmetric space of nonpositive curvature is simply the quotient of a non-compact Lie group and one of its maximal compact subgroups.

Parallel to that and with obvious interactions he also developed the complete theory of symmetric spaces (see II.C below) and a complete mastering of their geodesic behavior.

One can now see that the simply connected manifolds of constant sectional curvature are unique: they are the "round" spheres in the positive case, the Euclidian spaces for the zero case and the hyperbolic spaces Hyp^d for the negative case. Remember here that the first correct and complete definition of Hyp^d appeared for the first time in Riemann's 1854 text. Moreover the other space forms are "just" quotients of these by some suitable groups of isometries. These quotients will be studied in quite some detail in Chapter II. For the case $K = 0$, which is equivalent to locally Euclidean, one uses the wording *flat*.

C. Synge, Myers, Preissmann: the use of geometric tools

Back in the late 20's Heinz Hopf was working very hard on the links between curvature and topology, see his landmark paper (Hopf, 1932). In (Myers, 1941) the second variation formula is used to prove that the diameter can be bounded (optimally) when the manifold has a positive lower bound for the Ricci curvature (see 1.COMMENTS on I). The striking corollary is that the fundamental group is finite as soon as one has a positive lower bound for the Ricci curvature (just look at the universal covering with the lifted-up metric).

In the case of surfaces the second variation formula had already been exploited in (Bonnet, 1855) to show the diameter property for surfaces of positive (total, Gauss) curvature, which coincides with the Ricci curvature in dimension 2. For higher dimensions, this formula was introduced in (Synge, 1925) and (Schoenberg, 1932).

The second variation formula was again used in (Synge, 1936), together with the parallel transport along a periodic geodesic, to prove that positive sectional curvature, compactness and an even dimension force the fundamental group to be zero (or $\mathbf{Z}_2$ in the non-orientable case). A beautifully simple idea: move vectors by parallel transport along a shortest curve in any free homotopy class (which is necessarily a periodic geodesic). On coming back to the point of origin the vector is moved by an orientation-preserving orthogonal transformation, which in odd dimension always has at least one non-zero fixed vector. This yields a strip of curves that are arbitrarily close to the initial curve but whose lengths are smaller than that of the initial curve, hence a contradiction.

Preissman's paper (Preissmann, 1942–43) constitutes something of a *turning-point.* Apart from the theory of symmetric spaces, it contains on 40 pages all global results which were available on Riemannian Geometry in those years. Preissmann used Jacobi fields to rediscover the above triangle inequality of Cartan's for nonpositive curved manifolds. Moreover he used this triangle comparison to show that if the curvature is negative then abelian subgroups of the fundamental groups are necessarily cyclic. This was generalized only in (Gromoll & Wolf, 1971) and (Lawson and Yau 1972), see I. B. 4.

D. Hodge, harmonic forms and the Bochner technique: the use of Analysis

All the above results were obtained by geometric tools. A historical event, which is still of fundamental importance, was (Bochner, 1946). At that time the de Rham theorem, which links cohomology and exterior differential forms, was already available: (de Rham, 1931). Note that this theorem belongs to differential topology, for the moment Riemannian Geometry does not enter into it at all. The precise statement of the de Rham theorem appeared first in (Cartan, 1928b). Elie Cartan suspected that this result was true when he was working on the computation of the topology of symmetric spaces, see (Cartan, 1928b) and more generally that of homogeneous spaces, see (Cartan, 1929a) (or his complete works: the same applies, books excepted, to all other Elie Cartan citations below).

But, in addition, the Hodge theory of harmonic forms was available: (Hodge, 1941), although it was not until later that the solid foundations for this theory were laid, see (de Rham, 1954). It says that (via the de Rham theorem) the elements of the real cohomology of a compact Riemannian manifold are represented by exterior harmonic forms, and this representation is unique: for an exterior form ω harmonic means $\Delta\omega = 0$, where we, as usual, denote by Δ the Laplacian operating on numerical functions and on exterior forms, see IV and TOP. 6. This is a strong Riemannian condition on such an exterior form; the relation to cohomology is insured by the de Rham theorem.

Let us now use Ricci commutation formulas to permute the order of (covariant) derivatives: as seen in A the defect of the Schwarz lemma in classical differential calculus can here be computed solely with the help of the curvature tensor. One finally gets a formula evaluating the Laplacian of the square norm of such a harmonic p-form ω:

$$-\frac{1}{2}\Delta(||\omega||^2) = ||D\omega||^2 + \mathrm{Curv}_p(\omega,\omega)$$

where $D\omega$ is the covariant derivative of ω and Curv_p some universal algebraic term involving only the form ω (quadratically) and the curvature tensor (linearly). We now integrate over the compact manifold and see that such an ω cannot exist if Curv_p is positive definite, because the integral of a Laplacian is always zero by Stokes' formula. A short way to write the above formula is

$$\Delta = d \circ d^* + d^* \circ d = D^*D + \mathrm{Curv}_p.$$

This was first used in a global setting for $p = 1$, i.e. for 1-forms, in (Bochner, 1946), and very soon underwent a tremendous development, first for p-forms and thereafter for various vector bundles and elliptic operators. The story is still not finished, see the surveys (Wu, 1987), (Bérard, 1988), (Bourguignon, 1988) and their bibliographies. We will meet applications of analogous formulas using this *vanishing technique* in many places below. It was remarked in (de Rham, 1954), page 131, that the above formula for Δ with Curv_p in fact already appeared in (Weitzenböck, 1923), so that Bochner (Bochner, 1948) actually rediscovered it; note that Weitzenböck included no global application. We now give some details of the initial Bochner case.

This Curv_p term is especially nice when $p = 1$, since its value is $\mathrm{Curv}_1(\omega,\omega) = \mathrm{Ricci}(\omega,\omega)$. As a conclusion the positiveness of the Ricci curvature implies that the real 1-cohomology vanishes since the other two terms are nonnegative. This is certainly weaker than Myers' result. Yes, but it is such a completely new technique! Moreover one can use it immediately to take care of the nonnegative case: the positivity insures $D\omega = 0$, and a parallel transported form (i.e. with a zero covariant derivative) insures a local Riemannian product decomposition. See the digression in 1. COMMENTS on I.

Thereafter people tried to see what results could be obtained for other degrees p, see among others (Lichnerowicz, 1950) and (Bochner & Yano, 1953). But it was only in (Meyer, 1971) that it was made clear that it is the curvature *operator* which completely governs the term Curv_p: see the end of Section I.B.1; for Curv_p a very lucid exposition can be found in (Lawson & Michelsohn, 1989), Theorem 8.6.

It is important to see the philosophy here. The Laplacian Δ on exterior forms is what is called a *natural operator*, because it finally involves only Riemannian invariants: D, the covariant differentiation, D^*, its adjoint (defined by the global scalar product), and the curvature tensor. More generally for any operator S the product S^*S is always nonnegative by definition. Then one has $D\omega = 0$ as soon as $\Delta\omega = 0$. This scheme can and will be used for more general differential operators on various bundles, e.g. in I. B. 3 and TOP. 6.

E. Allendoerfer, Weil and Chern

In the late 20's Hopf thought about extending the so-called *Gauss-Bonnet theorem*, which says that for a compact surface M and its curvature K the Euler-Poincaré characteristic is given by

$$\chi(M) = \frac{1}{2\pi}\int_M K(m)\,dm$$

Consequently, if K has a given sign then χ has the same sign. In all books this formula seems to have been called the "Gauss-Bonnet theorem". This is quite surprising, since the notion of $\chi(.)$ did not exist then. It seems that this formula as such and for abstract surfaces first appeared partially in (Blaschke, 1921), page 109, and completely in (Blaschke, 1930), page 167 (we owe this historical information to Robert Osserman), however the formula was certainly around as part of mathematical folklore. For embedded surfaces, it is a combination of (Kronecker, 1869) and (Dyck, 1888), but Poisson in 1812 remarked that that integral was constant under variation and Rodrigues in 1815 used the Gauss map (not published at that time) to prove the formula in some cases. It seems that there is no detailed historical study of the Gauss-Bonnet theorem.

Hopf was looking at higher dimensions: when the sectional curvature (see 1. COMMENTS on I) has a given sign, does χ necessarily have the expected sign? Hopf managed to prove this only in the very special case of space forms (II.A) in (Hopf, 1925a) and also for hypersurfaces of Euclidean spaces in (Hopf, 1925b). And he probably guessed that in the general case it would follow from a generalization of the Gauss-Bonnet formula to higher dimensions. Such a formula was indeed obtained in (Allendoerfer & Weil, 1943). In dimension 4 it reads

$$8\pi^2\chi(M) = \int_M (||R||^2 - ||\mathrm{Ricci} - \frac{\mathrm{scal}}{4}.g||^2)\,dm$$

in invariant form. But using a nice basis the integrand becomes $K_{12}K_{34} + K_{13}K_{42} + K_{14}K_{23} + R^2_{1234} + R^2_{1342} + R^2_{1423}$ (with the obvious notations for the sectional curvature). This implies Hopf's conjecture as stated in (Chern, 1955). The invariant form of writing will be used in I.B.1 and in III.B. But starting with dimension 6 we will see that the formula is so complicated that it cannot be used to derive very many strong results in general (cf.(Bourguignon & Polombo, 1981)), mostly only some non-zero (vanishing) results: see TOP. 1.D. Starting with dimension 6 Hopf's question is still open, see the end of I.B.

In (Chern, 1944) Riemannian geometers were provided with a strongly conceptual proof of the Allendoerfer-Weil formula. Then of course Chern tried to do the same for the Stiefel-Whitney classes. But because they are derived from (real) Stiefel manifolds, they are torsion classes and therefore not directly accessible via exterior forms. Reading Ehresmann he learned that complex Stiefel manifolds have no torsion and this led him to discover the "Chern classes", which are cohomology invariants defined for any complex vector bundle over a manifold. They generalize the Euler-Poincaré characteristic and are, for "real" manifolds, the Pontryagin classes $P_{4k}(M)$: (Chern, 1946). His proof yielded integral formulas for those classes, with integrands involving universal polynomials in the curvature tensor. As above for the Euler-Poincaré characteristic, the formula for P_1 in dimension 4 is useful, see III.C for Einstein manifolds. Higher dimensions yield only general results of "non-zero" type but, using algebraic examples of the curvature tensor, it is proved in (Bourguignon & Polombo, 1981) that those formulas are "useless" or at least disappointing for the Riemannian geometer when using only pointwise estimates of the curvature tensor.

It is important to know that there are *no other* such universal Riemannian integral formulas which yield topological invariants. This was a query of Gelfand's, proved finally in (Gilkey, 1974) (see also (Abrahamov, 1951)). See (Gilkey, 1995) for an up-to-date text and also TOP. 6.

This does not prevent Chern classes from being fundamental. Recall that they are defined for any bundle, not only the tangent one: (Chern, 1946). They are today a building tool in algebraic geometry and in the heat kernel. But what also counts is the fact that there is an integral formula which is *polynomial and universal* in the curvature. See (Berline, Getzler & Vergne, 1992), (Gilkey, 1995) and TOP. 6.

A basic formula fact was discovered in the first edition (in 1956) of (Hirzeburch, 1966). A suitable combination of Pontryagin classes yields the signature for $4k$-dimensional manifolds. Read via de Rham theorem this signature is the linear algebra signature of the quadratic form which is defined by the cup product of cohomology $2k$-classes. Hence this important fact: there is an integral formula involving the curvature for the signature. See most importantly (Gilkey, 1995) and (Berline, Getzler & Vergne, 1992). See also TOP. 6 for secondary characteristic classes and the η-invariant.

F. Existing tools and a brief look at the new ones

From the above one sees that the existing tools in 1950 were only the following ones: the golden triangle and its rules, and the equation for Jacobi fields, which is the equation giving the transverse field to a one-parameter family of geodesics. The basic fact is that the equation is linear (of second order) and involves only the curvature tensor whereas the equation for geodesics themselves is definitely not linear in general. It cannot be integrated explicitly for most manifolds (not even for ones given explicitly), and even so the metric problem is often not finished. A general philosophy is that the exact (computable) solutions of a variational problem are also hard to handle. So one bypasses this difficulty by using various tricks. The curvature, on the other hand, can be computed explictly from the metric, which explains the importance of the topics involving it.

In addition, one had the first and second variation formulas. The first is given by angles of vectors at the extremities, the second needs an integral along a geodesic where a sectional curvature term enters. The first variation was used to prove that there exists at least one periodic geodesic in every non-zero free homotopy class of a compact manifold. It seems that this result was part of mathematical folklore, for example it is used in (Hadamard, 1898), where moreover uniqueness is proved in the negative curvature case. The first variation formula implies the strict triangle inequality and is used very often, typically in I and V. The second variation formula gives for a geodesic the second derivative of the lenghts of a family of curves as the integral along the geodesic which mixes the sectional curvature and the deviation of the strip from a parallel one, more precisely the norm of the covariant derivative of the transverse vector. Results that were available from Analysis were Hodge theory and the Bochner technique.

The very geometric notion of convexity came up quite recently for both subsets and functions in a Riemannian manifold, see its use in the non-compact situation in I.B.1 and I.B.4.

For the geodesic behavior, a basic tool is the Morse theory (Morse, 1934) of the global calculus of variations, see the digression in V.A. Before 1950 this extremely general and powerful theory still did not yield substantial results in Riemannian Geometry. This was not because of the theory in itself but because algebraic topology was not far enough advanced. It will be of basic use in V, see the digression in V. A.

✦

More about *tools.* We will not give a systematic exposition, but will merely give an indication of them at the time they are used. Tools used by Riemannian geometers are of two kinds: those they created themselves for their needs and the preexisting ones that they borrowed, and had to adapt in some instances.

Riemannian Geometry being "differentiable", one can risk saying caricaturally that *"half of it is Analysis".* The reader might have noticed that many names mentioned are those of Analysts: Jacobi, Hadamard, Poincaré, Elie Cartan, Morse, Chern, Bochner, Rauch, among others. This fact can also be remarked in the continuation. We mention now some important tools coming from Analysis which have appeared since the 50's. We are certainly biased and again insist on the fact that we are really sticking to Riemannian Geometry inside the huge world of differential geometry. The reader could consult the following surveys: (Yau, 1987), (Schoen, 1991).

One basic tool is Geometric Measure Theory (*GMT*). The initial idea was that of minimal surfaces, but GMT was the first to offer results which were general enough. It will be used for example in TOP. 1.B, TOP. 10.C and I.B.3. The major impetus came from (Federer & Fleming, 1960) and things crystallized in (Federer, 1969). See more on this tool in TOP. 10.C.

There is also the control of functions of a special type, like distance functions, harmonic functions, eigenfunctions of the Laplacian (in particular the first one). The two basic facts are: in polar coordinates the formula giving the Laplacian of a function depending only on the polar radius involves only the logarithmic derivative of the angular volume element θ ("solid angle"): in particular if d is a distance function (it can be a Busemann one "from infinity") its Laplacian is $\Delta d = -\frac{\theta'}{\theta}$. And this is upper bounded when the Ricci curvature is lower bounded by Bishop's theorem (TOP. 1.A). The second formula is Bochner's formula above as applied to an arbitrary function: written in full detail it reads $-\frac{1}{2}\Delta(||df||^2) = ||\text{Hessian}(f)||^2 - \langle d\Delta f, df\rangle + \text{Ricci}(df, df)$. One can apply it to a distance function and thus obtain, from the Ricci curvature, more control on the differential of its Laplacian $d(\Delta f)$, and/or to some suitably chosen harmonic functions. We will see that these two facts are heavily used, not only for the spectrum, but also to obtain metric comparison theorems with only a lower Ricci bound: see e.g. the digression in I.A.2.

Other tools are existence theorems for harmonic maps. The topic started with (Eells & Sampson, 1964). Today it is of basic use, see TOP. 7.

Harmonic coordinates were systematically introduced in (Jost & Karcher, 1982) and are now indispensable: see the digression in I.C and 1. COMMENTS on I. We note that harmonic coordinates were used a long time ago by theoretical physicists, see (Einstein, 1916) and (Lanczos, 1922). See the formula for Δg_{ij} in the digression of 1. COMMENTS on I, then III.C for the use of harmonic coordinates to prove that Einstein manifolds are real analytic, and the basic use of them in I. C. 1 and 2.

✦

Quite a new concept is that of deformation techniques. This means, on a manifold, a partial differential equation, like the heat equation, describing an evolution. They are used to prove the existence of desired objects. The technique is to prove first that the deformation equation (typically a partial differential equation of parabolic type) has a solution for all time points t, and secondly that it converges to "the" (or "a") desired object as t tends to infinity. See TOP. 7 for harmonic maps and V.A for obtaining periodic geodesics. If one deforms suitably chosen functions in the complex case this is the technique suggested in (Calabi, 1954) to obtain Kähler Einstein manifolds. These deformation equations are quite natural: they consist of deforming an initial object in the best way for reducing the *defect* with respect to the imposed condition.

✦

Calabi's equation was a particular case of the more recent and more abstract technique consisting of deformations in the space of all Riemannian metrics on a given manifold, see III.C. In the general case the deformation is really for Riemannian metrics, not at the numerical function level. Very often one uses quite a new tool, the Ricci flow. That is to say one considers a one-parameter family of metrics $g(t)$ and the PDE:

$$\frac{\partial(g(t))}{\partial t} = -2\mathrm{Ricci}(g(t))$$

This equation appears in slightly different forms where the right-hand term is normalized (in various ways depending on the authors) by adding the scalar curvature and playing with both coefficients of Ricci and scalar curvature, the point being to keep the volume fixed. The difficulty is that this equation is not really parabolic, because the right side is only "elliptic degenerate". We will meet various results later on, a partial survey is Section 4 of (Besse, 1994). The Ricci flow was first used by Hamilton in dimensions 3 and 4, see the results in I.B.1 and 2. In fact

Hamilton used it also to recover the standard conformal representation theorem, see the end of the story in (Chow, 1991). One of the most promising uses would be to attack the various conjectures on manifolds of dimension 3 and 4, see TOP. 8.

See also TOP. 3.A for results on manifolds with holonomy G_2 and $Spin(7)$ which are obtained by such a deformation technique, but with completely different equations. See also I.B.1 and I.C.2.

G. Existing examples and a brief look at the new ones

Another very important aspect of Riemannian Geometry is that of *examples*. This might not be obvious but some people even say that to feel, to understand what a Riemannian manifold is, cannot be achieved without being familiar with quite a few examples and we will mention them in time, even though only briefly. The following books lay special emphasis on examples in an expository way: (Gallot, Hulin, & Lafontaine, 1990), (Besse, 1987), (Petersen, 1998a). Here are the examples which were, to the best of our knowledge, known in 1950 and whose important geometric properties were more or less understood: first, following Riemann, the hyperbolic spaces Hyp^d in arbitrary dimension. On the compact side first of all of course the spheres. Then the complex projective spaces $\mathbf{C}P^n$ with their Fubini-Study metric: (Fubini, 1903) and (Study, 1905).

Those $\mathbf{C}P^n$ were put into the general framework of all symmetric spaces by Elie Cartan (see section II.C). This includes the symmetric spaces of rank one, which are, on the compact side, the spheres, the various $\mathbf{K}P^n$ for $\mathbf{K} = \mathbf{C}$ or $\mathbf{H}$ (the quaternion field), and of the Cayley numbers **Ca** only $\mathbf{Ca}P^1 = S^8$ and $\mathbf{C}aP^2$ are permitted. The Cayley projective plane $\mathbf{Ca}P^2$ is beautiful, we like to call it the *panda* of Riemannian Geometry. Its projective lines are 8-dimensional spheres and satisfy the projective plane axioms. Cartan suspected that this result was true in (Cartan, 1939), page 354. One had to wait until 1951 for a solid projective construction of it by Freundenthal, which was never published. For this intricate history of the panda, see Chapter 3 of (Besse, 1978) .

On the non-compact side, Cartan gave an exposition of the dual generalized hyperbolic spaces $Hyp\mathbf{K}P^n$. Higher rank symmetric spaces were also completely classified by him, they include in particular all the Grassmann manifolds $Grass_{\mathbf{K}}(p, q)$ as well as their nonpositive curvature analogues, plus a very restricted list. For the non-simply connected quotients of the above, see the entire Chapter II. Except for the spherical case and for surfaces compact quotients were not known to exist in 1950 besides the dimension 3 hyperbolic examples of (Löbell, 1931) .

The complete theory of symmetric spaces (geometry, classification, see more in II. C) was built up by Elie Cartan in the very short interval of 1926–27, in a series of papers from (Cartan, 1926b) to (Cartan, 1927). See more in II.C.

The geometry of the geodesic flow was understood: for surfaces of revolution (Clairaut circa 1730), for ellipsoids (Jacobi in 1832) see an exposition in (Klingenberg, 1982), Section 3 (these geodesic flows are called "integrable" in modern jargon), for symmetric spaces see (Cartan, 1927). One should add Morse's theorem quoted in V.A. But one still does not have a complete mastering of the metric of ellipsoids, see the end of TOP. 4.

A trivial but basic concept is the product of two Riemannian manifolds. The relation to holonomy groups (see TOP. 3) and the global product result in the complete simply connected case are in (de Rham, 1952) and (Borel & Lichnerowicz, 1952) (we make this an exception to the 1950 rule). In the case of symmetric spaces this notion of product and holonomy group was known to Cartan and was a basic tool in his work we mentioned above.

✦

This hides the, sometimes extremely hard, difficulty one has in building up a Riemannian manifold with imposed properties. By this we most often mean the control of the curvature and/or of the basic geometric invariants: volume, diameter, injectivity radius (see I.A.1).

There is quite a variety of techniques for building up examples. It starts with Algebra. First to come are *homogeneous* spaces G/H of Lie groups with H compact: they always admit invariant (called homogeneous) metrics and there are explicit formulas for computing the curvature, see also III.C. But their discrete quotients are much much harder to build up. For example building up general space forms was only in the air in the 50's and, moreover one had to await (Borel, 1963) for a general result. Note that it was more Number Theory than geometry.

Then comes the theory of *Riemannian submersions.* This means a submersion $M \to N$ between two Riemannian manifolds which should fulfill the following condition: the metric in M determines, at every point, an orthogonal complement (called horizontal) of the space tangent to the fiber; at every point the restriction of the projection to this horizontal space is required to be a Euclidean isometry between it and the tangent space at its image in the basis. In this case the formulas of (O'Neill, 1966) enable us to compute the curvature of the total space if one knows the curvatures of the basis and of the fibers, how the fibration is twisted and finally how the fibers are embedded (the best case is when the fibers are totally geodesic submanifolds). For example topological surgery can be Riemannianly controlled by these formulas. A now classical example is that for positive scalar curvature: I. B. 3. Particulary important Riemannian submersions are the *warped products.* Definitions here vary with authors. When the basis is one-dimensional, the metric will appear in coordinates as $g = dt^2 + f(t).g_{d-1}(t)$. If g_{d-1} is independent of t this is a rotationally symmetric metric, a generalization of surfaces of revolution. O'Neill's formulas then become much simpler. More recent still are the *double* warped products. For books on these formulas see (Cheeger & Ebin, 1975), (Besse, 1987), (Sakai, 1996) and (Petersen, 1998a). Warped products are of basic use in (Cheeger & Colding, 1996) (and subsequent works of these authors), where they serve as smooth rigid models for studying the structure of manifolds with a lower bound on the Ricci curvature.

Other examples are those obtained by conformal changes: g becomes $f.g$ where f is a numerical positive function. Although they are extremely specialized when $d \geq 3$ those changes are very useful sometimes and the formulas for the curvatures are extremely simple.

Mixing these two techniques one can compute the curvature of Riemannian manifolds of high cohomogeneity. This means manifolds which are almost homogeneous, i.e. the isometry group is large in the sense that its generic orbits have low codimension. The main difficulty is that of the singularities which occur at the places where the dimension of the orbit is not the maximal one (the case of the poles of surfaces of revolution is typical). This was used for example in (Page, 1979) and (Bérard Bergery, 1982) to build up Einstein manifolds. As one will see below these kinds of examples have now become more and more numerous and are often used as counterexamples.

Deep analysis, in particular strong results for various types of partial differential equations, is fundamental. See for example Chapter III and what was said about the new tools from Analysis.

Another approach is quite recent. It turned out in the 60's that some results (of algebraic topology, then for complex manifolds and even more recently of Riemannian Geometry) on a manifold cannot be obtained by just working on the manifold itself. One has to study various *bundles* over that manifold, principally vector bundles. Some bundles are canonically attached to a Riemannian manifold, like those of exterior forms, but it is also important to build various "twisted" bundles. This consideration has become a basic tool, we will devote TOP 6 to it. A special mention should be given to spinors. A fairly complete reference is (Lawson & Michelsohn, 1989), see also (Gilkey, 1995) and (Berline, Getzler & Vergne, 1992).

Last but not least: "pure" geometry. One can see it as consisting of two parts. The first part is geometry on the manifold itself. Most often one needs to, in the Riemannian context, adapt techniques coming from differential topology, surgery in particular. In this case one needs to control the curvature, an example is (Gromov and Lawson 1980a), used in I.B.3. One also finds various geometric controls of various volumes during surgery and other topological constructions, in relation to TOP. 1.E. We will mention other examples in the continuation, but Gromov's construction of a sphere with almost negative curvature (see details in (Buser & Gromoll, 1988)) and a generalization in (Bavard, 1987) show how high the degree of sophistication can be when trying to build up examples with only purely geometric tools. See the end of I.B.4.

Complexity theory is entering slowly into Riemannian Geometry, see 8. 11 of (Semmes, 1996a) for the case of a bounded geometry (TOP. 5) and the end of V.A.

1 Comments on the main topics I, II, III, IV, V under consideration

As already explained above we made a choice of five main topics, namely

I: curvature and topology (Hopf's urge)
II: looking for a construction and classification of space forms
III: looking for distinguished metrics, in particular Einstein ones
IV: study of the eigenvalues and the eigenfunctions of the Laplacian (in brief: study of the spectrum)
V: study of periodic geodesics, more generally the geodesic flow
TOP: at the end we will mention briefly some interesting and important topics, but not in as much detail

TOPIC I started with Hadamard and was pursued by H. Hopf, Synge, Myers, Preissmann and Bochner. The subject exploded with Rauch's work in 1952 and has not levelled off since. We will devote a large part of our text to it. It was a very strong incentive. Considering the huge harvest of results we had a very hard time organizing the material. We will make a first division into three parts: pinching, curvature of a given sign, finiteness theorems (and more). The reader will judge if that division is too artificial. Within each part the idea is also threesome, divided into sectional, Ricci and scalar curvatures. But moreover we will at the same time follow the dominant stream of today's research in Riemannian Geometry: to obtain results with fewer and fewer hypotheses, the simplification being that of having ever weaker invariants in the statement and/or even to replace pointwise norms by integral norms.

In particular one has recently obtained more and more impressive results using only Ricci curvature bounds. The motivation is thus: in dimension d the Ricci curvature has as many parameters as a metric: $d(d+1)/2$, whereas the sectional curvature depends on $d^2(d^2-1)/12$ parameters. For $d=2$ this is not enough information, but for $d \geq 4$ this can be considered as "too much" information. We saw above in 1.F one basic formula involving only the Ricci curvature, the second one is in the digresssion below.

✦

An important digression: Curvature is a very complicated invariant/How does one see curvature today?

The curvature tensor appears naturally as endomorphisms $R(x,y)$ of the tangent space attached to couples (x,y) of tangent vectors. A naively simple idea is that of the *sectional curvature* attached (by means of the curvature tensor) to a 2-dimensional tangent space P (a tangent *plane*) which is the scalar $K(P) = K(x,y) = R(x,y,x,y)$ when x,y is any orthonormal basis of P. Note that the form $R(x,y,z,t)$ is multilinear of degree 4 ("quadrilinear" or "biquadratic") and enjoys the following symmetries: it is anticommutative both in x,y and in z,t, one has $R(x,y,z,t) + R(x,z,t,y) + R(x,t,y,z) = 0$ and $R(x,y,z,t) = R(z,t,x,y)$. Geometrically the interpretation of the sectional curvature is wonderful: compute the length of the small metric circles generated by the geodesics starting with a speed vector in P. Then $K(P)$ measures exactly the defect of that length as compared to the Euclidean one; more precisely it gives the first non-trivial term in the limited expansion into a function of the radius.

But despite the symmetries of the curvature tensor the function K as a numerical function defined on the Grassmann manifold of the planes at a point is still almost a complete mystery: where are the critical points (planes), what are its critical values and in particular its extremal ones? Things are trivial in dimensions 2 and 3 but starting with dimension 4 one has only one partial result in (Singer & Thorpe, 1969) for 4-dimensional Einstein manifolds. A *notation* like $\boldsymbol{K} < \boldsymbol{a}$ for a Riemannian manifold will mean that the sectional curvature is smaller than the scalar $\mathbf{a}$ at every point, and that this holds for every tangent plane at that point.

The fact that the knowledge of the curvature tensor and the knowledge of the sectional curvature are equivalent is important theoretically but it is hard to use it explicitly. Important to know is the dimension of the space of curvature tensors (pointwise): $d^2(d^2-1)/12$. It is also very important to realize that the curvature tensor, in general, does not determine the metric, even though that dimension is so huge (when $d > 3$) in comparison to the number, equal to $d(d+1)/2$, of parameters for a metric: see the digression in I.C.1. The sectional curvature, on the other hand, almost always does determine the metric; the difference between the curvature tensor and the sectional curvature is that the sectional curvature involves the metric.

Computing the curvature tensor explicitly in a coordinate chart $\{x_i\}$ involves the first two derivatives of the metric (say the $g_{ij} = g(\partial/\partial x_i; \partial/\partial x_j)$) but the expression is very complicated. In short, however, one has in any coordinate system: $R_{ijkh} = \frac{1}{2}(\partial_{ik}g_{jh} + \partial_{jh}g_{ik} - \partial_{ih}g_{jk} - \partial_{jk}g_{ih}) + Q(\partial g, \partial g)$, where Q is quadratic in the first derivatives of g. From this formula one sees that one cannot recover the second derivatives of g from R, and this accounts exactly for the difference between $(d(d+1)/2)^2$ and $d^2(d^2-1)/12$. Roughly speaking, therefore, the curvature (tensor or sectional) can be seen as a "twisted Hessian of the metric". At the origin of the *normal coordinates* of Riemann, those which are given by the geodesics issuing from a given point as origin, one then has the treacherous formula $\partial_{ij}g_{kh} = \frac{1}{3}(R_{ikjh} + R_{ihjk})$. This does not explain but at least makes very plausible the comparison theorems of I.2: one can control a function when one has a control on its second derivative. But there is a long way to go to be able to really state the results and prove them.

It turned out (this will be seen in detail in I.C.1) that those very geometric and natural normal coordinates are in fact badly suited for coordinate changes when one controls only the curvature. This is because the equation for Jacobi fields in 0.B appears to be of second order but this is only after having integrated the equation for geodesics. The discovery of (Jost & Karcher, 1982) was that harmonic coordinates (0.F) are the well-adapted ones. With them one does not lose any derivatives. They are quite simple to define and build up, being just nicely chosen linearly independent harmonic functions (see I.C.2 and IV).

✦

If the reader does not like biquadratic forms as algebraic objects or the still not unveiled sectional curvature, he should be immediately more enthusiastic about the Ricci curvature. It is nothing but the trace (with respect to the metric) of the sectional curvature: for a unit vector x its Ricci curvature $\mathrm{Ricci}(x)$ is the sum of the sectional curvatures $K(x, x_i)$ where $\{x, x_2, \ldots, x_d\}$ is any orthonormal basis complementing x. One can also consider it as the mean value of the sectional curvature through a given vector; this is a quadratic form and then has the same number of parameters as the metric itself. It has been a vital lead in Riemannian Geometry to deduce as much as possible about the metric from its Ricci curvature. This will be seen amply in the continuation. One thing which really helped to unravel the situation was the fact that, in *harmonic* coordinates, (0.F), the Laplacian Δ of the metric (see IV for the Laplacian) is exactly the Ricci curvature except for terms Q of lower order (which are quadratic):

$$\Delta(g(\partial/\partial x_i; \partial/\partial x_j)) + Q(g, \partial g) = -\mathrm{Ricci}(\partial/\partial x_i; \partial/\partial x_j)$$

Then one has at one's disposal the entire classical Analysis to get information on functions and their Hessian from the knowledge of their Laplacian. For distance functions and Ricci curvature see 0.F.

Another of the main points about the Ricci curvature was in fact discovered in (Bishop, 1963): a lower bound on the Ricci curvature gives an upper (optimal) bound on the volume of metric balls. This will turn out to be fundamental for topological implications, but even more so for convergence and precompactness results. The proof is quite simple: the Ricci curvature is a trace and traces are derivatives of determinants (volumes). For all this see TOP. 1.A and B. In fact the Ricci curvature yields the first non-trivial term in the asymptotic expansion of the volume form along a geodesic (solid angle). Bishop's result is that it can be integrated with a lower bound. On the other hand, it definitively cannot be integrated with an upper bound as soon as $d \geq 4$. *Scalar curvature* is the trace of the Ricci curvature, it is a scalar (at every point). Infinitesimally it gives the first non-trivial term in the asymptotic expansion into a function of the radius of the volume of balls centered at a given point. But this time it cannot be integrated with a lower or an upper bound.

✦

TOPIC II is in fact the most natural one: study the Riemannian hierarchy. Riemann's idea was to generalize Euclidean geometry. So we look next for geometries which are successively farther away from Euclidean ones. Stated in other words: relax the Euclidean axioms more and more. This is the space form problem. It goes back to Killing who convinced Klein of the importance of this question during a lecture he gave. This explains partly the classical phrasing of "Clifford-Klein spaces".

Here is the place to comment on two sections (wings) of Riemannian Geometry, namely those of positive and negative curvature. We will see more than once below how much these two sections differ, often paradoxically. In the case of space forms the difference is the following: briefly speaking, it is child's play to build up space forms of positive curvature and they are "easily" classified. Negative space forms are very hard to build up, they always require Number Theory even if in disguise. In exchange, negative curvature space forms, once they are built up, enjoy many optimal results, see I.B.4 and V.

Positive curvature space forms, on the other hand, are still very mysterious. For example there are almost no reasonable invariants which are either explicitly known for them or which characterize them: see III.B for the minimal volume, TOP. 1.C for the embolic volume. And, as laid out in III.C, there are non-uniqueness results for Einstein metrics on them in some cases, but not a single case of uniqueness is known. It starts with S^4.

TOPIC III is satisfactory only for surfaces, in this case thanks to the concept of conformal representation. For higher dimension, it started with a question of Calabi's in 1950. The first general answer to that question came only in (Yau, 1978). Among the few results obtained since, let us mention (Hamilton, 1982). In particular the number one question "does any compact manifold of dimension larger than 4 admit an Einstein metric or not?" is almost completely untouched and even experts have no guess whether the answer is yes or no. One has only local non-deformation results.

TOPIC IV, which can be termed "Riemannian manifolds as quantum mechanics objects", started seriously only with the combination of (Milnor, 1964) and (McKean & Singer, 1967). But this subject then witnessed a real explosion. The links between IV and V have been well developed since (Colin de Verdière, 1973), a fact which could be surprising considering what comes next. The point in IV can be described as building up a Fourier analysis on Riemannian manifolds and studying it. This is fundamental in some pure geometric questions. Let us mention in classical geometry the use of spherical harmonics for studying the kissing number of spheres, see Chapter 13 of (Conway & Sloane, 1993) and in our text the control of distance functions in the digression in I.A.2.

TOPIC V, which can be termed "Riemannian manifolds as classical Hamiltonian mechanics objects", enjoys satisfactory answers for "generic" (*bumpy*) Riemannian manifolds, the strictly general case remaining quite mysterious. Results started to appear with Poincaré and Hadamard, but despite the initial efforts by Birkhoff, and even more efforts by Morse, one had to wait until the late 60's for a reasonable harvest. The case of surfaces is much better understood. It is only for the negative sectional curvature case that one has good, in fact excellent, results: they began with (Hopf, 1939) for surfaces and (Anosov, 1967) for the general case.

I Curvature and Topology

Since Heinz Hopf in the late 20's this topic has been and still remains the strongest incentive for research in Riemannian Geometry. Most books on Riemannian geometry give some results on this topic but none covers it completely, so we will rely on surveys and references. Some books which give special attention to this topic are (Cheeger & Ebin, 1975), (Sakai, 1996) and (Petersen, 1998a), see also (Petersen, 1998b). We explained at the beginning of 0.F why so much emphasis is given to curvature.

A. Pinching Problems

1. Introduction

Recall that manifolds of constant sectional curvature are locally isometric to spheres, to Euclidean spaces or to hyperbolic ones so that, if compact, they are compact quotients of the sphere, the Euclidean space or the hyperbolic space by a discrete isometry group with no fixed points. For more on these *space forms* see II.A. To put this result on a firm foundation was one of Heinz Hopf's tasks in the 30's and one of the motivations for the Hopf-Rinow theorem. The following question now suggests itself: assume that a compact manifold has a sectional curvature that does not vary too much (one will say that the manifold is "*pinched*"). Can one deduce from this that the underlying manifold is topologically (or more even – differentiably) identical with one of the above space forms? After normalization we are left with three cases: the pinching question around $\sigma = +1, 0, -1$. We assume that the *sectional curvature satisfies* (everywhere and for every tangent plane) $|K - \sigma| \leq \varepsilon$ and look for an ε which will lead to the conclusion that we are on a topological (or, better even, diffeomorphic to a) space form. Note that, for the zero case, some normalization is needed, which is usually done by requiring the diameter not to be too large. This restriction is needed because stretching the metric by a large factor will make the whole curvature go to zero. We will discuss the positive pinching problem in quite some detail because of its historical importance. Pinching is put into an interesting general perspective in (Gromov, 1990).

Back to the general pinching problem we will see that the answer has surprisingly different answers for the three possible signs. This will serve as a first illustration of the basic differences between space forms of different signs. There is a solution covering all three cases together, but one needs, besides the pinching of the sectional curvature, both an upper bound on the diameter and a lower bound on the volume. This was achieved in a unified way in (Fukaya, 1990), see Theorem 15.1, and covers prior results: the positive case was solved in 1951 by Rauch, see below, the zero case was solved in (Gromov, 1978a) and the negative case in (Gromov, 1978b).

There are more general types of pinching, the most general being that for symmetric space forms. A program of research on this topic was started by Rauch, for references and intermediate results see (Min-Oo & Ruh, 1979; Min-Oo & Ruh, 1981), but these texts have still not been understood completely. Informative texts are (Gromov, 1990) and (Petersen, 1996). For Ricci-pinching see the very end of 4 below.

2. Positive Pinching

The following quite natural question was repeatedly put forward by H. Hopf, in particular when Rauch (an analyst and an expert in Riemann surfaces) visited Zürich in the late 40's. Noting that the standard sphere is the only simply connected manifold of constant positive sectional curvature and bearing in mind some heuristic principle of semi-continuity, one can hope to be able to prove that if the sectional curvature is close to a positive constant, the underlying (simply connected) manifold will still be the sphere. This was indeed proved in (Rauch, 1951) with a pinching constant (i.e. the ratio of the lower to the upper bound of the sectional curvature) of roughly 3/4. Rauch's paper was seminal in two respects. First of all, it gave a control on the metric on both sides: if the curvature is smaller than Δ, then the metric (in a sense that of course needs to be made more precise) is larger than the metric in the sphere of constant curvature equal to Δ (at least in regions small enough). And mutatis mutandis for a curvature larger than δ. The case where $\Delta = 0$ was known to Cartan and Preissmann as we saw in 0.B and 0.C. Secondly, to get a global result, Rauch made a subtle geometric study in which he proved that, under the pinching assumption, one can build (by a sort of analytic prolongation using exponential maps) a covering of the manifold by the sphere.

We comment now on Rauch's result and the results of his followers. First we comment on the tools, thereafter on what assumptions are made about the curvatures (sectional or "only" Ricci), and finally on optimality. We will mainly give an historical overview, surveys are (Shiohama, 1990) and (Abresch & Meyer, 1997).

✦

The basic role of the *injectivity radius* (as linked to the cut-locus: TOP. 4) was made precise in (Klingenberg, 1959). The injectivity radius $Inj(g)$ of a Riemannian manifold (M, g) is the largest real number such that every pair of points whose distance is less than it is joined by a unique shortest geodesic (segment). In particular every open metric ball of radius $Inj(g)$ (or smaller) is diffeomorphic to $\mathbf{R}^d$. By the elementary foundations of Riemannian Geometry, there is such a positive number for every point (called the injectivity radius at that point). Moreover this number is continuous as a function of the point and in particular it is positive if the manifold is compact. We will see below how fundamental the injectivity radius is. In the positive curvature case two results of Klingenberg's tell us that, in even dimension and with simple connectedness, the double hypothesis $0 < K \leq 1$ always implies that $Inj(g) \geq \pi$ (think of the sphere). The proof uses Synge's result. This injectivity radius bound is definitely wrong in odd dimensions, see below. But it remains valid even for odd dimensions if moreover $1/4 < K \leq 1$. This remains true even for $1/4 \leq K \leq 1$, but the proof is more difficult. The key idea is due to Klingenberg, but some gaps still had to be filled. This was done in (Cheeger & Gromoll, 1980), a preprint of which was circulated as early as 1972. See also (Klingenberg & Sakai, 1980). For both results the use of Morse theory is fundamental, see the "long homotopy" below. One combines this with comparison theorems for Jacobi fields. We will see below in Section C.1 that a control on the injectivity radius from below in all generality definitely requires *four* conditions: an upper and a lower bound on the sectional curvature, an upper bound on the diameter and a lower bound on the volume. Note that Rauch's proof implies, under his pinching constant, Klingenberg's lower bound on the injectivity radius.

✦✦

A digression: Comparison theorems. The best possible *local* control for the metric from $\Delta = \sup K$ and $\delta = \inf K$ follows directly (for Δ or δ of arbitrary sign) from Rauch's estimate for Jacobi fields. Even though other techniques are now available, see below, the geometric metric two-way bounds obtained from $\sup K$ and $\inf K$ for (large or small) triangles are fundamental. Later Rauch, by integration along curves, obtained bounds for (small) triangles (i.e. within the injectivity radius). One calls these results *RCT* (Rauch comparison theorems).

They are to be distinguished from *Toponogov's* theorem: (Toponogov, 1959), (Toponogov, 1964b). The remarkable fact in Toponogov's theorem is that, for a lower bound on the sectional curvature, one can "ignore the injectivity radius or the cut-locus" (TOP. 4). This definitely cannot be the case for a Toponogov result with an upper curvature bound: just look at a very small periodic geodesic in a flat torus. However this is possible in the negative curvature case if moreover one has simple connectedness: see Cartan's result in 0.B and I.B.4 below. The Toponogov theorem is simple to state: for $K \geq 0$ it just says that every triangle is infra-Euclidean: $\boldsymbol{a}^2 \leq \boldsymbol{b}^2 + \boldsymbol{c}^2 - \boldsymbol{2bc} \cos \alpha$. For $K \geq k$ it is the analogous inequality but as applied to the classical formulas for triangles in the space of constant curvature equal to k, namely the spheres or hyperbolic spaces when $k > 0$ or $k < 0$ respectively. The relevant formulas are the classical ones of spherical or hyperbolic trigonometry.

One should be very careful not to expect too much from these comparison theorems. They are great in the sense that a double control on the sectional curvature gives us a double control on the metric. For example one recovers the classical fact that, locally, spaces of constant sectional curvature are isometric to the standard ones. But for variable curvature, things are different. Here are for example two cases where care is needed. The first problem is that the curvature tensor does not determine the metric in general; the case of constant curvature is a clear exception. This is a whole new topic in itself and started only fairly recently, we will come back to it in the digression of C.1. The second problem is that of pinching around zero, see A.3 below.

Rauch's proof of *RCT* was quite painful, as well as the proof of Toponogov's theorem. It was discovered in (Gromov, Lafontaine & Pansu, 1981a) (the "little green book") that the important idea was the control of the distance functions (distance to a point, to some hypersurface, to some periodic geodesic or from infinity: Busemann functions in B.2). The key idea of Gromov's was to look at the ODE for the second fundamental form of the level hypersurfaces: this is a Ricatti equation (this is not too surprising since any second order linear equation – in particular the Jacobi field equation – can be reduced to one of Ricatti type). These points are described well in (Karcher, 1989), thereafter in the lectures notes of (Eschenburg, 1994), and also in the books (Sakai, 1996) and (Petersen, 1997a). But heuristically, if one thinks of the sectional curvature as "the Hessian of the metric" (see the digression in 1. COMMENTS on I), such results are not really too surprising.

The Toponogov triangle theorem needs essentially a lower bound for the sectional curvature. But these days one can make use of more and more results of metric type which were obtained previously using the sectional curvature and are now valid under bounds just on the Ricci curvature. We saw in 0.F why such a hope was not too unrealistic. Some landmarks are (Calabi, 1958), (Cheeger & Gromoll, 1971) and (Abresch & Gromoll, 1990). A text that can be quite astonishing for the reader is (Colding, 1996c). One finds there a statement yielding both an upper and a lower bound for triangles and this holds using only a lower bound for the *Ricci* curvature (of arbitrary sign). The result is of L^1-type and implies that most of the (*thin*) triangles are OK for what one wants to do with them. We state this explicitly:

"In a Riemannian manifold of dimension d with nonnegative Ricci curvature, for every $\varepsilon > 0$ there exists some $\eta(d,\varepsilon) > 0$ such that for any $R > 0$, for any $r \leq \eta R$ and any points p, q with $d(p,q) > 2R$ one has

$$\frac{1}{\mathrm{Vol}(U(B(q,r)))} \int_{v \in B(q,R)} |d(\gamma_v(r),p) - d(\gamma_v(0),p) \\ - r\langle v, \mathrm{grad}(d(p,.))\rangle|\, dv) \leq \varepsilon R,$$

where $U(B(q,R))$ denotes the unit tangent bundle over the ball $B(q,R)$".

There are analogous formulas when the lower bound for the Ricci curvature is any real number, as well as an L^1-formula for the angles. Two remarks are now in order. The triangles should have at least one side which is large with respect to another. But in exchange, they are controlled on both sides. The reader may find it surprising that *both* sides are controlled by just a lower bound on the Ricci curvature. The heuristic explanation is the following: the second variation formula and Myer's trick (see 0.C and below) lead one to the conclusion that a long *segment* forbids any too large sectional curvatures along it: a segment being an absolute minimum the second variation along it cannot be negative. This two-sided control using only a lower bound appears in fact in the rigidity theorems in B.2 below for non-compact manifolds with nonnegative sectional and Ricci curvature, where infinite segments (lines) are used. Colding likes to think of his result as a *predictability* one. Given a point p, if one is given the initial velocity of a geodesic which starts close enough to q and is not too long, then one knows, to a fairly high degree of probability, i.e. in a local integral fashion, where the end of this geodesic is as seen from p. In the positive Ricci case, there is a stronger result, namely an L^2-formula and this is, moreover, valid for practically any triangle (not necessarily thin).

Colding's results are of L^1- and L^2-type but optimal; in (Dai & Wei, 1995) one can find Toponogov-type theorems with only a lower bound on the Ricci curvature, but then the bounds differ exponentially from those of the comparison space.

The tools used here are those explained in 0.F: playing Analysis, in particular with distance functions and suitably chosen harmonic functions. Colding's formula introduced a new point of view which is put into a very general framework in various texts quoted below, see in particular for a systematic exposition (Cheeger & Colding, 1997a), (Cheeger & Colding, 1997b), (Cheeger & Colding, 1998). The philosophy could be the following: the interplay between the lower bound on the Ricci curvature and the Laplacian on functions plays a role analogous to that of *RCT* and the Toponogov theorems in the case of a lower bound on the sectional curvature. The philosophy (see also section C.2) is that the sectional curvature is, up to some twisting, the *Hessian* of the metric, whereas the Ricci curvature is only the *Laplacian* of the metric. Playing with the sectional curvature is an ODE game, with the Ricci curvature it is a PDE one.

✦✦

We now come to the optimal constant in the (positive) pinching problem. As early as 1961 (Klingenberg, 1961) one knew that a strict (1/4)-pinching is enough to insure that the manifold is the sphere. In the literature, this is called the *sphere theorem*. And one cannot do better since the $\mathbf{K}P^n$ (with $\mathbf{K} \neq \mathbf{R}$) have a curvature betweeen 1/4 and 1. The classical proof consists simply in covering the manifold with two topological balls (which are metric balls in the construction, but beware of the fact that metric balls are not always topological ones when their radius is too large, typically beyond the injectivity radius). But only topological spheres can be covered by two balls (Reeb's theorem, see (Milnor, 1963), page 25). To find two such balls one takes two points whose distance equals the diameter and applies the Toponogov theorem to prove that the two balls centered at those points with a radius just a little larger than half the diameter do have a radius smaller than the injectivity radius. In fact the proof can be made precise in order to prove directly the desired homeomorphism, see (do Carmo, 1992), page 287. For "sphere theorems and below", see, besides the following, the survey (Abresch & Meyer, 1997).

An interesting proof is due to Gromov and is explained in (Eschenburg, 1986). It is in fact very close in spirit to Rauch's original proof. Look at balls centered at a fixed point: then just beyond $\pi/2$ the boundary becomes concave (this was proved and used by Rauch). With some work-effort and using positive curvature, one shows that the outside of that concave ball can be contracted to a point just by "following the normal field". This proof does not use Toponogov's theorem or Klingenberg's injectivity radius bound, this bound being in fact a corollary of the proof.

The sphere theorem triggered geometers to go further. We will now mention the essential developments; surveys on the topic are (Shiohama, 1990) and (Wu, 1990).

✦

The sphere theorem as stated above involves two dramatic provisos. The first one can be stated as follows: a covering by only two balls characterizes spheres but only topological spheres. Now there exist topological spheres with *differentiable* structures that are different from the standard one (so-called "exotic" spheres). Moreover some exotic spheres are obtained exactly by gluing two half-spheres together along their equator, but with a weird identification. So we are led to the "differentiable pinching problem" (since the simply connected manifolds of constant positive curvature are *the* standard spheres, see 0.B).

The second problem is to see what can be done for the non-simply connected case. For sure, looking at the universal covering, one can be sure that the manifold downstairs is a quotient of a sphere, but this does not say that the topology is that of a space form. The problem is the *equivariant* pinching question. It was solved progressively, a definitive result is (Im Hof & Ruh, 1975), but the pinching constant is still very close to 1. For the non-simply connected case in relation to the diameter see (Flach, 1994).

But work is still being carried out on the pinching constant, the latest reference for the simply connected (diffeomorphism) case is (Suyama, 1995) with a pinching independent of the dimension: 0.654. See also the references there for intermediate results. Nobody knows if 1/4 is possible or not. Worst of all: there is no exotic sphere known which has positive curvature, only nonnegative ones are shown to exist in (Gromoll and Meyer 1974). For more on this, see (Grove & Wilhelm, 1997).

Historically the differentiable pinching problem was solved by Calabi (unpublished) and in (Shikata, 1967) by methods similar to those which will appear in Section C below, but yielding no explicit pinching constants. Then Gromoll did the following (Gromoll, 1966): looking back at the proof "with two balls" centered at diametrically opposed points, he uses the Toponogov theorem and an extra control on parallel transport to show that the gluing together of the two balls along their boundary $(d-1)$-spheres is close to the identity. Then differential topology implies a total diffeomorphism. Gromoll's pinching constants were given explicitly, but they tend to 1 as the dimension tends to infinity.

The method of (Ruh, 1971) was completely different: the sphere was seen "from outside". The fact that the curvature is strongly pinched enables us to construct a line bundle on the manifold M^d which resembles the normal bundle of the standard sphere in $\mathbf{R}^{d+1}$ far enough to finally yield an embedding from M into $\mathbf{R}^{d+1}$. This embedding yields a sphere which is close to the standard one and hence diffeomorphic to it. There are other techniques, in (Otsu, 1993) for example one uses thin triangles. For lower Ricci and volume bounds, see below.

There is also the recent "pinching" result of (Grove & Wilhelm, 1995) which yields manifolds diffeomorphic to the standard sphere. It uses in the hypothesis a pure metric invariant called $\mathbf{pack}_{d+1}(\boldsymbol{g})$. For a Riemannian manifold (M^d, g) it is defined as one half of the maximum of the minimum non-zero mutual distances between $d+1$ points: the maximum is to be taken over all such sets of points in M. When one considers only two points instead of $d+1$ one has the definition of (half) the diameter. The result is then the following: if the sectional curvature is such that $K \geq 1$ and $pack_{d+1} > \pi/4$ then we have diffeomorphism with the standard sphere. The proof uses Alexandrov geometry (see TOP. 9) and is also a method "from outside" just like Ruh's method, except that it just uses the metric to get a suitable embedding. A very informative survey of these kinds of results (and of some others below involving the *radius*) is (Grove, 1992). Another type of result was that of (Otsu, Shiohama, & Yamaguchi, 1989): if $K \geq 1$ and the volume of the manifold is close enough to that of the standard sphere, then one has diffeomorphism. But now one can obtain this using only the Ricci curvature, see below.

✦

It is now natural to ask what happens at exactly 1/4 or a little below 1/4. There is a survey: (Abresch & Meyer, 1997). We will only give the essential points here. First, for exactly 1/4 , if we are not on a sphere, then we are necessarily on a $\mathbf{K}P^n$ which is moreover endowed with the standard metric: (Berger, 1960). This kind of theorem is called a *rigidity* result. One can think of it as a susceptibility result: the slightest touch on the standard metric of $\mathbf{K}P^n$ (with $\boldsymbol{K}$ not $\boldsymbol{R}$) will force the curvature range to be strictly outside the interval $[1/4, 1]$.

But today one can go a little bit below $1/4$. There is an $\varepsilon(d) > 0$ such that a $(1/4 - \varepsilon)-$pinching implies that one is on a sphere or on a $\mathbf{K}P^n$: (Berger, 1983). The proof was not only incomplete, see the survey, but moreover the ε was not given explicitly since it was obtained by the compactness result in C.2 and a contradiction. In (Abresch & Meyer, 1994) and (Abresch & Meyer, 1996) one has explicit pinching constants $\varepsilon(d)$. But their work yields the optimal topological answer only in odd dimensions. For even dimensions one only gets a piece of strong topology information which yields "almost only" the $\mathbf{K}P^n$, see the references for more details. The proof is very geometric, using Toponogov control of triangles on the one hand and Morse theory and parallel transport on the other to control the injectivity radius. The idea of the proof is as follows. If the injectivity radius were too small some very small periodic geodesic would exist. The simple connectedness enables us to deform it through a point by a "long critical homotopy". The homotopy has to be long because the mountain pass point is a conjugate point and of index exactly 1. But the curvature conditions finally yield a contradiction. The best $\varepsilon(d)$ is unknown, the present one is of the order 10^{-5} but in view of B.1 any guess is useless.

We have to stop here with the pinching constant. In fact, to go further below $1/4$ would mean merely looking for a classification of all manifolds (since any manifold has some Riemannian structure) or at least for manifolds of positive curvature: see B.1 and C.

✦

But the game is far from finished as one would like to get results using weaker hypotheses. An important event was (Grove & Shiohama, 1977). Their result was that one still gets a sphere when one only requires the curvature to be bounded below by 1 and the diameter to be larger than $\pi/2$: the upper bound on the curvature disappears. This is perhaps not too surprising, as we saw in Colding's comparison theorem in the digression above, because of results of Myers' type. The technique was new; it still made use of the Toponogov theorem, but the main point was being able to go *beyond* the injectivity radius. The trick is to use the distance function $d(p, .)$ from a point p. One says that a point q is *critical* for $d(p, .)$ if for any direction v at q there exists a segment (a shortest geodesic) from p to q whose speed vector w at q forms an angle with v that is not larger than $\pi/2$. Heuristically this means that one can get closer to p in any direction. Two extreme examples of critical pairs are the following: first when the distance between p and q is equal to the diameter. This was exactly the idea of the proof of the sphere theorem above. The second case is when q is the antipodal point of p on a geodesic loop with origin p.

If q is not critical then, on the contrary, one can find some direction in which $d(p,.)$ is strictly increasing. Then, as above, one succeeds in covering the manifold with two topological balls: like in Morse theory one can "push things without changing the topology" as long as one does not meet a critical point: within the contractibility radius, like for the injectivity radius, balls are *contractible.*

The study of what happens for the limiting case when the diameter is equal to $\pi/2$ was far advanced in (Gromoll & Grove, 1987) but is still not finished, more details are to be found in (Wilhelm, 1996). The point is to look at points in the cut-locus (see TOP. 4) and the fibration this determines on the unit sphere; then one has to show, if possible, that those fibrations are in fact isometric to the Hopf standard fibrations.

The *critical point* technique is now a basic tool in many instances. It was put into quite a general framework in (Greene & Shiohama, 1981a) and (Greene & Shiohama, 1981b). Surveys of this technique and its applications are (Cheeger, 1991), (Grove, 1985), (Meyer, 1989), (Grove, 1990), (Greene, 1990), and (Karcher, 1989); see also various places below.

✦

In fact the Grove-Shiohama result is of a flavor very similar to Myers' theorem ((Myers, 1941), see 0.C), which says the following. Assume that the Ricci curvature is bounded below by $d-1$. Then the diameter is bounded from above by π (think of the standard sphere) and one can look for a Myers-type diameter pinching result. The first thing to do is to look whether the equality characterizes the standard sphere. This can be done in many ways, the first appearance is (Cheng, 1975) but it is trivial with the Gromov-Bishop theorem, see TOP. 1.A. In (Anderson, 1990a) it was shown that a pinching result *cannot* exist, see also (Otsu, 1991). This was achieved by constructing suitable examples with $Ricci \geq d-1$ and a diameter closer and closer to π, with tools from Riemannian submersions. On the other hand, with an additional positive lower sectional curvature bound, one can arrive at the desired conclusion: (Perelman, 1995b).

But if the diameter is replaced by the *volume* then there is first of all the result of (Shiohama, 1983): there exists an $\varepsilon(d)$ such that Ricci $\geq d-1$ and volume $> \beta(d) - \varepsilon(d)$ imply homeomorphism with the sphere, but with an additional lower (negative) bound on the sectional curvature (of course here $\beta(d)$ denotes the volume of the standard sphere S^d). In (Perelman, 1994a) homeomorphism with the sphere is obtained without any additional condition. The proof is very geometrical. In brief it combines techniques of critical point theory, the contractibility of balls into concentric ones of larger radius and playing smartly with algebraic topology (see B.1 and also the notion of contractibility functions at the end of TOP. 5). Gromov's view of it is that a "non-trivial homology generates large *shadows*" and hence restricts the volume too strongly. Needless to say that the injectivity radius is ignored. By using the full force of $K \geq 1$ and volume $> \beta(d) - \varepsilon(d)$ the same conclusion was proved in (Otsu, Shiohama & Yamaguchi, 1989), in this case with an explicit ε. The technique is to build up an embedding into $\mathbf{R}^{d+1}$, it is an "outer" method. But all these results have now been superseded by (Cheeger & Colding, 1997a). In its Appendix 1, diffeomorphism with the standard sphere is proved under a lower Ricci bound and a volume close to the canonical one (moreover the volume bound can be computed explicitly). This is only the tip of the iceberg, whose total content will be revealed in the future. We will see below some other results in that vein. We advise the reader to read the introduction and Appendix 2 of (Cheeger & Colding, 1997a) for a survey of ideas, results and a program concerning Ricci curvature. Among other tools Colding's L^2-Toponogov type theorem mentioned in the digression above is fundamental. The fact that "thin" triangles are useful, as they were first of all for the splitting theorem which appears in B.2 below, can be explained heuristically if one sees sphere theorems as "splitting spheres into two balls". A systematic tool introduced by the authors for the general study of manifolds with a lower bound on the Ricci curvature is that of suitable warped products (0.G).

A recent result on pinching with "distances" is (Colding, 1996a). There the diameter is replaced by a very close but still stronger metric invariant, the *radius.* The radius looks very similar to the diameter but is surprisingly more powerful. It is the minimum of the radius of a metric ball which contains (and is hence equal to) the whole manifold. Think of a very thin ellipsoid: then the radius is close to half the diameter. It is proved in (Grove & Petersen, 1993) that $K \geq 1$ and radius $> \pi/2$ imply homeomorphy to a sphere. Colding's result is that Ricci $\geq d - 1$ and a radius close enough to π imply homeomorphism with the sphere (how close enough it must be can be stated explicitly). The idea of the proof is to show that the Gromov-Hausdorff distance (see C.2) between the manifold and the standard sphere can be made arbitrarily small. This is done by squeezing suitable balls into the manifold and defining with them a map to the sphere. Then one uses Colding's predictability result (see the digression in A.2) as applied to the unit tangent bundles of these balls. Repeatedly using Bishop-Gromov estimates for the volume of balls (TOP. 1.A), volume arguments complete the proof.

One should mention also the *pointwise pinching* problem: the curvature is required to be squeezed in a given ratio at every point, but the global sup and inf need not be squeezed. Due to the lack of space we just refer to the almost final set of results of (Margerin, 1991, 1993, 1994) for detailed statements and the very subtle control of the deformation technique. A nice survey of the *deformation technique* along the Ricci flow (see 0.F) is Section 4 of (Besse, 1994). See the solution of the (1/4)-case in B.1. For an exposition of some of these results in book form see (Hebey, 1997).

✦

But one would like hypotheses that are weaker again. The drive started with sectional curvature, then one tried to replace it by Ricci curvature and now finally by *integral norms* for various curvatures. We give only a few references: (Gallot 1987), (Anderson and Cheeger 1991), (Gao 1990a), (Gao 1990b), (Petersen & Wei, 1996b), (Anderson, 1990c) and the book (Petersen, 1998a). Note that the work to do here is more Analysis than geometry. See also C.1.

3. Pinching around zero

The space forms with zero curvature are flat tori or finite quotients of them (see II). Can one prove the following: there exists some $\varepsilon > 0$ such that if a manifold has a diameter bounded by 1 and a curvature bounded on both sides by ε and $-\varepsilon$, then this manifold admits a flat structure? Contrary to the positive case the answer in the present case is *no*. There are other manifolds which are known to be "almost flat", namely the so-called "nilmanifolds". They can be defined as compact quotients of nilpotent Lie groups. Topologically they are nothing but successive fibrations, whose fibers are always a circle and end in a point. They admit, for any $\varepsilon > 0$, a metric with a curvature between $-\varepsilon$ and ε and a diameter equal to 1. Heuristically, this almost flatness is possible because a circle (a curve) has no curvature. Then one just applies the formulas for Riemannian submersions (0.G), see also C.3.

The basic paper (Gromov, 1978a) proved the opposite: the solution of the pinching problem is true, but it is not flat manifolds but nilpotent ones (in fact precisely the so-called *infranilmanifolds*, namely the finite quotients of nilmanifolds) that are the answer. The complete result appeared in (Ruh, 1982a). It says that there exists some universal $\varepsilon(d) > 0$ such that this is the conclusion for manifolds with $-1 < K < 1$ and diameter $< \varepsilon(d)$. A detailed proof of Gromov's result is given in (Buser & Karcher, 1981). Techniques are purely geometric but a complete proof is quite long and involved. Two good sketches of it are (Sakai, 1996), pages 319 up, and (Fukaya, 1990), §8 and §9. Using Toponogov triangle control, one studies in detail the fundamental group made up of isometries of the universal covering. The curvature being extremely small, the exponential map is a covering at very large distances and moreover not much different from an isometry. Then the elements of the fundamental group almost commute, as can be seen by controlling also the parallel transport (remember the golden triangle: the curvature is related to the parallel transport around infinitesimal parallelograms). Using the control on the commutators of the fundamental group one can finally prove that the group has to be nilpotent. Except for the fact that the best $\varepsilon(d)$ has yet to be found, this result is then optimal.

4. Negative Pinching

Here the answer is an even stronger *no* than for the zero case. First in (Gromov & Thurston, 1987), then in (Farrell & Jones, 1989a), one finds many manifolds with a curvature arbitrarily close to -1 which can never have a metric of constant curvature. One cannot hope either for a general result around 1/4 for the negative $Hyp\,\mathbf{K}P^n$, as shown in (Farrell & Jones, 1994). The techniques used are very geometric. One uses smart gluings along totally geodesic hypersurfaces in one instance. In the other, one forms connected sums with exotic spheres and then controls the curvature with the formulas for the curvature in Riemannian submersions. With the first type of construction the resulting topology can be quite sophisticated, but with the second technique one gets only manifolds homeomorphic to space forms. Note that, from Section C.1 onwards, larger and larger diameters are enforced. For real analytic constructions, which are a harder game, see (Abresch & Schroeder, 1996).

There is however the following optimal result. Assume that M is such that $-4 \leq K \leq -1$ and that $\pi_1(M)$ is group-isomorphic to the π_1 of a compact space form of negative curvature which is moreover not of constant curvature (i.e. not real hyperbolic but of rank 1). Then M is isometric to that space form. This appeared in (Ville, 1985) for dimension 4, for the complex case independently in (Hernandez, 1991) and (Yau & Zheng, 1991). For the quaternionic and the Cayley case this follows from (Hernandez, 1991), (Corlette, 1992) and (Gromov, 1991c). Compare this rigidity result with the one in A.2.

✦

All of the above concerned various hypotheses to get space forms (especially ones of constant curvature) under various pinching conditions. These conditions almost always first of all, at least on one side, involved sectional curvature and the aim was to get space forms. They always started with $|K - \sigma| < \varepsilon$. Let us now suppose we want to do the same for the Ricci curvature. Then the first thing to do is to discover *which* are the manifolds with constant Ricci curvature. This will be the topic of the entire Section III. C. Such manifolds are called *Einstein* manifolds. Roughly speaking the answer, starting with dimension 5, is almost completely open (the reader will or will not agree with me after reading that section). But it is still natural to ask if a strong enough Ricci pinching on a given manifold implies for this manifold the existence of some Einstein metric; moreover such a kind of result could be a clue in the search for these wild animals.

There is a very good answer in (Anderson, 1990b): there exists a universal ε depending only on d, i, D and σ such that a d-manifold with a diameter smaller than D, injectivity radius larger than i and Ricci pinching $|\text{Ricci} - \sigma| < \varepsilon$ always admits an Einstein metric. The ε is not given explicitly since it is obtained by a contradiction proof using general compactness results (see C.2).

B. Curvature of a given sign

We are going to see again how much the picture differs for the positive and negative sign respectively.

1. The positive side: Sectional Curvature

Relevant surveys are (Gromoll, 1990), (Abresch & Meyer, 1997), Appendix A, and (Greene, 1997). Concerning this topic, we are still in an almost completely mysterious situation. We will now describe the only known examples of (compact) manifolds with *positive* sectional curvature: because of Myers' theorem, except for a finite (although not trivial) classification problem, we can essentially afford to look only at the simply connected case. First come the spheres, the $\mathbf{K}P^n$ and a few other homogeneous spaces of dimensions 6, 7 and 13 respectively.

Among them the *Aloff-Wallach* manifolds $W_{p,q}$ are especially interesting: (Aloff & Wallach, 1975). They are suitably defined homogeneous metrics on the quotient of the Lie group $SU(3)$ by the "(p, q)- circle": this means that the direction of the circle is given by the coordinate point (p, q) in the integral lattice defining a maximal torus of $SU(3)$. Recall that all maximal tori of $SU(3)$ are of dimension 2 and conjugate and note that this "universal lattice" is the regular hexagonal one. Within the desert-like area of manifolds of positive curvature, Aloff-Wallach manifolds are in fact a fascinating family. We mention *three* of their properties. First, there are infinitely many homotopy types of Aloff-Wallach manifolds $W_{p,q}$, depending on the couples (p, q) (and there are even some pairs which are homotopy equivalent but not diffeomorphic). Secondly, they carry for every (p, q) a metric with a positive lower pinching not much different from 16/29.37: see (Huang, 1981). Thirdly, they also admit homogeneous Einstein metrics and if the volume is normalized to 1 the set of Einstein constants (which are all positive) is infinite: see (Wang, 1982) and (Wang & Ziller, 1986). Note first of all that these Einstein metrics are not of Aloff-Wallach type; in particular they have sectional curvature of both signs. Another of their properties is that the set of those positive Einstein constants, when the volume is normalized to one, does not converge to a smooth metric: this shows that the Palais-Smale "C-condition" (see (Jost, 1995)) is not valid for Riemannian metrics, at least not in the ways known to be possible today: see III.C for this type of question.

There is a complete classification of positive curvature homogeneous spaces: there are no other homogenous spaces of positive curvature apart from the above ones: (Bérard Bergery 1976). It was discovered in (Wilking, 1997) that, even though this result is true, there was some curious mistake in the proof, finally compensated by the fact that some space missing in the list of (Berger, 1961) turns out to be isometric to one of the Aloff-Wallach spaces.

✦

Only four other types of examples (not homogeneous) are known. They are in dimensions 6 and 7 ((Eschenburg, 1992), (Taimanov, 1997)) and in dimension 13: (Bazaikin, 1996). The latter ones belong to a family closely related to the Aloff-Wallach examples. Pinching constants for these various spaces have funny coincidental properties which are explained in (Taimanov, 1997). Taimanov's study is pursued in (Püttmann, 1997) where the pinching constant 1/37 appears and is explained via Taimanov's deformations of Aloff-Wallach metrics. The starting point is to embed Aloff-Wallach manifolds as totally geodesic submanifolds in the 13-dimensional examples. Moreover there is enough "room" (transversality) to deform the metric quite a lot and still keep the totally geodesic property. One even gets a series of metrics which converge to a smooth one with a pinching constant equal to 1/37. This supported the conjecture that the best pinching constant for Aloff-Wallach manifolds is 1/37 and a complete proof of this is given in (Wilking, 1997).

No other manifolds of positive curvature are known, except of course small enough deformations of the preceding ones.

If we turn to manifolds of nonnegative curvature, the situation is not really any better. As examples we have all Riemannian homogeneous compact G/H's when the metric downstairs on G/H is induced by a biinvariant metric on G, in particular symmetric spaces of "positive" type (see II). Note that these do not make up a long list, since one knows how to classify all Lie subgroups of compact Lie groups. One has also the Riemannian products of the above known manifolds of positive and of nonnegative curvature. Besides those, to our knowledge the only examples are some exotic spheres and the connected sum of two symmetric spaces of rank one: see (Cheeger, 1973) and (Gromoll & Meyer, 1974). Those examples are constructed by a clever mixture of large actions (low cohomogeneity) of isometry groups and the Riemannian submersion technique to compute the curvature.

✦

But one knows at least that not every compact, simply connected manifold can carry a metric of nonnegative curvature (a fortiori of positive curvature). First came the basic theorem of (Gromov, 1981c): manifolds of nonnegative curvature have a sum of Betti numbers (for every field the way the proof goes) that is universally bounded by the dimension. Hence we are left with only a finite number of (integral) *homology types* (in a weak sense). The tool is somewhat, but not in reality, analogous to Morse theory. It requires subtle algebraic topology arguments, based on the compressibility of balls and the topological "content" of them. The basic lemma is that, by Toponogov's theorem for triangles, a succession of critical points for the distance function (see A.2) to a given point whose distances grow at least at the rate of a geometric progression, has to be *finite* (the cardinality being universal in the dimension). In short "critical points cannot be too far away". To prove this the idea is as follows: the initial speed vectors from the point p under consideration to two different critical points, when their distances are a lot different, have to form an angle that is large enough. This means putting onto the unit sphere at p a large number of points whose spherical mutual distances are all larger than a given number: the cardinality of such sets must be bounded from above. A rough (but not too bad by the way) estimate can be obtained just by computing a measure of spherical caps. For refined estimates, which form a central problem in pure and applied mathematics, namely that of spherical codes, see (Conway, 1995) or the bible (Conway & Sloane, 1993).

Then one has to play with distance functions to different points. The reader can look at the various presentations, involving also simplifications of the original text: (Cheeger, 1991), (Meyer, 1989) and the text (Grove, 1990) which explains in very short time what is happening. Today the bound for the Betti numbers is huge, the conjecture is that their sum is bounded by 2^d with equality only for tori (then necessarily flat, see below at least two reasons for that).

Gromov's result was extended in (Abresch, 1985) to the non-compact complete case. Playing with concentric balls the smaller of which is contractible in the larger one is an essential tool today, for example in the Perelman text quoted above and for the finiteness results in C.1, see the end of TOP. 5.

Do not hope now to extend these results to the Ricci curvature. Except for the first one (Bochner's result) there are examples of nonnegative Ricci curvature manifolds with arbitrarily large Betti numbers: this was achieved simultaneously in (Anderson, 1990c) and (Sha & Yang, 1991). In (Sha & Yang, 1991) one constructs connected sums, using delicate metric surgery on double warped products. An extra difficulty here is that one needs to spread the modifications all over the manifolds during the gluing operation. In (Anderson, 1990c) one uses models from (Gibbon & Hawking, 1978). This construction was put into the more general framework of the Schwarzschild metric in (Anderson, 1992b). More recently Perelman (Perelman, 1997) constructed examples showing that the Betti numbers need not be bounded when the Ricci curvature is bounded from below even if both the volume is bounded from below and the diameter is bounded from above (contrast this with the finiteness theorems in Section C.1), see also (Wei, 1998).

✦

The above still does not tell us the least thing about the following question: is there any difference between positive and nonnegative sectional curvature? A baffling remark in (Yau, 1982), page 670, is that one does not know of any (compact) simply connected manifold with nonnegative curvature for which one can prove that it *does not* admit a metric of positive curvature. For example Gromov's bound on the Betti number does not make any difference between positive and nonnegative curvature. Yau starts with Hopf's conjecture on $S^2 \times S^2$, see the last paragraph of B.5. For the non-simply connected case, Rong's results below provide a partial answer.

✦

Positive and nonnegative sectional curvature is a case where *non-compact* manifolds enjoy optimal results, so that we make an exception to our rule. Moreover this is fundamental when studying compact manifolds with an infinite fundamental group. In (Gromoll & Meyer, 1969b) it is proved that a complete non-compact manifold of positive curvature is diffeomorphic to $\mathbf{R}^d$. The basic geometric notion in non-compact manifolds is that of a *ray* (half-line): it is a geodesic defined on $[0, \infty[$ with a segment on every subinterval. There are in some sense at least as many rays as "ends" of the manifold. The basic fact is that any ray has associated with it a strongly convex subset, namely the set of points which are at a negative distance (suitably normalized) from the point at infinity on that ray. We will come back to these objects called *Busemann functions* in Section 2 below. They were defined and used first in (Busemann, 1955), but they were introduced and systematically studied in the Riemannian case in (Greene & Shiohama, 1981a) and (Greene & Shiohama, 1981b). In (Gromoll & Meyer, 1969b) the technique introduced enables one to prove that the total group of isometries is still compact (this is a classical and quite trivial result in the compact case, an isometry being determined by its effect on a single orthonormal frame); but this is no longer true for the positive Ricci case: (Wei, 1988).

In (Cheeger & Gromoll, 1972) it is proved, with a more subtle use of convex functions, that a complete manifold of nonnegative curvature is a vector bundle over a compact totally geodesic submanifold of positive curvature. Such a submanifold is called a *soul*, it need not be unique (think of a flat cylinder). This theorem was recently made more precise in (Perelman, 1994b), consult this text for its references to intermediate results. Recent expositions in book form are found in (Petersen, 1998a) and (Sakai, 1996).

✦

Another kind of positivity is now in order. Topological spheres can be characterized (up to homeomorphism today for $d \geq 5$) by a suitable positive curvature condition, namely that of the *curvature operator*: (Micallef & Moore, 1988). The curvature operator is the symmetric endomorphism of the 2-exterior tangent bundle defined by the curvature tensor. The proof uses strong geometric analysis, namely the existence of minimal harmonic maps of S^2 into our manifold, see TOP. 7. Then one analyses the second variation. This will finally yield the fact that the homotopy groups of the manifold all vanish in dimensions smaller than that of the manifold, and we will now use the solution of Poincaré's conjecture for $d \geq 5$. Using Bochner's technique for exterior forms, (Meyer, 1971) yielded only real homology spheres. For $d = 4$ one even has a characterization of spheres up to diffeomorphism: (Hamilton, 1986). The tool this time is the deformation of Riemannian metrics, see the comments in 0.F.

An interesting byproduct of Micallef and Moore is the solution of the "pointwise" (1/4)-pinching problem: we assume that at every point m one has, for all the sectional curvature *at m*: $\sup K/\inf K < 4$ (and $\inf K > 0$). Then the manifold is homeomorphic to the sphere. Note that such a result is completely out of the reach of *RCT* and/or Toponogov type results. In (Ruh, 1982b) this result was obtained with non-explicit pinching, using deep Analysis and the results of (Min-Oo & Ruh, 1979) mentioned at the very begining of A.1.

But we are left now with three questions: how about diffeomorphism in every dimension, the equivariance (the non-simply connected case) and the case of just nonnegativity? None of these questions has been completely solved up to now. It seems for the moment that the right curvature condition, especially if one wants to take care of the nonnegative case, is a more subtle question than that of the curvature operator. This is the contribution of the book-sized (Margerin, 1991, 1993, 1994) and we refer to Margerin's various references for a precise definition of which curvature deviations from constant sectional curvature should be considered. Margerin's contributions started with (Margerin, 1984a), (Margerin, 1984b), the most recent one is (Margerin, 1994). The equivariance is automatic since the technique is the Ricci flow and the results are, in some sense, weaker and stronger than the positivity of the curvature operator.

The question of going right to the zero level has been only partially solved to this day, the hope "of course" is to be able to prove that only products of spheres and $\mathbf{K}P^n$ can appear. This is almost known, due to various authors, see (Chen, 1990) and (Petersen, 1998a) for references. The answer is: There appear only, except spheres, symmetric spaces and manifolds biholomorphic to $\mathbf{C}P^n$.

✦

Back to a "classification" of the manifolds of positive curvature, we are still lacking any general structure statement. We quote here *four* different approaches to attacking the problem. The *first* and the most promising is that of (Rong, 1996a). Thanks to the finiteness theorem in C.1, in a given *even* dimension, there are only finitely many diffeomorphism types of manifolds with $0 < \delta \leq 1$. If the dimension is odd, we no longer have Klingenberg's lower bound π for the injectivity radius. The first thing to do is to look at the odd-dimensional case, always here for a given dimension and positive pinching δ. One of the results in (Rong, 1996a) is that, modulo a finite number of cases and in the simply connected case, one can modify the Riemannian metric, creating a new one which is still of positive curvature, but which now admits a circle acting freely and by isometries. Since the quotient is also of positive curvature by the Riemannian submersion formulas, one has then reduced the classification to that of the even-dimensional case and the classification of some circle bundles over them (always modulo a finite number). For this kind of result see also (Tuschmann, 1997). The initial idea is to look at the conditions in Cheeger's finiteness theorem (C.1) and to assume that things do not work, namely that we have a very small volume. But then we ought to be in the collapsing case (see C.3). Thereafter one applies the structure results for the collapsing situation and the main point now is to change the metric to a new one which satisfies the condition above. This is achieved in particular using the Ricci flow (see 0.F), but note that the pinching is now smaller than δ.

For the non-simply connected case, recall first that by Synge's theorem in 0.C, this is again an odd-dimensional story. The fundamental group π_1 is finite by Myers (0.C) and in (Gromov, 1978a) one finds an upper bound for its number of generators which depends only on the dimension, by using a smart basis for π_1 as acting on the universal covering and Toponogov's theorem. But we would also like an upper bound for the possible group structures as a function of the pinching. An old conjecture of Chern's was that every abelian subgroup of the π_1 of a manifold of positive sectional curvature is cyclic. In (Rong, 1996a), by the very technique used for the proof above, it is proved that this is true, but only up to an index which is bounded by a constant $w(d, \delta)$, which is universal in the dimension and the pinching δ. Rong's conjecture today is that $w(d, \delta)$ can be chosen independently of δ. With a much deeper analysis of the collapsing case, (Rong, 1996b) gives a partial result in that direction, namely that there are $w(d)$ and $w'(d, \delta)$ such that π_1 either has a finite cyclic subgroup of index less than $w(d)$ or is of an order less than $w'(d, \delta)$. In (Rong, 1997a) things are more refined. Finally Chern's conjecture is disproved in (Shankar, 1998), using the $W_{1,1}$-manifold seen at the beginning of this section.

Of course, always looking for a complete classification, the basic question is: look at all manifolds M^d that are of a given dimension and admit a positive curvature metric and call $\delta(M)$ their best pinching constant (see below if in any doubt). Define $\delta(d)$ as the infimum of all these $\delta(M^d)$: is $\delta(d)$ positive?

In another direction an interesting result was conjectured in (Gromov, 1988): nonnegative curvature manifolds "look like convex bodies, like ellipsoids". The precise notion is that of various *widths* for a Riemannian manifold: these are purely metric invariants. This conjecture was proved in (Perelman, 1995a).

✦

We will now give some details of the third approach to nonnegative curvature, which we will meet again in V.A and V.B. This approach is concerned with *rational homotopy theory*, see (Grove & Halperin, 1983) and TOP. 6.A for the context. In (Felix & Halperin, 1982) one considers the rational homology $H_*(M; \mathbf{Q})$ of manifolds and the following very strong dichotomy is proved: either the Betti numbers of the loop space Ω of M grow exponentially (heavy use of this will be made in V.A) or the sum of the rational Betti numbers of M^d is bounded by 2^d. By definition, the first class is that of rationally hyperbolic manifolds and the second class is that of rationally elliptic ones, whose definition is very simple: the homotopy groups $\pi_p(M^d)$ are required to be finite for every $p > 2d - 1$. A conjecture of Bott's is that manifolds of nonnegative sectional curvature are rationally elliptic. Please compare this with the 2^d-conjecture derived from Gromov's theorem on Betti numbers (this time over any field). The complete classification of elliptic manifolds is still not finished. For more on this approach, leading to the "double soul" problem, see (Petersen, 1996), Problem 21.

✦

A last approach would be to look at the *best pinched* metric on a given manifold admitting a metric of positive curvature. There is an optimal pinching δ (of course $< 1/4$ to be of interest). At least in even dimensions, the hypothesis $\delta_i \leq K \leq 1$ for a sequence of δ_i converging toward the upper bound δ satisfies the hypothesis of convergence in C.2: the diameter is bounded above by Myers' theorem and the volume from below because of Klingenberg's result on the injectivity radius. So we have an optimal metric, presently only belonging to the class $C^{1,\alpha}$. It seems there is no text that studies this approach, the main difficulty being that it seems very hard, at least today, to use the pinching condition. More precisely when one looks for extrema one can always first compute the first differential. This is possible for differentiable objects, and the sectional curvature, for a given tangent plane, is of course differentiable for families of metrics (see the formulas and more in (Bourguignon, 1973)). But the curvature is such a weird animal that the points where the sectional curvature is extremal have still not been understood even in dimension 4 (unless the manifold is Einstein: (Singer & Thorpe, 1969)), recall the digression in 1. COMMENTS on I.

✦

The last question concerns *moduli*: assume a manifold admits at least one metric of positive curvature. What does the set of all such metrics look like? To this day, we know only that it may be non-connected, an example being the two Aloff-Wallach manifolds $W_{-4638661,582656}$ and $W_{-2594149,5052965}$, which are diffeomorphic but cannot be connected by a path of positive scalar curvature: see (Kreck & Stolz, 1993) and Section B.3 below.

2. The positive side: Ricci Curvature

For the Ricci curvature (in the compact case), it is *still* a mystery today what exactly the implications of its positivity are, the only exceptions are the finiteness of π_1, the special situation in dimension 3, see below, and of course the necessary conditions for positive scalar curvature. In fact no other condition is known – can for example any product manifold $S^2 \times N$ carry a positive Ricci curvature metric ? There is only a conjecture for a necessary condition in the simply connected case in (Stolz, 1996). This conjecture is supported by the following heuristic principle: in the same way as positive scalar curvature (see Section B.4 below) forbids harmonic spinors and then implies the vanishing of a topological invariant, the $\hat{A}$-genus, one can interpret positive Ricci curvature as forbidding "harmonic spinors on the loop space of the manifold for a suitable operator" and then some topological invariant would again have to vanish. If one excludes this conjecture, no other restriction is "known" today except for those induced by positive scalar curvature. A very informative text is Section 5 of (Gromov, 1991b): positive Ricci curvature has a strong implication, namely that the index of geodesics (see the digression in V.A) grows at least linearly with the length, so that one could hope to get information on the Betti numbers of the manifold via Morse theory. But precisely this hope is dashed by Gromov's theorem mentioned in V.A. In contrast, this is used in (Wilhelm, 1997) to obtain simple connectedness with the help of a systole condition (see TOP. 1.E).

But for the fundamental group one has very strong and optimal recent results for nonnegativity and just below zero. We saw already in 0.D an application of Bochner's technique. Precisely if the Ricci curvature is nonnegative the first real Betti number b_1 is bounded by the dimension, with equality holding only for flat tori. A homotopy result is that of (Milnor, 1968), to the effect that the fundamental group is of at most polynomial growth when the Ricci curvature is nonnegative: this is obtained from an argument counting balls by volume in the universal covering, using Bishop's result in TOP. 1.A to control the volume of these balls. Which groups are really possible is far beyond our knowledge today. We know only from (Gromov, 1981b) that the groups we are looking for are discrete subgroups of nilpotent ones. For the complete case, see (Wilking, 1998).

The challenge now is to relax the hypothesis and to go a little bit below zero. For what follows, there is the informative text (Gallot, 1998). There is a heuristically motivated hope for that case. Recall now Bochner's result in 0.C: since Ricci ≥ 0 implies $b_1 \leq d$, it is reasonable to expect that b_1, being an integer, will remain $\leq d$ when the Ricci curvature is not too strongly negative (with some normalization). And moreover equality should be expected only for flat tori. Partial results appeared in (Gromov, Lafontaine & Pansu, 1981a) and (Gallot, 1981), see 5.21 of (Gromov, 1998). Using the basic Colding comparison theorem (see the digression in A.2) one has today two very strong results in (Colding, 1996b) and (Cheeger & Colding, 1996). First there exists some $\varepsilon_1(d) > 0$ such that Ricci.Diameter$^2 > -\varepsilon_1(\mathrm{d})$ and $b_1 = d$ imply that the manifold is diffeomorphic to a torus ($d \neq 3$). Secondly there exists an $\varepsilon_2(d) > 0$ such that Ricci.Diameter$^2 > -\varepsilon_2(d)$ implies that the fundamental group is almost nilpotent. For the moment both ε_1 and ε_2 are not given explicitly, since they were obtained by a limit-contradiction argument. An application of the Cheeger-Colding result is to be found in (Paun, 1997).

The latter ε_2-result is incredibly much stronger than the "old" pinching-around-zero result in A.3. In fact this result was conjectured by Gromov and proved in the case of a lower bounded sectional curvature in (Fukaya & Yamaguchi, 1992). The proof involves a thorough study of collapsing under only a lower bound, which is done in (Yamaguchi, 1991), and the authors mentioned there that their technique could be extended to the Ricci case, modulo some reasonable "splitting" conjectures. The proof of the Ricci case is based on the framework of the proof of the sectional case, but also uses some new techniques arising from a thorough study of the geometry of non-compact Riemannian manifolds with a lower bounded Ricci curvature, in particular an extension to Gromov-Hausdorff limits of the splitting theorem of Cheeger and Gromoll. The initial idea is to study the *structure at infinity* of the manifold under consideration, this is explained in TOP. 5.

The ε_1-result is proved as follows. One starts with Gromov's proof that $b_1 \leq d$: on the universal covering of a suitable finite covering of diameter D, one constructs a basis of the deck transformations such that the generators do not displace the points too much, but every element displaces them enough (in order to avoid triangles being squeezed too much). Independently, on suitable balls, one constructs d harmonic functions which have L^2-small Hessians and have L^2-almost orthogonal gradients. One combines this with predictability arguments and a smart basis as above to obtain, when inf(Ricci.Diameter^2) tends to zero, a Gromov-Hausdorff limit which is a torus.

Last but not least, in dimension 3 at least, the situation is completely understood. In (Hamilton, 1982) it is proved, by the deformation technique along the Ricci flow (see 0.F), that any positive Ricci metric can be deformed into one of constant sectional curvature. Note that in dimension 3 the Ricci and sectional curvature have the same number of parameters, namely 6. In (Sha & Yang, 1993) it is proved that a simply connected manifold of dimension 4 admits a metric of positive Ricci curvature if it is homeomorphic to one which admits a metric of positive scalar curvature. Moreover the technique completely covers the nonnegative case: the possible manifolds are diffeomorphic to S^3, $S^2 \times \mathbf{R}$, or $\mathbf{R}^3$, or one of their quotients by a group of fixed-point-free isometries of the standard metric.

The nonnegative case versus the positive one is a little mysterious: there is no known manifold which has a metric with positive scalar curvature (see just below), which is simply connected and for which one can prove that there is *not* a single metric with positive Ricci curvature. This will seem less surprising if you look here at a fact from III.C: Ricci flat manifolds are very mysterious, one only knows *some* examples. Still (see C.3) they deserve to be called the *harmonic* Riemannian manifolds.

✦

Manifolds of positive and nonnegative Ricci curvature show a very nice and useful form of behavior when non-compact (it was heavily used already in the results above for the case of a lower bound on the Ricci curvature), so that we will here once again make an exception to our rule. Here the basic notion is no longer that of a ray but of a *line*, namely a geodesic defined on $]-\infty, \infty[$, which is a segment on any of its intervals. Assume there exists a line in a manifold with Ricci ≥ 0. Define along this line two distance functions (one from $+\infty$ and the other from $-\infty$, normalized of course: they are called *Busemann* functions). Then use the basic formula for the Hessian of a distance function (see 0.F); the Ricci condition finally implies that these two functions coincide and that their gradient is a parallel vector field. This forces the manifold to split as a Riemannian product with **R**. Looking now at the universal covering of a compact manifold with Ricci ≥ 0, one can prove the *rigidity* result: the universal covering of M^d *splits* as a Riemannian product of $\mathbf{R}^k \times M^{d-k}$ with some compact manifold M^{d-k}. So for example the fundamental group is extremely special as the situation is essentially reduced to the finite case. The above result is to be found in (Cheeger & Gromoll, 1971), see also (Eschenburg & Heintze, 1984). Before that a splitting was obtained in (Toponogov, 1964b), requiring more, namely the sectional curvature to be nonnegative. The proof consisted in looking at Toponogov's theorem "when one goes to infinity" along a line on both sides. Namely, one ultimately proves that equality has to hold in the comparison theorem and this forces all the transverse sectional curvatures to vanish.

A new technique for studying the complete nonnegative Ricci case was introduced in (Abresch & Gromoll, 1990). One proves that the topology is bounded under a diameter growth condition. The result can also be thought of as a weak quantitative generalization of the splitting theorem. The tool used was the first appearance of "thin" triangles. Colding's L^2-Toponogov theorem quoted in the digression in A.2 is a dramatic generalization. The strongest splitting result is the one for Gromov-Hausdorff limits, in (Cheeger & Colding, 1996), §6.

✦

Moduli for positive Ricci curvature are not well understood. A recent result is included in (Kreck & Stolz, 1993): some Aloff-Wallach manifolds have a set of positive Ricci curvature metrics whose number of connected components is infinite: in fact what is proved is that those metrics cannot be connected even within positive scalar curvature, see 3 below.

3. The positive side: Scalar Curvature

This subject forms one of the most beautiful chapters of recent Riemannian geometry (even though still not completely finished). Today we know exactly which simply connected manifolds are able to carry a metric of positive scalar curvature: the keystone is (Stolz, 1992), see the report (Rosenberg & Stolz, 1994) for the many intermediate results. Bring your simply connected compact manifold M to a topologist and he will first check for you whether that manifold admits a spinor structure (TOP. 6.B) or not, then, if the answer is yes, he will compute for you an invariant called the *α-genus.* Thereafter you, the geometer, will be able to find a metric of positive scalar curvature in two cases: first, always as long as the manifold is not spin, secondly, if it is spin, then if and only if the α-genus is zero. This complete classification was achieved through successive efforts of both geometers and topologists. Among interesting examples which admit no metric of positive scalar curvature are some exotic spheres.

The main geometric tool is that of (Gromov & Lawson, 1980a). It consists in proving, using Riemannian submersions, that any topological surgery of codimension smaller than 3 can be carried out in order to obtain positivity for the scalar curvature of the total manifold, provided that both manifolds coming under surgery themselves already have positive scalar curvature. We refer the reader to the survey (Rosenberg & Stolz, 1994). The first result, a necessary condition, goes back to (Lichnerowicz, 1963), see below for more. Another early partial result was (Lawson & Yau, 1974).

Another geometric tool was the use of *GMT* (Geometric measure theory, see section 0.F) to get stable minimal hypersurfaces: (Schoen & Yau, 1979). They proved that, in a manifold with scal ≥ 0, any cohomology class of codimension 1 can be realized by a hypersurface which admits some metric with scal ≥ 0. This metric is not the induced one but is obtained after a conformal change of it. This change with finally positive scalar curvature is made possible if one uses the second variation formula for stable minimal hypersurfaces. But this works only in dimensions smaller than 8, since this is needed in *GMT* to be sure to get submanifolds without any singularity. See a nice exposition of the ideas on pages 91–95 of (Gromov, 1991b).

Comments on the advantages of both tools figure on page 246 of (Rosenberg & Stolz, 1994) and in Section $5\frac{2}{3}$ of (Gromov, 1996). According to Gromov, a basic question is to find the geometric concept unifying both techniques, see more on this below.

✦

The above results do not complete the story. *First* the non-simply connected case is well advanced but still not finished and in fact it is related in part to the famous Novikov conjecture, one of the driving forces of recent topology, see (Gromov & Lawson, 1980b). See also (Rosenberg & Stolz, 1994), (Gromov, 1996) and the references there for more on this.

Secondly the topologist's description is not geometrical enough. Moreover, as we will see, the proof of the main result above is everything but geometry, except for the surgery control. On the contrary, the product of any manifold with the 2-sphere always carries a metric of positive scalar curvature (just shrink the 2-sphere far enough). Also minimal hypersurfaces of a positive scalar curved manifold carry a metric with the same property, at least "today" in low dimensions, this is the nice construction mentioned above. A geometric classification in that spirit is sketched in (Gromov, 1996).

So an interesting question today is to look for a purely geometric study of manifolds with positive scalar curvature. There is in fact a purely geometric statement in (Llarull, 1996), to the effect that a Riemannian metric g on S^d, such that $g \geq g_{\mathrm{can}}$ (this means everywhere, i.e. all the lengths for g are larger than or equal to those for g_{can}), necessarily has one point m such that $\mathrm{scal}(m) < d(d-1)$ (the value of the scalar curvature for g_{can} of S^d), unless g is the canonical metric. But today the proof uses all the techniques of spin geometry (see TOP. 6.B) of the proof sketched below. It would be great to have a geometric proof of Llarull's result, see also (Llarull, 1995).

Spinors do not really look like metric objects but it is interesting to know that in (Connes, 1994) the Riemannian metric $d(.,.)$ is computed using the Dirac operator. In (Connes, 1995–6) Riemannian geometry is put into a very general context.

Bishop's theorem on the growth of the volume of balls is wrong for scalar curvature, even locally. One has only an infinitesimal version of it, which is useless for global purposes. It would be basic, according to Gromov, to decide whether or not a C^0-limit preserves the nonnegativity of the scalar curvature, see III.D. Note that this is definitely wrong for the negative side by Lohkamp's result quoted in B.5 below. According to Gromov again, a starting point for proving this and understanding what is going on is the following: the nonnegativity of the sectional curvature is equivalent to the non-divergence of geodesics as compared to the Euclidean case. For scalar curvature one has, in a sense to be made more precise, a "non-divergence result", but only in at least one direction.

Thirdly we have the *trichotomy* problem. We are interested in a limit case of nonnegative (scalar) curvature. On the positive side, things have been solved, on the negative side also. A long time ago in (Kazdan & Warner, 1975a) and (Kazdan & Warner, 1975b), it was proved that any manifold can carry a metric of negative scalar curvature (this is of course superseded by section B.5 below). But what about in between? With a little more than the Kazdan-Warner results (see (Besse, 1987), Section 4.E for details), one knows that for the class of manifolds inbetween there is only a third possible case: that of manifolds which cannot carry a metric of positive scalar curvature but can carry a metric of zero Ricci curvature. We will meet this mainly open question again in III.C and TOP. 3. As said previously, the class of zero Ricci curvature manifolds is still very mysterious today.

✦

Fourthly the total set of positive scalar curvature metrics on a given compact manifold can be *non-connected.* We saw above that this implies examples of non-connectedness for both the total spaces of positive sectional and positive Ricci curvature. The first example appeared in (Hitchin, 1974) where the π_0 (the number of connected components) and the π_1 of the set of positive scalar metrics on a given manifold were shown to be non-zero in many cases. In (Gromov & Lawson, 1983) it is shown that π_0 is infinite for the sphere S^7. In (Kreck & Stolz, 1993) the subject is continued. They make use there of the Aloff-Wallach manifolds we met before: we quoted this both in B.2 and in B.3. For more examples one can consult among others the survey (Lohkamp, 1996a) (for both signs by the way) and (Lawson & Michelsohn, 1989), page 329. Note that non-connectedness is important but does not say much about the topological structure of the components: does it have a more or less trivial topology? The case of negative Ricci curvature in B.5 below will be strikingly different.

✦

At this point it is important both historically and for the future to look at the *proof* of the above results. An expository sketch can be found on pages 95–100 of (Gromov, 1991b). The tools are of three completely different types. The first one is the above surgery, 100% geometrical. The second part is pure algebraic topology, namely cobordism theory. This theory will tell you which manifolds can be built up by surgery with simple building blocks (among them some of the $\mathbf{K}P^n$). More precisely, one has to show that any manifold satisfying the conditions above can be obtained by surgery with building blocks which do have positive scalar curvature. All this will insure the sufficiency of the condition.

The necessity is a completely different story and requires a new tool. It started with (Lichnerowicz, 1963). This was historically the first condition obtained from the positivity of the scalar curvature (and this even though those were the times of the positivity of the sectional curvature!). Assume the manifold admits a spin structure (see TOP. 6.B and the general scheme in 0.D). Then a generalization of Bochner's technique is available, through the theory of elliptic operators, in order to yield *harmonic spinors*, namely those which vanish under the Dirac operator δ that one can define canonically on the spinor bundle over the manifold. For this Dirac operator, the new "Bochner's formula" is surprisingly simple here and reads in short: $\delta^2 = D^*D + \frac{\text{scal}}{4}$, to the effect that scal > 0 implies the non-existence of any harmonic spinor. Then the Atiyah-Singer index theorem (see TOP. 6) implies that some invariant called the A-genus has to vanish. Moving to the α-genus required some extra work which was achieved in (Hitchin, 1974).

Last but not least, we cannot resist quoting the following beautiful result: on a torus a metric of nonnegative scalar curvature has to be flat (Gromov & Lawson, 1980b). The technique of the proof is put into a general framework in (Lawson & Michelsohn, 1989), IV. §7.

4. The negative side: Sectional Curvature

This topic forms an entire world in itself and is quite satisfactory for Riemannian geometers. In fact, since van Mangoldt in 1881 and Hadamard in 1898 and 1901, results have been appearing in a steady stream. First came Cartan's work in the 20's, then E. Hopf's in 1939, Preissmann's in the 40's and then the results described below.

The Cartan-Hadamard theorem of the 20's, clarified by Hopf-Rinow, implies that nonpositive sectional curvature manifolds have a universal covering which is diffeomorphic to $\mathbf{R}^d$. This follows from the stronger fact that any pair of points is joined by a unique geodesic which is automatically a segment. So the simply connected case "looks like a hyperbolic and/or Euclidean geometry". Then apparently, for the naive observer, the only problem consists in studying the fundamental group π_1: its algebraic nature and classification, the differences between the negative and the nonpositive case, etc. But things turned out not to be so simple and in particular there is still a long way to go towards a characterization of possible groups. Also one can hope that the compact case will look similar to space forms, see II, but the case of negative pinching seen in A.4 may arouse suspicion.

In fact the subject enjoys many strong results on the one hand. But at the same time it appears more and more difficult to achieve a real understanding of what is going on for π_1. In particular it is definitely misleading to remain in the compact case, so that we will again make an exception to our rule. This is seen as one of the contributions by Gromov and this can be justified for the following reason. We will see below many nice results on the π_1 of compact negatively curved manifolds. But as yet *no* property is known which is satisfied by these but not by *hyperbolic groups.* This notion of hyperbolic groups was introduced in the founding paper (Gromov, 1987a). An exposition in book form is (Ghys & de la Harpe, 1990). The strength of this notion lies in its four equivalent definitions, ranging from group theory to Riemannian manifolds. Roughly speaking (and using the word metric for groups) those groups enjoy the same asymptotic isoperimetric behavior and the same large triangle inequalities as in classical hyperbolic spaces. All this is a strong incentive to leave, at least for the moment, the compact (or even the finite volume) case.

✦

We will now try to ask some questions and give solutions to them. A very informative text is (Pansu, 1990/91) as well as the survey (Eberlein, Hamenstädt, & Schroeder, 1990), but note that the authors admit to not being exhaustive, even on their 50 pages. Among the books on this topic (Ballmann, Gromov, & Schroeder, 1985) was fairly comprehensive. The more recent book (Ballmann, 1995) is a complete exposition of the "rank rigidity" for manifolds of nonpositive curvature and finite volume, see below. For various points of view on the topic see the book (Farrell, 1996). Let us just extend the above remark by noting that space forms appear in many instances in this survey. See II for the completion of our very partial and biased exposition. We will now follow Gromov's philosophy and refer, for a much broader context, to (Gromov, 1987a), (Gromov, 1993), see also the "beginner's text" (Gromov, 1991b).

As remarked in the introduction to (Eberlein, Hamenstädt & Schroeder, 1993), the topic involves various techniques coming from different fields of mathematics. Here we note interferences with the dynamical systems of V.B, the space forms of II and the harmonic maps of TOP. 7. In particular to simplify the exposition we have put as many dynamical results as possible into V and the rigidity results for space forms (of negative curvature) into II.

✦

The simply connected story starts naively by looking for a comparison of different metrics of negative curvature on $\mathbf{R}^d$. A helpful notion is that of *quasi-isometry.* Two metrics are said to be quasi-isometric if there exist two constants such that one metric is squeezed between the other multiplied by these two constants. It turns out that it is very difficult to decide whether two given metrics (say of negative curvature) are quasi-isometric or not, we will soon sense the difficulty. This notion itself arises quite naturally in the compact context, since any diffeomorphism between two compact Riemannian manifolds lifts up trivially to a quasi-isometry between their universal coverings. Now the universal covering of a compact negatively curved manifold satisfies $a \leq K \leq b < 0$. So we will forget about this now and start with our first question: classify up to quasi-isometries the metrics on $\mathbf{R}^d$ with $a \leq K \leq b < 0$ (this situation is a particular case of a "bounded geometry"). The situation turns out to be extremely difficult as soon as $d \geq 3$. For $d = 2$ just use the conformal representation and compute the curvature. One gets a Laplacian for the conformity function and the bounds on the curvature finally yield a bounded function. Hence when $d = 2$ all our metrics are quasi-isometric. The only similar case is when both metrics under consideration admit a full orthogonal one-point symmetry. This time the Jacobi field equation again yields a conformity function which is bounded.

But that is the end of the easy part. To realize that this is so just try to prove that the first two space forms which come under consideration, namely Hyp^4 and $Hyp\,\mathbf{C}P^2$, are *not* quasi-isometric. This is hard to see, it is a corollary of the Mostow rigidity theorem (see II.B). It seems very difficult to classify our spaces under our condition when one has $-4 < K \leq -1$ or $-4 \leq K \leq -1$. For example it was already hard to find an example which can never be placed in the $-4 \leq K \leq -1$-range (see (Mostow & Siu, 1980)). We meet here the difficulty of constructing *manifolds* of negative curvature (see also A.4). Even though one performs some geometry "afterwards" they always come finally from "Number Theory". On the contrary, as remarked in TOP. 9, singular objects of negative curvature represent, in some sense, most of the natural geometric objects (glue together heptagons or higher-gons, and probably the same might work – involving some additional difficulty – for higher dimensions, but this is not clear today). In any case one now has quite a large family (depending on real parameters) of manifolds that are pairwise non-quasi-isometric, this is in essence in (Pansu, 1989). Moreover those metrics are homogeneous.

✦

We stay with the difficulty at the $\mathbf{R}^d$-level with some nonpositive metric but look at it from the other side: can *any* discrete finitely generated group act by isometries with non-bounded orbits? There is no restriction known today, besides the trivial one obtained from the fact that there is a quotient which is a $K(\pi; 1)$: our group should have some non-zero homology in some dimension. No conjecture seems in view, see (Gromov, 1993). The above considerations play a role in Novikov's conjecture, which still pervades geometry and topology.

Conversely one has a good understanding of how special groups act. In (Kleiner & Leeb, 1997) it is shown that if the group of an irreducible symmetric group of rank ≥ 2 acts by quasi-isometries it has to act by isometries. This is wrong for the reducible case. See also (Schwartz, 1995).

✦

The reader may be interested in the *tools* used. First we have the global triangle comparison with hyperbolic geometry. This is a trivial extension of the Cartan-Preissmann super-Euclidean inequality seen in 0.B. This inequality is not enough, it has to be refined. This is done by introducing the notion of convexity for functions in a Riemannian manifold. Negative curvature implies strict convexity for many functions, e.g. Busemann ones, the distance between sides of a quadrilateral and the distance function on the square of the manifold.

Basically the smarter tools all consist in compactifying the situation, since working with infinity is hard. We are still in $\mathbf{R}^d$ with a nonpositive metric, most often with a lower bound for K. We will use a few results from the basic reference (Ballmann, Gromov & Schroeder, 1985) and from the survey (Eberlein, Hamenstädt & Schroeder, 1990). One defines the sphere at infinity $S(\infty)$ in this situation by an equivalence relation between the set of all rays. This was already done in Hadamard's language in (Hadamard, 1898) for $d = 2$. The first thing one gets on $S(\infty)$ is a topology, the cone-topology defined in an obvious way through rays. It does not depend on the choice of origin and quasi-isometries of S still induce homeomorphism of $S(\infty)$.

The next point is to have a measure/measures or even a metric/metrics on $S(\infty)$. The metric called "Tits metric" by Gromov is very special. It is always nowhere defined (as equal to $+\infty$) when the curvature is negative (more precisely $K \leq b < 0$) and this is linked with the visibility condition introduced in (Eberlein & O'Neill, 1973). Visibility between two points at infinity means that there exists a geodesic joining them, which is definitely not the case for example for two different directions in Euclidean spaces. More subtle are some other families of metrics one can define on $S(\infty)$. Quasi-isometrical changes induce quasi-conformal transformations for these metrics. When those objects depend on the base point, they are still changed in a reasonable way.

When the curvature is only nonpositive, Tits metrics become finite for some pairs of points. These finite parts are associated with *flat* parts which we will meet below. The totality is some kind of jig-saw (railway) type metric.

✦

We now turn to properties of compact manifolds of negative (resp. nonpositive) curvature and of their fundamental group. First to come are *volumes*. As will be the case for space forms in II, volumes are discrete (except in dimension 3) and in particular isolated from 0 for space forms of any type. A basic fact is (Heintze, 1976): when $-1 \leq K < 0$ the volume is bounded below by $c(d) > 0$, which depends only on the dimension. This is often called the Heintze-Margulis lemma, because it was obtained by Margulis in the special case of space forms. In both authors' works the proof is purely geometric, using a "collar" argument. The strong divergence of geodesics for negative curvature as well as the nature of hyperbolic isometries force the tube around the smallest periodic geodesic to have not too small a diameter and then one is done with the comparison theorem needed for the tube argument in I.C.1. So one can have some hope here, in the negative case, of having finiteness results with milder conditions than the four needed in the general Cheeger finiteness theorem in I.C.2. This was achieved in (Gromov, 1978b), is detailed in (Ballmann, Gromov & Schroeder, 1985) and improved in (Fukaya, 1984). We mention here a finiteness theorem of Gromov's which is true for *real analytic* manifolds but false in the differentiable case. It is extremely rare in Riemannian geometry to have such a kind of situation. The result is that if $-1 \leq K < 0$ for (M^d, g) and if the metric g is analytic then the sum of its Betti numbers is bounded by $c(d)$.Volume(g), where $c(d)$ is universal in d. This is optimal, as shown by examples: they are obtained by connected sums, a typical smooth but non-analytic operation, see (Ballmann, Gromov & Schroeder, 1985) for details of the proof.

Compact space forms of negative curvature were already hard to construct, see II. For variable curvature we saw some examples in A.4 but real analytic ones are harder to construct, see (Abresch & Schroeder, 1996).

✦

Next come *fundamental groups.* The first statement appeared in (Milnor, 1968): for compact manifolds negative curvature implies that the fundamental group exhibits exponential growth (see also (Svarc, 1955)). In (Gromov, 1978b) (see (Buser & Karcher, 1981) for details) a bound is obtained for the number of generators of this group. Milnor's tools are just *RCT* and Bishop's theorem (see TOP. 1.A). Gromov uses the triangle comparison theorem and then one only has to count points on the unit sphere whose mutual distances are bounded from below as in B.1 above.

The most general statement is (Farrell & Jones, 1989b), which is a topological version of Mostow's rigidity theorem (see II): let M be a hyperbolic (compact) manifold, i.e. a space form of dimension larger than 5 and let N be any topological manifold which is homotopically equivalent to M. Then M and N are homeomorphic. For an exposition in book form see (Farrell, 1996).

Next, the product situation: if the manifold is compact with nonpositive curvature and with a fundamental group which is algebraically a group product (with moreover no center), then the manifold itself is a Riemannian product (Gromoll & Wolf, 1971). The result of Preissmann's seen in 0.C was generalized in (Gromoll & Wolf, 1971) and (Lawson & Yau, 1972): in a compact manifold with $K \leq 0$, commuting in π_1 means flat parts, i.e. any free abelian subgroup of π_1 implies the existence of flat totally geodesic tori in the manifold. This can be seen as the converse of the Bieberbach theorem for the classification of flat space forms: II.

✦

For negative curvature we saw that, at least today, hyperbolic groups are the "final answer". The big problem now, in the compact manifold downstairs as well as for the fundamental groups, is to appreciate the difference between negative and nonpositive curvature. There are today some extremely strong results which show in essence that the two cases can be dramatically separated – stated in other words: to go from one to the other you have to make quite a jump. We give the latest results and refer to (Eberlein, Hamenstädt & Schroeder, 1993) for a survey and to (Ballmann, 1995) for the proofs.

We recall briefly that symmetric spaces (see II.C) have a *rank* which is the common dimension of their maximal totally geodesic flat submanifolds, which are moreover all conjugate under the isometry group. This is of course of interest only when the rank is ≥ 2, otherwise we just have geodesics. In particular every direction is contained in at least one such *flat*. At the infinitesimal Jacobi field level, rank ≥ 2 implies the following: given any unit vector, along the geodesic it generates there is a non-trivial *parallel* Jacobi field (orthogonal to the geodesic of course and non-zero). This never happens for generic manifolds. In a Riemannian manifold we define the *rank* of a unit vector to be the dimension of the space of parallel Jacobi fields along the geodesic it generates. The *rank* of a Riemannian manifold is the minimum of the ranks of its unit vectors. The strongest result today is the following "tout ou rien" result for compact Riemannian manifolds of nonpositive curvature. We assume any reasonable irreducibility condition. The theorem (for a manifold of nonpositive curvature) says the following: "either the rank of the space is ≥ 2 and we are then on a space form with its symmetric space metric or the rank is 1 and then the geodesic flow is ergodic on the subset of UM made up by the vectors of rank 1." Today it is still an open question whether this subset of rank 1 vectors is of full measure or not.

Note that the rank, a geometric invariant, can be at least conceptually computed from the fundamental group: (Eberlein, Hamenstädt & Schroeder, 1993). The existence of flats and what they really imply is a major issue today, see for example (Hummel & Schroeder, 1997).

The above dichotomy admits a lot of extensions with weaker hypotheses like for instance a finite volume to replace compactness and/or a group restriction, the so-called " duality condition", and also versions for metric spaces mimicking Riemannian manifolds of nonpositive curvature. For all this the reader can look at the given references and TOP. 9.

✦

Finally we end with the question of relaxing the condition $K \leq 0$ slightly from above. It is important to realize that without any extra hypothesis "everything is permitted". A construction of Gromov's was published in (Buser & Gromoll, 1988) and (Bavard, 1987). There, for any $\varepsilon > 0$ examples are constructed of metrics on spheres and other 3-dimensional manifolds whose curvature satisfies $K \leq \varepsilon$ and whose diameters are all smaller than 1. See also (Fukaya & Yamaguchi, 1991): the results here are that one can obtain restrictions on the fundamental group, namely that the universal covering is diffeomorphic to $\mathbf{R}^d$, but with an extra hypothesis, like $K > -1$ and an upper bound on the diameter.

5. The negative side: Ricci Curvature

We have here the strongest possible answer: (Lohkamp, 1994) is one of a series of papers in which it is shown that, caricaturally speaking, negative Ricci curvature "means nothing" or, equivalently, "permits everything". Worst of all: *any* metric can be continuously approximated by a metric of negative Ricci curvature (of course not with too high a degree of differentiability). The tools used stem from Analysis, they are inspired by the *h-principle* of (Gromov, 1986) and also the formula giving the variation of the Ricci curvature under deformations of metrics, where one can see that it is quite easy to lower the Ricci curvature.

Another very important point (compare e.g. with section B.2) is the fact that, on any manifold, the total set of negatively Ricci-curved metrics is trivial. That is much more than being just connected, namely contractible. We advise the reader to consult the very informative text (Lohkamp, 1996a).

However even though negative Ricci curvature does not have any topological implications, this does not prevent it from having metric structure implications. In (Bochner, 1946) it was proved that negative Ricci curvature forbids non-trivial infinitesimal isometries, in (Rong, 1997b) this is generalized as follows: the Ricci condition forbids any non-trivial pure F-structure (see section C.3), namely a torus action, whereas Bochner prevented only circle actions.

✦✦

A final note. Two of the three favourite questions that Hopf had been asking since the 30's are still completely open: does a given sign for the curvature imply a certain sign for the Poincaré characteristic? Does $S^2 \times S^2$ admit a metric with positive curvature or not? In 0.E we saw that Hopf's conjecture is true for $d = 2$ and $d = 4$, for $d \geq 6$ it is completely open. Both questions today still leave experts with no solid guess. But for the Kähler case and negative curvature, Gromov arrived at a solution in 1989, using the L^2-Hodge-theory, see (Cao & Xavier, 1998).The latest news for $S^2 \times S^2$ is in (Kuranishi, 1990). Even more itching is Yau's question mentioned in B.1. This question is most irritating, since Synge's theorem (see 0.C) trivially excludes $\mathbf{R}P^2 \times \mathbf{R}P^2$ (as well as many other products of manifolds). Rong's results quoted in B.1 exclude a lot more even. A recent general list of problems is to be found in (Petersen, 1996).

C. Finiteness, Compactness, Collapsing and the space of all Riemannian structures

The logical mathematical order in which to present the results below in C.1 and C.2 is the inverse of the one we are going to follow. This is the way things are presented in the surveys (Abresch, 1990), (Fukaya, 1990), in book form in Appendix 6 of (Sakai, 1996), in (Petersen, 1998a), in the second edition (Gromov, 1998) to appear of the pioneer book (Gromov, Lafontaine & Pansu, 1981), and in (Chavel, 1993). Historically the order was the reverse of this and this is the one we are going to follow.

1. Finiteness results

This topic started simultaneously in 1966 with (Cheeger, 1967) and (Weinstein, 1967). For reasons of clarity we present first the simplest case, which is the only one treated by Weinstein. In a desperate search for a classification of positive sectional curvature manifolds, Weinstein proved that there are only a finite number $N(\delta)$ of possible different homotopy types of even-dimensional manifolds M^d with $0 < \delta \leq K \leq 1$. Unfortunately $N(\delta)$ tends to infinity (polynomially) as δ tends to 0. Note that, in contrast, Gromov's theorem quoted above on Betti numbers of nonnegatively curved manifolds implies a weak bound, but only among *homology* types.

The proof of the homotopy finiteness is purely geometrical and consists in covering the manifold with convex balls whose number can be estimated simply as a function of δ with Bishop's bound for the volume (needing only the Ricci curvature and hence valid a fortiori for the sectional curvature, see TOP. 1.A). This is done using a standard, but still wonderful, purely metric trick: packing as many balls of radius r as possible together implies that the corresponding balls of radius $2r$ form a covering. Such packings are called *efficient*. They also have the extra advantage that one has a universal control on the number of balls which intersect a given one. Then the homotopy type is that of a simplicial complex with as many vertices as the above balls. An even dimension is required because in this case the convexity radius is bounded from below by $\pi/2$ thanks to Klingenberg's theorem seen in A.2.

We will afterwards look at the next three levels: *homotopy, homeomorphy* types and then *diffeomorphy* types. Note also that in any such finiteness result one can also ask for explicit bounds for the cardinality of those finite sets. We will see that in some cases there is no explicit such bound, because the proof is obtained by a contradiction argument based on the convergence of some infinite set.

We will very briefly recall here some facts concerning the three ratios homotopy/homology, homeomorphy/homotopy and diffeomorphy/homeomorphy. The first two ratios are in general infinite. Regarding the first one we just mention that even the classification of homology spheres is not yet finished. Regarding the second one, the theory of characteristic classes is an ideal tool, a lucid exposition is in (Milnor & Stasheff, 1974). The third ratio is always finite as soon as $d \geq 5$; some basic texts are (Hirsch & Mazur, 1974) and (Kirby & Siebenmann, 1977). These finite numbers can be computed explicitly using various pieces of topological information about the manifold under consideration. Dimension 4 is an exception, counter-examples can be found even among algebraic surfaces.

It is fundamental to realize that one cannot extend the above result to odd dimensions, for example we saw in B.1 that there is an infinite set of Aloff-Wallach manifolds $W_{p,q}$ that are homeomorphic but not pairwise diffeomorphic, see Theorem 3.9 of (Kreck & Stolz, 1993). But one can still get not only homotopy but even diffeomorphy finiteness, but under a stronger hypothesis. The pioneer work of (Cheeger, 1967; Cheeger, 1970a), independent of Weinstein's, yielded the following: the set of simply connected possible types of diffeomorphisms is finite within the set of Riemannian manifolds of a given dimension $d > 4$ satisfying the four conditions $K \geq a$, $K \leq b$, volume $\geq v > 0$, diameter $\leq D$ (where a, b, v, D are four fixed real constants such that v has to be positive whereas a and b can be of any sign). So one has finiteness not only for homotopy types but also for homeomorphisms and, starting from there, for diffeomorphisms and this does not require any assumption about the sign of the curvature.

One remark is now in order: in some instances one can effectively get rid of the condition volume $> v > 0$, for example when one works within the class of manifolds with a non-zero characteristic number: as remarked in (Cheeger, 1970a) this follows immediately from Chern's formulas mentioned in 0.E. Another instance is that of negatively curved manifolds, see Section B.4 above.

The way the proof works is important for the future. The first thing we need is to be able to cover the manifold with an efficient packing of convex coordinate balls, the cardinality of the packing being universally bounded (universal obviously means in a, b, v, D and the dimension d). This is of course done with efficient packings. An estimation of the number of balls can be achieved with the help of Bishop's volume estimate using the Ricci curvature (which is here derived from the sectional one), as long as one can get a lower positive bound for the *injectivity radius*. This is the first part of Cheeger's work: the injectivity radius has a universal explicit (positive) lower bound which is a function of only a, b, v, D and d. The original proof was a nice geometric argument (called the "butterfly") which made use of *RCT* and Toponogov. Today the best proof is that of (Heintze & Karcher, 1978): there is an estimate of the volume of the tube of a given radius around a periodic geodesic, using a refinement of the *RCT* estimate for the left side. These *four* bounds are necessary to control the injectivity radius, the four counter-examples needed are flat rectangles with sides $(\varepsilon, \varepsilon^{-1})$ or $(\varepsilon, 1)$, very flat ellipsoids of revolution and very strongly strangulated spheres.

The second part works by contradiction and shows that the number of homeomorphism types is finite (and if desired even explicit bounds for the number can be obtained). Finally one will have to control the coordinate changes between Riemannian coordinate balls for two different metrics. This is exactly what *RCT* can do for you when you know that $K \geq a$ and $K \leq b$, but note for the continuation that this is *only* a C^0-estimate. To go from homeomorphism to diffeomorphism types, Cheeger used the results available from algebraic-differential topology which were mentioned above: the number of possible different diffeomorphism types on a given topological and simply connected manifold is finite when $d > 4$. However later on Cheeger managed to prove diffeomorphism finiteness for any dimension (under these four bounds) without having to appeal to differential topology. One relevant reference is (Cheeger & Ebin, 1975) (Theorem 7.37), to be completed by (Peters, 1984).

✦✦

A digression: Does the curvature determine the metric? We insist on the fact that the curvature tensor does not determine the metric (except in very special cases): this may shock the naive reader since the number of parameters is extremely large as soon as $d \geq 4$. Bear in mind also that Cartan's result in 0.B requires knowledge not only of the curvature, but also of the geodesics as well as the parallel transport along them. The subject started quite recently: there are many examples of non-isometric Riemannian manifolds which admit diffeomorphisms that preserve their respective curvature tensors. A trivial case is to move the metric of a surface along the level lines of Gauss curvature equal to K. In higher dimensions things are of course more subtle, since the curvature not only has many, many parameters but is also a very weird algebraic object. For surfaces it was just a numerical function. A complete mastery of this has not been achieved to this day, see below and the bibliographies of the references given.

Reflecting more on the fact that the curvature tensor is "the Hessian" of the metric, one fact is that it is not completely exact. From the curvature tensor one cannot recover all the second derivatives $\partial_{ij} g_{kh}$, as is clear from the complicated explicit formula below in general coordinates, but can also be deduced from the fact that one needs $(d(d+1)/2)^2$ numbers whereas the curvature tensor has only $d^2(d^2-1)/12$ components. If at the origin of normal coordinates one has $\partial_{ij}g_{kh} = \frac{1}{3}(R_{ikjh} + R_{ihjk})$, then this is because these coordinates enforce certain relations between the $\partial_{ij}g_{kh}$. This agrees with Elie Cartan's philosophy seen in 0.B. But in general coordinates, $R_{ijkh} = \frac{1}{2}(\partial_{ik}g_{jh} + \partial_{jh}g_{ik} - \partial_{ih}g_{jk} - \partial_{jk}g_{ih}) + Q(\partial g, \partial g)$. The above counter-examples, which are referred to below, as compared to Kulkarni's theorem just below, can be described as "threading one's way" into the room left open between $(d(d+1)/2)^2$ and $d^2(d^2-1)/12$.

On the other hand there is the result of (Kulkarni, 1970), to the effect that, when the sectional curvature is not constant and the dimension is larger than 3, diffeomorphisms preserving the *sectional* curvature are isometries. This is not in contradiction to the above examples because the definition of the sectional curvature involves not only the curvature tensor but also the metric. For lower dimensions see (Ruh, 1985).

Another meaning can be attributed to the question "does the curvature determine the metric?". It consists in looking for metrics whose curvature tensor satisfies at every point some imposed purely algebraic condition. A typical example is to require the curvature tensor to be the same as that of a complex projective space. One would then in some cases expect (local) isometry and in general at least local homogeneity, as well as a description of the moduli. Since the founding papers (Ambrose & Singer, 1958) and (Singer, 1960), the field has developed and today uses various definitions and enjoys many results. The books (Tricerri & Vanhecke, 1983), (Berndt, Tricerri, & Vanhecke, 1995) and (Boeckx, Kowalski, & Vanhecke, 1996) can be used as surveys. See also (Prüfer, Tricerri & Vanhecke, 1996) and (d'Ambra & Gromov, 1991). The result in (Tricerri & Vanhecke, 1986) is of examplary simplicity: if the curvature tensor agrees with that of an irreducible symmetric space at every point then the manifold is locally symmetric and thereby isometric to this model. The result is local, it uses a formula from (Lichnerowicz, 1950), which expresses the Laplacian of the full square norm of the curvature tensor as $-(1/2)\Delta(||R||^2) = ||DR||^2 + \mathrm{Univ}(R,R,R) + Q(D(\mathrm{Ricci}))$, where $\mathrm{Univ}(R,R,R)$ is a universal cubic form in the curvature tensor itself and Q is quadratic in the covariant derivative of the Ricci curvature. Then the hypothesis implies immediately that $||R||^2$ is constant, that $D(\mathrm{Ricci}) = 0$ and also that $\mathrm{Univ}(R,R,R)$ since it is obvious in the symmetric case which is characterized by $DR = 0$. It seems to us that this basic formula is still not used as much as one would expect.

For the curvature *with* parallel transport see 0.B and TOP. 4 (Ambrose's problem).

Many people think that the curvature and its derivatives are the only Riemannian invariants. This is in fact true and classical when one demands the following strong condition: they have to be algebraic invariants which stem from the connection, see page 165 of (Schouten, 1954) and the references there. But things are dramatically different if one asks only for tensors which are invariant under isometries (called "natural"). Then there is not even any hope of getting any kind of classification, as explained in (Epstein, 1975). For more on this see (Munoz & Valdés, 1996).

✦✦

Back to finiteness there are two ways in which to improve upon Cheeger's result. The problem is to find a stronger control than just C^0 on coordinate changes using only the sectional curvature (without the metric). One way is to use *compactness* theorems, see Section 2 below. The other is to use the *center of mass technique.* The center of mass in Riemannian geometry can be obtained for convex balls by two equivalent definitions (hence its usefulness). The first way is to minimize the sum of the squares of the distances, the second one is to demand linear dependence between the speed vectors from the center of mass to the points under consideration. The first definition being purely metric and the second one involving tangent vectors, one automatically gets some kind of C^1-behavior from a metric one via *RCT.* For the center of mass technique and some applications see (Karcher, 1989), (Buser & Karcher, 1981), Chapter 8, and (Cheeger, Müller, & Schrader, 1984). But it should be noted that the existence of a unique center of mass and in the large for nonpositive curvature was already proved and used by Elie Cartan back in the 20's (see 0.B), and also that for the general case (but only locally of course, think of the standard sphere) by Calabi in his unpublished result, quoted in A.2, where he solved the differentiable pinching problem. The center of mass technique was the one used in (Peters, 1984): he put suitably chosen $d+1$ points into very small balls.

✦

Now back to what has been the favourite game since the 80's: look for results that require weaker and weaker hypotheses. The first kind of result is to be found in (Gromov, 1981c), where it is remarked that the technique used to bound Betti numbers for nonnegatively curved manifolds (see B.1) extends without any modification to manifolds with $K \geq k$ (for any real k) and diameter $\leq D$. More precisely the Betti numbers (over any field) of Riemannian manifolds of a given dimension are exponentially bounded, with the real number $(\text{diameter})^2.(-\inf K)$ as exponent. Then we have finiteness for homology types (in the simply connected case of course) with only bounds on $\inf K$ and the diameter. An improvement on this bound appeared in (Abresch, 1985).

A very new idea was that described in (Grove & Petersen, 1988). A priori it seems impossible to get rid of some kind of injectivity radius estimate in Cheeger type results in order to get contractible, or more even, convex balls. But remember Gromov's proof for the Betti number of nonnegatively curved manifolds (see B.1): one of the basic facts is that critical points for the distance function cannot "appear too far from the center". In (Grove & Petersen, 1988), using the center of mass technique and Toponogov's theorem, it is proved that the class of Riemannian manifolds of a given dimension satisfying $K \geq k$, volume $\geq v > 0$ and diameter $\leq D$ has only finitely many possible *homotopy* types. The cardinality is bounded explicitly in d, D, k and v. One would be done, "à la Weinstein" as in Section C.1, if one could just control the criticality radius using the Grove-Shiohama technique met in A.2. This is impossible, think of a cone near the vertex. The trick is that one can control the *mutual criticality* radius, namely the infimum of the distances between two points which are critical for each other; this is done by generalizing the tube argument met in Section C.1. Easy examples show that none of these three bounds can be removed from the statement.

In (Grove, Petersen & Wu, 1990–1991), using the same hypothesis the result was extended to finiteness for *diffeomorphism* types. There are, however, some dimensional restrictions where this result does not hold: when $d = 3$, due to the status of Poincaré's conjecture, and when $d = 4$ one has to be happy (today at least) with just homeomorphisms. We remark here that it is also an open question today whether the counter-examples (mentioned above) yielding infinitely many differentiable structures on some topological manifolds of dimension 4 can or cannot satisfy these bounds. It seems also that the cardinalities cannot be universally bounded today using the same kind of proof. But if one has bounds at the homeomorphism level, then one has them also at the diffeomorphism level. The proof definitely involves new ideas beyond the 1988 result. It makes fundamental use of convergence techniques from Section 2 below and a lot of algebraic topology for manifolds, in particular the technique called "controlled topology". This is strongly linked with the notion of uniform contractibility, see the end of TOP. 5.

In the same article there is this astonishing result: finiteness of homeomorphism types (hence diffeomorphism if in addition $d \neq 4$) for the class of manifolds of a given dimension $d \neq 3$ under only *two* purely numerical bounds: volume $\leq V$ and injectivity radius $\geq i > 0$. We will comment on this in TOP. 1.C in the *embolic story*. Before, homotopy finiteness was obtained under the same conditions in (Yamaguchi, 1988). The result by Grove, Petersen and Wu used essentially the basic Croke's local embolic result mentioned in TOP. 1.C and the ingredients for the proof of the above result.

Back to the Grove-Petersen-Wu finiteness result, (Greene, 1994) and (Greene & Petersen, 1992) throw an interesting light onto it, in particular explaining the role of the volume of tubes around (periodic) geodesics.

✦

Again there are results obtained by replacing pointwise bounds on the curvatures by various integral bounds. Precise statements are always quite elaborate and hence not stated here, see (Yang, 1992), (Petersen & Wei, 1996a), (Petersen, Wei & Ye, 1997), and in book form there is (Petersen, 1998a). One geometrical point was to control the volume of a tube around a periodic geodesic, this was done in (Petersen, Shteingold, & Wei, 1996). The proofs are not only geometrical but also involve a lot of Analysis.

✦

Now how about finiteness with bounds only on the *Ricci* curvature? We saw above that already Gromov's homology finiteness for $K \geq k$ and diameter $< D$ does not extend to Ricci $\geq r$. However this did not stop geometers from seeing what inf Ricci means, since the Ricci curvature is hopefully a "reasonable" curvature control. Results started with (Abresch & Gromoll, 1990) and very strong ones now exist. We make a partial choice among them and refer to the very informative survey (Anderson, 1994) and the references therein. Note that some of them are optimal with respect to the ingredients in view of the examples in (Perelman, 1997).

First, there is finiteness of diffeomorphism types in a given dimension provided that Ricci $\geq \lambda$ (of any sign), vol $< V$ and inj $\geq i > 0$: (Anderson, 1990b). Since one succeeded for sectional curvature in going beyond the injectivity radius, the urge is to do the same here. The story is not finished but we know this from the combination of (Anderson, 1990b) and (Anderson & Cheeger, 1991): in a given *odd* dimension there are only finitely many diffeomorphism types under the conditions $|\text{Ricci}| \leq \lambda$, vol $\geq v > 0$, diam $\leq D$ and $\int_M ||R||^{d/2} \leq \Lambda$. When the dimension is even it is compulsory (there are examples) to add orbifolds (see TOP. 9) to manifolds. Moreover in dimension 4 the $L^{d/2}$-bound on the full curvature tensor is not needed thanks to the Allendoerfer-Weil formula (in 0.E). It is not clear if it is reasonable to conjecture that this extra condition is not really needed. Note finally that all these results are obtained by contradiction, hence they do not yield explicit bounds for the cardinalities.

The idea underlying the above results is to prove $C^{1,\alpha}$-compactness. A fundamental fact is (see the digression in 1. COMMENTS on I) that the Ricci curvature is in harmonic coordinates, up to fist-order terms, equal to the Laplacian of the metric. But one has to control the *harmonic radius* (i.e. the largest balls on which harmonic coordinates are well defined and linearly independent, see the next section for more on this). When using the bound on $\int_M ||R||^{d/2}$, the difficulty lies in understanding how the curvature concentrates, this is achieved in (Anderson & Cheeger, 1991) by looking for singular points in sequences converging in the Gromov-Hausdorff space to an orbifold. Then one rescales the metrics around the singular points and shows that iteration of this process ends after a finite number of steps.

For the fundamental group, one has in (Anderson, 1990b) finiteness of group structures with a positive lower bound on the volume, an upper one on the diameter and a lower one on the Ricci curvature.

From B.3 it is clear that no finiteness statement is to be expected from bounds on the scalar curvature.

2. Compactness, Convergence Results

When climbing Jacob's ladder, even in fog, a question that arises quite naturally is that of looking at any kind of convergence and/or compactness within the set of Riemannian manifolds and/or of various subsets of it. Moreover we met before many problems where a convergence existence theorem would have been useful. Look for example at finiteness diffeomorphism theorems or even more simply at the differentiable pinching problem. It is intuitive that – of course in a sense to be made more precise – the set of differentiable stuctures is discrete. Then within a compact subset we have finiteness. Another example is the isolation of the standard sphere. Note that this would be a very pleasant proof but not very explicit, in particular it would not yield any precise pinching constant.

✦

Another motivation is that of III: look at some functional on the set (or some subsets) of Riemannian metrics on a given manifold M. Does there exist some Riemannian manifold that realizes the infimum of that functional (a "best", an extremal Riemannian structure on M)? Sadly enough, all the convergence results we are going to see are unable to give an answer to this question, see e.g. the minimal volume and the embolic volume in TOP. 1.C and D as well as Einstein manifolds in III.C.

✦

It seems that the first appearance of results in that direction was in (Shikata, 1967), where an isolation result was obtained for differentiable structures. Thereafter considerations of that kind were implicit in (Cheeger, 1970a). Then appeared the (unnumbered) theorem on page 74 of (Gromov, 1983b). The proof uses implicitly a convergence theorem that the author always used to take for granted. According to him, since some people doubted it or at least asked for more details he published a proof in (Gromov, Lafontaine & Pansu, 1981). The proof was still a little incomplete until (Katsuda, 1985) offered a complete proof. In 1985 various proofs of the optimal result (see below) started circulating, all using centers of mass and harmonic coordinates. Printed references are (Greene & Wu, 1988), (Peters, 1987) and (Kasue, 1989). We now have to state matters more precisely since the notion of convergence needs a topology.

✦

We will now present the notions and the related results in the order which is now the appropriate one and is moreover well suited for a generalization to more general geometric objects than Riemannian manifolds. This is in fact indispensable for understanding what is going on and moreover considerably simplifies the proofs quoted above. We will follow mainly (Fukaya, 1990) and refer to this text for precise definitions. Surveys are (Petersen, 1990), (Petersen, 1997) and the second edition which is to appear of (Gromov, Lafontaine & Pansu, 1981): (Gromov, 1998).

Things start with the *Gromov-Hausdorff distance* $\boldsymbol{d_{G-H}(X, Y)}$ between isometry classes of abstract compact metric spaces. Its definition and properties appeared first in (Gromov, Lafontaine & Pansu, 1981). The set of all compact metric spaces, denoted by *MET*, endowed with d_{G-H}, is Hausdorff and complete, this is an extension of the well-known analogous statement for compact subsets of $\mathbf{R}^d$.

Already for MET we have the Riemannian geometry result of (Gromov, Lafontaine & Pansu, 1981): inside (MET, d_{G-H}) the subset of Riemannian manifolds of a given dimension d, with diameter $\leq D$ and Ricci $\geq -(d-1)$ is precompact. The argument is just an "efficient packing". The cardinality can be bounded from above using precisely the improvements to Bishop's theorem that are due to Gromov and explained in TOP. 1.A. But of course the closure of such sets is not in the realm of (smooth) manifolds of dimension d, just think for example of flat tori.

The next topology to feature in our story is the Lipschitz one. In MET we call it $\boldsymbol{d_L}$. Its value $d_L(X, Y)$ is the best Lipschitz constant for the Lipschitz homeorphisms $f: X \to Y$ together with the constant of f^{-1}. Back in (Shikata, 1967), the following is proved , which we called a discreteness result in the motivations above. We work within the set $RM(d, a, b, D, v)$ of Riemannian manifolds of given dimension d with $a \leq K \leq b$, diameter $\leq D$ and volume $> v > 0$. Then if N is the d_L-limit of a sequence $\{M_i\}$ then for i large enough M_i is diffeomorphic to N. This is how Shikata solved the differentiable pinching problem.

But this is still not a convergence statement for *any* sequence (or some subsequence) in the set $RM(d, a, b, D, v)$. This is achieved in two steps. The first one appeared in (Gromov, Lafontaine & Pansu, 1981) and was completed in (Katsuda, 1985). It says that if $\lim_{i\to\infty} d_{G-H}(N, M_i) = 0$ then $\lim_{i\to\infty} d_L(N, M_i) = 0$. At this stage we do not know if the limit metric is a Riemannian metric. This requires more work and is to be found in the works of Gromov, Greene-Wu and Peters referred to above. We saw above in the digression in 1 that the difficulty lies in the fact that the control on the curvature gives a *metric* control but not a differentiable one in the metric itself. This is because *RCT* shows only C^0-behavior. The basic trick is to replace normal (geodesic) coordinates with apparently less geometric ones that are, however, in fact more suitable finally. Those are the *harmonic coordinates* (see 0.F and 1. COMMENTS on I): one chooses d harmonic functions which are linearly independent and satisfy some ad hoc boundary conditions. It works only within the *harmonic radius* which was first systematically introduced and controlled in (Anderson, 1990b). Such a control is fundamental for many results and we refer to the various references which we will come across. For sectional curvature bounds, it is easier than for Ricci ones.

The exact statement is the convergence of some subsequence in any infinite sequence $\{M_i\}$ in $RM(d, a, b, D, v)$ toward a Riemannian manifold (N, g) which is diffeomorphic to M_i for i large enough, *but* the Riemann tensor g is known in general to be only of class $C^{1,\alpha}$ (for any α in $(0, 1)$). The proof uses coverings with suitable balls whose number is controlled as explained in C.1 above and harmonic coordinates on them. One can show that this is optimal, one cannot in general hope for C^2-results, as a flat cylinder with two spherical caps shows.

If one wants to get smoother limits one uses *smoothing techniques*, which are important in other instances and were obtained first, using the Ricci flow (see 1. COMMENTS on III), in (Bemelmans, Min-Oo, & Ruh, 1984) and thereafter in (Abresch, 1988) and in (Shi, 1989). See further statements in (Fukaya, 1990). Smoothness means that the absolute value of the covariant derivatives of the curvature tensor of any order, especially the first one, which is technically very useful, can be made arbitrarily small. One of the main points of smoothing is that it respects most hypotheses and that one can for example add smoothness in $RM(d, a, b, D, v)$ etc. See also the norms in (Petersen,Wei & Ye, 1997).

✦

Like in some instances we met above one is tempted to obtain results of compactness (convergence) with weaker hypotheses, in particular with *only Ricci* curvature control. Convergence results where the curvature control is mostly Ricci $\geq -(d-1)$ started with (Gao, 1990a) and were followed by (Anderson, 1990b). The best result today is that of (Anderson & Cheeger, 1992): one has precompactness in the $C^{0,\alpha}$- topology (for any α in $(0,1)$) under the conditions Ricci $\geq r$, injectivity radius $\geq i > 0$ and volume $\leq V$. The link between finiteness and compactness is explained in (Anderson & Cheeger, 1992a) and is of quite a general nature. If moreover the k-covariant derivatives of the Ricci curvature are bounded in absolute value by suitable constants, one has precompactness in the $C^{k+1,\alpha}$- topology. See an extremely brief exposition in (Hebey & Herzlich, 1998). Note that these extra conditions can be achieved by smoothing as seen above. The proof uses essentially (with a lot of Analysis, e.g. Sobolev inequalities) harmonic coordinates and a study of the harmonic radius which can be bounded by the metric injectivity radius (this is not too surprising) together with a Ricci bound: (Anderson, 1990b). See also (Cheeger, Colding & Tian, 1997) and (Brocks, 1997).

The philosophy here, as noted in the digression of Section C.1, is a PDE game and a basic fact is the formula given in 1. COMMENTS on I: in harmonic coordinates, up to first-order terms, the Ricci curvature is the Laplacian of the metric.

For applications of convergence theorems we refer the reader to the various surveys above. We just note that in most cases, like for instance pinching theorems or "just below", the convergence theorem yields an ε that is not given explicitly. This still leaves much work to do to get "direct" proofs and explicit constants.

A final remark is in order. Most of the above results work for (complete) non-compact manifolds, provided one sticks to the *pointed* category. These results are essential for example when one studies the fundamental group as in B.

3. The set of all Riemannian metrics: Collapsing

The desire here is very natural: if a sequence of Riemannian manifolds does not converge nicely (as we have seen this is mainly because the injectivity radius tends to zero) then what is really happening? Do we have some limit space of some kind – e.g. still a manifold, but of a smaller dimension, or a reasonable generalization of Riemannian manifolds (see TOP. 9), or just some metric space? In some sense one is looking for a *compactification* of the set of Riemannian metrics (of a given dimension) and an understanding of what happens when one approaches the boundary. This of course makes sense only within suitable subsets, namely when one imposes some curvature bound(s) and some metric invariant(s). When there is no convergence towards a Riemannian manifold (of the same dimension, even permitting singularities) the situation is called *collapsing*. This is vague, we now study things more precisely. Here again the reference we follow is (Fukaya, 1990), at least up to 1990. An informative text is (Pansu, 1983–1984). The basic papers are (Cheeger & Gromov, 1986–1990), (Fukaya, 1987a), (Fukaya, 1989) and by the whole team: (Cheeger, Fukaya, & Gromov, 1992), whose introduction is very informative. For recent works on the set of all Riemannian structures on a given manifold, showing that its structure is quite sophisticated and that one might have to resort to *complexity theory*, see (Nabutovsky, 1996b). See also the *toy model* of it in (Nabutovsky, 1996c).

✦

The first situation we study arises when one drops the last condition volume $\geq v > 0$ in $RM(d,a,b,D,v)$ and works in $RM(d,-1,1,D)$ (an obvious notation) after the normalization from (a,b) to (–1,1), which is not restrictive when looking only for general statements. Next we have to study the situation where the volume tends to zero, which by Croke's local embolic theorem in TOP. 1.C implies that the injectivity radius tends to zero uniformly. The first thing to do is to find examples. We saw in the section about pinching around zero (see A.3) that nilmanifolds are members of our class. Note then that the *limit space* is the smallest possible one, namely it is reduced to a *point*.

The other basic general example appeared in (Gromov, 1983b): any manifold admitting a circle action without fixed points, or say fibered by S^1 over some other manifold N, collapses to N. This is just an application of Riemannian submersion formulas: take first a fixed Riemannian metric invariant under S^1 and make the fibers smaller and smaller keeping the "horizontal" metric component fixed. One gets the nilmanifolds as a particular case upon applying this trick to the successive S^1-fibrations which arise from the nilpotent structure. From the Hopf fibration $S^1 \to S^{2n+1} \to \mathbf{C}P^n$ one gets a collapsing of the sphere S^{2n+1} to $\mathbf{C}P^n$. A fortiori of course this extends to manifolds which admit a torus simple action. If you are familiar with the maximal tori in compact Lie groups, you will discover how to collapse the group $SO(2d)$ onto the homogeneous space $SO(2d)/T^d$.

A more elaborate type of example is the following. Take two 2-dimensional manifolds N and N' of the same dimension with a boundary which is a circle and take the products $M = S^1 \times N$ and $M' = S^1 \times N'$. These are manifolds with boundary, in both cases the boundary has the topological type of a torus T^2. Then glue M and M' along that T^2, not in the trivial way but *interchanging* the two circles in T^2 (exchange the parallels and the meridians). The resulting manifold does not in general admit a global S^1-action, only a local one on both parts, since on the common part they do not coincide but locally they agree as both being part of the same local 2-torus action. Then it is still possible to define a collapsing structure in $RM(3,-1,1,D)$ for this manifold. This example was put into a very general context in (Cheeger & Gromov, 1986–1990), where the notions of ***F****- and* ***T****-structures* were defined. Extremely important notions are those of *polarized* and *pure-polarized* F-structures. Detailed definitions are also to be found at the end of the survey (Fukaya, 1990). The existence of various structures of those types is linked with the topology of the manifold, but the question of how exactly is still partially unsolved, see the various references for precise statements. What is difficult is to decide whether one is within manifolds with a bounded diameter or not.

For example, for a general F-structure, one can build up only Riemannian metrics with $-1 \leq K \leq 1$ and an injectivity radius tending to zero. If the structure is moreover polarized, then one can get the volume to tend to zero, and if moreover it is pure-polarized then one really has a collapsing in $RM(d,-1,1,D)$. In (Cheeger & Gromov, 1986–1990) many results are obtained for the study of what happens to the various Riemannian invariants during the process of collapsing. The above theory is used also to study the existence of characteristic numbers and integral formulas for these when the manifold is no longer compact but still of finite volume: see (Cheeger & Gromov, 1985). For what happens with the spectrum (Chapter IV) see (Fukaya, 1987b).

✦

A byproduct of (Cheeger & Gromov, 1986–1990) is a *structure* statement for any Riemannian manifold. Namely there exists a universal number $\varepsilon(d)$ which depends only on the dimension d such that, given any complete Riemannian manifold M of dimension d and with $-1 \leq K \leq 1$, there exists an open set U (the *thin* part) of M with the following properties: on U there exists an F-structure (of positive dimension) and at points in $M \setminus U$ (the *thick* part) the injectivity radius is larger than $\varepsilon(d)$. The results on F-structures are typically used in (Anderson, 1996a).

✦

We now look at the structure of the collapsing itself when the limit set of $\{M_i\}$ is a (compact) manifold N. The answer appeared in (Fukaya, 1987a) and (Fukaya, 1989). We are in $RM(d, -1, 1, D)$ and assume $\lim_{i\to\infty} d_{G-H}(M_i, N) = 0$. Then, at least for large i, there exist maps $f_i\colon M_i \to N$ which are fibrations. The fibers are infranil-manifolds and every f_i is almost a Riemannian submersion. In (Cheeger, Fukaya & Gromov, 1992) the two points of view are united to get a very strong statement for collapsing. See this text for detailed statements and open problems in this topic. Finally the bold question of the nature of the d_{G-H}-closure of $RM(d, -1, 1, D)$ was addressed in (Fukaya, 1988). When one is interested in various closures on a *given* manifold things are much easier, see III.D.

✦

We now look at the boundary structure for subsets involving weaker curvature bounds. The first thing to do is to discard $K \leq 1$ as we did in Section 1 for finiteness results, still keeping $K \geq -1$ however. It is intuitively clear that limits can now have singularities, think for example of small fingers, bubbles, etc. The major discovery was that there exists a reasonable generalization of Riemannian geometry which is preserved under metric limits provided that $K \geq -1$. This is the notion of Alexandrov's space, see TOP. 9 for more on this. For the case of collapsing under only $K \geq -1$ with a smooth limit one has a strong structure result in (Yamaguchi, 1991).

✦

One now has the strong wish to find a control using only the *Ricci* curvature plus of course various metric invariants. This is a topic which is the focus of lively activity today and is hence hard to present concisely. Besides the references of the preceding two sections, we mention (Anderson, 1992b); (Cheeger & Colding, 1997a) marks the beginning of a series of papers which take into account previous results. Recall that Colding's L^2-Toponogov theorem (see the digression in A.2) plays the role of Toponogov's theorem when one has only a lower Ricci bound. To give a better insight we mention that those results are part of a program by Anderson and Cheeger, which includes the conjecture that "d_{G-H}-convergence in the presence of a lower Ricci curvature bound implies volume convergence". This has since been proved in (Colding, 1996b).

✦

A further way of weakening the assumptions is to use only *integral* bounds, sometimes mixed with various others. We just mention (some of them we have already met): (Gallot, 1986), (Gallot, 1987), (Yang, 1992), (Gao, 1990a), (Gao, 1990b), (Petersen, 1997), (Petersen & Wei, 1996b) and the expository texts (Anderson, 1990d) and (Petersen, 1998).

II The geometrical hierarchy of Riemannian manifolds: Space Forms

Defining, constructing and classifying space forms can be very hard and is still not finished. We will explain at the same time what we mean by this hierarchy. Here again the situation will be completely different according to the sign under consideration. There are some books and partial surveys: (Buser, 1992), (Wolf, 1972), (Gromov & Pansu, 1990) and (Vinberg, 1993). For the hyperbolic case (Benedetti & Petronio, 1992) and (Ratcliffe, 1994) are two excellent references. So we will give below only a few references, mainly to locate the dates. Do not believe that a complete mastering of the geometry of space forms has already been achieved. Amazing relations between geometry (volumes of polytopes, cross-ratios and their generalizations) and Number Theory are the focus of lively research and enjoy some beautiful results. We just mention Chapter 14 by Kellerhals of the book (Lewin, 1991) and the informative text (Oesterlé, 1992/93).

Once again we will work only within the compact case but the most natural and important cases are those of finite volume, see TOP. 5. Surveys on rigidity are (Gromov & Pansu, 1990), (Pansu, 1993–94) and the book (Farrell, 1996).

A. The constant curvature case

A fundamental property of Euclidean spaces is the congruence axiom: two triangles with equal respective sides are always deduced from one another by an isometry of the space. Let us say that these triangles cannot be distinguished: "they are the same". Another way to formulate this is to say that they are metric spaces which are *3-point* transitive. So the beginning of the hierarchy is formed by looking for all metric spaces with this property, first globally, then only locally. The global answer was known in the 20's and follows from Hopf-Rinow's and Cartan's philosophy, see for example the book (Cartan, 1928a) (the second edition (Cartan, 1946–1951) is also good of course). Look at infinitesimal triangles. Their infinitesimal shape will yield the sectional curvature and the spaces we are looking for are those of constant sectional curvature. The global answer is given by the simply connected ones, namely the spheres (with the canonical metric if the curvature equals

1), the Euclidean spaces and the hyperbolic spaces Hyp^d (most often normalized to a curvature equal to –1).

Local (non-simply connected but compact) answers are then the quotients of one of those three spaces by discrete groups of their isometry group. What is the state of affairs today? Things were quite clear by the late 30's for *surfaces.* We discard first the trivial cases of positive and zero sign. On the negative side it is easy to build up examples geometrically, one way is to take suitable triangles or polygons in the hyperbolic plane and make tessellations with them. But note that the group yielding the quotient is then hard to visualize. Another way is to glue together pantaloon pieces along their boundaries, provided they are closed geodesics, the pantaloon pieces are obtained by gluing together identical hyperbolic hexagons all of whose angles are equal to $\pi/2$. This latter way enables us to construct all examples, as seen by working back by dissection (this is explained perfectly in (Buser, 1992)). The other way is to use conformal representation for compact topological surfaces of any genus. Finally the complete classification was worked out by Teichmüller in the late 30's. On an orientable surface of genus γ, the constant curvature structures form a space with $6\gamma - 6$ parameters. Building those space forms by group theory is harder and is basically Number Theory, see for example (Vigneras, 1980a). The link between the algebra and the genus of the surface is already subtle.

Starting with dimension three, the difficulties appear formidable. We will describe things in a very sketchy way. For all dimensions the positive and the zero case were understood almost completely in the 60's and are detailed in the book (Wolf, 1972) (try to get the second edition). Contrary to what most people think the classification is not completely finished but this is more of an algorithmic problem. Moreover the rigidity problem has hardly been studied in the positive case, in contrast with the negative case to come now. To get some feel of the mysterious and hard component in the positive case see (Milnor, 1966). The flat case is based on the famous Bieberbach theorem (1911–12): the flat compact manifolds are tori or finite quotients of them.

The negative case is already difficult starting with dimension three. It was only in 1931 that examples were constructed in (Löbell, 1931). These were geometric, using tessellations with polyhedra in a three-dimensional hyperbolic space. In the 60's the existence of negative curvature space forms was more or less folklore, see in particular Selberg's work. More surprising: many papers appeared which studied these space forms without knowing of their existence. For a general result one had to wait until (Borel, 1963). It is important to know that Borel's construction is entirely Number Theory, the point being to find suitable discrete subgroups (called "arithmetic") of the isometry group of Hyp^d.

There is a good reason for that. In (Vinberg, 1984) it was realized that geometric tessellation constructions *cannot* exist in large dimensions. A question we are left with today is the exact value of the limit dimension, see (Vinberg, 1993).

✦

How about some kind of Teichmüller theory now (deformations, number of parameters)? The big event was (Mostow, 1968): space forms of constant negative curvature ($d > 3$) are *rigid.* As soon as the fundamental group is given as a group, one has Riemannian uniqueness, in particular on a given space form. So important and conceptual is this rigidity that a succession of ever simpler proofs appeared. The last one is nothing but a direct corollary of (Besson, Courtois & Gallot, 1995a) and (Besson, Courtois & Gallot, 1995b), to be found also in V.A and V.B. See also the survey (Besson, 1996), the expository texts (Pansu, 1996–97) and (Pansu, 1993–94) and the book (Farrell, 1997).

This rigidity does not imply a classification. Today one still does not know everything. On the one hand, we have the Borel type arithmetic examples (a precise definition of *arithmeticity* would be rather too long). On the other hand, in (Gromov & Piatetski-Shapiro, 1988) examples which are definitely *not* arithmetic are constructed in any dimension. Whether arithmetic or non-arithmetic ones are more numerous is the principal open question.

✦

An important issue for negative space forms is that of *volumes.* When $d = 2$ the Gauss-Bonnet theorem implies immediately that the set of volumes is an arithmetic progression. The existence of a universal positive lower bound for any dimension is shown in (Wang, 1972) where it is proved that starting with dimension 4 the set of volumes is discrete for space forms of any rank. In (Prasad, 1989) the volumes of all arithmetic space forms are computed explicitly. The case of dimension 3 was solved in (Jorgensen, 1977) and (Thurston, 1978) and the result is fascinating: volumes are isolated from zero (although the best value is still unknown today) but they have accumulation points located on a discrete scale. For more see (Gromov, 1981a) and (Ratcliffe, 1994), page 501. See the recent text (Goncharov, 1997) to get an idea of the depth of the problem.

B. Space forms of rank one

Pursuing down our hierarchy we now look for metric spaces which are "only" *2-point* transitive, i.e. there is always an isometry mapping one pair of points to another pair (provided of course that their respective distances are equal). Alternatively, an infinitesimal formulation is that we look for spaces which are *isotropic*, namely where all unit tangent directions are metrically equivalent. In the simply connected case they are all known. Discarding the preceding ones of constant curvature, one is left with the $\mathbf{K}P^n$ and their negative curvature analogues, denoted here by $Hyp\,\mathbf{K}P^n$. Their curvature range is from $1/4$ to 1 or from -1 to $-1/4$ after normalization. The $\mathbf{C}P^n$ together with the $Hyp\,\mathbf{C}P^n$ are the complex geometries corresponding to the spheres and the hyperbolic spaces. This classification is to be found in (Wang, 1952) for the compact case and in (Tits, 1955) for the non-compact one. It is still a long story to do the classification in detail, the best reference today is, to our knowledge, that of (Karcher, 1988) (see page 120). The difficult part of the proof is to show that the space under consideration is a symmetric space, see (Szabo, 1991) for a short proof. Then it obviously has to be of rank one, see C below.

We will now look for compact quotients of those simply connected spaces. In the positive case (spheres excepted) there are no such quotients (except up to $\mathbf{Z}^2$

because dimensions are even here and we then have Synge's theorem, see 0.C). We now turn to the negative curvature case. The complex case is particularly interesting because of the connections with complex analysis and algebraic geometry. General existence is again shown in (Borel, 1963), of course of arithmetic type. Non-arithmetic examples appeared in (Mostow, 1980), built up using subtle tessellating polyhedra in $Hyp\,\mathbf{C}P^2$ and in $Hyp\,\mathbf{C}P^3$, also later on in (Deligne & Mostow, 1986) by different and expensive Number Theory techniques. The question of deciding whether any non-arithmetic examples exist for the $Hyp\,\mathbf{C}P^n$ is still open today for higher n. But for the quaternionic and the Cayley case, for $Hyp\,\mathbf{H}P^n$ and $Hyp\,\mathbf{Ca}\,P^2$ it is proved in (Gromov & Schoen, 1992) that all compact quotients of these spaces are necessarily arithmetic. The proof uses hard Analysis, more precisely harmonic maps into manifolds with singularities, see TOP. 7.

There is a rigidity theorem as well for the compact quotients of the $Hyp\,\mathbf{K}P^n$ (Mostow), but today it can be obtained by the general result of (Besson, Courtois & Gallot, 1995a) already mentioned.

The search for spaces which are only *measure-isotropic* (so-called *harmonic* ones) started in the early 40's. There are many equivalent definitions, the first one says that at every point the solid angle (the infinitesimal measure along a geodesic starting from this point) depends only on the distance. The word harmonic was chosen because this condition is equivalent, at least locally, to the fact that the value at every point p of any harmonic function f ($\Delta f = 0$) is equal to its mean value on every distance sphere centered at p. Lichnerowicz conjectured in 1944 that such spaces are, at least locally, isometric to a space form of rank one, so that measure-isotropy will force metric isotropy to hold. After many intermediate results the conjecture was proved for the compact case by Szabo in 1990 and then disproved in the non-compact case by Damek and Ricci. All the references can be found in (Berndt, Tricerri & Vanhecke, 1995) and its bibliography.

C. Space forms of symmetric spaces of rank larger than one

Next after 2-point transitive spaces come the 1-point ones. But these are nothing more than *homogeneous* spaces. In some sense this category is too general, even though we will meet it many times below when constructing examples of Riemannian manifolds.

But there is a basic intermediate category, that of the *symmetric spaces*, discovered by Elie Cartan in 1926. These are the manifolds for which the geodesic symmetry around any point is (at least locally) an isometry. This implies immediately (local) homogeneity by composing those symmetries at pairs of points. But another equivalent definition (this has to be proven and is due to Elie Cartan) is that the curvature is invariant under parallel transport. It is the most natural homogeneity condition (think of the golden triangle in 0.A and of Ambrose's problem at the very end of TOP. 4). Elie Cartan showed that these (local) geometries can in fact be completely classified, a job done first at the level of Lie algebras. The result for irreducible ones is a very restricted list, containing infinite series and a finite number of exceptions in low dimensions. All members of the infinite series are spaces known in geometry: the simple Lie groups themselves, the Grassmann manifolds over the fields **R**, **C**, **H**, the set of complex structures, of Lagrangian subspaces, and a few others.

A symmetric space has attached to it an integer, called its *rank*. It is the common dimension of all maximal totally geodesic flat submanifolds. They are all conjugate under the group action. The rank one spaces are those of Section B above. These flat submanifolds are, in the simply connected case, tori for the compact case and Euclidean spaces for the non-compact case. This completely determines the geodesic flow (see V.B). In particular all the geodesics of the spheres and the $\mathbf{K}P^n$ are simple, closed (periodic) and of a common length.

These irreducible symmetric geometries split again into two classes: those of nonnegative and those of nonpositive curvature. Finding all the quotients in the positive case is a finite job, which was done back in (Cartan, 1927). For the negative case it was again in (Borel, 1963) that compact quotients were proved to exist for the first time, always by arithmetic considerations. But here, if there is a classification problem, it is one for number theoreticians because, in (Margulis, 1977), a *superrigidity* result was proven, which implies not only toplogical rigidity but also arithmeticity for the rank ≥ 2 case. Margulis' tools were expensive, today there are other ways to recover Margulis' result: see for example (Jost & Yau, 1990). Results like (Besson, Courtois & Gallot, 1995a) still do not yield the superrigidity.

✦

Now one can look at generalizations of Euclidean geometry with spaces more general than Riemannian manifolds. In fact one is more or less forced to do so when working in Riemannian geometry, this was seen in I.C.3: see TOP. 9.

III The set of Riemannian structures on a given compact manifold: Is there a best metric?

A. The problem

We still stick to the compact case. A question that clearly suggests itself is the following one: given a compact differentiable manifold, does it carry a *best*, or a family of best, Riemannian structure(s)? We saw in II.A that things work wonderfully in the case of surfaces. Any compact surface admits a metric of constant curvature. Moreover on a given surface they are completely classified (Teichmüller theory), forming a finite-dimensional "moduli" space. How about higher dimensions? This question was put to the author by Thom around 1960 and is the very first problem in the list of (Yau, 1990). In the low dimensions 3 and 4 the question is of primary importance since Riemannian geometry seems to be a basic tool for solving the various topological conjectures that are still open (see TOP. 8).

Today there is not a single satisfying general statement in a dimension larger than 4, there are only some partial results. The lower dimensions are better understood. One of the reasons which makes the problem difficult is the following one. To get a best metric it is reasonable to work with the condition $-1 \leq K \leq 1$. But we saw in I.C.2 that in order to obtain a limit we need moreover volume $\geq v > 0$ *and* diameter $\leq D$. Even when the topology of the manifold satisfies the volume condition, there is still no way of controlling the diameter.

Another scheme of attack is to look at a suitable numerical functional $F(g)$ defined on the space of metrics g and then search for those for which $F(g)$ reaches its minimum (or is at least *critical*, that is to say the derivative of $F(g)$ at g is zero for any variation of the metric). If the functional is scale-dependent, just normalize it. If criticality means anything to us, we can stop here. If we need more, we can then look for an extremum. For the general context see Chapter 4 in (Besse, 1987) , (Anderson, 1990d) and (Sarnak, 1997a). Sarnak considers in particular as a functional the determinant of the Laplacian, see IV.

The first thing to do is to study the space of all Riemannian metrics on a given compact manifold, but also its quotient by the set of all diffeomorphisms, since we are interested mainly not in metrics but in *structures* (metrics up to isometry). There is no reason for this (infinite dimensional) structure space to be smooth, but there is a nice *slice* result which gives a local structure and its "tangent space", see 12.C of (Besse, 1987). See also (Gil-Medrano & Michor, 1991). Concerning the topology of the set of all Riemannian structures on a given manifold we know only of (Bourguignon, 1975), (Nabutovsky, 1996c) and (Nabutovsky, 1996b): these latter references show that the situation might well be extremely sophisticated.

B. The minimal volume and Min$||R||^{d/2}$

It seems to us that the most natural definition of an optimal metric is the following: "*it is the least curved one*". So the obvious functional is $\int_M ||R||^{d/2}$ and we immediately attach to a compact manifold M the invariant $\mathbf{min}||\boldsymbol{R}||^{d/2}(\boldsymbol{M})$, which is the infimum of all the $\int_M ||R||^{d/2}$ when considering all Riemannian metrics on M. Note that the Gauss-Bonnet theorem shows that this functional is certainly not good in dimension 2, but we are interested precisely in higher dimensions. It turns out that, at least in dimensions larger than 4, the condition for g to be critical is not understood today: it is a partial differential equation for g and its curvature which is actually indecipherable. There are only results telling us that this invariant is zero in some cases and non-zero in some others. But the exact delimitation remains a mystery, as well as knowing if its vanishing is or is not equivalent to that of the minimal volume, see TOP. 1.D. The functional $\int_M ||R||^{d/2}$ is also encountered in finiteness theorems using the Ricci curvature: see I.C.1.

But the lower dimensions 3 and 4 are exceptions. For dimension 4 the generalization of the Gauss-Bonnnet theorem (Section 0.E) reads:

$$8\pi^2\chi(M) = \int_M (||R||^2 - ||\mathrm{Ricci} - \frac{\mathrm{scal}}{4}.\mathrm{g}\,||^2)\,\mathrm{dm}$$

Then, on a given M, our functional is an absolute minimum when the Ricci curvature of g is proportional to it: $\mathrm{Ricci(g)} = \frac{\mathrm{scal}}{4}\,\mathrm{g}$. This proves, by the way, that the characteristic of an Einstein 4-manifold is always nonnegative, see Section C. But the converse works only when $\chi(M)$ is nonnegative. The metrics whose Ricci curvature is proportional to the metric itself are nothing but what we are below going to call *Einstein* metrics. And we will study them in quite some detail in C, because it is the sole notion of a best metric which seems really systematically workable today in general dimensions.

In dimension 3 partial (but strong) results are obtained in (Anderson, 1996a) and (Anderson, 1996b), which pave the way to the Thurston geometrization conjecture.

✦

Three other functionals are more geometric (in fact two of them are curvature-free) and look interesting: they are the minimal volume, the systolic volume for some very special manifolds (see TOP. 1.E) and the embolic volume. The *minimal volume* of a compact differentiable manifold is the infimum of the volumes of all Riemannian metrics on this manifold which have a sectional curvature between -1 and 1. This is equivalent to looking at the infimum of $\sup |K|$ under the condition volume $= 1$. This is another way of looking at the "least curved" condition. The *embolic* volume is the infimum of the volumes under the condition that the injectivity radius is equal to π.

Despite such simple definitions results on them are so few that we discard them here and defer a more detailed account of MinVol to TOP. 1.D. For the systolic story, which is quite interesting in itself, see TOP. 1.E. Finally for the embolic story some recent references are (Croke, 1988) and (Grove, Petersen & Wu, 1990–1991) (see their Theorem C), see also I.C.1 and TOP. 1.C. The difficulty of studying these invariants can be illustrated just by pointing out that it is already beyond our knowledge whether they are zero or not. This is a place where Chern's integral formulas are useful but not enough (see 0.E and I.C.1).

However we mention here the main result of (Besson, Courtois & Gallot, 1995a) in V.B. One of its many byproducts is the fact that the best metric is known on compact space forms of negative curvature (rank 1): it is unique and is precisely this locally symmetric metric. This is true for more than only the minimal volume, see for this result V.B and TOP. 1.D.

In the case of a manifold which admits a metric of a given sign, a best metric would naturally be one with the best pinching. Except the cases met above in I.A.2 no answer to this question exists today and we already explained the difficulties at the end of I.B.1.

Finally, in (Nabutovsky & Weinberger, 1998) the critical diameters are studied under a bounded curvature, and it is shown that the space of Riemannian metrics on a given manifold can be a very subtle object.

C. The case of Einstein manifolds

We now turn to Einstein manifolds. The story started with (Hilbert, 1915). There, motivated by theoretical physics, the author computed the directional derivative in the space of all Riemannian metrics of the total scalar curvature

$$F(g) = \int_M \operatorname{scal}(g)\, dg.$$

Without normalization, critical implies for g that Ricci(g) $= 0$, which is too restrictive, see below. So we normalize the volume to some given constant (which does not matter) and get the condition "Ricci(g) is proportional to g": $\mathrm{Ricci(g)} = \frac{\mathrm{scal(g)}}{\mathrm{d}}\mathrm{g}$ (this is nothing but the classical technique of the Lagrange multiplier). Using the Bianchi identity the proportionality function is necessarily constant as soon as $d \geq 3$. Such a metric will be called *Einstein.* In dimension 3 this condition is too strong, it implies constant sectional curvature, so we discard this dimension from now on. In dimension 4, we saw that this is equivalent to minimizing the total square norm of the full curvature tensor. Note that Einstein manifolds come into the picture automatically when one looks at Ricci pinching, see the end of I.A.4.

The fact that the equation defining Einstein metrics is a PDE (because the number of parameters is $d(d+1)/2$ on both sides) gives rise to great hope, as does the fact that we have a smooth functional. However there is today no way of using any available kind of Morse theory to get some form of existence (of critical points). In fact experts are divided, some think it cannot because of an interval structure nature. Some others think a tool like Floer homology (see (Hofer, Taubes, Weinstein & Zehnder, 1995)) could help. See the end of I.B.1, Besse's book and below for more comments, and (Nabutovsky, 1996b), (Nabutovsky & Weinberger, 1997).

This Hilbert functional was considered from a new perspective in (Connes, 1995–6), where it appeared as the $(d-2)$-volume of the manifold, in particular an "area" in dimension 4.

✦✦

Digression: The Yamabe Problem. We will here give a short historical digression on a topic which is completely omitted elsewhere in this survey, except here and indirectly in TOP. 8. This is the Yamabe problem. In (Yamabe, 1960) the author tried, starting with any Riemannian metric g, to deform it conformally into an $f.g$ of constant scalar curvature, where f is a positive numerical function. It seems probable that the author's secret hope was to attack the Poincaré conjecture in dimension 3 (compare this with Hamilton's result in I.B.2). Going further in dimension 3 is the content of (Anderson, 1997) and references therein.

The problem that needed solving was a non-linear elliptic equation in the unknown conformity function f. This question from Analysis is extremely hard and has been studied extensively. The difficulty is that the equation is of the nonlinear type $\Delta f + \frac{d-2}{4(d-1)}\mathrm{scal} = \mathrm{f}^{(\mathrm{d}+2)/(\mathrm{d}-2)}$, where the unknown conformity function f comes with the "limit" exponent for the Sobolev embedding. For larger or smaller exponents standard results are available but they are of no use here. Today the Yamabe problem is almost completely understood thanks to basic contributions by Aubin and Schoen. We refer to the surveys (Besse, 1987), 4.D, (Hebey, 1993), (Hebey, 1995) and the references therein. We just note that the set of metrics of constant scalar curvature is always an infinite dimensional space as soon as $d > 2$ and this is definitely not what we want when looking only for a "best" metric, see also (Anderson, 1997).

✦✦

We now come back to Einstein manifolds. What is the state of affairs today concerning the existence, uniqueness and/or moduli of Einstein metrics? This is a whole subject in itself; the book (Besse 1987) is devoted entirely to it and was up-to-date at that time. The introduction (Chapter 0) gives a good resumé. Since 1987 a few papers have appeared on the subject, very recent ones are (Eschenburg & Wang, 1997), (Boyer, Galicki & Mann, 1994a), (Boyer, Galicki, Mann & Rees, 1997) and the survey (Anderson, 1994). We now give very briefly the main results and the main open questions.

In dimension 4 there are some topological obstructions. The first type uses the Allendoerfer-Weil formula, which yields among other conditions the nonnegativity of the Euler characteristic χ seen above in B. With a more refined analysis of the integrand one can obtain the inequality $\chi(M) \geq \frac{3}{2}|\tau(M)|$ between the characteristic χ and the signature τ: (Thorpe, 1969), see Chapter 6 of (Besse, 1987) for more. The second type uses the minimal volume of Gromov and its inequality with the simplicial volume (see TOP. 1.D). Combining both techniques one can see that in some sense "many 4-dimensional manifolds do not admit any Einstein metric": see (Sambusetti, 1996), which uses also techniques from (Besson, Courtois & Gallot, 1995a), and (LeBrun, 1996).

On the other hand, the most baffling open question is that, starting with dimension 5, no obstruction to the existence of an Einstein metric is known on any compact manifold. This means that it is not out of the question that any compact differentiable manifold (with this dimensional condition) can be endowed with an Einstein metric. It seems that experts have only one guess today, namely that most manifolds ($d > 5$) will admit Einstein metrics but with reasonable singularities: (Anderson, 1994).

We turn now to the existing known examples and results. First come the locally irreducible symmetric spaces (in particular generalized space forms). The reason is that the isotropy group is irreducible and cannot have, up to a scalar, more than one invariant quadratic form: use diagonalization. For homogenous but non-symmetric spaces, even though in theory the curvature is algebraically computable, it can be a very subtle question to find Einstein homogeneous metrics, and it can even be proved to be impossible in some cases. For more on this, see a series of papers by Wang and Ziller quoted in (Besse, 1987) and (Wang, 1992). Needless to say that we are far from a classification of homogeneous Einstein manifolds. A recent text is (Kowalski & Vlasek, 1993), which in particular gives a complete classification of the Einstein and homogeneous metrics on the Aloff-Wallach spaces $W_{p,q}$; we met those examples in I.B.1.

Then comes quite a large variety of non-homogenous examples, obtained by different techniques: spaces which have a group action and are "more or less almost homogeneous", fiber bundles of various types, twistor theory (see TOP. 6), 3-Sasakian manifolds (TOP. 3.A). Today examples are becoming more and more numerous. We only give some references to texts that appeared after Besse's book. The very recent (Böhm, 1996) is important in two respects. First it gives among others non-standard Einstein metrics on *even*-dimensional spheres. The technique uses cohomogeneity-one-manifolds and the equations of (Bérard Bergery, 1982) for obtaining an Einstein metric. Secondly, it presents examples of infinite sequences of Einstein metrics on given compact manifolds with fixed volume which do not converge to any metric, and this shows that the Palais-Smale "condition C" (Palais & Smale, 1964) is not satisfied in general for Riemannian metrics. For this condition see (Jost, 1995) and see other types of such "counterexamples" in I.B.1. Before, in (Jensen, 1973), non-standard homogeneous Einstein metrics were constructed on the S^{4n+3}-spheres.

The subject of *uniqueness and classification* is still in its infancy. The above examples show the complexity of the topic. We meet here our favourite paradox: one knows that uniqueness holds for the compact space forms which are quotients of Hyp^4 and $Hyp\,\mathbf{C}P^2$. For the latter it is proved in (LeBrun, 1995), where the tools used are the new Witten invariants (TOP. 8). For Hyp^4 it is a byproduct of the strong result of (Besson, Courtois & Gallot, 1995a), which is quoted in V.B and in TOP. 1.D. An exceptional case is that of $K3$-surfaces. Using ingredients of various horizons and the Kähler facts just below, one has a complete description of all the Ricci flat metrics on them: see details in (Besse, 1987), 12. K .

An interesting question is that of the *signs* of Einstein manifolds, i.e. the sign of the proportionality factor (i.e. of the scalar curvature). For example it is plus, zero or minus in the case of space forms of positive, zero and negative curvature respectively. In (Catanese & LeBrun, 1998) one finds examples of $4k$-dimensional manifolds which admit at the same time two Einstein metrics, one with positive and one with negative sign. In fact each one is a Kählerian manifold in two completely different ways and one applies the results just below.

Non-compact Einstein manifolds are also studied, see among others (Heber, 1997), (Hitchin, 1995) and (Lanzendorf, 1997).

✦

We now turn to the only domain where things are very satisfactory, namely for *Kähler* manifolds (see TOP. 3.B). Moreover this topic turned out to be important in mathematical physics. The basic remark is the following: using the complex structure one can transform the symmetric differential form which is the Ricci curvature into an exterior form (of degree 2) which is automatically closed. It was part of mathematical folklore in the 50's that, by Chern formulas and the de Rham theorem, this form belongs to the first Chern class of the Kähler manifold under consideration. So we have an immediate necessary condition for a Kähler manifold to admit any Einstein metric: the first Chern class should have some de Rham representative 2-form which (via the complex structure) is either zero, positive definite or negative definite. This condition is not all that strong. Many algebraic manifolds satisfy it and it can be checked by using techniques that are standard today (see e.g. (Hirzeburch, 1966), (Griffiths & Harris, 1978)), which enable one to compute their Chern classes (see TOP. 3.B).

We explain briefly here why things are "workable" for Kähler manifolds. In fact the Kähler structure g yields an exterior closed 2-form ω of complex type (1,1) and Kähler variations in the same cohomology class are easily seen to be necessarily of the form $\omega + \sqrt{-1}\,\partial\bar{\partial}f$, where f is a numerical function on the manifold. The partial differential equation needed for f to yield an Einstein metric is of Monge-Ampére type and then hopefully solvable.

This led Calabi to conjecture in the pioneer paper (Calabi, 1954) that as soon as one has insured that the Chern class is zero or definite, one can change the initial Kähler metric into a new one which is finally Einstein. More even: in the negative case one has uniqueness. It was only in (Aubin, 1970) that any progress was made and in (Aubin, 1976) and (Yau, 1978) that the following was proven. Aubin proved the existence in the negative case, Yau proved it in the zero and the negative case. Then in (Futaki, 1983) an obstruction was found for the positive case. In (Tian, 1990) one has a definite answer for complex surfaces ($d = 4$): necessary and sufficient conditions to get Einstein Kähler metrics of the positive type. For larger dimensions, the latest result is (Real, 1996). A conjecture is formulated in (Tian, 1996), see also (Tian, 1997) and (Bando & Mabuchi, 1987). For the zero case and a proof following the Ricci flow see (Cao, 1985).

Calabi-Yau existence results are extremely useful. First to prove the uniqueness of the Kähler structure on $\mathbf{C}P^n$ (for any n): (Yau, 1977). Besides Yau's existence result one needs the profound text (Hirzebruch & Kodaira, 1957). For $\mathbf{C}P^2$ one even has uniqueness of the complex structure. Another application is the inequality $c_1^2 - 3c_2 \leq 0$ between the first two Chern classes of any Kähler manifold. For this and more see 11.B in (Besse, 1987). A still mysterious application is the fact that, for a Kähler manifold, the vanishing of the two first Chern classes c_1 and c_2 implies that the manifold can be made flat and in particular is a finite quotient of a torus: see the references and comments on page 67 of (Bourguignon, 1993). The mystery lies in the fact that nobody knows a direct proof.

To stay in the Kähler Einstein domain, we mention (Hulin, 1995). The author used the very interesting notion of *diastasis* introduced by Calabi in 1953, see the expository text (Berger, 1993b) and TOP. 3.B. She went on to prove some very strong results for the complex submanifolds of $\mathbf{C}P^n$ which are Einstein. The diastasis has been rarely used up to now but it might still have a significant role to play in the future.

✦

Moduli of Einstein metrics are not bad, in the sense that we know they are always finite-dimensional and real analytic stratified spaces. But they are known explicitly only in extremely few cases, the most spectacular being that of $K3$-surfaces seen just above. For all this see Chapter 12 of (Besse, 1987). In (Anderson, 1992a) the moduli space of Einstein metrics for any compact manifold of dimension 4 is studied thoroughly for the three possible signs. Special attention is given to the closure, here orbifolds (TOP. 9) appear and in particular the above results for $K3$-surfaces are made more precise and (algebraic) surfaces with singularities appear on the scene.

Of special interest and completely mysterious is the case of Ricci flat metrics (the zero sign case), which one can call the *harmonic manifolds* in view of the formula in the digressions in 1. COMMENTS on I (not to be confounded with those in II.B). For all examples known today the holonomy group is a special one, namely $SU(n)$ and $Sp(n)$ in the Kähler case, then G_2 and Spin(7), which are very special, see TOP. 3.

Some final facts on Einstein manifolds: first of all they are, in a sense to be made more precise, always real analytic. This is a good example of an application of harmonic coordinates, see Section 5.E of (Besse, 1987) and the references therein. Secondly, many results more recent than 1987 exist on pinching and convergence for Einstein manifolds. Most of these are contained in the texts mentioned in the corresponding sections: the end of I.A.2 and I.C.1, 2. Non-compact Ricci flat manifolds are important and are studied exhaustively today, see TOP. 5 and the recent text (Cheeger & Tian, 1994).

In dimension 4 there is a notion weaker than that of Einstein, namely that of *anti-self-duality*. See the entire Chapter 13 of (Besse, 1987) and complement it with (Donaldson & Kronheimer, 1996), (Donaldson, 1996) and the expository papers (Gauduchon, 1992/93) or (Taubes, 1992). For the proof of Taubes' result special norms are introduced which are close to the various global norms we mentioned in different places in *I*.

D. Some topological closures

We come back to a strict interpretation of the title of the present section. In the space of all Riemannian metrics on a given M one looks, in the spirit of I.C.2, for the closure of the various subsets defined by some curvature condition. This is surveyed systematically in (Lohkamp, 1992). We just mention a few cases. The notations are obvious for those spaces, e.g. C^α-closure $(\text{Ricci}^{\geq\kappa}(M))$. Following (Gromov, Lafontaine & Pansu, 1981) the *RCT* theorems imply the strongest possible closure: d_{G-H} – closure $\mathrm{K}^{\geq \mathrm{k}}(\mathrm{M})) = \mathrm{K}^{\geq \mathrm{k}}(\mathrm{M})$ and the same is true for the case $\leq k$. This implies any desired C^α-smoothness. For a lower Ricci bound it is an open problem for d_{G-H} but true for C^0. For a lower scalar curvature bound things are a complete mystery, see I.B.3. For an upper bound, Lohkamp's result in I.B.5 can be written as C^0- closure$(\text{Ricci}^{\leq\alpha}(\mathrm{M})) = \{\text{all metrics on M}\}$. See (Gromov, 1991b), (Gromov, 1996) and also TOP. 9.

IV The Spectrum, the Eigenfunctions

Recall that on a Riemannian manifold we have a canonical differential operator Δ of second order operating on functions, namely just take the trace (with respect to the metric) of the Hessian (the second covariant derivative): $\Delta f = -\text{trace}_g(\text{Hess} f)$. Expressing this with the star-operator on differential forms (see TOP. 6.A) we get just $\Delta = - * d * d$. At the center m of normal coordinates this is written as $\Delta f(m) = -\sum_{i=1,\dots,d} \frac{\partial^2 f}{\partial x_i^2}(m)$. The Laplacian of Riemannian manifolds controls for them first the *heat* equation $\Delta f = -\frac{\partial f}{\partial t}$, where $f(m,t)$ is the temperature at the point m at time t. Also the *wave* equation $\Delta f = -\frac{\partial^2 f}{\partial t^2}$: the wave equation for the height $f(m,t)$ of "water" after time t at the point m, if you consider (M,g) more or less abstractly as a thin sheet of water. The equivalent language of geometrical optics may be preferred to the acoustic and/or tidal language and one may talk about the propagation of *light*. The wave equation can also be considered as describing (M, g) as an abstract vibrating object. Finally the Schrödinger equation uses *complex* valued functions. It is written as $h^2.\Delta f = i h \frac{\partial f}{\partial t}$ (where $i = \sqrt{-1}$ and h is the Planck constant). Very close in fact to the heat equation is *Brownian motion* on Riemannian manifolds, whose "propagation speed is the Ricci curvature". We can quote (Stroock, 1996), the text by Elworthy in (Diaconis, Elworthy, Nelson, Papanicolaou & Varadhan, 1985–87) and (Elworthy, 1988), (Pinsky, 1989), (Pinsky, 1990).

Despite its importance, the spectrum of Riemannian manifolds ("spectrum" is a short-cut for the study of the eigenvalues and the eigenfunctions of the operator Δ) does not seem to have interested mathematicians much before the texts (Minakshisundaram & Pleijel, 1949) and (Avakumovich, 1956). Some exceptions are the spherical harmonics and the computation of the spectrum of the complex projective space in (Cartan, 1931). Add the result of (Lichnerowicz, 1958) on λ_1, which will be commented on at the very end of this section. In brief, thanks to Δ, one has on every compact Riemannian manifold a well-defined Fourier analysis and, with the wave equation and the hard work of Hörmander, a Fourier transform. This is one reason, among others, for the success of the notion of Riemannian manifolds.

There are some books devoted to the topic. The first one was (Berger, Gauduchon & Mazet, 1971); some people still like it because it is a quick introduction to the topic. (Buser, 1992) is exhaustive for the case of Riemann surfaces (namely the curvature is a negative constant), (Chavel, 1984) and (Chavel, 1993) cover a good part of the "classical" material, then (Bérard, 1986), which, besides many things, contains a systematic bibliography, is up-to-date but only up until 1982, see also (Colin de Verdière, 1992). The books (Gilkey, 1995) and (Berline, Getzler & Vergne, 1992) concern the case of general elliptic operators, especially in connection with TOP. 6. (Hörmander, 1985) encompasses many things.

The *isospectral* problem was first around in the 60's, when the question of isospectrality was raised by Leon Green and soon answered in the negative in (Milnor, 1964), see (Buser, 1996) for a historical account. This paper launched a lot of activity to find examples of isospectral (non-isometric) manifolds, see below. But the paper that really triggered research was (McKean & Singer, 1967). The problem was to find out what information it is really possible to extract from the asymptotic expansion of the heat kernel obtained in (Minakshisundaram, 1953), see below. Apparently not much is extractable in a naive sense, but with the help of (Patodi, 1971) and the following paper in the same journal, people managed to link this with the already existing Atiyah-Singer index theorem. Then one got a complete set of results. The discovery of the η-invariant, see TOP. 6.A, was quite special, the Laplacian was, however, needed not only on functions but also on all exterior forms. This, in turn, triggered the development of the heat equation technique for various bundles over manifolds, as well as spinor bundles. We will not elaborate on this, instead we refer the reader to the books (Gilkey, 1995), (Berline, Getzler & Vergne, 1992) and (Lawson & Michelsohn, 1989). We met spinors in Section I.B.3 when the case of positive scalar curvature was investigated, see TOP. 6 for more. The techniques used for the above combine Analysis and topology.

We now come back down to the manifold itself and to the most natural question: describe the behavior of the spectrum as a subset of **R** under various conditions. First in general, then under some geometric conditions (like curvature, volume, diameter), finally as linked with the geodesic flow, in particular with ergodic theory for the geodesic flow and periodic geodesics. We will describe now what is roughly the state of affairs today.

Already in (Minakshisundaram & Pleijel, 1949), where a parametrix and the ζ-function of the spectrum $\zeta(s) = \sum_{\lambda \neq 0} \lambda^{-s}$ were used, one knew that the spectrum has an asymptotic behavior whose first term is given by the dimension and the total volume $\mathrm{Vol}(g)$ of the manifold:

$$N(\lambda) = \text{number of eigenvalues smaller than } \lambda$$
$$\sim_{\lambda\to\infty} \frac{\beta(d)}{(2\pi)^d}\mathrm{Vol(g)}\lambda^{d/2} + o(\lambda^{d/2})$$

(where $\beta(\mathrm{d})$ is the volume of the ball of radius 1 in $\mathbf{R}^d$)

This is the so-called Weyl's estimate, the name comes from (Weyl, 1911) (or his Complete Works), where this estimate is obtained for plane domains with boundary by cutting them into smaller and smaller pieces and using the minimax principle, see an exposition in III. C of (Bérard, 1986). But the second term of the asymptotic expansion is a completely different story, which we will address now.

In (Minakshisundaram, 1953) the so-called *heat kernel*

$$K(x,y,t) = \sum_i \exp(-\lambda_i t)\Phi_i(x)\Phi_i(y)$$

(where the λ_i are the eigenvalues and the Φ_i are corresponding orthonormalized eigenfunctions) was constructed in a way which yields immediately, when t tends to zero, an asymptotic expansion of the following form:

$$K(x,x,t) \sim \frac{1}{(4\pi t)^{d/2}}\left(\sum_{k=0}^{\infty} u_k(x)t^k\right)$$

The numerical functions $u_k(x)$ are given by universal (but not explicit) formulas involving, thanks to Elie Cartan's philosophy of normal coordinates (0.B), only the curvature tensor of (M,g) and its covariant derivatives at x. In particular

$$\sum_i \exp(-\lambda_i t) \sim_{t\to\infty} \frac{1}{(4\pi t)^{d/2}}(\mathrm{Vol(g)} + \mathrm{U_1 t} + \mathrm{U_2 t^2} + \ldots)$$

where $U_k = \int_M u_k(x)$.

This gave rise to great hope since the knowledge of the spectrum is equivalent to that of the series $\sum_i \exp(-\lambda_i t)$ (always converging for $t > 0$). Sadly enough these invariants U_k are of almost no direct use, except the first one which is the volume and the second one only in dimension 2. Indeed $u_1(x) = \frac{1}{6}\mathrm{scal}(x)$ and then the formula of the Gauss-Bonnet theorem (0.E) yields the Euler-Poincaré characteristic as a function of the spectrum. To recognize the weakness of the U_k in general, think of manifolds of constant curvature (space forms, e.g. flat manifolds). They will, by their universality, all have the same U_k for every k as soon as the volume is known. And the volume is certainly not enough to recover the manifold up to isometry (except in dimension 1 – it is already insufficient in dimension 2); see also Lohkamp's result below. However, although it is not of too much direct use, the question of what happens in higher dimensions was in fact a tremendous incentive. For example a query of the mid 60's was whether U_k is a topological invariant for every manifold of dimension $d = 2k$. This turned out to be false but see in TOP. 6 what other results one arrived at while working on this query. It is fair to say that the asymptotic expansion and its universal curvature form are of basic use in the finiteness and compactness results to be seen soon below.

Finally we remark that it is hard to extract any topological information from the spectrum (on functions) (except when the dimension is 2). To our knowledge there is no connection between the spectrum (only on functions) and the topology. But the story is different when using the spectrum of objects more general than functions, in particular all exterior forms, see TOP. 6. One reason for the case of dimension 2 being special is that the spectrum of functions determines the spectrum of all differential forms by duality (just use Hodge theory, 0.D and add the first Betti number for harmonic 1-forms).

If one knows the asymptotic behavior of the heat kernel for pairs of points, one can recover the metric by the formula in (Varadhan, 1967): $\lim_{t\to 0} t \log K(x, y, t) = -\frac{d^2(x,y)}{2}$. This formula is valid for points close enough, but at the cut-locus (TOP. 4) things explode, typically for antipodal points on the sphere: see (Malliavin & Stroock, 1996). This formula does not seem to be useful for the various problems we met.

✦

Coming back now to the asymptotic value of $N(\lambda)$, a classical theorem of Analysis yields the Weyl asymptotic from the first term in that of $\sum_i \exp(-\lambda_i t)$, but only with a "small o". Note here that $\sum_i \exp(-\lambda_i t)$ is, as we will see, "practically weaker" (when using only its asymptotic expansion in t near zero) than the classical ζ-function $\zeta(s) = \sum_{\lambda\neq 0} \lambda^{-s}$. In (Avakumovich, 1956) (only for dimension three) and (Hörmander, 1968) in general (in fact for any elliptic operator with the obvious modifications in the powers involved) it was shown that the next order term is in fact a "capital o": $O(\lambda^{(d-1)/2})$. This is the best possible result as seen on standard spheres. This last result will control the *gaps* in the spectrum: they cannot be too large, there is some regularity. But Hörmander's result yielded only a non-explicit $O(\lambda^{(d-1)/2})$, it was only in (Gromov, 1996) that an explicit control on the gaps in terms of the geometry was first obtained. Gromov needs upper and lower bounds for the curvature and a lower bound for the injectivity radius and the dimension needs to be odd. This is not surprising in view of I.C.2. The proof is extremely involved and indirect, this is in contrast to Hörmander's proof, based on the wave equation. The question of obtaining the desired control with the wave equation is still open today. Even in the case of flat tori a fine study of the gaps is extremely subtle, see (Sarnak, 1997b).

It is important to realize that an *a-priori*-control of the gaps requires some geometric conditions as (Colin de Verdière, 1987) proved the following basic fact: given any manifold of dimension three or more, one can always find on it a metric for which the beginning of the spectrum is any finite given subset of $\mathbf{R}^+$ (including any desired multiplicities). In (Lohkamp, 1996b) much more is done: any finite beginning of the spectrum can be given, but with strong extra conditions, like negative Ricci curvature but also the dramatic killing of the U_k above: one realizes any infinite "spectrum" by successive finite exhausting parts such that the so obtained sequence of metrics has all the U_{2k} tending to $+\infty$ and at the same time all the U_{2k+1} tending to $-\infty$.

This does not touch the question of finding sufficient conditions for a sequence to be the spectrum of some manifold, the question is studied, to our knowledge, only in (Omori, 1983).

✦

In the spirit of control theory like in the digression in I.A.2, one can expect to be able to get both upper and lower bounds for the eigenvalues from various conditions on the geometric invariants of the metric. There is a very satisfying answer which started with (Gromov, 1980) and was finished in (Bérard, Besson & Gallot, 1985). More references are to be found in the book (Bérard, 1986). Upper bounds are obtained using only the volume and a lower bound for the Ricci curvature. Lower bounds require a lower bound for the Ricci curvature and the diameter. Examples show that this is optimal as far as the ingredients are concerned, only the explicit constants leave room for improvement. Both bounds are in terms of $\lambda^{2/d}$, in agreement with Weyl's asymptotic. For the first eigenvalue λ_1 this sharpens considerably the preceding "classical" inequalities of Lichnerowicz and Cheeger, see the very end of this chapter.

The proof uses a function-symmetrization of the heat kernel, in order to compare it with the heat kernel of the comparison space of constant curvature. On a simple function, this technique was introduced in (Faber, 1923) and (Krahn, 1924). So in fact the result of (Bérard, Besson & Gallot, 1985) yields much more, since it controls $K(x, x, t)$ at every point and for all t. Another basic tool for the proof is the control of isoperimetric inequalities in a Riemannian manifold, which was achieved also in (Bérard, Besson & Gallot, 1985), controlling the so-called "isoperimetric profile" with the help of only a lower bound for the Ricci curvature and the diameter. These ingredients are optimal, only the best constants still need to be found. For more on the isoperimetric profile, see TOP. 1.B. For surfaces, there are relations with the area, this started with (Hersch, 1970), see now (Bär, 1997).

In the spirit of Chapter I the work (Gallot, 1987) improved upon the preceding results on the eigenvalues by replacing uniform lower bounds on the Ricci curvature with integral norms. The basic tool is still the isoperimetric profile and the work consists in extending the control of this using only integral norms on the Ricci curvature, see TOP. 1.

✦

The link with periodic geodesics came first in (Colin de Verdière, 1973). One uses here the heat equation with complex time, which is the Schrödinger equation in disguise. But the simpler and more essential wave equation explanation appeared just after in (Chazarain, 1974). Immediately afterwards came (Duistermaat & Guillemin, 1975). The first essential link is the following: the distribution $\sum_{k\geq 0}\exp(\pm i\sqrt{\lambda_k}\,t)$ (this is luckily never a converging series, unlike $\sum_k \exp(-\lambda_k t)$) has a *singular support* contained in the set $\mathcal{L}$ of the lengths of the periodic geodesics (including 0). A very caricatural heuristic explanation is that the waves emanating from a periodic geodesic with a good (bad) frequency in $2\pi N/\text{length}$ will give birth to some kind of resonance (a tidal wave). The explicit solutions for the $\mathbf{K}P^n$ are in (Bunke & Olbrich, 1994). If the set $\mathcal{L}$ is discrete and if the sets of periodic geodesics of a given length are "well organized", then there is a Poisson type formula: $\sum_{k\geq 0}\exp(\pm i\sqrt{\lambda_k}\,t) = \sum_{L\in\mathcal{L}} T_L$, where the T_L are distributions with a singularity in L which can described by an asymptotic expansion, see more in (Chazarain, 1974).

This is a result in just one direction. In the special case of space forms, one gets much more with the help of Selberg's trace formula: see (Iwaniec, 1975) and (Hehjal, 1976). But in general it is only in dimension 2 that results are very strong, see (Huber, 1961). The reason is that one needs to know not only the length of the periodic geodesic, but also what happens when one makes one turn around it. This involves two things: the Poincaré return map and the holonomy (parallel transport). Recently some progress in higher dimensions was made in (Guillemin, 1993), see also (Guillemin, 1996). There it is proved that the spectrum completely determines the symplectic invariants of the Birkhoff canonical form for the Poincaré map. A spectacular result, even though it does not solve the problem for manifolds all of whose geodesics are closed (see V.B), tells us that this property is equivalent to the fact that the eigenvalues are concentrated in intervals in arithmetic progression: (Colin de Verdière, 1979).

There are many links between the spectrum (for functions) and the length spectrum (the set of the lengths of all periodic geodesics). The belief which was expressed in (Balian & Bloch, 1972) is that the oscillations in the eigenvalue density in the spectrum are related in some sense to the length spectrum. The jumps could be understood, for example, roughly as deviations from Weyl's asymptotic law.

An interesting link between the spectrum and the periodic geodesics is the notion of *quasi-modes*. The story started in (Babich & Lazutkin, 1967) and is far from finished today, instead it remains quite mysterious, see (Colin de Verdière, 1977). Briefly speaking, one can associate with a given periodic geodesic (satisfying certain conditions) a series of numbers which approach quite a lot of eigenvalues. The idea of the proof is to build up approximate solutions of the wave equation which will propagate along the geodesic. It is natural to ask whether there are many cases for which one can obtain the whole spectrum in that way. The answer is that it is an exception and happens only when the geodesic flow is integrable. In general the hyperbolic zones between the KAM tori will yield a contradiction, the entire book (Lazutkin, 1993) is devoted to this topic.

✦

Those were *direct* problems: One knows the manifolds and some of its invariants. What can one say about the spectrum? *Inverse* problems are of the following form: one knows various things about the spectrum – what can one recover of the metric? This can also be seen as a "recognition" problem. The book (Andersson & Lapidus, 1997) can be used as a survey.

The first question is that of *uniqueness:* are two isospectral manifolds necessarily isometric? The first counter-example appeared in (Milnor, 1964): two flat tori of dimension 16, quotients of $\mathbf{R}^{16}$ by two lattices famous in Number Theory: E_{16} and $E_8 + E_8$. Further questions are the following ones: Is the set of isospectral metrics compact? Are generic Riemannian manifolds determined by their spectrum? Concerning an eventual compactness of isospectral sets there are today only partial results in dimensions 2, 3 and 4: (Osgood, Phillips & Sarnak, 1989), (Anderson, 1991) and (Brooks, Perry & Petersen, 1994). For topological finiteness of isospectral sets, see (Brooks, Perry & Petersen, 1990) and (Brooks, Perry & Petersen, 1992). One introduces the ζ-function: $\zeta(s) = \sum_{\lambda \neq 0} \lambda^{-s}$ and one can manage by Analysis to define its derivative $\zeta'(0)$ at the origin. This value is linked to a suitably defined *determinant* of the spectrum, which is formally defined as the infinite product $\prod_i \lambda_i$. Note that the determinant is studied as a functional when trying to obtain extremal metrics in (Sarnak, 1997a), see III.A. The curvature nature of the asymptotic expansion of the heat kernel is also of basic use. The determinant is also fundamental in (Osgood, Phillips & Sarnak, 1988), where the classical conformal representation theorem for compact surfaces is proved in a Riemannian way using the Ricci flow (see 0.F and (Chow, 1991) for the final proof). For Riemann surfaces (surfaces of constant curvature) there are beautiful explicit formulas relating this determinant to the geometry of the surface, see (Pollicott & Rocha, 1997).

The generic situation is completely open today, except within the very restricted class of Riemann surfaces. In dimension 2 the conjecture is the finiteness of isospectral sets: (Sarnak, 1997c). The subject of uniqueness is especially irritating, one can prove it today only for $\mathbf{R}P^2$, 2-dimensional flat tori and spheres up to dimension 6: (Tanno, 1980). The proof uses only Minakshisundaram's asymptotic expansion of the heat kernel, which is too weak when the dimension gets large.

For isospectrality the results extend from 1964 up until today with still much ongoing research. The main aim is to find more and more examples of less and less geometrically special isospectral manifolds, and even one-parameter isospectral deformations. A survey is (Bérard, 1989). Important intermediate texts were (Vignéras, 1980b), (Sunada, 1985). See (Gordon & Mao, 1994), (Gordon, 1994), (Pesce, 1992), (Gornet, 1998) and the references therein. Two major questions remain: Are *generic* Riemannian manifolds spectrally isolated (solitary)? Are isospectral manifolds of only finite topological type possible?

For some special cases one has finiteness or non-deformability results: for Riemann surfaces see Chapter 13 of (Buser, 1992), for "solitary surfaces" (Buser, 1996) and for negatively curved surfaces (Guillemin & Kazhdan, 1980). This last text puts together the fact seen above that the spectrum determines the length of periodic geodesics and the fact that negative curvature implies a lot of well distributed periodic geodesics. Then one uses a Fourier analysis of the derivative of the metric as a function on the unit tangent bundle. This beautiful technique does not seem to have been used much since.

This is the place to remark that the wave equation technique used in these results combines both *microlocal analysis* (which is essentially Analysis but in the tangent (phase) bundle) and *symplectic* geometry. Strictly speaking, the symplectic structure is defined on the cotangent bundle T^*M; on UM one has only what is called a contact structure and more precisely a *Sasakian* structure, see definition 2.1 in (Boyer, Galicki & Mann, 1994a).

✦

Another important and natural question is to know whether the ergodicity (V.B) of the geodesic flow does or does not imply a very good regularity for both the spectrum and the eigenfunctions. Is the spectrum very well distributed? For the sake of simplicity let us look only at the case of dimension 2 because then Weyl's asymptotic is linear: here $d/2 = 2/2 = 1$. One then searches for a probabilistic result on the variance of the quantities $|N(\lambda + L) - N(\lambda) - L|$, suitably averaged in λ and L.

It was conjectured in (Pandey, Bohigas & Giannoni, 1984) that the spectrum is that of a GOE (i.e. a Gaussian random symmetric matrix) when the geodesic flow is ergodic. Today there is no theoretical general answer. There are many numerical experiments by theoretical physicists, see also (Lazutkin, 1993). At the other end stands the integrable case, see (Sarnak, 1997b).

From some numerical experiments it seems that the distribution could be other than GOE for some arithmetic Riemann surfaces. This was mathematically proved in (Luo & Sarnak, 1996). However the geodesic flow is ergodic. So the belief today is that the distribution will still be GOE for *generic* Riemann surfaces. To explain heuristically why arithmetic forms form an exception, one remembers what was said above, namely that the jumps in the spectrum are linked with the structure of the length spectrum. But precisely the length spectrum of arithmetic forms is very "degenerate" in the sense that the lengths are given (after normalization) by integers. The asymptotic exponential behaviour (see V.A) then forces all these periodic geodesics to have very large multiplicities, hence the huge jumps in the length spectrum. Relevant texts are (Sarnak, 1995), (Luo & Sarnak , 1995), (Luo & Sarnak, 1996), (Rudnick & Sarnak, 1996) and the references therein.

✦

Here is another question: Are the nodal sets (the sets of points where the eigenfunction vanishes) evenly distributed? Even more important is the following one: Are the eigenfunctions evenly distributed? This has been proved up to now only for a subset of the eigenfunctions (but, however, of full density) in (Colin de Verdière, 1985), which follows ideas of (Schnirelman, 1974). See also (Zelditch, 1987) and (Zelditch, 1992) for the non-compact case.

This is linked with the question of "scars": in a numerical experiment, it was found that the nodal lines of some surfaces were, in some sense, accumulating along some periodic geodesics. But in (Sarnak, 1995) it is proved that this can never happen for arithmetic space forms (for a certain definition of a *scar*). This is an amusing paradox: the arithmetic case implies more regularity, at the same time it is a less common case (in the realm of space forms). The question of what the general state of affairs is still divides experts, since scarring is today almost purely experimental and the definition of scars varies according to authors. See (Rudnick & Sarnak, 1996), (Shimizu & Shudo, 1995) and the references therein.

For the volume of the nodal sets (i.e. the set of zeroes of an eigenfunction) a conjecture by Yau in 1972 on the asymptotic behavior of their volume was partially solved in (Donnelly & Fefferman, 1988). One has $c(g)\sqrt{\lambda} \leq \mathrm{Vol}(\Phi_\lambda^{-1}(0)) \leq c'(g)\sqrt{\lambda}$ for every eigenfunction Φ_λ attached to the eigenvalue λ. The story is not finished because today one first needs analyticity and secondly the constants $c(g)$ and $c'(g)$ are not explicit, whereas one would like to have for them an explicit estimation based on Riemannian invariants. See also (Donnelly & Fefferman, 1990). The intuitive idea behind Yau's conjecture was the following: eigenfunctions for λ behave roughly like polynomials of degree $\sqrt{\lambda}$, which is the case for the standard sphere, for which the eigenfunctions are the restrictions to the sphere of the harmonic polynomials of the Euclidean space.

✦

Another paradox is this: first, as we will see in the next chapter, the sole existence of periodic geodesics is an almost completely open question, whereas, at the same time, we have some very good general results for the spectrum. But in the case of ergodic manifolds, especially for manifolds of negative curvature and even more specially for space forms, one has on the contrary a very good control of the periodic geodesics: an asymptotic expansion of the length counting function, a regular distribution in space and even more so in phase. We have just seen that, in exchange, for these space forms both the eigenvalues and the eigenfunctions are still not completely understood. An open problem is the question of the multiplicty of these various lengths.

✦

One may ask oneself why we are so much interested in the asymptotic properties and not in the individual eigenvalues. There is more than one reason for that. The first one is that, most often, only asymptotic properties are accessible, recall Colin de Verdière's result above: any finite part of the reals can be realized as the beginning of the spectrum of some Riemannian manifold. Then in Physics asymptotics are highly significant in quantum theory. A basic question is to understand to what extent classical and quantum mechanics are linked. The hope is that the classical case is the limit of the quantum one when the Planck constant h tends to zero in the Schrödinger equation $h^2.\Delta f = ih\frac{\partial f}{\partial t}$. One sees that this amounts to studying what happens for large eigenvalues. The various studies connected with this type of question are called *semi-classical analysis.*

✦

On the other hand it is also extremely important to have a good lower control on the first non-zero eigenvalue λ_1 depending on the geometry. This is important for example in order to exclude some possible "resonances". The first inequality came with (Lichnerowicz, 1958) (see page 135): from a positive lower bound on the Ricci curvature one deduces a lower bound for λ_1 which is optimal. Equality holds only for the standard sphere (Obata, 1962). For more references see (Croke, 1982) which has more, namely a pinching type theorem. Lichnerowicz's proof used just Bochner's formula given in 0.F, but his bound appears in many contexts. A typical example of its application is that by the geometer when studying distance functions, as in Colding's triangle theorem seen in Section I.A.1. Thereafter came (Cheeger, 1970b), which was concerned with the isoperimetric profile (TOP. 1.B) and its minimum value h_c over $[0, \frac{1}{2}]$: $\lambda_1 > \frac{1}{4}h_c^2$ and had a great impact. Now the inequality of Bérard et al. seen above is much stronger and this is not surprising in view of Cheeger's inequality since the authors' estimation is obtained using their bound for the whole isoperimetric profile $h(t)$.

The nature of the first eigenvalue for Riemann surfaces and their locations with respect to $\frac{1}{4}$ is a fascinating and fertile topic, see a description in (Buser, 1992).

For other spectra and operators, see TOP. 6.A.

V Periodic Geodesics, the Geodesic Flow

A. Periodic geodesics

Here one has some good results but also some quite irritating unsolved questions (except in the negative curvature case). A very good survey is (Bangert, 1985) and there is the book (Klingenberg, 1978a) (Lemma 4.3.4 contains gaps, as do other statements, but there is a Russian translation with critical annotations). See also (Klingenberg, 1982). Note that many authors use the word "*closed*" instead of periodic, but this can be misleading when one thinks of geodesic loops. For the point of view of classical mechanics periodic geodesics play the role of the eigenfunctions of the Laplacian and their length that of the eigenvalues, hence the name "length spectrum" below. They represent the stationary moves, the length being the energy (its square, strictly speaking), so that their importance cannot be overestimated, they could even replace Fourier analysis. Hadamard said that they were "a kind of coordinate system in which all the other geodesics can be expressed".

First let us state what seems reasonable to ask. One would expect first, like for the spectrum of the Laplacian, the existence of infinitely many periodic geodesics on any compact manifold. This means "*geometrically distinct*", one does not consider as geometrically different two periodic geodesics which are both an iteration of a common periodic one, hence having the same geometric support (for the physicist they will be different, as having different lengths, hence different energy levels). Secondly, assuming such an existence, what is the growth of the counting length function? The *counting function* $CF(L)$ is the number of periodic geodesics of length smaller than L. One can expect different results depending on the geometry of the manifold. Note that, if the counting function shows exponential growth, then the iterates will not appear because they make only linear contributions to the counting function; this will be important in the continuation.

Before giving the positive existing results, we draw attention to the fact that it is not known today, in all generality, whether there is more than one periodic geodesic for every manifold of dimension larger than 2 (this starts with S^3), so that the counting function question does not even make sense. Also one thing is sure due to (Weinstein, 1970): in the general case, the periodic geodesics are not dense in the phase space. But they could still always be dense in the manifold itself, this is an open question.

✦

Two points will help the reader realize the difficulty of the problem and make clear that one cannot work purely with topology, i.e. the metric enters essentially into the game. First, the following surprising theorem in (Morse, 1934) shows how weird the set of periodic geodesics can be in an apparently very nice manifold. Given any length L (think of it as very large) there exists an $\varepsilon(L) > 0$ with the following property. Assume an ellipsoid in $\mathbf{R}^d$ has all its principal axes $a_1 < a_2 < \ldots < a_d$ satisfying $|1 - a_i| < \varepsilon(L)$, then the only periodic geodesics of length smaller than L are the $d(d-1)/2$ intersections with the 2-planes of coordinates. The proof given in (Klingenberg, 1982), Lemma 3.4.7, is just a subtle play with the Sturm-Liouville type equations for the coordinates of a geodesic. In some sense, except a finite number of them, all the other geodesics "disappear" as the ellipsoid approaches the sphere: they are "ghosts". This implies that it can be expensive to find them. Note also the paradox for the limit case, the standard sphere.

Secondly, an example due to Katok (you can find this in (Ziller, 1982), along with many generalizations), exhibits a Finsler (today still non-symmetric) metric on S^2 with only two periodic geodesics. But the free loop space structure is basically the same for Riemannian and for Finsler metrics. For Finsler metrics see TOP. 9.

✦

The initial positive fact is that, as soon as the manifold is compact, in any non-zero free homotopy class (these are the conjugacy classes of the fundamental group) there always exists at least one periodic geodesic: just take the shortest curve of the class (see 0.F). So there is (again) a radical difference between the simply connected case and the other extreme, the *negative curvature* one. Then we immediately have a huge number of periodic geodesics since π_1 is huge. And, in the case of space forms of negative curvature (those of rank 1), the results are beautiful. Back in (Katok, 1988) and (Katok, 1982) and thanks to conformal representation, it was proved that *any* metric on such a negative curvature surface has as many or more periodic geodesics than the constant curvature one. More precisely the counting function is *exponential*, with an exponent which is larger than that for the constant curvature case, unless we are precisely in this case. One can paraphrase this result by saying that these spaces are very susceptible animals, reacting very strongly: any time you touch them, even as slightly as possible, they develop exponentially more periodic geodesics. Inevitably "hidden" behind these statements are the various notions of *entropy*, see the digression in B below.

In (Besson, Courtois & Gallot 1995a) this result is generalized to any dimension. But today negativity of the curvature is still required. Things work for rank 1 symmetric spaces and their products but not today for higher ranks. This means as much suceptibility as described above. See B for details of the proof. The exact value of $CF(L)$ can be derived easily from the volume entropy, in the case of constant (negative) curvature in dimension d one has $\lim_{L\to\infty}\frac{1}{L}\log(CF(L)) = d-1$.

The way the periodic geodesics are distributed in those manifolds of negative curvature is a subtle question but more or less understood, see (Eberlein, Hamenstädt & Schroeder, 1990) and Section 3 of (Besse, 1994). One fact is that an even distribution characterizes space forms. See also for this and the next section the expository text (Pansu, 1990/91) and Chapter 20 of the fairly complete book (Katok & Hasselblatt, 1995).

As soon as one leaves the negative curvature realm, things are not understood (except what is said in I.B.4), for example the case of various metrics on tori seems intractable at the moment (see however (Bangert, 1988)). Via its conjugacy classes, the fundamental group will easily yield a counting function which is polynomial of degree equal to the dimension, but it might well be that the counting function is always exponential (at least for dimensions larger than 2).

✦✦

A digression: Geodesics joining two points. We comment here in an indirect way on the difficulty of obtaining periodic geodesics, and also give some interesting results concerning the two-point case. Morse's theorem above on ellipsoids was one remark. To sense the other difficulties, let us look first before at the collection of geodesics joining two given points p and q. Morse theory is fundamental for that. In (Serre, 1951) algebraic topology and Morse theory yielded an infinite number of geodesics between any pair of points of a compact manifold. They might be all covering a periodic one (like for the standard sphere), but this situation is an exception (if it is true for any pair of points, it probably implies that we are in a manifold all of whose geodesics are periodic, see below for this); moreover this never happens for bumpy metrics (see below). In (Morse, 1934) this conclusion was obtained only for topological spheres. For more on the counting of geodesics between two points see (Mentges, 1987). The proof of Morse-Serre goes as follows.

In brief: between two given points, Morse theory (see (Klingenberg, 1978a)) provides at least as many geodesics of index equal to k as the kth Betti number $b_k(\Omega)$ of the space Ω of all curves in the manifold joining these two points (to save time we will completely skip the problem of which field the Betti numbers are considered on, see the references for this); this space of curves is denoted by the fixed letter Ω because it is the same as the space of loops pointed at some (any) given point. The *index* is the number of conjugate points between the two extremities but it is of no consequence *here.* Now Serre's contribution was that for any compact manifold there is an infinite number of non-zero $b_k(\Omega)$. A minor technical point here is that Morse theory requires the two points to be not conjugate (along any geodesic joining them). Two points are said to be conjugate along a geodesic joining them if there is a non-zero Jacobi field which vanishes at both ends. This is of no consequence at the end since, given any point, the set of its conjugate points is of zero measure. But in the case of periodic geodesics the difficulty is inevitable and a major one, besides the others to be explained below.

Now Morse theory is useless as it stands for the counting function, since it gives information on critical points as a function of their index but *not of their value*. So there is apparently no way of using one's knowledge of the Betti numbers of Ω to obtain some information on the counting function $CF(p,q,L)$ which counts the number of geodesics that join two points p and q and are of a length smaller than t. The pioneer result is to be found in (Gromov, 1978c): for a Riemannian manifold there are two positive constants a and b such that, for all points p and q one has $CF(p,q,L) \geq a\sum_{k\leq bL} b_k(\Omega)$.

It remains now to elicit from algebraic topologists what they know about the $b_k(\Omega)$, and the results are wonderful, even though not completely finished. We saw in I.B.1 that manifolds fall only into two classes, those in the first class are called rationally hyperbolic, the others rationally elliptic. And the Betti numbers of the rationally hyperbolic ones grow exponentially, so that one has the following strong statement: for any metric on any rationally hyperbolic manifold and all pairs of points p, q the counting function $CF(p,q,L)$ grows exponentially, with an exponent which can moreover be estimated by the Betti numbers of the loop space. It is important to note that one is sure to count geometrically different geodesics, since the iterates have a length growing linearly. Since hyperbolic manifolds form the majority, say a generic manifold is hyperbolic, then one knows now that, generically in the realm of Riemannian manifolds, the counting functions $CF(p,q,tL)$ grow exponentially.

For manifolds that are not necessarily hyperbolic, one knows from (Ziller, 1977) for symmetric spaces and (McCleary & Ziller, 1987) for general homogeneous spaces, that, for any metric, the couting functions grow polynomially, but if one wants to be sure of non-covering ones, one should subtract 1 from the degree of the poynomial, since the coverings grow linearly. For symmetric spaces the degree of the polynomial growth of the Betti numbers of the loop space is precisely equal to the rank.

✦

There is now a completely different way of estimating the counting functions, namely the formula which says that the double integral of the counting function is exponential with an exponent equal to the (topological) entropy of the geodesic flow: (Mañé, 1997) and (Paternain & Paternain, 1994). The other entropies will be defined below in Section B. Define the *geodesic entropy* $\boldsymbol{h}_{\mathbf{geod}}$ as

$$h_{\mathrm{geod}}(M,g) = \lim_{L\to\infty} \frac{1}{L} \int_{M\times M} \log(CF(p,q;L))\, dp\, dq,$$

where the counting function $CF(p,q,L)$ is the number of geodesics that join p and q and are of a length smaller than L. Then both Mañé and the Paternains proved that $h_{\mathrm{geod}} \leq h_{\mathrm{top}}$. The other inequality $h_{\mathrm{geod}} \geq h_{\mathrm{top}}$ is a particular case of a more general inequality for smooth flows in (Przytycki, 1980). The proof of the first inequality is strongly Riemannian, working on the product manifold $M \times M$. There the set of geodesics joining two points in M becomes the set of geodesics initiating orthogonally from the diagonal. To obtain this result, start from the middle point of the geodesic and go an equal length in both directions. This "naive" idea was already used in (Grove & Petersen, 1988). To get the entropy at the end one uses a strong theorem of (Yomdin, 1987) (see also (Gromov, 1987b)), where the computation of the entropy uses differentiability. Note that h_{geod} is for the Riemannian geometer a definition of the entropy of absolute simplicity. It is not clear today whether, possibly in some cases like negative curvature or genericity, one can just use the same definition but with only two fixed points (not taking a mean value). Or whether one can at least do so for almost all pairs of points. This is certainly not true in general because in (Burns & Paternain, 1996) a large set of metrics was found on S^2 for which the strict inequality

$$\limsup_{L\to\infty} \frac{1}{L} \log(CF(p,q;L)) < h_{\text{top}}$$

holds for an open set of pairs (p, q).

Note finally that the two results are quite distinct, since there is in general no direct link between the entropy and the topology. This is shown in (Lohkamp, 1998), where it is proved that any manifold admits some metric of positive topological entropy. But Gromov's result above, joined with the above formula, shows that any metric on a rationally hyperbolic manifold necessarily has positive entropy: this was known already in (Paternain, 1992).

✦✦

For obtaining periodic geodesics "à la Morse" following the scheme above, one has to replace the space of curves joining two points by the space of all closed curves in the manifold. Serre's theory still formally applies but there are two difficulties. The first one concerns the "non-degeneracy" of a periodic geodesic, which here plays the role of non-conjugacy for pairs of points. This can be overcome by considering only *bumpy* metrics, for which all desirable conditions are satisfied: all periodic geodesics are non-degenerate, their lengths are different, their indices are different, etc. For a precise definition and the result that bumpy metrics are dense in a reasonable sense in the set of all metrics, see (Klingenberg & Takens, 1972), (Anosov, 1983), (Klingenberg, 1978a) and (Rademacher, 1994a). Even if this difficulty is overcome there is the fact that this infinite number of periodic geodesics so obtained might be only the covering of a finite number of them! The second difficulty is that the space Ω^* of maps from the circle S^1 to a manifold is much more subtle than the set of pointed loops Ω. The job to do in algebraic topology is already treacherous. Many mathematicians made wrong statements concerning the topology of the space of closed curves in a manifold. Moreover, even a perfect knowledge ot this topology would not be enough, in view of the Morse and Katok examples, since the fact that one is "only" in a Finsler manifold does not make any difference at the level of the topology of the space of closed curves.

So the job is now to look more carefully *simultaneously* at Morse theory and at the geometry of the manifold. Indices here matter a lot. One should *decouple* indices and homology classes, looking at the behavior of the indices of the various iterates of periodic geodesics. The two basic tools here are the following: first, according to (Bott, 1956) and (Gromoll & Meyer, 1969a), the index of the iterates of a given periodic geodesic grows linearly plus or minus a fixed constant. The second tool is provided in (Gromov, 1978c), see also in book form (Gromov, 1998): for a bumpy metric $CF(L) \geq \frac{a}{L} \sum_{k \leq bL} b_k(\Omega^*)$. Here now is what can be done today: compared with the above $CF(p, q, L)$ case, the bumpiness is needed to insure the non-degeneracy and the division by L is needed to take care of the coverings, but note again that in the exponential case the $\frac{1}{L}$ will be swallowed by the exponential. This statement is improved upon in (Ballmann & Ziller, 1982) by replacing the sum by the sup of the Betti numbers.

Now it turns out that all the growth results above for the Betti numbers of the loop space Ω are valid also for the space Ω^*. So one has the following *double generic* result: for a generic (namely a rationally hyperbolic) manifold and a generic (bumpy) metric on it $CF(L)$ grows exponentially, moreover the exponent can be estimated with the Betti numbers of Ω^*. For bumpy metrics on homogeneous spaces, in particular globally symmetric ones, one has polynomial growth. It is an open question today whether the above results remain valid without bumpiness.

✦

We turn now to the *general case*, i.e. the metric is not necessarily bumpy. Then matters become extremely difficult for the reasons explained in general above, it is even extremely difficult to get just an infinite number of geometrically different periodic geodesics.

The case of surfaces is not too bad, the only problem arises for the *sphere*, the only simply connected compact surface, and this problem was addressed in (Poincaré, 1905). For the real projective plane $\mathbf{R}P^2$, one looks at its spherical two-sheet covering and projects to downstairs the periodic geodesics above. The problem of getting an infinite number of non-null-homotopic ones still remains open today. Despite his efforts Poincaré could not show the existence of at least one periodic geodesic. Birkhoff proved it in 1917 and this made him famous overnight. He succeeded by using a minimax principle applied to "tapestries" of the sphere and by reducing this infinite dimensional problem by approximating curves by broken geodesics (using the injectivity radius). This is now a finite-dimensional story and results can be obtained with the help of classical compactness results. This was the birth of Morse theory. Note that the Birkhoff geodesic is not necessarily *simple* (i.e. without self-intersection), see (Calabi & Cao, 1992).

In (Lusternik & Schnirelman, 1929) the existence of at least three simple geodesics was claimed, still on S^2. The topic is very treacherous. Things seem to be OK now after many incomplete proofs, see (Taimanov, 1993) and the analysis there. It is fair to say that the "intermediate" results gave true and important facts. Note in recent works the use of a deformation technique involving Analysis (see also 0.F), see the references on page 209 of (Jost, 1995). The idea was first used in the plane in (Gage & Hamilton, 1986). Deforming a plane curve by normal variations which are proportional to the curvature makes the curve look more and more like a circle ("the" curve of constant curvature). In a Riemannian manifold the curve will (hopefully, this has to be proved) have as a limit a curve of zero geodesic curvature, i.e. a geodesic. The equation is a parabolic PDE for which the existence of solutions for small time is quite easy, the greater part of the job consists in showing first the existence for any time and secondly, a good geometric behavior of the limit curve, see (Grayson, 1989).

The important event was the combination of (Bangert, 1993) and (Franks, 1992), which yielded at least an infinite number of periodic geodesics. This made popular writers rejoice: "a million rubber bands around a potato". Since the result was made more precise in (Hingston, 1993) one has even known that the counting function grows at least as a constant times $L/\log L$. There is also the problem of the real projective plane. But taking the sphere covering of it and bearing in mind that there is an infinite number of periodic geodesics on the sphere, those geodesics will project down and again yield an infinite number. There remains the problem of seeing whether there is an infinite number of periodic geodesics which are not null homotopic. The answer is not known today, but is probably yes following the proofs of Bangert and Franks (courtesy of Rademacher). To this day there is still not a single explicit surface of the type of the sphere, the projective plane, the torus or the Klein bottle for which the asymptotic order of the counting function is known exactly. No expert seems to have any guess even what the answer to the following question is: is it always polynomial, exponential, or generically exponential? Note that you can discard the question when a manifold, e.g. a surface of revolution or an ellipsoid, possesses a one-parameter continuous family of periodic geodesics (then necessarily locally of the same length).

✦

For higher dimensions, one had to await (Lusternik & Fet, 1951) just to get one periodic geodesic. The proof is in the Birkhoff spirit. There is no generalization of the Lusternik-Schnirelman three geodesics result to higher dimensions today, only with the strong extra hypothesis of $\frac{1}{4}$-pinching, see (Ballmann, Thorbergsson & Ziller, 1983) and do not forget Morse's theorem above. Thereafter came (Gromoll & Meyer, 1969a): one is sure to get infinitely many periodic geodesics as soon as the Betti numbers of the space of closed curves of our manifold are not bounded. The beautiful idea is to use the fact that the index of the iterates of periodic geodesic grows only very close to linearly when one iterates it. By contradiction a finite number of periodic geodesics will then force the Betti numbers to be bounded. But, as explained above, one has to handle the case of degenerate geodesics. The authors succeeded in doing so by a technical "tour de force", unfortunately this achievement is useless for overcoming the bumpiness when one desires an estimation of the counting function using the Betti numbers of Ω^*, this is one of the main questions today.

This leaves us with the topological problem of deciding when this Betti number condition is true, and this is part of the problem of classifying the rationally elliptic manifolds. The story is still not completely finished today, but it is almost true that the only exceptions are the positive curvature space forms, the spheres and the $\mathbf{K}P^n$. What a paradox: all the geodesics of these manifolds are periodic for their canonical metric! So we are left with the first example: on the three-dimensional sphere one still does not know whether there is, for any metric, always more than one periodic geodesic. For periodic geodesics of manifolds of the rational homotopy type of spheres or the $\mathbf{K}P^n$, see (Rademacher, 1994b).

In fact a real understanding of the situation requires more technicalities. One knows that, as soon as a periodic geodesic is of the so-called "twist type", there is an infinity of periodic geodesics in any tubular neighborhood of it, this was shown in (Birkhoff & Lewis, 1933). Moreover the counting function grows at least at the rate of prime numbers. So one of the main questions is perhaps not about an infinite number of geodesics but the following: does there exist on S^2 a metric all of whose geodesics are hyperbolic? The same question arises in fact for every simply connected manifold in any dimension, and one can also ask the same for only an open set of such metrics. Hyperbolicity for a periodic geodesic means that the Poincaré return map (met also in IV) has no eigenvalues of modulus equal to one, ellipticity means that all eigenvalues are of modulus one, the twisting condition involves a little more.

✦

In the absence of a general statement, we look at the *bumpy* situation on *any* manifold, i.e. one that is not necessarily rationally hyperbolic. Recently it was proved in (Rademacher, 1994a) that one can decouple the indices of periodic geodesics by small perturbations of the metric. Then one knows that, at least in the *generic* case, there always exist infinitely many periodic geodesics on any manifold.

Moreover the proof yields more. If one adds the validity of the conjecture above on the haunting $\mathbf{K}P^n$ and combines the results of Gromoll-Meyer, Hingston, Rademacher and Moser, then the counting function grows at least as a constant times $L/\log L$ in the generic case. The factor $L/\log L$, as well as the one above for surfaces, is not mysterious, it comes from the fact that one has to look finally at prime numbers in a given arithmetic progression. Dirichlet's theorem gives precisely such an order of magnitude. But, except in the rationally hyperbolic case above, nobody seems to have any guess whether things always (or in the generic case) grow faster, say like a polynomial of a degree possibly related to the dimension, or even exponentially, etc.

We finally mention quite a special result, which throws some light on both the difficulty of the problem of periodic geodesics and the necessity to use *complexity theory* when dealing with Riemannian manifolds. In (Nabutovsky, 1996a) one finds special finitely generated discrete groups which have the following property: for any Riemanian metric on any manifold whose fundamental group is such a group the counting function for periodic *contractible* geodesics grows exponentially.

B. The geodesic flow (geometry and dynamics)

The *geodesic flow* is the set made up of the collection of the G^t (where t is any real number) which act on the unit tangent bundle UM by traveling along geodesics of a length equal to t. More explicitly: $G^t(v)$ is the speed vector $\gamma_v'(t)$ at the point $\gamma_v(t)$ of the geodesic γ_v whose initial speed vector at 0 is $v = \gamma_v'(0)$. A fact that is fundamental for this entire study is that (cf. Liouville's theorem) the flow respects the canonical contact structure of UM, hence the attached measure. For the importance of contact structures in Mechanics, see (Arnold, 1988) and (Arnold, 1978). The symplectic structure is defined on the cotangent bundle T^*M (isomorphic to TM for Riemannian manifolds). So we have a natural and geometric dynamical system in Riemannian geometry. The inevitable invariant is the *entropy*, but there are two difficulties concerning this. First the definition is never very short, secondly there is more than one definition of entropy, see the digression at the end of this section and the general references given there.

Problems are of a direct or inverse (recognition) type. The direct one: knowing certain things about the manifold, what can we say about its geodesic flow? Inverse problems are of the following type: assuming the geodesic flow is known, what can be said about the geometry of the manifold? In particular there is the uniqueness problem: if two manifolds M and N have the same geodesic flow (the precise definition leads instead to the "conjugacy of geodesic flows") are they then isometric? A partial survey is Section 3 of (Besse, 1994) and a very detailed reference is (Eberlein, Hamenstädt & Schroeder, 1993). *Conjugacy* means the existence of a map $UM \to UN$ which commutes for every t with the two geodesic flows $G^t_M\colon UM \to UM$ and $G^t_N\colon UN \to UN$. Commutation with the projections $UM \to M$ and $UN \to N$ is not demanded and would be too strong to be of any interest. This means that, when flying, you are interested in the trajectory of the plane itself, but not in looking through the window to see above which point of the earth you are.

(Hopf, 1939) gave the first direct result: the geodesic flow of a compact surface of negative curvature is *ergodic* (this was Eberhard Hopf, not Heinz Hopf). Ergodicity means for a dynamical system that almost every trajectory is everywhere dense. It is equivalent (thanks to Birkhoff's main theorem) to the following extremely practical and useful definition: "space and time averages coincide". For entropies see the definitions at the end of this section and the references there. A much weaker notion is that of topological transitivity: there exists at least one trajectory which is everywhere dense.

In fact Hopf's result had a precursor in (Morse, 1921a) and (Morse, 1921b), where topological transitivity was obtained and secondly, even better, a coding for the trajectories. *Coding* is extremely important, it enables people to study the geodesic flow by looking only at the model called the "discrete shift". We mention only some recent references: (Parry & Pollicott, 1990), (Katok & Hasselblatt, 1995) and (Katok, 1996). H. Hopf's result was extended to any dimension in (Anosov, 1967) where it was proved moreover that the geodesic flow is stable (one can trace this type of question about "structural stability" to (Hadamard, 1901)). Anosov's proof is quite involved, even today. The best exposition is included in the appendix by Brin in (Ballmann, 1995).

In (Katok, 1982) for surfaces, and in (Besson, Courtois & Gallot, 1995a) for any dimension, there is a very strong statement: when one looks at various metrics on a locally symmetric space form of negative curvature (i.e. of rank 1), the topological entropy of the geodesic flow reaches a minimum for the locally symmetric metric and only for this metric. This strong result has many corollaries, we saw one of them for the case of Einstein metrics in III. Another one concerns the inverse problem of the counting function, see A above, these manifolds can also be recognized by counting their periodic geodesics (at least in the negative curvature case). As seen above in A one can present this as a "susceptibility" result.

✦

Below we will come back in detail to that result. We first address two questions stemming from Hopf's and Anosov's results. First: is negative (or nonpositive) curvature really needed to get ergodicity? Does one really need a manifold with a large fundamental group? Recent results seem to say no. We know now that there exist ergodic metrics on S^2 (Donnay, 1988) but it is still an open question whether one can obtain ergodic metrics in the case of positive curvature (a convex surface). There are real analytic convex surfaces (close to ellipsoids) with positive topological entropy: (Paternain, 1993). More generally there are C^2-deformations of ellipsoids in any dimension with positive topological entropy and exponential growth of the counting functions: (Petroll, 1996), see also (Knieper & Weiss, 1994). And one knows more: any compact manifold has some ergodic metric according to (Lohkamp, 1998). For what is meant by "chaos" see the end of the section. Matters are unsolved for the measure (metric, Liouville) entropy. It seems that the main question (see also A above) is to know whether S^2 can or cannot carry a metric all of whose geodesics are hyperbolic. This never happens for a $\frac{1}{4}$-pinching: (Ballmann, Thorbergsson & Ziller, 1982) and this implies that one will have to go quite far to find such a metric.

The second question is an inverse (recognition) problem in the spirit of IV. Does the length spectrum (the set of the lengths of all periodic geodesics) determine the Riemannian manifold? In all generality it certainly does not as we have counterexamples of isospectral surfaces and for surfaces the knowledge of the length spectrum and the knowledge of the ordinary spectrum are equivalent. The situation is radically different for the *marked length* spectrum. This means one "remembers" for lengths which free homotopy class they come from. And for manifolds of nonpositive curvature the equal marked length spectrum condition is equivalent to the conjugacy of the geodesic flow. Then many partial but strong results are available, in particular (Otal, 1990a). We refer the reader to Section (9.2) of (Eberlein, Hamenstädt & Schroeder, 1993) for more details, but note that in general one can have the same marked length spectrum and yet not have isometry: (Gornet, 1996).

✦

Let us now give some more details of (Besson, 1996) and (Besson, Courtois & Gallot, 1995a). In the various statements we will always assume the volume is normalized. The authors address the more general problem of maps of non-zero degree but we simplify matters by working on a given manifold (namely we pick up the identity map). Now let(M, g_0) be a compact space form of *negative* curvature (hence locally symmetric and of rank 1 by definition). The theorem is that for any other metric g on M the volume entropy $h_{\mathrm{vol}}(g)$ satisfies the inequality $h_{\mathrm{vol}}(g) \geq h_{\mathrm{vol}}(g_0)$ and equality holds only if $g = g_0$. Since the inequality $h_{\mathrm{top}} \geq h_{\mathrm{vol}}$ always holds with equality holding for nonpositive curvature, the same result is valid for h_{top}.

The very definition of the volume entropy and Bishop's theorem (TOP. 1.A) imply immediately the minimal volume result (see TOP. 1.4) in the case of constant sectional curvature. The result on counting periodic geodesics comes from the estimation of their counting function with the topological entropy and the fact that $h_{\mathrm{vol}} = h_{\mathrm{top}}$ in the nonpositive curvature case. The result on Einstein 4-manifolds (see II.B) is derived from the formula in 0.E for the characteristic, see the original text for other corollaries.

The more natural and conceptual proof would be to prove that the entropy is a strictly convex functional on the space of all Riemanian metrics and that the locally symmetric metric is a (the) critical point. Unfortunately this program does not work as such except in a conformal class, see (Robert, 1994). The original proof was quite involved but used nice techniques, like the center of mass for the structure of the sphere at infinity (see I.B.4) and the technique of calibration (see TOP.3.B). In (Besson, Courtois & Gallot, 1995b) the proof is greatly simplified by introducing the Patterson-Sullivan measures on the sphere at infinity: with these it is easier to work on the center of mass. Finally the result yields an extremely simple proof of Mostow's theorem (see II.A and B) for the case of rank 1. For an expository text, see (Pansu, 1996/97).

The theorem also shows that conjugacy of the geodesic flow to that of a space form implies isometry: one can recognize the compact space forms of rank one (and of negative type) by their geodesic flow. This is now the place to mention another recognition procedure for those forms among negatively curved manifolds.

✦

Basic to the proof of Anosov's main result are the stable and unstable foliations. For the Riemannian geometer they are the vector fields normal to a family of spheres centered at the same point at infinity. Equivalently one can prefer to look at Jacobi fields which are zero at infinity when given an initial condition: the stable one when they are zero at $+\infty$ and the unstable one when they are zero at $-\infty$. These foliations are always defined for negative curvature and are of course smooth for space forms. Historically they appeared first in (Hadamard, 1901). But smoothness when one looks from infinity is known to be a difficult game in Analysis. And this is dramatically illustrated now: if the curvature is negative then the smoothness of the Anosov foliations implies that we are metrically on a space form. This is a very strong recognition result, just by smoothness. After intermediate results (see (Eberlein, Hamenstädt & Schroeder, 1993)) this was obtained by combining (Besson, Courtois & Gallot, 1995a) and (Benoist, Foulon & Labourie, 1992).

✦

Back to the geodesic flow, in particular for tori, there were partial results, see e.g. (Croke, 1992). We mention the recent proof of a long standing conjecture stemming from the result in (Hopf, 1948) in dimension 2, to the effect that a metric without conjugate points on a torus (of any dimension) has to be flat: (Burago & Ivanov, 1994). We also recall here the strong results of I.B.4 which study the hard problem of distinguishing between manifolds of negative curvature and those of nonpositive curvature, see also Finsler spaces in TOP. 9.

In all the above results the proofs combine more or less classical results of ergodic theory with various geometric techniques. The notion of Busemann function was already mentioned previously. Variations of them can be described caricaturally as follows: one looks at the universal covering of the manifold of nonpositive curvature under consideration. It looks topologically like $\mathbf{R}^d$. One defines on it various kinds of structures "at infinity" and looks at the metric "from infinity". We met this technique already in Section I.B.4.

✦

The extreme opposite case is that of space forms of positive curvature, the spheres and the $\mathbf{K}P^n$. They have a beautiful geodesic flow, since all the geodesics are periodic, simple and of the same length (this is understood for their canonical metric): see II.C. They can be called the *harmonic oscillators* of Riemannian geometry. We saw in IV that this property of all geodesics being closed can be characterized by the fact that the spectrum of the Laplacian concentrates in intervals which are in geometric progression. However the *inverse* problem is almost completely open. The spheres and the $\mathbf{K}P^n$ behave very differently. Back in (Zoll, 1903) one finds on the two-dimensional sphere many examples of metrics all of whose geodesics are periodic (not isometric to the standard metric). They have been classified completely only for surfaces of revolution, (Besse, 1978), Theorem 4.13, and for metrics close enough to the standard one in (Guillemin, 1976) (see also Section 4.H of (Besse, 1978)). For higher dimensional spheres a very partial technical result is (Kiyohara, 1984), but the classification question is left untouched.

For the reader who might underestimate the difficulty of the subject, we mention a few facts. For more details and for some references see Besse's book. First, even though it is trivial to show that the lengths are all multiples of a smallest one, it is true but hard to show that the lengths are finally bounded. Worse: even though there are plenty of examples for which the lengths are different (not of the iterates of course), e.g. lense spaces and more generally any space form of curvature 1, it is an open question whether all the lengths are equal in the simply connected case. There is also the question of simple or self-intersecting periodic geodesics. The only result we know of is on S^2: (Gromoll & Grove, 1981) shows that then in fact all the lengths are equal and the geodesics are all simple. But the proof is very much two-dimensional and uses the three geodesic result of Lusternik-Schnirelman mentioned above.

On the other hand no metrics on any $\mathbf{K}P^n$ are known all of whose geodesics are periodic (except of course the standard one, see II.C). The case $\mathbf{K} = \mathbf{R}$ has been solved: only standard metrics have that property. This is a direct corollary (look at the universal covering) of the result on spheres all of whose cut-loci are reduced to a point, to be seen in TOP. 4. For the other **K**, we have only a partial result, i.e. this is true for metrics close enough to the standard one: (Tsukamoto, 1981). This is used in (Kiyohara, 1987) to prove the spectral isolation (solitude) of the $\mathbf{K}P^n$: the link with the spectrum is insured by the results quoted above concerning the connection between periodic geodesics and the spectrum. Other results of this "infinitesimal rigidity" type are to be found in (Gasqui & Goldschmidt, 1994) and (Gasqui & Goldschmidt, 1997).

✦

In the field "geometry and dynamics" a special mention should be given to (d'Ambra & Gromov, 1991), see also TOP. 9. Some geodesic flows are very special, they are called *integrable*. This is the case for surfaces of revolution as well as ellipsoids. For more on these, see (Spatzier, 1990).

✦✦

Digression: Entropies in Riemannian Geometry. Entropy is a difficult concept. Heuristically it is not difficult: it is nothing but the measurement of the exponential factor of anything you want to compute in a dynamical system: the loss of information when time goes by, the dispersion (divergence) of trajectories, etc. Strict definitions are less simple. Moreover there is more than one type of entropy. Two are standard: the metric (measure theoretic) and the topological one. Up to quite recently the labeling was ridiculous: metric entropy requires only a measure to be defined and is really a measure notion, topological entropy needs a metric (even if it is finally independent of it) and is really a topological notion. The third type of entropy (volume entropy) makes sense only for manifolds whose universal covering is "huge". Metric (measure theoretic) entropy was historically the first. General references are the following: up to 1982 (Walters, 1982) was the standard and efficient book, now there is (Katok & Hasselblatt, 1995). See (Sinai, 1976) for its very informative style, besides (Katok & Hasselblatt, 1995) and (Mané, 1987) especially for the smooth case. For references to results we refer mainly to the bibliographies there. For the geodesic entropy, see above.

Entropy is defined, on a compact space X endowed with a measure, for *discrete* dynamical systems $f: X \to X$, i.e. the map f is a measure preserving homeomorphism. In the case of the geodesic flow one takes $X = UM$ as the unit tangent bundle and f as $\boldsymbol{f} = \boldsymbol{G}^1$ (the geodesic flow for a length equal to 1). It does not matter that the "length" 1 looks rather special since in fact one is interested only in what happens for large iterates f^k and this means essentially that $t \to \infty$.

The simplest one to define is the *volume entropy* in compact Riemannian manifolds (M, g). Let M^* be its universal covering. We look at the metric balls $B(p, R)$ in M^* and set

$$h_{\mathrm{vol}}(g) = \lim_{R \to \infty} \frac{1}{R} \log(\mathrm{Vol}(\mathrm{B}(\mathrm{p}, \mathrm{R}))),$$

which is easily seen to exist and to be independent of the base point $p \in M^*$. Unfortunately it is of no interest unless M^* (that is $\pi_1(M)$) is "huge". For "smaller" fundamental groups and a polynomial renormalization, see (Babenko, 1992).

For a general dynamical system $f: X \to X$ endow X with any metric and look at the iterates f^k of f. One gets new metrics d_n for every integer n by setting $d_n(x,y) = \sup_{0 \leq k \leq n} d(f^k(x), f^k(y))$. Now let $N_n(\varepsilon)$ be the minimum number of ε-balls in the metric d_n needed to cover X and define the topological entropy to be

$$h_{\text{top}}(X;f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log(N_n(\varepsilon)).$$

It is easy to check that this exists and does not depend on the choice of metric on the compact manifold under consideration. We saw in *A how to compute it with geodesics joining two points.*

Finally the *metric entropy* (more and more frequently now called *measure theoretic* or just *measure*), which was historically the first to be defined, was defined by a mixing definition which was, however, quite lengthy and moreover almost impossible to compute explicitly in given situations. We still use an auxiliary metric which can be ignored in the end. This time we define $N_n(\varepsilon, \delta)$ to be the minimum number of ε-balls (still for the metric d_n) needed to cover some subset of X whose complement has a measure less than δ. Then one checks that the following definition makes sense:

$$h_{\text{met}}(X;f) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log(N_n(\varepsilon, \delta)).$$

These entropies enjoy many inequalities and properties in various instances. First, always $h_{\text{top}} \geq h_{\text{met}}$; then $h_{\text{vol}} \leq h_{\text{top}}$ and equality holds as soon as the curvature is nonpositive: see (Manning, 1979) and references there for intermediate results. The link with periodic geodesics is the following (see (Margulis, 1969)): if the curvature is negative then

$$h_{\text{top}} = \lim_{L \to \infty} \frac{1}{L} \log(CF(L)).$$

Do not be led to believe that positive entropy implies ergodicity, even locally, it only implies "some kind of local chaos". Metric (measure) entropy is the strongest invariant. For example positive metric entropy in a Riemannian manifold implies the existence of a set of positive measure where a geodesic is everywhere dense. This is definitely not the case for the topological entropy. In Riemannian geometry one also calls the metric (measure) entropy of the geodesic flow the *Liouville entropy*: $h_{\text{Liouville}}$.

We end with a formula giving explicitly a piece of Riemannian quantitative information on divergence. This is the formula of (Ballmann & Wojtlowski, 1989) for the geodesic flow of compact manifolds of nonpositive curvature:

$$h_{\text{Liouville}} \geq \int_{UM} \operatorname{trace}(\sqrt{-\mathrm{R}_v})\, \mathrm{dv},$$

where the integral is on unit vectors v and R_v denotes the linear map $u \to R(v,u)v$ in terms of the curvature tensor. The trace is taken with respect to the Riemannian metric. Moreover equality holds only for locally symmetric spaces. This formula was preceded by weaker ones, see the references in (7.3) of (Eberlein, Hamenstädt & Schroeder, 1993). In (Foulon, 1997) there is a generalization of the above formula to Finsler spaces (*TOP. 9*).

It is interesting to compare the formula above with the special "entropy" introduced by Hamilton for proving the standard conformal representation theorem for surfaces with the Ricci flow (see 0.F), namely this entropy is defined as $\int_M K \log K \, dm$, where K is the Gauss curvature: see (Chow, 1991).

TOP. Some other Riemannian Geometry topics of interest

1. Volumes

A. Bishop's Theorem

In (Bishop, 1963) manifolds M with Ricci $\geq (d-1)k$ are studied where k can be of any sign or zero. It is proved that the volume of any ball $B(p, R)$ of radius R is bounded by the volume of "the" ball $B^*(R, k)$ in the simply connected space form $M^*(k)$ of constant curvature equal to k and of the same dimension. This was stated for balls which lie within a coordinate ball. The following statement was stronger: the ratio $\mathrm{Vol}(B(p, R))/\mathrm{Vol}(B^*(R, k))$ is nonincreasing with R. The original proof was in the spirit of *RCT*. Modern proofs are simpler, they use the distance function "à la Gromov", see (Eschenburg, 1994) for a very detailed and nice proof. Despite its simplicity, the importance of Bishop's theorem should not be underestimated. We saw above many applications of it and the importance is underlined strongly by Gromov's following double remark. In (Gromov, Lafontaine & Pansu, 1981) it is proved (even if it is not difficult once Bishop's theorem is taken for granted) that the statement remains valid for any metric ball, that is to say one can go "beyond the injectivity radius". Then a lower bound on the Ricci curvature yields first an upper bound for the volume of balls. But the non-increasing property now also gives a lower bound once the diameter or the volume of the manifold is known, since the equality $\mathrm{Vol}(B(p, \mathrm{diameter})) = \mathrm{Vol}(M)$ always holds trivially. This is the key to many results, the most spectacular being the precompactness of I.C.2. Such a two-sided bound on the volume of balls using only a lower bound on the Ricci curvature should be compared with Colding's formula seen in the digression of I.A.2.

B. The isoperimetric profile

In IV we used the fact that there is for general Riemannian manifolds an isoperimetric inequality involving only the diameter and the lower bound on the Ricci curvature. The *isoperimetric profile* $h(t)$ of a compact Riemannian manifold (M^d, g) is defined as the map $h: [0, 1] \to \mathbf{R}_+$ where $h(t)$ is the infimum of the measures of the boundaries $\partial(\Omega)$ of all domains Ω in the manifold whose volume is $\mathrm{volume}(\Omega) = t.\mathrm{volume}(g)$. Note that $h(1-t) = h(t)$ (look at the complements). The domains have to be reasonable and then the measure of their boundary is any classical $(d-1)$-dimensional measure. It is in general impossible to know what the explicit domains realizing the infimum are in explicit (standard) manifolds. Typically they are not metric balls, so that Bishop's inequality is not of any help. What is important, and is very often enough for applications, is to have a lower bound for $h(t)$. Even a lower bound on the Ricci curvature (together with the diameter) is enough as was discovered in (Gromov, 1980). This was made more precise in (Bérard, Besson & Gallot, 1985), the case of a negative Ricci lower bound being more tricky because of the lack of an optimal comparison manifold. In the positive case one of course compares things with spheres. A description of the proof, which is very important to know as it is a very new technique, is given below. Moreover Gromov's result (called Lévy-Gromov in the literature), which is apparently pure Riemannian geometry, triggered a lot of applications in Banach spaces as well as in probability theory : see the report (Ledoux, 1992/93).

The basic tool here is *GMT* (see TOP. 10.C). It insures for us first the existence – under some reasonable extra conditions – of a submanifold (smooth or with reasonable singularities) which realizes the minimal volume in its homology or homotopy class. Regarding the isoperimetric profile it insures for us the existence of extremal domains. Gromov's idea was to fill up the "inside" of the domain Ω with the geodesics normal to its boundary $\partial(\Omega)$. One then gets an inequality between volume(Ω), volume$(\partial(\Omega))$ and the mean curvature η of $\partial(\Omega)$. This inequality, which is to be found in (Heintze & Karcher, 1978), generalizes Bishop's one and applies to the sheaf of geodesics orthogonal to a given hypersurface, involving only the mean curvature of the hypersurface and the lower Ricci bound. Here η is constant because *GMT* insures that the boundary is smooth almost everywhere and then its mean curvature is constant by the first variation formula for hypersurfaces. But this is apparently a dramatic situation since we have no information on this constant η. Gromov's amazing trick is to look both at Ω and its complement $M \setminus \Omega$ and then one is done because the effect is just to change η to $-\eta$.

For the amateur with regard to problems that are naive to state, seemingly obvious or at least simple but still open, we mention here the question of computing the exact value (and also if possible the shape of the optimal domains) for our preferred manifolds: the spheres S^d and the $\mathbf{K}P^n$. One can add the cubic flat tori. For the sphere the isoperimetric profile (then called "the isoperimetric inequality") has been known since (Schmidt, 1948–1949). Schmidt by the way also solved the problem for hyperbolic geometry (the simply connected space forms of constant negative curvature). The optimal domains are, as expected, metric balls. The tool used is symmetrization and today the best reference is (Burago & Zalgaller, 1988).

But symmetrization does not work any longer, not globally for the real projective spaces $\mathbf{R}P^d$ and not even locally for the other $\mathbf{K}P^n$. The problem has to this day been solved only for $\mathbf{R}P^3$ in (Ritoré & Ros, 1992). The solution is as follows: for small t the optimal domains are metric balls but, starting at a number t which is trivial to evaluate, the tubes around a periodic geodesic (a projective line) give a better answer. The proof for $\mathbf{R}P^3$ is complete because one is able to completely classify the constant mean curvature surfaces in $\mathbf{R}^3$ (these surfaces moreover have a nonnegative second variation and are then called "stable"). A reasonable conjecture is the following: in all the $\mathbf{K}P^n$ the isoperimetric profile is given by the succession of the tubular neighborhoods around a point, a projective line, a projective plane, etc.

In fact the isoperimetric profile is a subtle object, in (Pansu, 1997b), (Pansu, 1997a) one can find surprising results. In particular this one: in general this profile will not be smooth at zero.

✦

Another important control is that of λ_1 using Cheeger's more brutal older "h-constant" (which is the infimum h_c of $h(t)$ for t in $[0, 1/2]$): (Cheeger, 1970b). His inequality reads $\lambda_1 > \frac{1}{4}h_c^2$ and is optimal by (Buser, 1978) but never attained by a smooth object, see the very end of Chapter III.

The formula in (Savo, 1996) and (Savo, 1997) studies the volume of various tubes and then covers many results of the above type with a very nice proof: the second derivative with respect to the radius of the tube is computed and linked with the Laplacian.

Recently, in the spirit of many results in Riemannian geometry (some of them were met above), the article (Gallot, 1987) succeeded in controlling the isoperimetric profile using only integral bounds on the Ricci curvature, see this text for precise statements. There are many corollaries: finiteness theorems (see I.C.1) and a control on the spectrum (see IV). Moreover Gallot also controls the volume of tubes around hypersurfaces. A key tool, as we saw in I.C, would be to control the volume of tubes around geodesics. Using weaker hypotheses, in particular an integral Ricci bound, this was achieved in (Petersen, Shteingold & Wei, 1996). However examples by Eguchi-Hanson show that the volume of tubes cannot be controlled with only the Ricci curvature.

✦

A special mention should be given to the *nonpositive* (sectional) curvature case. The conjecture is the following: in a simply connected manifold of nonpositive curvature the isoperimetric profile is *super-Euclidean.* This means that for any compact domain Ω the ratio $\mathrm{Vol}^d(\partial(\Omega))/\mathrm{Vol}^{d-1}(\Omega)$ is larger than or equal to its value for balls in $\mathbf{R}^d$. Today matters are still mysterious. For surfaces this conjecture was probably stated by Paul Lévy in a talk at Hadamard's seminar in 1926, then immediately proved by André Weil, using conformal representation: (Weil, 1926). In (Croke, 1984) it was also proved for $d = 4$ using integral geometry and there is no hope of making it work in any other dimensions. For $d = 3$ this is the result of (Kleiner, 1992). Then in (Cao & Escobar, 1995) it was proved for piecewise linear (PL) manifolds of CAT(0) (see TOP. 9 for those manifolds) of any dimension. However it is not clear today whether any (smooth) Riemannian manifolds of nonpositive curvature can be approximated by PL-manifolds of CAT(0). The main difficulty is that *GMT* is no longer of use in a straightforward way: because of the lack of compactness, optimal domains can "escape at infinity".

✦

In Analysis the Sobolev inequalities are a basic tool, and they are used for many results quoted in this text. For a numerical function $f: M \to \mathbf{R}$ on the Riemannian manifold (M^d, g) and the integers (p, q) with $\frac{1}{p} + \frac{1}{d} = \frac{1}{q}$ they say that one always has the following inequality between the function and its gradient for the global L^p and L^q norms:

$$||f||_p \leq A||df||_q + B||f||_q$$

In most cases the Riemannian geometer is interested not in just having some A and B but in a control of them using the curvature, etc. The problem of finding the optimal A and B are a lot different. For B it is basically the work of Gallot, see the various references to this author above. For A it is the work of Aubin, see for example (Aubin, 1982). See the recent reference (Hebey, 1996) and its bibliography. We just mention that the control of the isoperimetric profile is basic to both works.

C. The embolic volume

This was alluded to in III.B. The simplest numerical and curvature-free functional invariant on the set of compact differentiable manifolds M is the embolic volume, namely the infimum $MinEmb(M)$ of the ratio $\mathrm{Vol}(g)/Inj^d(g)$, where g runs through all possible Riemannian metrics on M. Today one does not know whether there is some kind of convergence result for MinEmb which will insure the existence of a minimal metric on M, even if one admits metrics with (reasonable) singularities. However the situation is not completely desperate thanks to the following three results. First the very strong finiteness theorem of (Grove, Petersen & Wu, 1990–1991) says that (at least when $d \neq 3, 4$) the possible diffeomorphism types with MinEmb $< A$ are always finite, so that the various real numbers MinEmb can be considered as a nice *scale* using Riemannian geometry to evaluate the degree of complexity of a manifold. The second result is (Berger, 1980): MinEmb(M) is always larger than or equal to that of the standard sphere, namely $\beta(d)/\pi^d$ (where $\beta(d)$ denotes the volume of the standard sphere of dimension d) and equality holds for $\mathrm{Vol}(g)/Inj^d(M)$ only in the case of constant curvature. In particular any embolic volume is always positive. The third result is the isolation result in (Croke, 1988a): there is some $c > 0$ such that for any manifold M which is not a topological sphere one has MinEmb$(M) > \beta(d)/\pi^d + c$. It seems an interesting question to relate MinEmb(M) in various ways to the different topological invariants of M. Finally we cannot resist telling the reader that the exact value of the embolic volume for the $\mathbf{K}P^n$ is completely unknown. If it were equal to the value obtained for the standard metric, this would solve (or at least help a lot towards a solution of) the problem of characterizing the $\mathbf{K}P^n$ by their geodesic flow (see the end of V.B).

Of basic importance in various domains of Riemannian geometry is the local embolic result of (Croke, 1980), which tells us that, when the injectivity radius of the manifold is given, every ball with a radius of half the injectivity radius (or smaller) has a volume bounded from below by a constant which is universal in the dimension (and the radius). This was used above in many finiteness and convergence theorems of I.C. It is an open question whether the value of this universal constant is or is not always simply equal to the value it takes on the standard sphere. This of course would be the optimal value.

✦

D. The minimal volume

One natural functional to evaluate when one wants to get the "*least curved*" metric on a compact manifold is Gromov's *minimal volume*. *MinVol*(M) is the minimum of the volumes of all Riemannian metrics on M such that $-1 \leq K \leq 1$. Here this functional can be zero, think of flat tori or even nilmanifolds. More generally it is zero whenever the manifold M admits a sequence of collapsing metrics, see I.C.3. Thinking back to the finiteness results in I.C.1 for manifolds with $a < K < b$, diameter $< D$ and volume $> v > 0$, the importance of deciding if MinVol is positive or zero cannot be exaggerated. This is because $a \leq K \leq b$ can always be written as $-1 \leq K \leq 1$ after normalization when one is not interested in precise values or when one is looking for a *best metric* on a given manifold (see III). Finally a positive minimal volume is an insurance against collapsing. There is no general survey of the minimal volume, but one of the sections of (Fukaya, 1990) is devoted to it.

Apart from the case of surfaces where the Gauss-Bonnet theorem (0.E) tells us everything, the minimal volume remained quite mysterious up to 1996. Let us take a quick look at the state of affairs today. We saw previously that the main point is to know when the minimal volume is zero and when it is positive. We first remark (following Cheeger) that the minimal volume is positive as soon as the manifold has some non-zero characteristic number: this is thanks to the Chern integral formulas for characteristic classes seen in 0.E. Note that this works only in even dimensions and mainly in dimensions which are multiples of 4. In (Gromov, 1983b) it is proved that the minimal volume is non-zero for space forms (see II) of negative type and hence for many other manifolds, using various functorial properties of the minimal volume. But note that those are always highly non-simply connected. Today one still does not know any *odd-dimensional simply connected* manifold for which one can prove that its minimal volume is positive.

On the other hand (Gromov, 1983b) exhibited many manifolds with MinVol $= 0$. He started with manifolds where there is a free action of the circle, the first example (besides the obvious flat space forms) of course being the odd-dimensional sphere where the circle action is that of its Hopf fibration. Next, this was extended to various manifolds which laid the foundations for the study of collapsing, see I.C.3. So we are left with the basic question whether there are more manifolds with MinVol > 0 or more with MinVol $= 0$. The fact that MinVol $= 0$ implies the existence of an F-structure (see I.3.B) is not enough. We will see now that the recent work (Cheeger & Rong, 1996) seems to show that "most" manifolds have a minimal volume of zero.

A basic inequality in (Gromov, 1983b), see also (Besson, 1996), relates the minimal volume MinVol(M) to the *simplicial volume* $||M||$: for any manifold M^d

one has $\mathrm{MinVol}(M) > c(d)||M||$ with $c(d)$ universal in the dimension. The simplicial volume $||.||$ is defined in a purely topological way, namely by writing the fundamental class as a sum of simplices with real coefficients. However, do not expect to find many manifolds with MinVol > 0 with the help of this inequality: in fact the simplicial volume is such a deep invariant that, today, only manifolds admitting a metric of negative curvature, products of them and connected sums of them with anything else can be shown to have a non-zero simplicial volume. Many attempts have been made to prove that (compact) locally symmetric spaces of any rank, not only of rank 1, have a non-zero simplicial volume, see (Savage, 1982).

✦

Let us finish the MinVol story by addressing three questions. The *first* one is the following: what does the subset of the reals made up of the minimal volumes of all manifolds of a given dimension d look like? Is it a discrete set? For surfaces the Gauss-Bonnet formula tells us that our subset is the arithmetic progression $2\pi\mathbb{N}$. Discreteness at 0 is equivalent to the isolation of 0 (existence of a *gap*). This is true in dimension 3 and is stated implicitly in (Cheeger & Gromov, 1986–1990). On the other hand the proof of the isolation of 0 in dimension 4 obtained in (Rong, 1993) is extremely involved and not completely geometric. It uses the η-invariant (see TOP. 6) in a basic way. But for $d \geq 5$ it is an open question whether there is or is not a gap (one of Gromov's conjectures). In (Cheeger & Rong, 1996) the existence of a gap is proved when one restricts the problem to manifolds whose diameter is bounded. Namely, for any d and D there exists an $\varepsilon(d, D)$ such that $\mathrm{Vol} < \varepsilon(d, D)$, $|K| \leq 1$ and Diameter $< D$ imply MinVol $= 0$. By Cheeger's finiteness theorem (I.C.1), this implies moreover the discreteness of the set of all MinVol when the diameter is bounded. The problem is to know whether a gap and/or discreteness can be preserved as D tends to infinity. The proof uses collapsing theory (I.C.3) and consists in constructing, when $\mathrm{Vol} < \varepsilon(d, D)$, a polarized F-structure on the manifold. When the manifold is simply connected the result is an application of the previous collapsing results of Cheeger-Fukaya-Gromov, where one can obtain a pure-polarized structure.

The *second* question is the computation of the minimal volume of "standard" manifolds, e.g. space forms of different types. Matters are again dramatically different according to the sign of the curvature. For positive or nonnegative curvature not a single minimal volume (when it is non-zero of course) is known. It starts with S^4. The only known fact is a local result for the even-dimensional spheres in (Ville, 1987). Conversely, for negative curvature (hyperbolic) space forms one has an unbelievably strong result. As a corollary of the main theorem in (Besson, Courtois & Gallot, 1995a) (we already saw more than one application of it in III; the result is detailed in V.B) one has that any metric on a compact hyperbolic space form has a volume larger than or equal to that of the hyperbolic metric under the sole condition $\mathrm{Ricci} \geq -(d-1)$. Moreover it is equal to it if and only if the metric is the hyperbolic one. So we know once again that the hyperbolic metric on a negative space form is "the best one".

The last and *third* question concerns the relation of MinVol to the functional $g \to \int_M ||R||^{d/2}$ (see III.B). The minimum of this functional for various metrics on a given compact M is called $\mathbf{Min}(||\boldsymbol{R}||^{d/2})$ and, except for the case $d = 2$, it is of interest because of the Gauss-Bonnet formula. It plays a role in I.C. Trivially it is universally bounded by MinVol so that MinVol $= 0$ implies $\mathrm{Min}(||R||^{d/2}) = 0$. The converse should be obvious with the convergence theorem in $RM(d, -1, 1, v, D)$ since we can always normalize our sequence to within $-1 \leq K \leq 1$ and argue by contradiction. But we forgot about the diameter. Today the question of whether this converse is true or not remains open, despite many attemps.

E. The systolic story

In A above the game was to control volumes of different kinds with a control derived from the metric, e.g. curvatures of various types. The systolic story is entirely different: one wants to have inequalities for volumes which are independent of the metric, depending only on the topological structure of the manifolds. Here the results are basically complete, all major questions have been solved. Only a few points (which might be very hard however) remain open. The story, which we will now relate, is mainly Gromov's work. An almost complete survey is (Berger, 1993a) – add (Gromov, 1992a), which contains a lot of information, for a survey in book form see Chapter 4 together with Appendix D of (Gromov, 1998). For the use of systoles for the problem of characterizing the Jacobians of complex surfaces, see (Buser & Sarnak, 1994a) and (Gromov, 1992a); this characterization is the Schottky problem, see (Beauville, 1987).

The work on this topic started with Loewner's theorem (1949, unpublished). Define as the *systole* of a two-dimensional Riemannian torus (T^2, g) the infimum of the lengths of closed curves which are not contractible. Let Sys(g) be this positive real number. Then the area Area(g) of our torus satisfies the inequality $\mathrm{Area}(g) \geq \frac{\sqrt{3}}{2}\mathrm{Sys}^2(g)$ and equality holds only for the flat regular hexagonal torus. The proof is not too hard and rests essentially on conformal representation: when averaging the conformity function under the translations of the flat underlying structure the systole can, by its very definition, only increase and the area decreases by the Schwarz inequality.

✦

The challenge is obvious: generalize this to surfaces of higher genus, then to (non-simply connected) manifolds of higher dimension and also to higher dimensional systoles. Call $\mathbf{Sys}_k(\boldsymbol{g})$ the infimum of the volumes of all homologically nontrivial k-dimensional submanifolds (for some Riemannian metric g on M) of M. In this broader context René Thom's question to the present author in the 60's was to find out when there is some universal inequality between the different systoles, the total volume of M^d being nothing but $\mathrm{Sys}_d(g)$. In short we know today that we have an almost complete positive answer in the case of the 1-systoles and a negative answer for higher dimensional systoles. Let us be more precise in the next paragraph.

✦

For the 1-systole and surfaces of higher genus an answer appeared in (Accola, 1960) and independently in (Blatter, 1961), but the lower constant for $\mathrm{Area}/\mathrm{Sys}^2$ was very poor, that is to say it went against intuition since it tended to zero extremely fast with an increasing genus whereas one on the contrary expects some increase with an increase of the genus. For the special case of the real projective plane the expected optimal answer was obtained "à la Loewner" in (Pu, 1952). Nothing more appeared on the subject before (Gromov, 1983a), which gave first an optimal answer for surfaces (at least asymptotically with the genus γ) with a lower bound $c\gamma/\log^2\gamma$ (with c universal). The result is the same for the homotopy systole as for the homology systole. Secondly, for the homotopy 1-systole and for any dimension d Gromov proved the existence of a positive lower bound for $\mathrm{Vol}/\mathrm{Sys}^d$ for any reasonable manifold, namely for a class containing all acyclic ones and some more, like $\mathbf{R}P^d$. The technical definition is that of an *essential* manifold.

Gromov's proofs are very involved. The one for higher dimensional manifolds introduces many interesting new geometric invariants for Riemannian manifolds: the *filling radius* and the *filling volume.* Like the minimal and the embolic volumes their explicit values for standard manifolds are very difficult to compute explicitly. The latest state of affairs for the important filling radius is to be found in (Katz, 1983) and (Katz, 1991): this radius is known for the spheres, the real projective spaces and the complex projective plane, but still unknown for the remaining $\mathbf{K}P^n$. These invariants will certainly have a great role to play in Riemannian geometry in the future. For example the filling radius is used in (Greene & Petersen, 1992) to improve upon the recent finiteness theorems mentioned in I.C.1. The simplest filling volume, namely that for the circle S^1, was only obtained in (Katz, 1998).

Back to Gromov's proof, the game is to deduce various inequalities between the above invariants (even with a non-optimal constant) and relate them finally to the systole. The proof uses an important trick: embed a Riemannian manifold in a Banach space using the distance functions to various points (see TOP. 9) (even if during the proof one makes systematic use of finite dimensional approximations). The proof of optimal (in the ingredients) systolic inequalities for surfaces of high genus is even harder, it uses the notion of *simplicial volume* and a diffusion process for simplicial cycles. These two ingredients are basic to the results we mentioned on the minimal volume in (Gromov, 1983b), see D above.

We are left with three questions. The first one is to find the optimal value and to decide whether "extremal metrics" exist and which they are. For this, besides the "Filling paper" (Gromov, 1983a), see (Calabi, 1992) and (Gromov, 1992a). The second one is to characterize the manifolds for which we have such a universal inequality. This is done in (Babenko, 1992) for the *homotopical* $\mathrm{Sys}_{1,\pi}$ associated with the fundamental group. Such a result can be seen as a "systolic" characterization of a topological property. This is also valid for the *homological* systole $\mathrm{Sys}_{1,H}$ (but does not figure in the text), just be careful about the non-orientable case.

✦

For higher dimensional systoles, negative results started to appear in (Gromov, 1992a), a recent reference is Appendix D of (Gromov, 1998). There were some intermediate results, Gromov's example of $(1,3)$-softness on $S^1 \times S^3$ is basic and consists simply of metrics on $S^1 \times S^3$ obtained from $[0,1] \times S^3$ by identifying $\{0\} \times S^3$ with $\{1\} \times S^3$ after enough twisting with the Hopf fibration. We now have an extremely large category of negative examples in (Babenko & Katz, 1997) and (Babenko, Katz & Suciu, 1998), see also Katz's appendix in (Gromov, 1997). One says that a manifold M^d is *systolically* $(\boldsymbol{k}, \boldsymbol{d}-\boldsymbol{k})$*-soft* (or *free*) if the infimum of the quotients $\mathrm{Vol}(g)/\mathrm{Sys}_k(g)\mathrm{Sys}_{d-k}(g)$ is zero for all metrics g on M^d. The above authors proved softness in the following cases: first, for any orientable $(k-1)$-connected M^d where $d \geq 3$, $k < d/2$ and k is not a multiple of 4, one has $(k, d-k)$-softness. Secondly, for simply connected $M^{d=2n}$ with $d \geq 6$ one has (n,n)-softness. Startling examples are the $S^k \times S^k$ for any $k \geq 3$ and $\mathbf{H}P^2$. The $(4,4)$-softness of $\mathbf{H}P^2$ is most surprising since the projective lines really fill up the whole space in the most geometric fashion possible and should have permitted us (sniffing all around) to prevent softness. The "freedom" can be very large, for example on $S^3 \times S^3$ one can even use metrics that are homogeneous. It is not clear today if the various topological restrictions are really necessary in addition to the dimensional ones. In fact *no* example of a *hard* inequality is known as soon as the involved systoles are 2-dimensional or higher dimensional. However dimension 4 might be an exception. In (Babenko & Katz, 1997) it is shown that the $(2,2)$-softness of $S^2 \times S^2$ and that of $\mathbf{C}P^2$ are equivalent.

An important parallel worth mentioning is (Besicowitch, 1952) (see Chapter 7 of (Gromov, 1983a)), this concerns manifolds with boundary (see TOP. 9). The author proves that for any Riemannian metric the volume of a cube has a volume larger than or equal to the product of the infima of the lengths of the curves joining two opposite pairs of faces. But for a cylinder, there are metrics with ever decreasing volume even when both the infima of the lengths of the curves joining the two boundary disks and the areas of the disks whose boundary belongs to the cylinder itself are larger than one.

The techniques used for the above results are completely geometric. One builds up submanifolds which are "roomy" like in a Besicowitch cylinder, but are constructed like Gromov's $S^1 \times S^3$ counterexample above. Then one can find associated exterior forms which are calibrating (see TOP.6.A) to insure that the systoles are lower bounded. All the above softness results can be interpreted as *instability* results for the homology, see (Babenko & Katz, 1997) for more on this.

✦

One can still talk about the 1-systole on any manifold being the length of the shortest periodic geodesic, even if it is contractible. For surfaces one has a curvature-free inequality, still not optimal, in (Croke, 1988b). For higher dimensions, one is more or less forced to enter some curvature term, see (Wilhelm, 1997), (Rotman, 1997).

2. Isometric embedding

Since (Whitney, 1936) one has known that there are no more abstract manifolds than topological submanifolds of Euclidean spaces. Since (Nash, 1956) one has known that abstract Riemannian manifolds are nothing more than submanifolds of Euclidean spaces endowed with the induced Riemanian metric. The game of finding the optimal codimension is hard. Questions here can be local or global or they can address real analyticity. For the general case a good part of the book (Gromov, 1986) is devoted to it. We refer the reader to this and just mention a few points.

A spectacular case is the following one: any abstract metric on S^2 of positive curvature can be realized by a convex surface. This was achieved through successive efforts: in (Lewy, 1938) for the analytic case, in (Nirenberg, 1953) for the twice differentiable case. A related result for the case of polyhedra (tending to the limit) was deduced back in 1941–42 by A. D. Alexandrov. For more precise and the most general statements in connection with singular Riemannian manifolds, see the book (Pogorelov, 1973).

It is important to note that the differentiability assumptions are basic to this game and to the general existence result above. For example any compact Riemannian manifold M^d admits a local C^1-isometric embedding in $\mathbf{R}^{d+1}$ and a global one in $\mathbf{R}^{2d}$ (a dimension which is already needed topologically). This was achieved in (Kuiper, 1955) and (Nash, 1954). The idea is related to that of the Lebesgue handkerchief for developable surfaces: clever crumpling and creasing. This result can be compared with Lohkamp's on negative Ricci curvature in I.B.5. A result that is very interesting to visualize is that of (Bleecker, 1995). This text first extends Kuiper's result to one-parameter families, then uses it to obtain C^1-isometric deformations of, for example, ellipsoids in $\mathbf{R}^3$ which are prolate enough. Paradoxically, these deformations *increase* the volume. The wording of the author is "finely corrugated wrinkling". This of course does not exist under a C^2-condition since Gauss' theorem would imply convexity and hence rigidity by the theorem quoted in TOP. 10.B.

3. Holonomy groups and special metrics: another (very restricted) Riemannian hierarchy, Kähler manifolds

Since the parallel transport is the most natural geometric operation in Riemannian manifolds, we ask the following question: are there any manifolds for which the parallel transport preserves some "extra" structure, besides obviously the Euclidean metric of the tangent spaces? Manifolds could then, in some sense, be classified by those structures. A typical case is the Kähler one, it was made clear in (Lichnerowicz, 1955) that the manifold being Kählerian is equivalent to the complex structure being preserved by parallel transport.

A. Holonomy groups

Such a classification, at least in a special case, was Elie Cartan's hope in (Cartan, 1925), where he introduced the notion of *holonomy* groups for general relativity theory purposes; then he did some calculations up to dimension three in

(Cartan, 1926a). The holonomy group of a Riemannian manifold is simply the subgroup of the full orthogonal group created by the parallel transport along all possible loops based at a given point. Changing the point does not change the structure of the group, the only possible difference is that the manifold may be simply connected in one case and not in the other. Then one has to distinguish between the full and the restricted holonomy group (use only contractible loops). Surveys of holonomy groups are Chapter 10 of (Besse, 1987) and the book (Salamon, 1989).

Holonomy groups were completely forgotten (or might at least have been found too hard to study) between Elie Cartan's time and the 50's. They came back under study in (Borel & Lichnerowicz, 1952). Matters here are in some sense completely local, holonomy groups make sense even for non-complete manifolds. The only problem left is this still unbelievably open question: is the holonomy group of a compact manifold compact? Since, by Borel-Lichnerowicz, the restricted holonomy is always compact, the question arises of course only in the non-simply connected case. The answer is yes for many cases of the classification below, but the answers are indirect: one first uses the fact that most groups in the classification have a *finite* normalizer in the full orthogonal group. Secondly, for the Ricci flat cases, one uses the splitting theorem, see Theorem 6 in (Cheeger & Gromoll, 1971).

The question to look at next is the reducibility of the group. Things were finally cleared up in (de Rham, 1952). The reducibility of the group implies quite easily that the metric is locally a product. Then simple connectedness implies moreover a global product but this is harder to prove.

Next, in (Lichnerowicz, 1955) (pages 258–261) it is proved that the Kähler condition is equivalent to the holonomy group being a subgroup of the unitary group $U(n)$ for M^{2n}. Moreover the special unitary group $SU(n)$ characterizes Ricci flat Kähler metrics. Before Yau's solution of Calabi's conjecture in 1978, no (compact) manifold with holonomy $SU(n)$ was known (see the end of III.C).

How about other irreducible subgroups of the orthogonal group? The result of (Berger, 1953) (see (Simons, 1962) or the surveys indicated above for a much better proof) says the following. Note first that the result is completely local, no completeness is needed. Satirically speaking the holonomy group is almost always the full orthogonal group in general, or the unitary group for the Kähler case. So there exists a two-step hierarchy: the general case and the Kähler one. There are however a few exceptions to this hierarchy. The first exception is formed by the space forms of the general type, namely of any locally symmetric space, so that our new hierarchy is finally not too much different from the one in II.

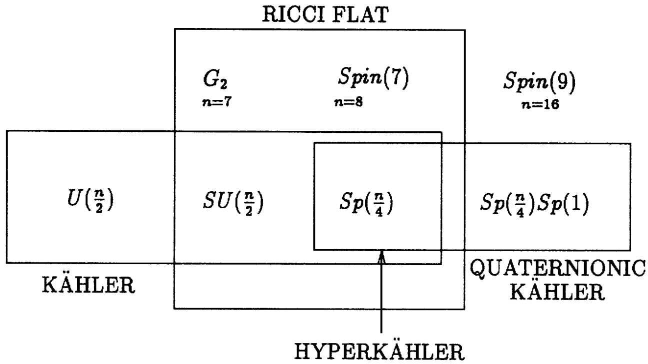
Then come two other exceptions in general dimensions: the first one is the case of manifolds called *quaternionic* (even though they do not really admit a quaternionic structure): their holonomy group is that of $\mathbf{H}P^n$ or $Hyp\mathbf{H}P^n$, which is the group $S(Sp(n) \times Sp(1)) \subset SO(4n)$. It is an open question today whether there exist many quaternionic manifolds besides the locally symmetric ones, no compact one is known today. Recent references are (Le Brun & Salamon, 1994), (Boyer, Galicki & Mann, 1994a) as well as the entire Proceedings (Gentili, Marchiafava & Pontecorvo, 1994). The recent notion of *3-Sasakian* manifolds is very useful for constructing examples.

The second exception is the case of the group $Sp(n) \subset SO(4n)$. Those manifolds are called *hyperkähler*. Paradoxically there are quite a few such manifolds. The first complete (non-compact) example appeared in (Calabi, 1979). Compact ones can be constructed using the solution of Calabi's conjecture and some algebraic geometry: (Beauville, 1983). In practice they are Kähler manifolds with more than one Kähler structure, in fact they really admit a quaternionic structure. They are today encountered in mathematical physics, in particular in the theory of "mirror manifolds": (Voisin, 1996). Recent references for them are (Salamon, 1996), the report (Hitchin, 1991/92), (Besse, 1987), (Biquard & Gauduchon, 1996) and the book (Greene & Yau, 1997).

The naive reader should be warned against thinking that hyperkähler manifolds are the quaternionic analogue of complex manifolds and enjoy coordinates like complex ones: holomorphy ("quaternionorphy"), quaternionic derivatives, etc. This can, sadly enough, happen only in the flat case (then the holonomy is zero and not $Sp(n)$). This result, which is due to Fueter, who deduced it prior to the 40's, can be found worded in modern language in (Salamon, 1982), see also (Sommese, 1975) and (Batson, 1992). However in (Joyce, 1997) the development of a theory of "quaternionic functions" on hyperkähler manifolds was achieved.

The list in (Berger, 1953) ended with three exceptions in dimensions 7, 8 and 16. The case of dimension 16 was that of the subgroup $\mathrm{Spin}(9) \subset SO(16)$. But this group has to be struck off the list because of (Alekseevskii, 1968), where it is proved that this group forces such strong curvature relations to hold that in fact it implies local isometry with the panda $\mathbf{Ca}P^8$ or its negative analogue, which is a symmetric space, see (Brown & Gray, 1972) for a more detailed proof. One is left with two possible groups which are $G_2 \subset SO(7)$ and $\mathrm{Spin}(7) \subset SO(8)$ respectively. It was only in (Bryant, 1987) that the first examples appeared, sadly enough only local ones. Complete ones appeared in (Bryant & Salamon, 1989). Compact ones exist now in (Joyce, 1996a) and (Joyce, 1996b). It is not clear whether manifolds with these holonomy groups are numerous or scarce. Joyce's construction is extremely expensive but very nice. It uses deformations of metrics, singular manifolds and the Atiyah-Singer theorem. For example, for G_2, the idea is to start with a manifold with a reasonable G_2-structure on its frame bundle. Such a structure has a *torsion* which measures the defect of it as compared to being parallel transported. Then one uses a one-parameter deformation technique to get finally a metric with vanishing torsion.

Note that, just like as $SU(n)$ and $Sp(n)$, holonomy G_2 and Spin(7) force the manifold to be Ricci flat, see the end of section III. There is also a strong relationship between these special holonomy groups and parallel spinors: see TOP. 6.B. This is due to the fact that both G_2 and Spin(7) are, contrary to $SO(d)$, simply connected and hence induce a canonical spin structure on the manifold. Using parallel objects one can define the case of G_2 (or Spin(7) respectively) very simply by just requiring the manifold to have an exterior form of degree 3 (or 4 respectively) that is invariant under parallel transport (i.e. with a covariant derivative of zero). The picture below is taken from (Salamon, 1989).



Recently holonomy groups have become important in mathematical physics: see (Fröhlich, Granjean & Recknagel, 1997) for a systematic presentation of this hierarchy from the viewpoint of theoretical physics and also (Andersen, Dupont, Pedersen & Swann, 1997). The holonomy group classification is also useful when studying nonpositive curvature manifolds, see Section I.B.4, and on the nonnegative side for the characterization of symmetric spaces among compact Kähler manifolds by their bisectional curvature, see (Mok, 1988) and I.B.1.

We mention here an important heuristic meaning of the holonomy classification, whose consequences have still not been exhausted: *objects (tensors) that are invariant under parallel transport do not exist in general in a Riemannian manifold.* On the contrary, when one does exist it implies some extremely strong restrictions for the Riemannian structure. The reason is that the full orthogonal group has "no invariant" besides the trivial ones. On the other hand, as soon as an object is invariant, it has to be very special, e.g. the Kähler form, etc.

B. Kähler manifolds

This is an entire topic in itself and we will have to be extremely brief. We know of no recent survey. A classic is the book (Weil, 1958), thereafter there is (Wells, 1980). Although motivated by algebraic geometry, (Griffiths & Harris, 1978) is invaluable for Kähler geometry. See also Chapter 2 of (Besse, 1987), moreover Besse's book contains many facts on Kähler manifolds, in particular a detailed study of the homogeneous ones. The book (Amoros, Burger, Corlette, Kotschik & Toledo, 1996) (see also (Toledo, 1997)) is concerned entirely with the fundamental group of Kähler manifolds but it can be used indirectly, with its references, as a survey or a partial survey of the "Riemann-Kähler" domain. For more general complex geometry one can consult the three volumes of (Bedford, d'Angelo, Greene & Krantz, 1991).

An analysis of the historical contribution of Kähler himself can be found in (Bourguignon, 1993). Of course part of the interest of Riemannian geometers was to add the Kähler hypothesis in order to attack many problems too difficult to solve in the general Riemannian context. However the motivation was not purely pride, since Kähler manifolds appear most naturally in algebraic geometry. More even: Kähler manifolds have recently become extremely important in mathematical physics: see for example (Voisin, 1996), the references there and (Fröhlich, Granjean & Recknagel, 1997). Just be careful that Kähler manifolds are almost never to be considered as "complexifications" of "real" manifolds. However they are of course wonderful to work on, since we have the holomorphic calculus for them.

By definition a Kählerian manifold is one with a complex structure (this means in particular that the coordinate changes are holomorphic for the complex coordinates) together with a Riemannian metric which has the best possible *link* with this complex structure, namely the complex structure is first compatible with the metric but moreover invariant under parallelism. This is equivalent to the condition for the holonomy group to be included in the unitary group and is hence equivalent also to requiring the existence of a 2-form of maximal rank and of zero covariant derivative. An equivalent definition is the following (but the proof is a little tricky): we have a complex structure J with a Riemannian metric g such that both are compatible: $g(Jx, Jy) = g(x, y)$ for all x, y. Then one gets an exterior 2-form ω, called the *Kähler form*, defined as $\omega(x, y) = g(x, Jy)$. Then the manifold is Kählerian if and only if ω is closed: $d\omega = 0$.

To many people Kähler manifolds look like algebraic ones. There is a good reason for that: a celebrated theorem of Kodaira asserts that this is the case under the sole condition that, via de Rham, the form ω belongs to the rational cohomology (or, equivalently, it has rational periods). For a proof in book form see (Griffiths & Harris, 1978) or (Wells, 1980).

Beware also that Riemann-Cartan normal coordinates can never be complex, except in the flat case. But a fundamental concept (even if trivial) is heredity: a complex submanifold of a Kählerian one is also Kählerian. The parallelism of the complex structure J immediately implies the curvature relation $R(Jx, Jy) = R(x, y)$ as endomorphisms. Note that a Kähler manifold is then canonically a *symplectic* manifold, so that the whole of symplectic geometry becomes available. Recent books on the topic are (McDuff & Salamon, 1995), (Hofer & Zehnder, 1994) and also (Audin & Lafontaine, 1994). Another strong point is that the Kähler notion is heriditary not only for (complex) submanifolds but also for many algebraic geometry operations, in particular that of blowing-up. Note also that ω and its exterior powers ω^k are "calibrating", see TOP. 6.A.

Gromov's complaint was that the above definition is not all that geometric: (Gromov, 1992b). A rarely used geometrical tool is the *diastasis.* Invented in (Calabi, 1953), it is some kind of adapted metric (in fact it should be viewed more as a potential), but in general it is defined only locally. All points of interest concerning this tool can be found in (Hulin, 1995) and its bibliography. In this text the diastasis is used to study Einstein manifolds (see III.C).

Concerning curvature, two notions besides the sectional curvature are natural for Kählerian manifolds: the first one is holomorphic curvature, namely the sectional curvature of the real 2-planes which are complex lines. Stated explicitly these are the $K(x, Jx)$ (where $||x|| = 1$). Metrics of constant holomorphic curvature are locally isometric to $\mathbf{C}^n$, $\mathbf{C}P^n$, $Hyp\mathbf{C}P^n$ (please use normalization): this is a standard result now and is one very special case among the general type of questions raised in the digression of I.C.1. It was stated in (Bochner, 1947), see (Igusa, 1954) or (Hawley, 1953) for a detailed proof. A weaker notion is that of bisectional curvature: $B(x, y) = R(x, Jx, y, Jy) = R(x, y, x, y) + R(x, Jy, x, Jy)$ and equal in particular to $K(x, y) + K(x, Jy)$ if $\{x, y, Jy\}$ is orthonormal. A very strong result is that of (Siu & Yau, 1980) and, independently, (Mori, 1979): positive bisectional curvature implies that the underlying manifold is $\mathbf{C}P^n$. The nonnegative case is settled in (Mok, 1988): one can only have biholomorphy with $\mathbf{C}P^n$ and isometry with Hermitian symmetric spaces, plus of course products and coverings.

For complex manifolds the theory of exterior forms is richer, they (at least the pure ones) can be associated with a *type* (p, q) which refers to their representation in terms of complex coordinates. But one has more, namely that the exterior differentiation can also be split into two: ∂ and $\bar{\partial}$. The holomorphic forms are those swallowed by $\bar{\partial}$. If moreover the manifold is Kähler then the (total) Laplacian $\frac{1}{2}\Delta$ coincides with both the partial Laplacians coming from ∂ and $\bar{\partial}$. Via the Hodge-de Ram theorem and the exterior product with ω (which is of course harmonic), this yields a lot of information. Let us first mention the increase of the Betti numbers up to the intermediate dimension: $b_{p+2} \geq b_p$. Another simple example is that positive Ricci curvature forbids the existence of holomorphic forms of any degree.

Secondly, the Sullivan theory mentioned in TOP. 6.A is much stronger here. It is proved in (Deligne, Griffiths, Morgan & Sullivan, 1975) that the real homotopy of a compact Kähler manifold is a formal consequence of the real homology ring (compare this with I.C.1). It is also easy to imagine now that the Bochner vanishing technique will yield a lot of strong results under various assumptions about the curvature in the Kähler case. This is indeed the case and is used heavily for various bundles over a Kähler manifold, especially (complex) line-bundles. There is an immense literature, see the classic (Hirzeburch, 1966), (Griffiths & Harris, 1978) and thereafter, among others, (Siu, 1980) and (Sampson, 1986).

Can one tell by the spectrum of a Riemannian manifold whether it is a Kähler one? We are far from it. Using the asymptotic expansion some special results were obtained in (Gilkey, 1973a). Then (Gromov, 1992b) was more ambitious. There are some good results for λ_1, see (Lichnerowicz, 1958), where a lower bound for λ_1 depending only on a positive Ricci lower bound is given and (Bourguignon, Li & Yau, 1994), where an upper bound is given which uses the volume for algebraic manifolds.

4. Cut-loci

A question that suggests itself on a Riemannian manifold is to ask when two points p, q are joined by more than one segment (a segment is a shortest path, hence supported by a geodesic). There is a technical problem here, which partly explains the difficulty of the topic.

Assume compactness for simplicity. Starting from a given point p a geodesic will be a segment (realizing the distance) up to a certain value, called its *cut-value* (certainly smaller than or equal to the diameter). The *cut-locus of* p is by definition the set of these cut-points. But at such a point two things can happen (simultaneity is possible): a second segment may appear or/and the point may be conjugate to p, i.e. the exponential map may not be of maximal rank. This is the "dichotomy" for the cut-locus. The cut-locus of a point p is always the topological closure of the set of points joined to p by more than one segment, see page 589 of (Warner, 1965). The injectivity radius at p is the (positive) distance between p and its cut-locus. On a compact manifold the injectivity radius, introduced in I.A.2, is the infimum of the various injectivity radii at p when p runs through the manifold. It is a positive number and we saw above that it is often a basic tool (cf. I.C) and an important invariant (cf. III.B). A fairly detailed description of the cut-locus in book form is to be found in Chapter III of (Sakai, 1996), in the last chapter of (do Carmo, 1992) and in (Kobayashi, 1989).

We now address the question of the structure of the cut-locus. Contrary to many topics we have considered in Riemannian geometry, results on the cut-locus have appeared in a steady stream. But these results are very few in number and quite far apart in time. The cut-locus was first considered for surfaces in (Poincaré, 1905), where it was called the "ligne de partage". Then Elie Cartan used it for the topology of Lie groups in (Cartan, 1936) and studied it also in (Cartan, 1946–1951) (it seems that he was unaware of Poincaré's and Myers' results). For surfaces it was well studied in (Myers, 1935b), (Myers, 1936) and (Whitehead, 1935), where the expressions cut-locus and conjugate locus were introduced (and by the way the general existence of local convex balls was proved). Next, it appeared explicitly in the proof of the sphere theorem in (Klingenberg, 1959). One then had to wait until (Wall, 1977) and (Buchner, 1978) to get structure results for the generic and the real analytic case. At the same time (Gluck & Singer, 1978) showed that in general the cut-locus can be very bad, typically non-triangulable for an open set of points. Very recently the cut-locus made a strong comeback. Essentially "cut-loci are not that bad", more precisely their Hausdorff dimension is in a direct relation to the smoothness of the metric, see (Itoh & Tanaka, 1997) and the references therein.

But still the cut-locus remains mysterious and basically unknown on a given manifold. The sole exception is the case of symmetric spaces, see (Sakai, 1978) and (Takeuchi, 1979). See for space forms (Hebda, 1995). The cut-locus on a two-dimensional ellipsoid (even on one of revolution), however, is still not known. For an umbilic it is reduced to its antipodal umbilic. For all other points it is believed to be a topological segment. But this latter assertion depends on the scandalously unproved Jacobi "statement": the conjugate locus of a non-umbilical point m of an ellipsoid has exactly four cusps. Geometrically this conjugate locus is the *envelope* of the geodesics emanating from m.

Inverse results for the cut-locus are completely unknown apart from one exception: the cut-locus of every point is always reduced to a single point (Wiedersehensmannigfaltigkeiten) only for the standard sphere, see in book form Chapter VI of (Sakai, 1996). This result was obtained for surfaces in (Green, 1963) and was a conjecture of Blaschke's in the 20's. For higher dimensions it is the content of Appendix D of (Besse, 1978). A more general inverse problem could be stated as follows: which are the manifolds all of whose conjugate loci are submanifolds, in particular of codimension larger than 1? The problem is untouched today, even with stronger conditions, see the sections concerning the Blaschke conjecture in (Besse, 1978). A very itching question, which is a very specialized case of the above, is the following one: which are the Riemannian manifolds for which the diameter is equal to the injectivity radius? This occurs for the (standard) spheres and all the $\mathbf{K}P^n$ (with the standard metric). Are there any others? The question is raised in a general setting in (Besse, 1978), where it is called the Blaschke conjecture. It is not too hard to prove that the cut-locus of any point is a submanifold of constant dimension (that of a **K**-hyperplane) and it is also true, but more subtle, by algebraic topology, that it is almost only the $\mathbf{K}P^n$ that can appear as topological manifolds. We have just seen that the question has been solved for the sphere. The question is also solved for the sphere covering an $\mathbf{R}P^d$: there are no others. But for $\mathbf{K} = \mathbf{C}$, **H** or **Ca** the question is open.

We now come back to the question raised in 0.B about Cartan's local statement: parallel transport and the curvature determine the metric. In (Ambrose, 1956) the global problem is raised where one knows the parallel transport and the curvature for all geodesic segments starting in a given point. Ambrose conjectured that this knowledge is enough for a complete determination of the metric. This is linked with the structure of the cut-locus and is still an open problem. It was solved recently for surfaces in (Itoh, 1996) using (Hebda, 1994).

A final remark is in order: the cut-locus is essentially a Riemannian notion if one expects a reasonable form of behavior. As soon as one goes to more general metric spaces things can become very very wild. See for example (cf. TOP. 9) (El Alaoui, Gauthier & Kupka, 1996) for Carnot-Caratheodory spaces, (Shiohama & Tanaka, 1992) for Alexandrov surfaces and (Zamfirescu, 1996) (as well as the references therein) for the boundary of convex bodies. For Riemannian manifolds with boundary see the references in TOP. 9.

5. Non-compact manifolds

In various instances we already mentioned previously some extensions of results for compact manifolds to complete ones. It is clear that one should restrict oneself to some kind of non-compact Riemannian manifold to have some hope of obtaining non-trivial results. Let us mention in particular the following restrictions: replace compactness by *finite volume*, ask for *asymptotic behavior of some given type at infinity*, e.g. quadratic decay, quadratic curvature decay, volume behavior, etc. Note that non-compact manifolds occur necessarily when studying manifolds with infinite fundamental groups.

A typical example is that of manifolds with nonnegative Ricci curvature. In the case of nonnegative sectional curvature we saw in I.B.1 a perfect structure theorem (both a splitting and a soul theorem) by Cheeger-Gromoll and the bounded topology result of Gromov-Abresch. Note that these results are valid without any extra condition on the geometry (boundedness or its behavior at infinity). But if one asks the same question for nonnegative Ricci curvature, the splitting theorem is still valid but there is no structure theorem and there is no bounded topology result as seen in examples. This implies that results on nonnegative Ricci curvature on compact manifolds should use extra hypotheses, typically some kind of growth. The results one expects could be of a different nature. Corresponding to various structures on the sphere at infinity in the case of negative curvature (I.B.4), it seems that the right structure to be defined (even though not unique in general) is the following. The idea is to look at the *structure at infinity* by considering, for a given (M,g), the sequence of the (pointed) manifolds $(M,p,r_i^{-1}.g)$ when the "radii" r_i tend to infinity. This idea appeared first for the case of discrete groups in (Gromov, 1981b).

The convergence theorems in C.2 extend easily to the category of pointed manifolds, see (Petersen, 1997). In particular one can extract a subsequence that converges toward "some" limit "cone" (a cone at infinity, denoted with a slight abuse of notation by M_∞); this cone need not be unique and might also depend on p. In various instances, one can prove that M_∞ is a volume cone, or better a metric cone, and finally sometimes the Euclidean space itself. Here is a small selection of results from among many recent ones: §7 of (Cheeger & Colding, 1996), (Cheeger & Colding, 1997a) and their successors. An exemplary conjecture is that by Anderson and Cheeger to the effect that, if a cone at infinity is isometric to $\mathbf{R}^d$ and the Ricci curvature is nonnegative, then the manifold itself is isometric to $\mathbf{R}^d$; this conjecture has since been proved in (Colding, 1996b).

But we insist that non-compact manifolds are, to some respect, more important than compact ones. This is why our partial survey should definitely be complemented in some way or another by the reader. The reader may discover a way of doing so in the references given. Non-compact manifolds appear together with surfaces, especially space forms, where the eigenvalue behavior with respect to the value 1/4 is even more delicate than for compact ones, see for example (Luo, Rudnick & Sarnak, 1995).

We saw in I.B.2 and I.B.4 that some non-compact manifolds appear naturally in the case of compact manifolds with a bounded Ricci curvature, of negative or of nonpositive curvature, when one looks at their universal coverings. But they are also used heavily in different instances, like in collapsing with diameters that are *not bounded* or, more generally, when one drops the diameter bound in various situations (see I.C and the references there). See also the very geometric text (Babenko, 1992).

A natural case is that of *finite volume.* Many results we met above extend to this case, involving a little or even a lot of pain, depending on the question. It is even impossible already to give a short list of references, so that what follows is very biased. Note just, to start with the naturalness of the topic, that the famous modular domain $SL(2,\mathbf{R})/SL(2,\mathbf{Z})$ is of finite volume for its hyperbolic canonical metric (but also has singularities): see (Luo, Rudnick & Sarnak, 1995). We also mention the problem of extending the integral Chern formulas to the finite volume case. Surprisingly, this turned out to be an extremely hard subject, even for surfaces, where it was first studied in (Cohn-Vossen, 1935). See (Cohn-Vossen, 1936) for the problem of extending the Gauss-Bonnet theorem. However a lot of work remained to be done for surfaces. We refer to (Shioya, 1992) and the intermediate references therein, and just mention that many topics introduced in I.B.4 (for negatively curved manifolds) play a role here. For higher dimensional formulas, then involving the characteristic χ but also the characteristic (Pontryagin) numbers, the job was started in (Cheeger & Gromov, 1985): in the non-compact case one can get *irrational* characteristic numbers. More results are to be found in the recent text (Rong, 1995), see also the survey (Lück, 1996).

Finite volume is an especially strong condition in the negative curvature realm, where most results valid for compact manifolds can be extended with mostly not too much pain to the case of a finite volume. For space forms and Mostow's rigidity see (Farrell & Jones, 1989b), for the general case see (Ballmann, Gromov & Schroeder, 1985) and of course (Eberlein, Hamenstädt & Schroeder, 1993).

Another condition is that of a *bounded geometry*. It arises naturally for coverings of compact manifolds and homogeneous spaces, it appeared for example in (Greene, 1978) and now forms a huge universe. The question is studied in (Semmes, 1996a) in connection with the fairly recent notion of complexity.

✦

An interesting question is to know when a complete Riemannian manifold can admit a non-constant harmonic function ($\Delta f = 0$) or, even more, a positive spectrum. This possibility is linked directly with various curvature properties. The founding text was (Yau, 1975), where positive Ricci curvature was shown to be enough to exclude the possibility of non-constant harmonic functions (this is a "Liouville type" result). Then results did not stop appearing, we refer only to recent texts and their bibliographies: (Colding & Minicozzi II, 1997) and (Yu, 1997). See also (Benjamini & Cao, 1996) for connections with sectional curvature.

✦

In the non-compact realm, various results address the question of obtaining results when the structure at infinity satisfies some condition, like being asymtotically Euclidean (with an asymptotic order to be made more precise). See among others (Shen, 1996) and the references therein. There are also *gap* results, which forbid various compact metrics to be extended by completely flat ones at infinity (unless they are already flat everywhere), etc., see the introduction to (Lohkamp, 1996a) and (Greene & Wu, 1990) as well as what was seen in I.B.4. Many results are to be found in (Eberlein, Hamenstädt & Schroeder, 1993), like for example asymptotically harmonic manifolds, see also (Funar & Grimaldi, 1977) and (Unnebrink, 1997). A strong result is (Cheeger & Tian, 1994), it is used for the results of Cheeger and Colding in I.B.2.

There are many techniques in non-compact Riemannian geometry besides the study of the structure at infinity seen above and the compactification in I.C.4. Besides the use of the sphere at infinity in various ways, we mention the very geometrical *chopping* technique of (Cheeger & Gromov, 1989). This is an exhaustion technique where, as one goes to infinity, the boundaries of the successive compact pieces keep a bounded second fundamental form and are also volume-controlled. We also mention the notion of manifolds with a uniform contractibility function, which extends the notion of the injectivity radius: the definition demands the existence of a function $f(r) \geq r$ such that every ball of radius r is contractible in the ball which has the same center and a radius of $f(r)$, see (Greene & Petersen, 1992) and the references therein.

Very interesting also is the technique for proving the "mass conjecture" in (Schoen & Yau, 1981) (see also the expository paper (Kazdan, 1981/82)): using GMT one looks at minimal hypersurfaces in the manifolds and studies what they develop into when their boundary goes to infinity. For other viewpoints see the references in (Cao, 1996).

✦

On compact manifolds all homology and cohomology theories more or less coincide. In the non-compact case matters are much more subtle. A helpful notion for Riemannian geometry is that of L^2-Betti numbers. There are the surveys (Pansu, 1996) and (Lück, 1996). We just mention that the "only-compact" motivated Riemannian geometer should consider this notion with respect. A striking example is (Gromov, 1991a), where L^2-Betti numbers are used to solve H. Hopf's conjecture mentioned above (at the very end of I.B) for the Kähler case: a compact Kähler manifold M^{2n} of negative curvature has a characteristic whose sign equals $(-1)^n$.

6. Bundles over Riemannian manifolds

We introduce bundles in the order of their Riemannian geometric character. First canonical or almost canonical ones (spinors), then those obtained by twisting canonical ones, then Yang-Mills fields. The most natural ones are those of *exterior differential forms* met in 0.D. A reference in book form is (Gilkey, 1995), but add also (Berline, Getzler & Vergne, 1992) for the index theorem in a very general context and (Lawson & Michelsohn, 1989) for the spinors. Of basic importance are the characteristic classes for various bundles over general differentiable manifolds, so one should look at their connections with Riemannian geometry.

A. Exterior differential forms (and some others)

Exterior differential forms exist on any differentiable manifold, their degrees extend from 0 to the dimension d of the manifold. The corresponding vector spaces are denoted by $\boldsymbol{\Omega^p M}$. The basic link is the exterior differential $d\colon \Omega^p \to \Omega^{p+1}$ with $d \circ d = 0$ (this collection is called the *differential complex* of M). The *closed* forms are the ω with $d\omega = 0$. This leads, as seen in 0.D, to the de Rham theorems via Stokes' formula. Although not a Riemannian story, one should know that, even though this only yielded the real Betti numbers for the topologist, more can now be extracted solely from the exterior differential complex, as was discovered in (Sullivan, 1977), for example some topological finiteness type results (see I.C.1), see also the rational homotopy theory in I.B.1, an application in the digression in V.A and TOP. 3.B. What more can a Riemannian geometer now ask and do? Of course he can look for interesting relations between the metric and exterior forms.

A basic fact is the existence of the $*$-operator $*\colon \Omega^p \to \Omega^{d-p}$. It is involutive or antiinvolutive: $* \circ * = (-1)^{d(p+1)+1}$ and needs an orientation. The form $*\omega$ is defined by $*\omega(x_{p+1},\dots,x_d)=\omega(x_1,\dots,x_p)$ for any positive orthonormal basis $\{x_1,\dots,x_d\}$. For example on an oriented manifold the $*$ of the constant function 1 is the *volume form*, namely the d-form which takes the value +1 on any positive orthonormal set (of tangent vectors). Its existence is equivalent to orientability.

It is natural to look at the canonical norms $||\omega||$ and, bearing in mind the de Rham theorem, to look (if it exists) at the minimum of the L^2-norm $\int_M ||\omega(m)||^2\, dm$ for ω running through a given cohomology class. The basic theorem of Hodge and de Rham (met in 0.D) says that this minimum is attained by a unique form in the class which is characterized by $\Delta\omega=0$, where the Laplacian Δ on forms is $d\circ d^*+d^*\circ d$, with the adjoint $d^*:\Omega^p\to\Omega^{p-1}$ of d being defined as $(-1)^{d(p+1)}(*\circ d\circ *)$. There is no need for an orientation since one uses $*$ twice. *Harmonic* forms ω are those with $\Delta\omega=0$. Stokes' formula implies that $\Delta\omega=0$ is equivalent to $d\omega=0$ (closed forms) and $d^*\omega=0$ (co-closed forms). The second condition is exactly where the Riemannian structure enters.

Concerning harmonic forms one can wonder about the "inverse problem" of whether harmonic forms are special among closed differential forms or not. This is a tricky question and was attacked first for 1-forms and solved in (Calabi, 1969). For higher degrees see (Farber, Katz & Levine, 1996).

One can also wonder about the fact that, except for 1-forms, exterior forms are not well suited in general for computing volumes when one restricts them to submanifolds. But there are some exceptions, as was first noticed in (Wirtinger, 1936). In a Kählerian manifold the Kähler form ω enjoys the following property: for any orthonormal pair x, y one has $\omega(x,y)\leq 1$ with equality holding only for complex lines: $y=Jx$, and this works also mutatis mutandis for the exterior powers $\wedge^p\omega$. Using Stokes' formula along with the fact that ω is closed, one can see that this implies immediately that any complex submanifold of a Kählerian manifold has an absolute minimal volume in its homology class (this is much stronger than being only a minimal submanifold and is called stability). Wirtinger used his result for various applications, for example to compute explicitly the volume of any algebraic submanifold of a given degree in $\mathbf{C}P^n$.

The above inequality may hold for some forms of degree p on some Riemannian manifolds: $\omega(x_1,\dots,x_p)\leq 1$ for every orthonormal p-tuple. This property is called *calibration* (calibrating form, calibrated manifold) in (Harvey & Lawson, 1982). As above the straightforward property of a calibrating form ω is that, if a submanifold N^p has a volume equal to $\int_M\omega$, then it is an absolute minimum in its homology class, in particular a stable minimal submanifold. For this it is necessary and sufficient that ω takes the value 1 on each tangent space to N^p. Calibration was used in (Berger, 1972) to prove the systolic inequality for the standard metric of $\mathbf{K}P^n$ for $\mathbf{H}$ and $\mathbf{Ca}$ (for $\mathbf{C}$ this is the work of Wirtinger). In (Harvey & Lawson, 1982) a general theory of calibration is given. In particular some generalizations of Wirtinger's are presented, in the sense that calibration is also linked with some PDE in the same way as complex submanifolds can be defined as holomorphic, namely by $\bar\partial=0$. Recently calibration was used for systolic softness (freedom) in (Babenko & Katz, 1997) (see 1. E) and in (Besson, Courtois & Gallot, 1995a) in a Hilbert framework.

✦

We now present the two outcomes of spectral considerations for exterior forms which have a Riemannian geometry flavor. It seems that, today at least, apart from the two results below and of course the knowledge of the real Betti numbers via Hodge harmonic form theory, there are no other known implications for topology that one can deduce from spectral considerations on the totality of exterior forms. The Kähler case is more fertile as seen above in I.B.1. (McKean & Singer, 1967) started off a firework of results which yielded a deep understanding and many applications. We will describe it briefly, a complete reference is (Gilkey, 1995) (this second edition is very much up-to-date). Roughly speaking, what happens is the following: we look back at the asymptotic expansion in IV for $\sum_i \exp(-\lambda_i t)$ with the integrals $U_k(x)$, which are universal in the curvature tensor (this will always mean including its covariant derivatives). People despaired at the fact that $U_{d/2}$ is not a topological invariant as soon as $d > 2$. Since they also have a canonical Laplacian we can do the same (it is not too much more expensive and appeared first in (Gaffney, 1958)) for p-exterior forms and get the pointwise invariants, denoted by $u_{p;k}(x)$, appearing in the asymptotic expansions of the corresponding heat kernels for the term in t^k. They are still universal in the curvature but differ in general with various values of p. Their integrals over M will be denoted by capitals: $U_{p;k}$; the eigenvalues of the p-spectrum will be denoted by $\{\lambda_{p;i}\}$.

Now let us perform the alternate double sum $\sum_p \sum_i (-1)^p \exp(-\lambda_{p;i}\, t)$. As both the operators d and d^* commute with the Laplacian Δ they transform eigenforms into eigenforms. Now the Hodge decomposition theorem, which decomposes any form into a harmonic part, a closed part and a coclosed part, shows that everything will disappear in this alternate summation except at the harmonic level: there the zero eigenvalue $\lambda_{p;0}$ has a multiplicity equal to the p^{th} Betti number b_p. So everything should disappear also in the alternate sum of the corresponding asymptotic expansions for any k, except when $k = d/2$. Hence the alternate pointwise sums $\sum_s (-1)^p u_{p;k}(x)$, when integrated over M and added up after multiplication with t^k, will *identically* yield the constant $\sum_s (-1)^p b_p = \chi(M)$. This explains McKean and Singer's dream: a fantastic pointwise cancellation might well take place in the pointwise $u_{p;k}$ to yield the forced integrated cancellation. This was indeed proved in (Patodi, 1971) .

The rebound came first in (Gilkey, 1973b) then in (Atiyah, Bott & Patodi, 1973). One studies the Patodi cancellation result but puts it into the framework of successively more and more general bundles equipped with suitable elliptic operators including the Dirac one on spinors and then uses Gilkey's results. It then turns out that those structures are numerous enough to yield all elliptic operators, giving a new proof of the index theorem (see C below). The use of the theory of invariants "à la Gilkey" and functorial behavior is important. The (final) result is obtained by using this technique for more general statements. The harvest here is larger: Hirzebruch's signature theorem (see 0.E) can be obtained in this way and of course this new insight yields many results in differential topology. Research in this domain is still actively ongoing, see the two books already mentioned. One point in this philosophy is that "pointwise cancellation" shows that *local type index* theorems can exist. But Riemannian geometry is quite far away. However here is the second byproduct of the rebound: the η-invariant.

The main trick in the founding papers (Atiyah, Patodi & Singer, 1975–1976) is to obtain the characteristic $\chi(M)$ not as the alternating sum of the zero-eigenvalues of the various Laplacians on the exterior forms of a given degree on (M, g), but at one stroke as the index of the first order operator $B = d - d^*$ as acting on the total set of exterior forms on M (one just has to be careful to put the right signs in front of B). The eigenvalues of Δ are of the form λ^2, where λ is an eigenvalue of B, but different signs are possible here. Hence the function

$$\eta(s) = \sum_{\lambda \neq 0} (\mathrm{sign}(\lambda))|\lambda|^s$$

makes sense for suitable s. In a strict sense (as usual for this kind of function) $\eta(0)$ is not defined but with some extra work one can still make some sense out of it. It is then called the *η-invariant* of (M, g) and measures the "spectral asymmetry". This invariant is especially interesting for manifolds with boundary. For a $4k$-manifold M' with a $(4k - 1)$-boundary M (and provided that locally at the boundary the metric is a product) one can express the signature $\mathrm{sign}(M')$ by the integral formula $\mathrm{sign}(M') = \int_{M'} L(R) - \eta(M)$, where L is the universal curvature integrand for Hirzebruch's signature seen in 0.E. This invariant has many applications when looking at the subtle problem of the non-existence of pointwise invariant integration formulas for the "signatures". Besides the original papers we refer the reader to (Atiyah, Donnelly & Singer, 1983) and (Gilkey, 1995). There are also connections with the secondary characteristic classes below, also with the A-genus when spinors are being examined. An application of the η-invariant for 3-manifolds is the isolation result of (Rong, 1993) for the minimal volume in dimension 4 seen in TOP. 1.D., a recent result is (Bunke, 1995b). The η-invariant is also used in Number Theory: (Atiyah, Donnelly & Singer, 1983).

Another invariant based on the spectral analysis of differential forms is to be found in (Ray & Singer, 1971). The result is that, from the sum $\sum_p (-1)^p . p . \zeta_p'(0)$ built up with the ζ_p-functions associated with the spectrum of the p-differential forms of all degrees p, one can recover a topological invariant. Ray and Singer conjectured that their invariant coincided with the topological invariant called the Reidemeister torsion and gave some evidence for that. The conjecture was proved independently in (Müller, 1978) and (Cheeger, 1979). The proof is very involved and was one of Cheeger's motivations for the study of the spectrum on some singular manifolds, see TOP. 9.

✦

There is no reason to stick with exterior forms and not to look at other tensors. Lichnerowicz introduced the theory of canonical and special Laplacians in (Lichnerowicz, 1961). Particularly important is his Laplacian for symmetric bilinear differential forms, as they can be interpreted as infinitesimal variations of a Riemannian metric. For Hodge-de Rham type theorems concerning this, see Section 12.C of (Besse, 1987), where it is used to study deformations of Riemannian structures, i.e. metrics up to isometries (diffeomorphisms). Lichnerowicz' Laplacians are *natural*, in the sense explained in 0.D.

B. Spinors

When it exists, the *spinor bundle* is "almost canonical". For all what follows and more on "spin geometry" see (Lawson & Michelsohn, 1989), and add for some parts (Gilkey, 1995), (Berline, Getzler & Vergne, 1992). We are going to present things in a very caricatural way. The basic idea is the following: Riemannian geometry has a Euclidean structure on every tangent space, so that, if the space is moreover oriented, the special orthogonal group $SO(d)$ is "the basic object". Besides the fundamental representation on $\mathbf{R}^d$, the special orthogonal group has a basic one on the exterior algebra and, using tensor products, one can construct all the others. So apparently the Riemannian geometer should not look any further than for those. But the Lie group $SO(d)$ is not simply connected and its (nontrivial and twosheeted when $d > 2$) universal covering group $\mathrm{Spin}(d)$ has an extra representation in a "canonical" complex vector space, the space of spinors of the initial Euclidean space.

✦

First let us give some purely algebraic facts. The canonical quadratic form on $\mathbf{R}^d$ gives birth to the *Clifford algebra* Cliff(d) of dimension 2^d. This algebra, only as a vector space, is canonically isomorphic to the exterior algebra $\wedge(\mathbf{R}^d)$. Now there is a basic link between Spin(d) and Cliff(d) to the effect that one finally obtains two things in a canonical way. First, a complex vector space $\Sigma(d)$ of dimension $2^{[d/2]}$ (where $[n]$ denotes the integral part of n). This space shows up when one tries to write Cliff(d) as an endomorphism algebra End(?) and it is called the (complex) spinor space of $\mathbf{R}^d$. Secondly, because Spin(d) can be realized as a subgroup of the group made up of the unit elements of the algebra Cliff(d), one finally gets a representation of Spin(d) on the space $\Sigma(d)$. The theory is complicated by the fact that this representation is irreducible only when d is odd but splits into non-isomorhic ones denoted by $\Sigma^+(d)$ and $\Sigma^-(d)$ of half the dimension when d is even.

Real spinors also exist, but the situation is more complicated with them and is to be considered 8. It is a basic fact that this game cannot be played without having fixed a quadratic form, because the universal cover of the special linear group $SL(d, \mathbf{R})$ does not admit any faithful finite dimensional represention, except those in the exterior algebra.

✦

Since all the above construction depends only on oriented Euclidean geometry, namely a (the) positive definite quadratic form on $\mathbf{R}^d$, the game is to do this twostep construction for any oriented Riemannian manifold (M^d, g). Since the (principal) bundle of oriented frames $P_{SO(d)}M$ has $SO(d)$ as a structure group one first tries to define a universal double covering of this bundle, demanding moreover of course compatibility with the group covering $\mathrm{Spin}(d) \to SO(d)$. This is not always possible and a necessary and sufficient condition for this is the vanishing of $w_2(M)$, the second Stiefel-Whitney class of M: (Haefliger, 1956). Up to isomorphism, the spinor structures so obtained are classified by $H^1(M, \mathbf{Z}_2)$. They are for example unique in the simply connected case. However one in general keeps to a unique notation, namely $P_{\mathrm{Spin(d)}}M$ to denote any spin-bundle covering $P_{SO(d)}M$. Now, attached to any spin structure on the manifold, see C below, the representation of Spin(d) on $\Sigma(d)$ automatically produces a (complex) vector bundle denoted by $\Sigma(M,g)$ and called the ("a") spinor bundle on (M,g). A section of it is what is called a *spinor field* on M. Still using general facts on bundles (C below) the above construction and the Levi-Civita connection yield a canonical connection D on $\Sigma(M)$.

The third point, a major one, is that, using D, one can define on $\Sigma(M)$ a canonical differential operator of degree one, called the *Dirac* operator and *denoted by* δ. In even dimensions it exchanges the spinors $\Gamma(\Sigma^+)$ and $\Gamma(\Sigma^-)$. Now for δ^2 there exists a Bochner-Weitzenböck type formula which is $\delta^2 = D^*D + \frac{1}{4}\mathrm{scal}$ (the Lichnerowicz formula) and was already used heavily in I.B.3. Compared with the formulas for Δ involving the collection of the Curv_p (see 0.D) this looks a priori a disaster. But it is just the opposite: with much less information on the curvature, one still gets information on the topology with the Hodge-type theorem for δ, namely for harmonic spinors σ, i.e. those satisfying $\delta\sigma = 0$.

The history of spinors is fascinating. It started with Elie Cartan when, in (Cartan, 1913), he gave a complete classification of complex irreducible representations of simple Lie groups: besides the expected orthogonal and exterior representations he found an extra one which brought a new space to life. Moreover he prophetically indicated how this representation could generate all others. Then, completely independently, Dirac invented spinors on physical grounds (hence their name) in 1928 and defined the Dirac operator not for manifolds but just for the Minkowski space $R^{3,1}$. The link was built by Cartan in 1937, but he concluded his book (Cartan, 1937) by noting the "impossibility" of constructing a satisfying theory: this is so because coordinate changes with general Riemannian metrics on manifolds are only linear (not orthogonal) and we saw above that the universal cover of the special linear group has no finite dimensional represention apart from those derived from the genuine linear group. The construction of spinor bundles and of the Dirac operator on them is hard to date precisely, it seems that they were more or less part of mathematical folklore. But a historical year was 1963 which saw the appearance at the same time of the index theorem, which among others yielded the fact that the index for the Dirac operator is the Hirzebruch Â-genus (hence an integer), and of Lichnerowicz' scalar curvature formula above. For this history and more on spinors see the introductions to (Lawson & Michelsohn, 1989) and to (Berline, Getzler & Vergne, 1992), as well as the postface "Spinors in 1995" by Bourguignon to "The algebraic theory of spinors and Clifford algebras" in (Chevalley, 1997).

We saw "the" basic applications of this in I.B.3, for more see the books above. But the basic idea is that, when one defines much more general spin-type-bundles, one can often prove that $\frac{1}{4}\mathrm{scal}$ is the dominant term. Other applications are the following: some proofs of the index theorem, the construction of special holonomy groups, see TOP. 3.

A strong warning: as opposed to the case of the Laplacian and harmonic forms, the kernel of the Dirac operator δ depends on the choice of metric on a given manifold. The dimension of this kernel (the space harmonic spinors) can change, as was first discovered in (Hitchin, 1974). Worst of all: any spin-manifold can carry a metric with some non-trivial harmonic spinor (at least today in dimensions $4n+3$): (Bär, 1996). However, harmonic spinors are conformal invariants: (Hitchin, 1974). The dependence on the metric is not easy to control because when one varies the metric, the bundle also changes. In (Bourguignon & Gauduchon, 1992) a detailed computation of everything in this context is presented, in particular the first variation of the Dirac operator and of its eigenvalues (see the references therein for intermediate results). Another type of result concerns the first eigenvalue of the Dirac operator, in the vein of IV, regarding this long history we just mention a recent reference: (Hijazi, 1995).

Finally, for the very geometrically minded reader, we mention that in (Connes, 1994) one finds a formula giving the distance of two points using the Dirac operator, which has the advantage of carrying over to the non-commutative setting.

✦

Not much more complicated to define are the **Spin**$_c$ structures, they exist on many more manifolds than spin ones. In particular they exist on any 4-dimensional manifold, see TOP. 8 below for some references to some recent very significant applications. Beware of the notation: $\mathrm{Spin}_c(d)$ is not the complex group $\mathrm{Spin}(n,\mathbf{C})$ but "only" $(\mathrm{Spin}(d)\times S^1)/\mathbf{Z}_2$.

C. Various other bundles

The theory of bundles can be found in most books on differential geometry and in detail in (Kobayashi & Nomizu, 1963–1969). From the topology viewpoint, the classics are (Steenrod, 1951) and (Husemoller, 1975). Bundles E are differentiable manifolds equipped with a map $E \to B$ onto the basis B and a typical fiber F such that the base B is covered with chart domains such that the counter-images in E are products with F. On their intersections, these bundle charts should be smooth on the fibers and most often preserve some given structure. For principal bundles the fiber is acted on simply and transitively by a group, and chart changes have to be group automorphisms. For vector bundles they should of course be linear maps. An important point is the possibility, when given a principal bundle together with an action of the group on some object (e.g. a representation in a vector space), of deducing canonically from it an associated bundle, for example such a construction was basic to the canonical construction of the spinor bundle above.

For bundles one has the notion of connection, that of curvature, of parallel transport and of holonomy. The idea of connection is the same as the one explained in 0.A, namely one wants to be able to compare (infinitesimally) two fibers and to develop some kind of differential calculus. Of course, the connection that one uses should preserve, in a reasonable sense, the various structures the bundle carries. In particular, Riemannian vector bundles appear when a positive quadratic form is given on every fiber (with smooth dependence on the base point). For vector bundles one has characteristic classes and the formulas of Chern type are still valid. Relevant books are (Hirzeburch, 1966), (Berline, Getzler & Vergne, 1992) and (Gilkey, 1995).

There exist some secondary characteristic classes: we mentioned in 0.E that there are no other integral formulas involving the curvature besides Chern's. In (Chern & Simons, 1974) new subtle invariants were introduced. They occur in various bundles, they are not "downstairs" Riemannian objects. Although these are typically topologically trivial their connections are not and the parallel transport yields those new invariants. In (Cheeger & Simons, 1985) they were put downstairs onto the manifold where they live as "differential characters". These new invariants have become more and more important, in particular recently in mathematical physics and in Number Theory: (Gillet & Soulé, 1994). In book form one can consult (Berline, Getzler & Vergne, 1992) and (Gilkey, 1995), Section 3.11.

✦

Yang-Mills fields first appeared in the late seventies because of a demand by theoretical physicists. A book appeared as early as 1979: (Atiyah, 1979). But since then the question has witnessed a dramatic increase and is now a topic in itself, with many subtopics. Yang-Mills' question can be stated roughly as follows: on a given compact Riemannian manifold, most often a "standard" one like the sphere, one picks out some interesting vector bundle over it, some "bundle-Riemannian" metric on it and some connection preserving this metric. Then one looks for the "least twisted" one, typically in the L^2-sense for the curvature of this connection. This is a problem of calculus of variations and leads, for the curvature of the bundle, to a condition which looks like the harmonicity of the curvature. A rich harvest is possible on Yang-Mills fields because what one looks for is like looking for harmonic forms, only that the harmonicity condition is on the curvature. There are results like those of Hodge-de Rham. Besides their natural differential geometry setting, they are basic to theoretical physics (*gauge theory*). Surveys and/or recent references are (Donaldson & Kronheimer, 1996), (Donaldson, 1996) and (Andersen, Dupont, Pedersen & Swann, 1997). Intermediate references were (Freed & Uhlenbeck, 1984), (Bourguignon & Lawson, 1982).

We just mention here that there is a drastic difference between Yang-Mills structures on various bundles and the condition for this manifold to be Einstein (see III). In the Yang-Mills game one keeps the metric downstairs fixed and varies the connection in the bundle, while the Einstein condition consists in *coupling* the Riemannian structure of the basis with the tangent bundle connection. To do both at the same time is very difficult today as we have seen.

✦

Twistor theory belongs more to Riemannian geometry than Yang-Mills fields, even though there are strong links between them. Twistor spaces are mostly bundles with compact fibers. They are constructed from a Riemannian manifold using its metric and some given additional structure, e.g. a complex structure, a special holonomy, etc. In dimension 4 every Riemannian manifold has a canonical twistor space above it, with fiber S^2. The main point is that twistor spaces have a richer structure than their basis, e.g. Kählerian. In many cases they have purely geometric applications. One application is the classification of minimal surfaces in standard spheres of high dimension done in (Calabi, 1967). They are also of basic use for finding holonomy groups of some types (TOP. 3), but also to construct Einstein manifolds, see for example (Hitchin, 1995). Besides a brief account in (Lawson & Michelsohn, 1989), one can consult in book form (Besse, 1987). An intermediate reference is (Atiyah, Hitchin & Singer, 1978). Twistor-like considerations appear also in 3-Sasakian manifolds in connection with quaternionic holonomy: TOP. 3.A.

✦

General vector bundles are of course only differential geometry at the beginning. The fact that the basis is Riemannian gives in particular a measure and also a canonical connection on the tangent bundle. What makes vector bundles interesting is the fact that the so-called *K-theory* is an algebra made up of all complex vector bundles over some given manifold. So one has succeeded in bringing in an algebraic tool to study differentiable manifolds, the maps between them, etc. For example, one of the important facts about the Dirac operator is that it represents the fundamental class in K-theory. For real vector bundles one had to use the more subtle *KO-theory*.

For general Riemannian real vector bundles, the idea of Atiyah and Singer was to extend to any such bundle the spinor construction of B (a sketch was given for the complex case). The topological construction is the same as for the tangent bundle. One can consider both the complex and the real case. Moreover Riemannian connections have associated with them a canonical Dirac operator and a generalized Lichnerowicz formula, which is again surprisingly simple, even for twisted bundles: see II.§8 of (Lawson & Michelsohn, 1989). For various vanishing theorems as used in I.B.3 an important fact is that the scalar curvature of the base manifold appears separately from the curvature term of the vector bundle under consideration. Finally these generalized Dirac operators became essential for obtaining a better understanding of the index theorem below. This was achieved through various contributions of Atiyah-Bott, McKean-Singer, Patodi and Gilkey: see the introduction to (Berline, Getzler & Vergne, 1992). This book is devoted precisely to the index approach with generalized spinors and Dirac operators.

✦

If one now very broadly generalizes the Laplacian acting on functions and on exterior forms by considering any elliptic operator on some vector bundle, there exists a very deep and universal result, the *Atiyah-Singer index theorem* which was a great event in 1963. This was a query of Gelfand's who conjectured the topological invariance of the index, see (Atiyah & Singer, 1963) for historical references and intermediate results. The index of an elliptic operator is the dimension of its kernel minus the dimension of the kernel of its adjoint (cokernel). The theorem is that this integer, on a compact manifold M, is equal to a number computed from only two things: first the Todd class of M, which can be expressed in terms of the set of its Pontryagin classes, secondly an invariant computed with the symbol of the elliptic operator considered as acting on the various Chern classes of the vector bundles on which the operator is defined. In the case of differential forms one just recovers the characteristic via the Hodge-de Rham theorem. Applications of the index theorem are numerous (and far from finished), for instance they yield the integrality of various invariants which are a-priori only real numbers. In the "pure Riemannian case" we saw above in I.B.3 the case of the scalar curvature. References are (Gilkey, 1995), (Berline, Getzler & Vergne, 1992) and the references therein.

✦

Starting with Witten in 1982, with Quillen in 1985 and with Bismut in 1986, most of the above various notions for bundles were raised at a *super* level, e.g. superspace, superconnection, supersymmetry, etc. In particular the Bismut Levi-Civita superconnection is fundamental for bringing the "super" concepts into Riemannian geometry. Since then things have kept developing and have drastically enlarged the panorama and changed the view, new concepts include connections, Laplacians and Dirac operators, asymptotic expansions of the heat kernel and the index theorem. The book (Berline, Getzler & Vergne, 1992) is the ideal reference for all this. For the holomorphic side, see (Bismut, Gillet & Soulé, 1988).

7. Harmonic maps between Riemannian manifolds

For the amateur of categories, systematically introducing maps between Riemannian manifolds is quite a natural idea, if not a must. It was realized a long time ago that geodesics do not only minimize the length $\int_{[0,1]} ||f'(t)||\, dt$ but also the *energy* $\int_{[0,1]} ||f'(t)||^2$. The quadratic character, as opposed to the absolute value, makes the computation of the variational derivatives much easier. Moreover this energy can be interpreted as follows: it concerns a map f from the interval into the Riemannian manifold and is defined with only the Riemannian structure of it. More generally one can attach to a map f between two Riemannian manifolds (M, g) and (N, h) its *energy* $\boldsymbol{E}(f)$, which is the integral over M (for its canonical measure) of the trace, with respect to g, of the quadratic form which is defined as follows: the derivative $f'(m)$ induces on every $T_m M$ a quadratic form which is the inverse one, by f', of h on $T_{f(m)}N$. A map f is said to be *harmonic* if its energy is critical among all maps from M to N.

This very general notion was introduced in the founding paper (Eells & Sampson, 1964). The case of surfaces is quite special because harmonic maps are directly related to minimal surfaces, thanks to the conformal representation. But in higher dimensions there is in general no direct connection between minimal submanifolds and harmonic maps, only in some special cases.

Since then, a rich history of this notion has developed, regarding the natural questions as well as applications. The reader will find references, in particular to surveys, in the book (Eells & Rato, 1993). At the time of its publication, the book-sized (Eells & Lemaire, 1988) was a very systematic, informative and complete survey. Important are various regularity results. In some cases harmonic maps are defined in a more general context than the purely Riemannian one (see also TOP. 9), see the books (Jost, 1995), (Helein, 1996) and for applications to Riemann surfaces (Jost, 1996).

Harmonic maps are a basic tool today in Riemannian geometry as is very well illustrated in the report by Eells-Lemaire. For example see their use for space forms in II.B, where the notion had to be extended to manifolds with singularities, for manifolds with a positive curvature operator in I.B.1, for nonpositive curvature manifolds (I.B.4) in (Jost & Yau, 1990). Harmonic maps are especially useful for studying Kähler manifolds, in particular their fundamental groups, see (Amoros, Burger, Corlette, Kotschik & Toledo, 1996) and the references in 3.B.

8. Low dimensional Riemannian Geometry

Even though dimension 2 is wonderful, we already saw above that dimensions 3 and 4 have to be treated with care. For example many finiteness theorems mentioned in I.C.1 start to hold only in dimension 5 or more. On the other hand, in some instances one has very strong results in dimensions 3 and 4. For dimension 3, there is the "topological" survey (Scott, 1983).

One of the points of interest of Riemannian geometry in low dimensions is the hope of using Riemannian tools to solve topology and differential topology problems which still haunt topologists. Recent reports are (Donaldson, 1996) and the books (Donaldson & Kronheimer, 1996) and (Morgan, 1996). They present in particular the basic tools: Yang-Mills theory, twistors, anti-self-duality in dimension 4 and the Spin_C theory of (Seiberg & Witten, 1994), see the end of TOP. 6.B.

For the "best metric" approach based on functionals and their critical points, and the link with the geometrization program of Thurston, see the various references to Anderson we gave in Chapter III, in particular (Anderson, 1997), as well as (Kapovich & Leeb, 1996) . Finally we recall here the two Hamilton results, using the Ricci flow, for the Ricci curvature in dimension 3 (I.B.2) and for the curvature operator in dimension 4 (I.B.1).

9. Some generalizations of Riemannian Geometry

A first generalization is that of *Riemannian manifolds with boundary.* Except for the η-invariant above, we never mentioned them. For the general case one can see (Alexander, Berg & Bishop, 1990) and the references therein. For the relation to positive scalar curvature see (Lawson & Michelsohn, 1984). For the cut-locus see (Alexander, Berg & Bishop, 1993) and (Alexander & Bishop, 1997).

We mention a problem which, besides its importance in itself, appears in many practical problems. Assuming some kind of convexity on the boundary $\partial(M)$ of a Riemannian manifold (M, g), one can define a good distance $d(p, q)$ between points of $\partial(M)$. To what extent does this distance d as a map $\partial(M) \times \partial(M) \to \mathbf{R}^+$ determine the "inside" metric g (up to isometry-diffeomorphism of the inside)? Thinking a while of earthquakes, tomography, X-rays and scanners, one sees the practical importance of such a kind of problem. In an indirect way it was also met in I.B.4 and in the "gap" results in TOP.5. References are (Otal, 1990b), (Michel, 1994), (Croke, 1991), (Arconstanzo, 1994) along with its references, and (Bourdon, 1996). See also the Besicovitch results at the end of TOP. 1.E.

✦

Back to generalizations, the subject exploded recently and results are now appearing in a steady stream so that we will give a very concise overview. Some cases appeared sporadically before, today strong incentives come from the wish to obtain a deeper understanding of Riemannian geometry in various instances, for instance that of I.C, namely to look at limits of a sequence of Riemannian manifolds. Interesting generalizations of Riemannian geometry can be of different types, since of course looking at general metric spaces would be too general to obtain reasonably deep results, for example to define objects playing the role of tangent vectors, angles, curvature. We will try to guide the reader through this new realm by suitable recent references and of course survey type ones when they exist. Concerning references of a general type we know only of (Berestovskii & Nikolaev, 1993).

A special mention should be given to Gromov's recent *mm-spaces* introduced in Chapter $3\frac{1}{2}$ of (Gromov, 1998), these are metric spaces endowed with a measure reasonably compatible with the metric, namely the metric should be a measurable function on the product. One of their properties is that one can work efficiently on the geometry of the set made up of all these spaces, in particular one can study what happens when the dimension gets large, a fundamental problem in statistical mechanics. Such mm-spaces are very general, the same applies to various spaces which appear in the work of Semmes, see (Semmes, 1994). We turn now to more direct (and less general) generalizations of Riemannian manifolds.

✦

Orbifolds were introduced in (Satake, 1956) and (Baily, 1957) in algebraic geometry and they were revived in Riemannian geometry by (Thurston, 1978). Essentially they are quotients of Riemannian manifolds by isometries when one permits fixed points and moreover requires the isotropy group to be finite. This is typical for the case of space forms. But they arose in more subtle situations, for example as limit spaces of some subsets of Riemannian manifolds with a Ricci curvature bounded from below as well as when working on Einstein manifolds, see (Anderson & Cheeger, 1992), (Anderson, 1992a), (Anderson, 1994) and see also (Ballmann & Brin, 1995). For orbifolds and quaternionic holonomy see (Galicki & Laswon, 1988), for orbifolds in the Kähler realm see (Tian, 1991).

✦

Spaces with an isolated *conical singularity* look quite simple, but for applications it is important to see what happens to the spectrum (see IV). More (or less) generally one can think of studying Riemannian manifolds by approximating them by *PL* (piecewise linear) ones, metrically the pieces are all flat and "the curvature is concentrated (distributionally)" at the vertices, the edges, etc. The PL-manifolds are the very natural locally Euclidean version of general Riemannian manifolds but with singularities. What happens to the curvature and to various formulas is the object of (Cheeger, Müller & Schrader, 1984) and (Cheeger, Müller & Schrader, 1986), see (Lafontaine, 1985/86) for an expository text. This is in fact a very subtle subject, we saw one case in TOP. 1.B for the isoperimetric profile of nonpositive curved manifolds, see also the references given in I.B.4 in connection with hyperbolic groups.

An extensive study of the spectral behavior of singular spaces was made by Cheeger in a long series of papers: we refer to (Cheeger, 1983) and its bibliography. This work was the outcome of the proof in (Cheeger, 1979) of the conjecture by Ray-Singer concerning the Reidemeister torsion (see TOP. 6.A). One of its byproducts concerned Analysis, namely results on the diffraction of waves meeting some kind of obstacles, another one was an explicit solution of the heat equation on standard spheres; for both see (Cheeger & Taylor, 1982).

✦

Alexandrov spaces are a wonderful class because they enjoy two simultaneous properties. First they arise naturally at the boundary of the space of all Riemannian manifolds whose curvature is bounded (only) from below. Secondly they share many properties with (smooth) Riemannian manifolds. There are many texts which are more or less of survey type: (Perelman, 1994c), (Berestovskii & Nikolaev, 1993), (Reshetnyak, 1993), (Burago, Gromov & Perelman, 1992), (Otsu & Shioya, 1994) and (Yamaguchi, 1992). We now give an extremely brief account.

These spaces appeared first in 1948 in A. D. Alexandrov's work on convex surfaces, but in 1957 Alexandrov made a more systematic study. Then came the founding paper (Burago, Gromov & Perelman, 1992) and in between (Gromov, Lafontaine & Pansu, 1981).

In I.C we had the desire to look at the boundary of suitable subsets of the set of all Riemannian manifolds (compactness and convergence results). The description was possible (to some extent) for a bounded curvature, say $-1 \leq K \leq 1$. But for $K \geq k$ we had only finiteness theorems and no description of collapsing. The solution lies in the theory of Alexandrov spaces with a curvature bounded from below.

On the one hand, Alexandrov spaces with a curvature bounded from below are defined very simply. They are first locally compact metric spaces which are *length spaces*, namely the metric coincides with the infimum of the lengths of curves joining the two given points. Note here a question of terminology. In (Gromov, Lafontaine & Pansu, 1981) the wording was "length space", in (Gromov, 1987a) the wording "geodesic space" was used for the case when moreover any pair of points can be joined by at least one shortest path. Busemann spoke of "intrinsic" metrics in (Busemann, 1955). Here such a length space is called Alexandrov with curvature $\geq k$ if it enjoys (besides completeness of course) everywhere a local Toponogov comparison theorem (see I.A.2) with the standard space form of constant curvature equal to k. One can then prove a global Toponogov theorem but also much more. Only with such a mild condition one has a notion of angles, a notion of a tangent cone which is itself an Alexandrov space of curvature ≥ 0 and can be defined in many equivalent ways. The points where this cone is not Euclidean are the singular points and their set is of Hausdorff codimension at least 2, so that our spaces are almost everywhere locally Riemannian manifolds. Note however that the singular set can be everywhere dense, just think of a suitable limit of convex polyhedra. Important for generalizations of Riemannian results in the spirit of I.A and B is the fact that the notion of critical points for distance functions is well defined and that regions without critical points can be deformed into each other.

On the other hand, because the definition of curvature $\geq k$ is purely metric within the set (MET, d_{G-H}) introduced in I.B the limit points of Riemannian manifolds with $K \geq k$ are automatically Alexandrov spaces with a curvature $\geq k$. Hence the results above constitute the answer to our main query and any further results on those spaces will be a progress in the study of the boundary of the set of Riemannian manifolds with $K \geq k$. The following recent references give an idea of how active the research on this topic is. (Grove, 1992) extends some results of I.A, in (Shiohama & Tanaka, 1992) cut-loci and distance spheres are studied, in (Yamaguchi, 1992) the collapsing structure result of I.C is generalized; various types of results generalizing those in I can be found in (Petersen, 1996), for example the bound for Betti numbers (see I.B.2) of nonnegatively curved manifolds (see the references therein).

In (Cheeger & Colding, 1997a), (Cheeger & Colding, 1997b) and (Cheeger & Colding, 1998) many of the above results are generalized to "limit spaces with a Ricci curvature bounded from below". These spaces occur of course as limit spaces, in the Gromov-Hausdorff space, of Riemannian manifolds with the same Ricci condition.

✦

In the opposite direction to the above one can look for a large, but not too large, class of spaces generalizing Riemannian manifolds of negative curvature. A helpful notion is that of *hyperbolic spaces*, which where introduced and studied in (Gromov, 1987a) (not to be confounded with hyperbolic space forms, which are only a very very special case). This is a game where the interesting case is when the "curvature" is bounded from above. To get an idea of the spirit of the game see I.B.4. After the publication of the founding paper (Gromov, 1987a) the topic developed into an entire world of its own. We just mention some references: (Gromov, 1993) and the books (Ghys & de la Harpe, 1990) and (Bridson & Haefliger, 1998).

✦

Here the analogous condition to that of Alexandrov spaces with a curvature bounded from below, but this time involving an upper bound on the curvature, is the **CAT(k)** condition. Say that we work in simply connected spaces. The **CAT(0)** condition requires that one has, for every triple of points, the triangle inequality discovered by Cartan and rediscovered by Preissmann(see 0.B and C). For

CAT(k) it is the same condition but the comparison is no longer performed with respect to Euclidean spaces but with respect to hyperbolic spaces of curvature k. The acronym CAT stands for Cartan, Alexandrov, Toponogov. One of the difficulties in hyperbolic spaces is that they can have branch points, they can even be graphs. Think also of a one-sheet hyperboloid converging toward the total cone and look at the origin. Conversely, one sheet of the two-sheet hyperboloid associated with the same asymptotic cone will converge to a conical point (of positive curvature), a nice Alexandrov space.

Examples are to be found among the PL-manifolds (piecewise linear, i.e. Euclidean). For PL-surfaces the CAT(0) condition is what you expect: at every vertex the sum of the angles of the triangles meeting there has to be larger than or equal to 2π. For higher dimensions it is not all that easy to state this condition in explicit terms. The general CAT(k) condition goes the reverse way to the Alexandrov case. It was mentioned above how difficult it is to understand what happens to the curvature and to various formulas, see also (Bourdon, 1996).

✦

Other recent newcomers in the generalization of Riemannian geometry are the *Carnot-Caratheodory metric spaces.* Regarding them one can say in a caricatural sense that singularities occur at every point and are very strong. They are the subject of the book (Bellaïche & Risler, 1996). More precisely one is given at every point m in some manifold M^d some vector subspace $V(m)$ of the tangent space $T_m M$ of a constant dimension $k < d$. One endows those subspaces with a Euclidean structure. The metric is then defined as follows: the distance $d(m, n)$ between two points, which is the infimum of the lengths of all curves whose speed vector at every one of its points q should always belong to $V(q)$. Hence the other name for this topic: *sub-Riemannian* geometry. This distance can be infinite but it is always finite when the distribution $V(.)$ is wild enough, that is to say "completely non-integrable", in some sense the complete opposite of the foliation case. Carnot-Caratheodory spaces are basic to *control theory* since in this setting only special paths are permitted due to various restrictions imposed by practical situations. In street traffic, the geodesics are what you have to do to park and unpark in a narrow slot. See the recent (El Alaoui, Gauthier & Kupka, 1996) and the introductory text (Pelletier & Valère Bouche, 1992). For the naive reader or a naive PDE person (Bryant & Hsu, 1993) is extremely informative.

✦

I have been keeping back *Finsler geometry* for the middle of this section. This is not historically correct. As is very well explained in (Spivak, 1970), where Riemann's dissertation is analyzed, Riemann in fact generalized Euclidean geometry by introducing Finsler manifolds. These are differentiable manifolds M where some Banach structure $g(m)$ (i.e. a convex body symmetric around the origin of $T_m(M)$) is given at every point m. With this, lengths and hence a metric can be defined as well. Moreover Riemann wrote this prophetic sentence: "we will now stick to the case of ellipsoids (quadratic forms) because, if not, the computations would become very complicated". One can also relax the symmetry condition, see for example the Katok type examples mentioned in V.A.

There are only few geometrical objects as natural as Finsler manifolds, they arise for instance in any kind of "Mechanics" as soon as the energy is no longer quadratic. However the destiny of Finsler geometry was different from that of Riemannian geometry. It is clear that I am biased regarding this subject but I think that, up until very recently, there were very few strong typically Finsler results which were not already Riemannian. And one can also say, of course only when it makes sense, that there were few results that were true in Riemannian geometry but not in Finsler geometry. Up to very recently we knew two such results: those by Leon Green and by Burago-Ivanov in V.B. One counterexample appeared in (Busemann, 1955), where non-flat Finsler tori without conjugate points were constructed. The other one is to be found in (Skorniakov, 1955), where it is proved that any system of curves on $\mathbf{R}P^2$ satisfying the axioms of projective geometry (and some obviously needed reasonable smoothness) is that of the geodesics of some Finsler (non-Riemannian in general, thanks to Desargues' theorem) structure. Green's theorem figures in V.B and in TOP. 4. An interesting text is (Vérovic, 1996) where it is proved that the result of (Besson, Courtois & Gallot, 1995a) (quoted in I.B.4, III and V) can be either true or false for Finsler manifolds. The inconclusive answer "yes or no" is due to the fact that Finsler manifolds can be assigned different notions of measure in their unit bundle. Conversely, most of the results in the "filling paper" (Gromov, 1983a) are in fact valid for the Finsler case.

Here is a personal comment. Finsler spaces look more general than Riemannian metrics. They are so in some sense. But in another sense they are more specialized. In an affine (finite dimensional) real vector space a (symmetric) convex body has priviliged directions (points), e.g. those where the affine curvature is critical, etc. Conversely, an ellipsoid is *infinitesimally isotropic*, all the directions (the points) are equivalent. This is for us one reason for the importance of Riemannian geometry.

There are of course many places where one can find information on Finsler manifolds, which are experiencing a strong revival at present. A recent reference is (Bao, 1995). The note (Foulon, 1997) is a perfect quick introduction to the modern point of view, see the formula at the end of V. We mention the fact that Finsler manifolds have a curvature, but what is implied by constancy of this curvature is not clear today without any extra assumptions. The result of the aforementioned note is again a rigidity result for compact negative curvature manifolds. The positive case remains mysterious, but (Bryant, 1996) throws some light on it. (Alvarez, Gelfand & Smirnov, 1997) characterizes Finsler metrics for which the geodesics are the straight lines (Hilbert's fourth problem), see also (Shen, 1998).

✦

Riemannian foliations also form a topic of their own, as do totally geodesic foliations of Riemannian manifolds: relevant books are (Tondeur, 1988) and (Godbillon, 1991). For the relation to scalar curvature see Chapter $1\frac{7}{8}$ of (Gromov, 1996) and see also in (Connes, 1994) a new way of viewing foliations.

✦

Lorentzian manifolds are completely similar, at least at the start, to Riemannian ones. They are 4-dimensional manifolds that have everywhere a definite quadratic form, but one of signature $(+,+,+,-)$. More generally *pseudo-Riemannian* geometry is of arbitrary dimension and has nondegenerate quadratic forms of any signature. At the beginning they have many things in common, but this is definitely misleading at long range, as was remarked by the physicist C. N. Yang with the help of a beautiful picture: (Yang, 1980). The picture can also be seen on page 11 of (Besse, 1987). The golden triangle is the main point that both objects have in common. Then questions and results diverge. Lorentzian "geometry" derives its inspiration mainly from General Relativity. Books on Lorentzian geometry are (Beem & Ehrlich, 1981), (Hawking & Ellis, 1973), (Misner, Thorne & Wheeler, 1973), (O'Neill, 1983) and (Sachs & Wu, 1977). A special mention should be given to (d'Ambra, 1988) (see also (d'Ambra & Gromov, 1991)), where it is proved that the isometry group of a compact and simply connected Lorentzian manifold is itself compact (presently analyticity is needed but this looks only technical), for more see (Adams & Stuck, 1997). For surfaces (Weinstein, 1996) studied a Lorentzian conformal concept which is analogous to that of Riemann surfaces as opposed to that of Riemannian surfaces. (Christodoulou & Kleinerman, 1993) is of Riemannian flavor, see the expository paper (Bourguignon, 1990/91). Recently pseudo-Riemannian geometry has appeared in various contexts, see for example (Benoist, Foulon & Labourie, 1992) and (Kühnel & Rademacher, 1995).

✦

Infinite-dimensional Riemannian geometry is still very young. The book (Lang, 1972) was the first to treat Riemannian geometry systematically in an infinite dimensional setting. Then appeared (Klingenberg, 1982) and recently the second edition (Lang, 1996) and the reader can consult these books. See also (Bourbaki, 1969), (Gil-Medrano & Michor, 1991) and Chapter $3\frac{1}{2}$ of (Gromov, 1998). We will mention only a few aspects.

The *first* one is to use suitable embeddings of Riemannian manifolds in the standard Hilbert space. This can be done for example by using suitably normalized eigenfunctions of the Laplacian, see (Besson, Courtois & Gallot, 1991) and (Bérard, Besson & Gallot, 1997) for the setting and applications to finiteness results.

One can also embed any Riemannian manifold M in the Banach space of the continuous functions $C^0(M)$ by the "trivial" map made up of the various distance functions, namely $m \to d(m,.)$. This childishly simple map turned out to be basic to the systolic inequality of (Gromov, 1983a), see TOP. 1.E.

The *second* most natural desire is to define a notion of infinite-dimensional Riemannian manifolds by starting with some infinite-dimensional differentiable manifold M (whatever it means) and endowing every tangent space T_mM with some Hilbertian structure. This seems easy. In our view, it is in fact difficult, at least today, for the following reason: for a simple Hilbertian structure one gets in general non-complete metric spaces. With more complicated ones (typically in the

Sobolev range) one has completeness but then the geodesics seem to have no good and/or useful geometrical properties, their equation is too complicated, see (Ebin, 1970) and (Ebin, 1972). The text (Gil-Medrano & Michor, 1991) addresses the question of constructing a manifold with the set of all Riemannian metrics on a given manifold.

In Appendix 2 of (Arnold, 1978) the group of diffeomorphisms of a manifold is given an (infinite-dimensional) Riemannian structure which is proved to be always of negative curvature.

Path-space Riemannian geometry is very important, but many things concerning it are only conjectural, since a solid framework is still missing. For example for more on this in the context of positive Ricci curvature see (Stolz, 1996). There is also the informative paper (Stroock, 1996).

✦

Finally special attention should be given to (Connes, 1995–6): this is the beginning of a complete program designed to put most (if not all) geometries into a very general framework. The framework is that of algebras of operators and is called "noncommutative geometry". In this framework, according to various suitable additional axioms one can recover almost any kind of geometry, including of course Riemannian ones. The next step in the program will be to generalize every concept one could wish, e.g. curvature. For the metric itself, this has been done and we saw in TOP. 6.B that one can compute the distance with the help of spinors. See also III.C for the Hilbert functional in this context.

10. Submanifolds

Among other motivations, Riemannian geometry came into existence when one was trying to generalize the geometry of surfaces in $\mathbf{R}^3$. There is no reason not to consider the geometry of submanifolds inside a given Riemannian manifold. Because Euclidean geometry is hereditary for subspaces, it follows that Riemannian geometry is also *hereditary* for submanifolds (but of course the metric is not the induced one in the rough metric sense), this is one more reason why Riemannian geometry is important. We do not know of any systematic survey of this topic, even in the special and much studied case of submanifolds of Euclidean spaces. A few books give the general equations for a submanifold of any codimension in a Riemannian manifold (the so-called Gauss and Codazzi-Mainardi equations), among them Chapter VII of (Kobayashi & Nomizu, 1963–1969), Chapter XX (see Section 20.14.8) of (Dieudonné, 1972) and Chapter 7 of (Spivak, 1970) present the foundations of the theory of submanifolds in a Riemannian manifold. But there is a huge number of local and global results. Recent surveys are (Terng, 1990), (Terng & Thorbergsson, 1995), (Dillen & Verstraelen, 1992), (Palais & Terng, 1988), see also (Willmore, 1993). In particular *isoparametric hypersurfaces* deserve a special mention as they relate different topics to each other. The founding text is (Cartan, 1939), for isoparametric submanifolds in an infinite dimensional setting see (Heintze & Liu, 1997).

The naive geometer will ask what constitutes a generalization of affine (Euclidean) and/or projective subspaces in Riemannian geometry. The basic remark is that a general Riemannian manifold admits *no* submanifold which is stable under geodesy (i.e. local geodesics in it are geodesics in the ambient one and this is certainly not the case in general). Such submanifolds are called *totally geodesic.* Exceptions are extremely few and among them are the symmetric spaces of rank ≥ 2, see II.C. We saw in I.B.4 how the fact of having "flats" is very powerful in some instances, see among others (Samiou, 1997). An extreme case occupied people at the beginning of the century: the axiom of free mobility. Matters were completely clarified by Elie Cartan in (Cartan, 1928a) (one can also consult the second edition (Cartan, 1946–1951)), where he proved that only space forms of constant curvature satisfy this axiom.

A most naive problem is the following: what is the convex envelope of k points in a Riemannian manifold of dimension $d \geq 3$? Even for three points and $d \geq 3$ the question is completely open (except when the curvature is constant). A natural example to look at would be $\mathbf{C}P^2$, because it is symmetric but not of constant curvature. The only text we know of that adresses the question is (Bowditch, 1994).

A. The case of surfaces in $\mathbf{R}^3$

Surfaces in $\mathbf{R}^3$ were the main object of differential geometers in the preceding century, the bible was (Darboux, 1887, 1889, 1894, 1896) or (Darboux, 1972). Many things were put into a modern setting in (Spivak, 1970), a survey is (Burago and Zalgaller, 1992). Even for compact surfaces the subject enjoyed many results in our half of the century. A famous recently still open problem is Willmore's conjecture, see (Willmore, 1993).

In 1955 Alexandrov proved that embedded surfaces with constant mean curvature have to be round spheres, and he proved this for any topological type. For the simply connected cases (topological spheres) this was Hopf's "old" result (Hopf, 1951). Besides the book by Burago-Zalgaller, one could consult (Hopf, 1983) and (do Carmo, 1976). These surfaces are physically the *soap-bubbles* and Hopf proved that they are really round spheres. The problem left open concerned the possibility of immersions, which are impossible for the topological type of a sphere. An important event was (Wente, 1986), where immersed tori of constant mean curvature were constructed. This started a field of research which is still not finished, see (Pinkall & Sterling, 1989). A recent point of view is illustrated in (Kamberov, 1997), where spinors enter into the picture.

In higher dimensions, Alexandrov's theorem is still valid for embedded hypersurfaces: only round spheres have a constant mean curvature. But for immersions matters are radically different: there are many non-spherical immersed topological spheres S^{d-1} of constant mean curvature in $\mathbf{R}^d$: see (Eells & Rato, 1993) for references; the topic is directly related to harmonic maps.

An old result of Hilbert's says that there is no immersed (an embedding is not even needed for the conclusion) surface isometric to the hyperbolic plane. This means constant Gauss curvature and completeness. This might explain why hyperbolic geometry could not be discovered in $\mathbf{R}^3$. This result was generalized in the 50's by Efimov in a profound paper, see (Klotz Milnor, 1972) and (Burago & Zalgaller, 1992): there is no immersion in $\mathbf{R}^3$ of a complete surface of curvature $K \leq a < 0$. But there is still a deep (naive) problem: is there any complete surface of negative curvature that stays in a bounded region of $\mathbf{R}^3$? Some people call the preceding question "Hadamard's conjecture" because Hadamard took the non-existence for granted. An analogous conjecture, that by Calabi-Yau, was that there cannot exist a complete minimal surface that stays in a bounded region. In (Nadirashvili, 1996) both conjectures are smashed – one constructs a surface living in a bounded region which is at the same minimal and of negative curvature. The tool consists in applying Weierstrass' formulas for minimal surfaces.

Some questions which are still open concern the isometric non-trivial deformations of surfaces in $\mathbf{R}^3$. First globally: all non-deformation results are for convex surfaces and non-congruence examples are C^∞-surfaces. Locally it looks obvious that any piece of a surface can be deformed (with a lot of parameters: take a tennis ball with some piece removed). In fact the question is unbelievably hard, we refer to the survey in (Burago & Zalgaller, 1992). We mention here that there are examples of pointed surfaces which cannot be deformed no matter how small the neighborhood of the point is. See also Bleecker's result at the end of TOP. 2.

A special mention should be given to the case of zero mean curvature, namely the minimal *surfaces* (*Plateau's problem*). This old topic has undergone some tremendous developments. There are many good surveys and even entire books on the topic. The most recent one is (Dierkes, Hildebrandt, Küster & Wohlrab, 1992), others are (Nitsche, 1996), (Jost, 1991), (Struwe, 1988) and (Osserman, 1996); see also GMT in C below and in 0.F.

B. Higher dimensions

An important remark, which is not very well known despite being classical, is in order. Topic A is important not only for historical reasons but because of the following. When $d \geq 4$ take in $\mathbf{R}^d$ some hypersurface which is generic. Then it is easy to see that the sectional curvature (hence the isometry type) determines the second fundamental form. Classically this implies congruence (this is a purely local statement). The proof works in the following way: in a basis which diagonalizes the second fundamental form, the Gauss equations imply that, for the sectional curvature, one has $K(e_i, e_j) = \eta_i \eta_j$, where the η_i are the principal curvatures (this is exactly the generalization of the theorema egregium in 0.A). And now, if one knows the three products ab, bc and ca, then one knows a, b, c as long as none is zero. So in the generic case the metric of a hypersurface automatically also determines its second fundamental form. The generalized Gauss-Codazzi equations show that the metric and the second fundamental form together completely determine the embedding (see any of the books quoted at the beginning of this section). So in some sense there is no "congruence versus isometry problem" left. Of course the above philosophy applies also to hypersurfaces in general Riemannian manifolds M^d as soon as $d \geq 4$. One can find the above remark back on page 237 of (Killing, 1885) and in (Thomas, 1935), or see page 42 of Volume II of (Kobayashi & Nomizu, 1963–1969).

Matters are completely different when $d = 3$, the Gauss curvature (hence the isometry type) yields only the product of the two principal curvatures. Then there is plenty of space for a fascinating game, not finished as just seen above. Let us look for example at rigidity: two compact strictly convex hypersurfaces which are abstractly isometric are congruent (deduced by a global isometry of the whole $\mathbf{R}^d$). For $d > 3$ it follows from the above remark, but when $d = 3$ the result remains true, but the proof is much more sophisticated and is due to Cohn-Vossen and Herglotz: for references and more see (Berger & Gostiaux, 1988), Section 11.4 and/or (Klingenberg, 1978b), 6.2.8.

Having realized this, many topics still remain in this field, especially global questions – we already mentioned some of them above. *Integral geometry* is very interesting, but works only (with one exception to come) in Euclidean spaces and space forms. See the surveys (Schneider & Wieacker, 1993), (Langevin, 1997) and add the classic (Santalo, 1976), in particular Chern's kinematic formula. In general Riemannian manifolds integral geometry also works but only for hypersurfaces and geodesics starting from them. It is basic to Croke's local isoembolic inequality of TOP. 1.C and the isoperimetric inequality for nonpositive curvature manifolds in TOP. 1.A: see (Croke, 1980) and (Croke, 1984).

C. Geometric measure theory and pseudo-holomorphic curves

Submanifolds enjoying various strong geometric properties are basic in some instances. Two seem to be important today. First the dramatic appearance of *GMT* (geometric measure theory, see 0.F) with its harvest of results provided Riemannian geometers with almost all the existence theorems they could dream of when forgetting about *minimal* (or constant mean curvature) objects. This was used above, in Schoen-Yau's aproach for positive scalar curvature in I.B.3 and for the isoperimetric profile in TOP. 1.B. GMT is an extremely hard subject, in the founding book (Federer, 1969) as well as in many other articles on the subject it is very hard to thread one's way through in a reasonable time. Our advice is to read first (Morgan 1988), then (Simon, 1983).

The second tool appeared in the pioneer paper (Gromov, 1985). A recent survey in book form is (Audin & Lafontaine, 1994). Although those pseudo-holomorphic curves are mainly used in symplectic geometry, one meets them in Riemannian geometry, see for example the article of Labourie in the aforementioned book and (Gromov, 1992a).

Bibliography

Abrahamov (1951). On the topological invariants of Riemannian spaces obtained by integration of tensor fields. Doklady Akad. Nauk. SSSR, 81, 125–128.

Abresch, U. (1985). Lower curvature bounds, Toponogov's theorem, and bounded topology. Ann. scientif. Ecole norm. sup., 18, 651–670.

Abresch, U. (1988) Über das Glätten Riemannscher Metriken. thesis, Bonn.

Abresch, U. (1990). Endlichkeitssätze in der Riemannschen Geometrie. Jber. d. Dt. Math-Verein., Jubiläumstagung, 152–176.

Abresch, U. & Gromoll, D. (1990). On complete manifolds with nonnegative Ricci curvature. J. of the Amer. math. Soc., 3, 355–374.

Abresch, U. & Meyer, W. (1994). Pinching below 1/4, injectivity radius, and conjugate radius. J. differential geometry, 40, 643–691.

Abresch, U. & Meyer, W. (1996). A sphere theorem with a pinching constant below 1/4. J. of differential geometry, 44, 214–261.

Abresch, U. & Meyer, W. (1997). Injectivity radius estimates and sphere theorems. In Comparison geometry, MSRI publications, Cambridge University Press.

Abresch, U. & Schroeder, V. (1996). Analytic manifolds of nonpositive curvature. Table ronde de géométrie différentielle 1, pp. 1–67, Luminy, Séminaires et congrès 1, Soc. Math. France

Accola, R. (1960). Differential and extremal lengths on Riemann surfaces. Proc. Nat. Acad. Sci. USA, 46, 540–543.

Adams, S. & Stuck, G. (1997). The isometry group of a compact Lorentz manifold, I and II. Inventiones math., 129, 239–261 and 263–287.

Alekseevskii, D. (1968). Riemannian spaces with exceptional holonomy groups. Functional analysis and applic., 2, 97–105.

Alexander, S. & Bishop, R. (1997). Thin Riemannian manifolds with boundary. Math. Annalen.

Alexander, S., Berg, D. & Bishop, R. (1990). The distance geometry of Riemannian manifolds with boundary. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54, part 3 (pp. 31–36). Los Angeles: American math. soc. 1993

Alexander, S., Berg, D. & Bishop, R. (1993). Cut-loci, minimizers and wave fronts in Riemannian manifolds with boundary. Michigan math. j., 40, 229–237.

Allendoerfer, C. & Weil, A. (1943). The Gauss-Bonnet theorem for Riemannian polyhedra. Trans. Americ. math. Soc., 53, 101–129.

Aloff, S. & Wallach, N. (1975). An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures. Bull. of Americ. math. Soc., 81, 93–97.

Alvarez, J., Gelfand, I. & Smirnov, M. (1997). Crofton densities, symplectic geometry and Hilbert's fourth problem. Univ. cath. Louvain, Belgique.

Ambrose, W. (1956). Parallel translation of Riemannian curvature. Annals of math., 64, 337–363.

Ambrose, W. & Singer, I. (1958). On homogeneous Riemannian manifolds. Duke math. j., 25, 647–669.

Amoros, J., Burger, M., Corlette, K., Kotschik, D. & Toledo, D. (1996). Fundamental groups of compact Kähler manifolds. American math. soc.

Andersen, J., Dupont, J., Pedersen, H. & Swann, A. (Ed.). (1997). Geometry and physics. Marcel Dekker.

Anderson, M. (1990a). Metrics of positive Ricci curvature with large diameter. Manuscripta math., 68, 405–415.

Anderson, M. (1990b). Convergence and rigidity of metrics under Ricci curvature bounds. Inventiones math, 102, 429–445.

Anderson, M. (1990c). Short geodesics and gravitational instantons. J. of differential geometry, 31, 265–275.

Anderson, M. (1990d). Degeneration of metrics with bounded curvature and applications to critical metrics of Riemannian functionals. In R. Greene & S.-T. Yau (Ed.), Differential geometry, part 3 (pp. 53–79). Los Angeles: American math. soc. 1993

Anderson, M. (1991). Remarks on the compactness of isospectral sets in low dimensions. Duke math. j., 63, 699–711.

Anderson, M. (1992a). The L^2 structure of moduli spaces of Einstein metrics on 4-manifolds. Geometric and functional analysis, 2, 29–89.

Anderson, M. (1992b). Hausdorff perturbations of Ricci-flat manifolds and the splitting theorem. Duke math. j., 68, 67–82.

Anderson, M. (1994). Einstein metrics and metrics with bounds on Ricci curvature. In International congress of mathematicians, 1 (pp. 443–452). Zürich: Birkhäuser.

Anderson, M. (1996a). Scalar curvature and geometrization conjectures for 3-manifolds. In Comparison geometry (pp. 47–80). MSRI publications, Cambridge university press.

Anderson, M. (1996b). Extrema of curvature functionals on the space of metrics in 3-Manifolds. In Calculus of variations and PDE.

Anderson, M. (1997). The static vacuum Einstein equations and degenerations of Yamabe metrics on 3-manifolds.
Anderson, M. & Cheeger, J. (1991). Diffeomorphism finiteness for manifolds with Ricci curvature and $L^{n/2}$ curvature bounded. Geometric and functional Analysis, 1, 231–251.
Anderson, M. & Cheeger, J. (1992). Ca compactness for manifolds with Ricci curvature and injectivity radius bounded below. J. differential geometry, 35, 265–281.
Andersson, S. & Lapidus, M. (Ed.) (1997). Recent progress in inverse spectral geometry. Birkhäuser.
Anosov, D. (1967). Geodesic flows on closed Riemannian manifolds with negative curvature. Proc. Stekhlov Inst., 90.
Anosov, D. (1983). On generic properties of closed geodesics. Math. USSR Izv., 21, 1–29.
Arconstanzo, M. (1994). Des métriques finslériennes sur le disque à partir d'une fonction distance entre les points du bord. Commentarii math. helvetici, 69, 229–248.
Arnold, V. (1978). Mathematical methods of classical mechanics. Springer.
Arnold, V. (1988). Geometrical methods in the theory of ordinary differential equations, second edition. Springer.
Atiyah, M. (1979). Geometry of Yang-Mills fields. Pisa: Academia naz. dei Lincei and Scuola norm. superiore.
Atiyah, M., & Singer, I. (1963). The index of elliptic operators on compact manifolds. Bulletin of the A.M.S., 69, 422–433.
Atiyah, M., Bott, R. & Patodi, V. (1973). On the heat equation and the index theorem. Inventiones math., 19, 279–330.
Atiyah, M., Donnelly, H. & Singer, M. (1983). Eta-invariants, signature defects of cusps, and the value of L-functions. Annals of math., 118, 131–177.
Atiyah, M., Hitchin, N. & Singer, I. (1978). Self-duality in four-dimensional geometry. Proc. royal soc. London, 362, 425–461.
Atiyah, M., Patodi, V. & Singer, I. (1975–1976). Spectral asymmetry and Riemannian geometry I, II, III. Math. Proc. Cambridge phil. soc., 77–78-79, 43–69, 405–432, 71–99.
Aubin, T. (1970). Métriques riemaniennes et courbure. J. of differential geometry, 4, 383–424.
Aubin, T. (1976). Equations du type de Monge-Ampère sur les variétés kählériennes compactes. C. R. Acad. Sc. Paris, 283, 119–121.
Aubin, T. (1982). Nonlinear analysis on manifolds. Monge Ampère equations. Springer.
Audin, M. & Lafontaine, J. (Ed.) (1994). Pseudo-holomorphic curves in symplectic geometry. Birkhäuser.
Avakumovich, V. (1956). Über die Eigenfunktionen auf geschlossenen Mannigfaltigkeiten. Mathematische Zeitschrift, 65, 327–344.
Babenko, I. (1992). Asymptotic invariants of smooth manifolds. Russian acad. sc. Izv. math., 41, 1–38.
Babenko, I. & Katz, M. (1997). Systolic softness of orientable manifolds. Annales scientifiques Ecol. norm. sup.
Babenko, I., Katz, M. & Suciu, A. (1998). Volumes, middle-dimensional systoles, and Whitehead products.
Babich, & Lazutkin, V. (1967). On eigenfunctions concentrated at a closed geodesic. In Problems in math. phys. N2 Leningrad University.
Baily, W. (1957). On the embedding of V-manifolds in projective space. Americ. j. of math., 79, 403–430.
Balian, R., & Bloch, C. (1972). Distribution of eigenfrequencies for the wave equation in a finite domain. Annals of Physics, 69, 401–447.
Ballmann, W. (1995). Lectures on spaces of nonpositive curvature. Birkhäuser.
Ballmann, W. & Brin, M. (1995). Orbihedra of nonpositive curvature. Publications math. IHES, 82, 169–209.
Ballmann, W. & Wojtlowski, M. (1989). An estimate for the measure theoretic entropy of geodesic flows. Ergodic theory and dynamical systems, 9, 271–279.
Ballmann, W., Gromov, M. & Schroeder, V. (1985). Manifolds of nonpositive curvature. Birkhäuser.

Ballmann, W., Thorbergsson, G. & Ziller, W. (1982). Closed geodesics on positively curved manifolds. Annals of math., 116, 213–247.

Ballmann, W., Thorbergsson, G. & Ziller, W. (1983). Existence of closed geodesics on positively curved manifolds. J. of differential geometry, 18, 221–252.

Ballmann, W. & Ziller, W. (1982). On the number of closed geodesics on a compact Riemannian manifold. Duke math. j., 49, 629–632.

Bando, S. & Mabuchi, T. (1987). Uniqueness of Einstein Kähler metrics modulo connected group actions. Advanced studies in pure math., 10, 11–40.

Bangert, V. (1985). Geodätische Linien auf Riemannschen Mannigfaltigkeiten. Jahresbericht der deutschen math. V., 87, 39–66.

Bangert, V. (1988). Mather sets for twist maps and geodesics on tori. In U. Kirchgraber & H. Walther (Eds.), Dynamics reported (pp. 1–56). Teubner, Wyley.

Bangert, V. (1993). On the existence of closed geodesics on the two-sphere. International J. math., 4, 1–10.

Bao, D. (1995). Finsler geometry. In D. Bao, S.-S. Chern & Z. Shen (Ed.), Joint summer research conference on Finsler geometry, 196. Seattle, Washington: American math. society. 1996

Bär, C. (1996). Metrics with harmonic spinors. Geometric and functional analysis, 6, 899–942.

Bär, C. (1997). Heat operator and zeta-function estimates for surfaces.

Batson (1992). Quaternionic complexes. J. geometry and physics, 8, 29–52.

Bavard, C. (1987). Courbure presque négative en dimension 3. Compositio math., 63, 223–236.

Bazaikin, J. (1996). On a certain family of closed 13-dimensional Riemannian manifolds of positive curvature. Siberian math. J. 37, 1068–1085.

Beauville, A. (1983). Variétés Kählériennes dont la première classe de Chern est nulle. J. differential geometry, 18, 755–782.

Beauville, A. (1987). Le problème de Schottky, in Séminaire Bourbaki 1986/87, n°675. Astérisque, 152–153, $1_1 - 112$.

Bedford, E., d'Angelo, J., Greene, R. & Krantz, S. (Ed.). (1991). Several complex variables and complex geometry, Santa Cruz 1989. Proc. symposia in mure math. n°52, American math. Soc.

Beem, J. & Ehrlich, P. (1981). Global Lorentzian geometry. Dekker.

Bellaïche, A. & Risler, J.-J. (1996). Sub-Riemannian geometry. Birkhäuser.

Bemelmans, J., Min-Oo, & Ruh, E. (1984). Smoothing Riemannian metrics. Math. Zeitschrift, 188, 69–74

Benedetti, R. & Petronio, C. (1992). Lectures on hyperbolic geometry. Springer.

Benjamini, I. & Cao, J. (1996). Examples of simply-connected Liouville manifolds with positive spectrum. Differential geometry and its applications, 6, 31–50.

Benoist, Y., Foulon, P. & Labourie, F. (1992). Flots d'Anosov à distributions stables et instables différentiables. J. american math. soc., 5, 33–74.

Bérard, P. (1986). Spectral geometry: direct and inverse problems LNM 1207. Springer.

Bérard, P. (1988). From vanishing theorems to estimating theorems; the Bochner technique revisited. Bull. Amer. Math. Soc., 19, 371–406.

Bérard, P. (1989). Variétés riemanniennes isospectrales non isométriques. In Séminaire Bourbaki, 177–178 (pp. 127–154). Paris: Soc. math. France.

Bérard, P., Besson, G. & Gallot, S. (1985). Sur une inégalité isopérimétrique qui généralise celle de Paul Lévy. Inventiones math., 80, 295–308.

Bérard, P., Besson, G. & Gallot, S. (1997). Embedding Riemannian manifolds by their heat kernel.

Bérard Bergery, L. (1976). Les variétés riemanniennes homogènes simplement connexes à courbure strictement positive. J. math. pures et appl., 55, 47–68.

Bérard Bergery, L. (1982). Sur de nouvelles variétés riemanniennes d'Einstein. Publications de l'Institut Elie Cartan (Nancy), 4, 1–60.

Berestovskii, V. & Nikolaev, I. (1993). Multidimensional generalized geometry. In Encyclopedia of math. sciences Geometry IV, 70, Springer.

Berger, M. (1953). Sur les groupes d'holonomie des variétés riemanniennes. Bulletin Soc. math. France, 83, 279–330.

Berger, M. (1960). Les variétés riemanniennes (1/4)-pincées. Ann. Scuola. norm. sup. Pisa, 14, 161–170.
Berger, M. (1961). Les variétés riemaniennes homogènes normales simplement connexes à courbure positive. Ann. scuola norm. sup. Pisa, 15, 179–2546.
Berger, M. (1972). Du côté de chez Pu. Ann.Scient. Ec. Norm. Sup., 5, 1–44.
Berger, M. (1980). Une borne inférieure pour le volume d'une variété riemannienne en fonction du rayon d'injectivité. Ann. Inst. Fourier Grenoble, 30, 259–265.
Berger, M. (1983). Sur les variétés riemanniennes pincées juste au-dessous de 1/4. Ann. Institut Fourier, 33, 135–150.
Berger, M. (1993a). Systoles et applications selon Gromov. In Séminaire Bourbaki (pp. 279–230), Astérisque 216, Société Mathématique de France.
Berger, M. (1993b). Encounter with a geometer: Eugenio Calabi. In P. de Bartolomeis, F. Tricerri & E. Vesentini (Ed.), Conference in honour of Eugenio Calabi, Manifolds and geometry (pp. 20–60). Pisa: Cambridge University Press.
Berger, M. & Gostiaux, B. (1988). Differential geometry: manifolds, curves and surfaces. Springer.
Berger, M., Gauduchon, P. & Mazet, E. (1971). Le spectre d'une variété riemannienne. Springer.
Berline, N., Getzler, E. & Vergne, M. (1992). Heat kernels and Dirac operators. Springer.
Berndt, J., Tricerri, F. & Vanhecke, L. (1995). Generalized Heisenberg groups and Damek-Ricci harmonic spaces. LNM 1598, Springer.
Besicowitch, A. (1952). On two problems of Loewner. J. London math. soc., 27, 141–144.
Besse, A. (1978). Manifolds all of whose geodesics are closed. Springer.
Besse, A. (1987). Einstein manifolds. Springer.
Besse, A. (1994). Some trends in Riemannian geometry. In M. Artin, H.-P. Kraft & R. Remmert (Eds.), Duration and change: fifty years at Oberwolfach (pp. 71–105). Springer.
Besson, G. (1996). Volumes and entropies. In M. Lovric, M. Min-Oo & M. Wang (Eds.), Riemannian geometry (pp. 1–22). American math. soc.
Besson, G., Courtois, G. & Gallot, S. (1991). Volume et entropie minimale des espaces localement sym'etriques. Inventiones math., 103, 417–445.
Besson, G., Courtois, G. & Gallot, S. (1995a). Entropies et rigidités des espaces localement symétriques de courbure strictement négative. Geometric and funct. Analysis, 5, 731–799.
Besson, G., Courtois, G. & Gallot, S. (1995b). A simple proof of the rigidity and the minimal entropy theorem No. 314). Institut Fourier, Grenoble.
Biquard, O. & Gauduchon, P. (1996). La métrique hyperkählérienne des orbites coadjointes de type symétrique d'un groupe de Lie complexe semi-simple. C. R. acad. sciences Paris, 323, 1259–1264.
Birkhoff, D. & Lewis, D. (1933). On the periodic motions near a given perioidic motion of a dynamical system. Annali di mat., 12, 117–133.
Bishop, R. (1963). A relation between volume, mean curvature and diameter. Notices of the Amer. math. Soc., 10, 364.
Bismut, J.-M., Gillet, H. & Soulé, C. (1988). Analytic torsion and holomorphic determinant I, II, III. Communic. in math. physics, 115, 49–78, 79–126, 301–351.
Blaschke, W. (1921). Vorlesungen über Differentialgeometrie, I (first edtion). Springer.
Blaschke, W. (1930). Vorlesungen über Differentialgeometrie, I (third edition). Springer.
Blatter, C. (1961). Über Extremallänge auf geschlossenen Flächen. Comment. Math. Helvetici, 35, 153–168.
Bleecker, D. (1995). Isometric deformations of compact hypersurfaces. Geometriae dedicata, 64, 193–227.
Bochner, S. (1946). Vector fields and Ricci curvature. Bulletin of Amer. math. Soc., 52, 776–797.
Bochner, S. (1947). Curvature in Hermitian metrics. Bull. Amer. Math. Soc., 53, 179–195.
Bochner, S. (1948). Curvature and Betti numbers. Annals of math., 49, 379–390.
Bochner, S. & Yano, K. (1953). Curvature and Betti numbers. Princeton University press.
Boeckx, E., Kowalski, O. & Vanhecke, L. (1996). Riemannian manifolds of conullity two. World scientific
Böhm, C. (1996). Sequences of Einstein metrics with bounded volumes.

Bonnet, O. (1855). Sur quelques propriétés des lignes géodésiques. C. R. Acad. Sc. Paris, 40, 1311–1313.

Borel, A. (1963). Compact Clifford-Klein forms of symmetric spaces. Topology, 2, 111–122.

Borel, A. & Lichnerowicz, A. (1952). Groupes d'holonomie des variétés riemanniennes. C. R. Acad. Sciences Paris, 234, 1835–1837.

Bott, R. (1956). On the iteration of closed geodesics and the Sturm intersection theory. Communications in pure and applied math., 9, 171–206.

Bourbaki, N. (1969). Fascicule de résultats des variétés. Hermann.

Bourdon, M. (1996). Sur le birapport au bord des CAT(-1)-espaces. Publications math. IHES, 83, 95–104.

Bourguignon, J.-P. (1973). Some constructions related to Hopf's conjecture on product manifolds. In S.-S. Chern & R. Osserman (Ed.), Summer Institute on Differential geometry, 25 (pp. 33–37). Stanford: American math. soc.

Bourguignon, J.-P. (1975). Une stratification de l'espace des structures riemanniennes. Compositio math., 30, 1–41.

Bourguignon, J.-P. (1988). The "magic" of Weitzenböck formulas. In C. Berestycki Ekeleland (Ed.), Variational methods, (pp. 251–271). Paris: Birkhäuser.

Bourguignon, J.-P. (1990/91). Stabilité par déformation non-linéaire de la métrique de Minkowski (d'après D. Christodoulou et S. Kleinerman). In Séminaire Bourbaki, Astérisque 201-202-203 (pp. 321–358). Paris: Société math. France.

Bourguignon, J.-P. (1993). Eugenio Calabi and Kähler metrics. In P. de Bartolomeis, F. Tricerri & E. Vesentini (Ed.), Manifolds and geometry, (pp. 61–85). Pisa: Cambridge University Press.

Bourguignon, J.-P. & Gauduchon, P. (1992). Spineurs, opérateurs de Dirac et variations de métriques. Communications in math. phys., 144, 581–599.

Bourguignon, J.-P. & Lawson, B. (1982). Yang-Mills theory: its physical origins and differential geometry aspects. In Seminar on differential geometry Princeton university press.

Bourguignon, J.-P. & Polombo, A. (1981). Intégrands des nombres caractéristiques et courbure: rien ne va plus dès la dimension 6. J. of differential geometry, 16, 537–550.

Bourguignon, J.-P., Li, P. & Yau, S.-T. (1994). Upper bounds of the first eigenvalue for algebraic submanifolds. Commentarii math. Helvetici, 69, 199–207.

Bowditch, B. (1994). Some results on the geometry of convex hulls in manifolds of pinched negative curvature. Commentarii math. helvetici, 69, 49–81.

Boyer, C., Galicki, K. & Mann, B. (1994a). The geometry and topology of 3-Sasakian manifolds. J. für reine und angew. Math., 455, 183–220.

Boyer, C., Galicki, K. & Mann, B. (1994b). Quaternionic geometry and 3-Sasakian manifolds. In G. Gentili, S. Marchiafava & M. Pontecorvo (Ed.), Quaternionic structures in mathematics and physics. Trieste: SISSA Trieste.

Boyer, C., Galicki, K., Mann, B. & Rees, E. (1997). Compact 3-Sasakian 7-manifolds with arbitrary second Betti number

Bridson, M. & Haefliger, A. (1998). Metric spaces of non-positive curvature. Springer.

Brocks, R. (1997). Ein Kompaktheitssatz fr Mannigfaltigkeiten mit unteren Schranken für Riccikrümmung und Konvexitätsradius.

Brooks, R., Perry, P. & Petersen, P. (1990). Finiteness of diffeomorphisms types of isospectral manifolds. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54, part 3 (pp. 89–94). Los Angeles: American math. soc.

Brooks, R., Perry, P. & Peteresen, P. (1992). Compactness and finiteness theorems for isospectral manifolds. J. reine and ang. Math., 426, 67–89.

Brooks, R., Perry, P. & Petersen, P. (1994). Spectral geometry in dimension 3. Acta mathematica, 173, 283–305.

Brown, R. & Gray, A. (1972). Riemannian manifolds with holonomy group Spin(9). In Differential geometry, in honour of K. Yano, (pp. 41–59). Tokyo: Kinokuniya.

Bryant, R. (1987). Metrics with exceptional holonomy. Annals of math., 126, 525–576.

Bryant, R. (1996). Finsler structures on the 2-sphere satisfying K = 1. Contemporary math., 196, 27–41.

Bryant, R. & Hsu, L. (1993). Rigidity of integral curves of rank 2 distributions. Inventiones math., 114, 435–461.
Bryant, R. & Salamon, S. (1989). On the construction of some complete metrics with exceptional holonomy. Duke math. j., 58, 829–850.
Buchner, M. (1978). The structure of the cut-loci in dimension less than or equal to six. Compositio math., 37, 103–119.
Bunke, U. (1995). On the gluing problem for the eta-invariant. J. of differential geometry, 41, 397–448.
Bunke, U. & Olbrich, M. (1994). The wave kernel for the Laplacian on the classical locally symmetric spaces of rank one, theta functions, trace formulas and the Selberg zeta function. Annals of global anal. and geom., 12, 357–405.
Burago, Y. & Ivanov, S. (1994). Riemannian tori without conjugate points are flat. Geometric and functional analysis, 5, 801–808.
Burago, Y. & Zalgaller, V. (1988). Geometric inequalities. Springer.
Burago, Y. & Zalgaller, V. (Ed.). (1992). Geometry III. Encyclopedia of math. sc. 48, Springer.
Burago, Y., Gromov, M. & Perelman, G. (1992). Alexandrov's spaces with curvatures bounded from below I. Russian mathematical surveys, 47, 1–58.
Burns, K. & Paternain, G. (1996). On the growth of the number of geodesics joining two points, p. 720, in Ledrappier et al. (ed.), 1st international conference in dynamical systems, Pitman Res. Notes Math. Ser. 362.
Busemann, H. (1955). The geometry of geodesics. Academic Press.
Buser, P. (1978). Über eine Ungleichung von Cheeger. Math. Zeitschrift, 158, 245–252.
Buser, P. (1992). Geometry and spectra of compact Riemann surfaces. Birkhäuser.
Buser, P. (1996). Inverse spectral geometry on Riemann surfaces. In S. Andersson & M. Lapidus (Eds.), Recent progress in inverse spectral geometry. Birkhäuser.
Buser, P. & Gromoll, D. (1988). On the almost negatively curved 3-sphere. In T. Sunada (Eds.), Geometry and Analysis on manifolds, LNM 1339, Springer.
Buser, P. & Karcher, H. (1981). Gromov's almost flat manifolds. Société math. France. Astérisque n° 81.
Buser, P. & Sarnak, P. (1994). On the period matrix of a Riemann surface of large genus (with an appendix by Conway and Sloane). Invent. math., 117, 27–56.
Calabi, E. (1953). Isometric embedding of complex manifolds. Annals of Math., 58, 1–23.
Calabi, E. (1954). The space of Kähler metrics. In Proceedings of the International Congress of Mathematicians (pp. 206–207). Amsterdam
Calabi, E. (1958). An extension of E. Hopf's maximum principle with an application to Riemannian geometry. Duke math. j., 25, 45–56.
Calabi, E. (1967). Minimal immersions of surfaces in Euclidean spheres. J. Differential Geometry, 1, 111–125.
Calabi, E. (1969). An intrinsic characterization of harmonic one-forms. In D.C.S. a. S. Iyanaga (Eds.), Papers in Honor of K. Kodaira (pp. 101–117). Princeton University Press.
Calabi, E. (1979). Métriques kählériennes et fibrés holomorphes. Annales scientifiques Ecol. norm. sup., 12, 269–294.
Calabi, E. (1992). Extremal isosystolic metrics for compact surfaces. In A. Besse (Ed.), Table ronde de géométrie différentielle, Séminaires et congrès 1 (pp. 167–204). Luminy: Soc. Math. France.
Calabi, E. & Cao, J. (1992). Simple closed geodesics on convex surfaces. J. Differential Geometry, 36, 517–549.
Cao, H.-D. (1985). Deformations of Kähler metrics to Kähler Einstein metrics on compact Kähler manifolds. Inventiones math., 81, 359–372.
Cao, J. (1996). The Martin boundary and isoperimetric inequalities for Gromov-hyperbolic manifolds, No. 251, Univ. Notre-Dame, dept. of math.
Cao, J. & Escobar, J. (1995). An isoperimetric theorem for PL-manifolds of non-positive curvature.
Cao, J. & Xavier, F. (1998). Kähler parabolicity and the Euler number of compact manifolds of non-positive sectional curvature.

Cartan, E. (1913). Les groupes projectifs qui ne laissent invariante aucune multiplicité plane. Bull. soc. math. France, 41, 1–44.
Cartan, E. (1925). Sur les variétés à conexion affine et la théorie de la relativité généralisée. Ann. sc. Ecole norm. sup., 42, 17–88.
Cartan, E. (1926a). Les groupes d'holonomie des espaces généralisés. Acta math., 48, 1–42.
Cartan, E. (1926b). Sur une classe remarquable d'espaces de Riemann. Bull. Soc. math. France, 214–264.
Cartan, E. (1927). Sur certaines formes riemanniennes remarquables de géométries à groupe fondamental simple. Annales scientif. Ecole norm. sup., 44, 345–367.
Cartan, E. (1928a). Leçons sur la géométrie des espaces de Riemann. Gauthier-Villars.
Cartan, E. (1928b). Sur les nombres de Betti des espaces de groupes clos. C. R. Acad. sciences Paris, 187, 196–198.
Cartan, E. (1929a). Sur les invariants intégraux de certains espaces homogènes clos et les propriétés topologiques de ces espaces. Annales Soc. math. polonaise, 8, 217–252.
Cartan, E. (1929b). Groupes simples clos et ouverts et géométrie riemannienne. J. math. pures et appl., 8, 1–33.
Cartan, E. (1931). Leçons sur la géométrie projective complexe. Gauthier-Villars.
Cartan, E. (1936). La topologie des espaces représentatifs des groupes de Lie. L'Enseignement math., 35, 177–200.
Cartan, E. (1937). La théorie des spineurs. Hermann.
Cartan, E. (1939). Sur quelques familles remarquables d'hypersurfaces. Math. Zeitschrift, 45, 335–367.
Cartan, E. (1946–1951). Leçons sur la géométrie des espaces de Riemann, seconde édition, Gauthier-Villars.
Catanese, F. & LeBrun, C. (1997). On the scalar curvature of Einstein manifolds.
Chavel, I. (1984). Eigenvalues in Riemannian geometry. Academic Press.
Chavel, I. (1993). Riemannian Geometry: a modern introduction. Cambridge University Press.
Chazarain, J. (1974). Formule de Poisson pour les varietes riemanniennes. Inventiones math., 24, 65–82.
Cheeger, J. (1967) Comparison and finiteness theorems for Riemannian manifolds. PhD, Princeton University.
Cheeger, J. (1970a). Finiteness theorems for Riemannian manifolds. American J. of math., 92, 61–74.
Cheeger, J. (1970b). A lower bound for the smallest eigenvalue of the Laplacian. In Symposium in honour of S. Bochner, (pp. 195–199). Princeton University Press.
Cheeger, J. (1973). Some examples of manifolds of nonnegative curvature. J. differential geom., 8, 623–628.
Cheeger, J. (1979). Analytic torsion and the heat equation. Annals of math., 109, 259–322.
Cheeger, J. (1983). Spectral geometry of singular Riemannian spaces. J. of differential geometry, 18, 575–667.
Cheeger, J. (1991). Critical points of distance functions and applications to geometry. In J. Cheeger, M. Gromov, C. Okonek & P. Pansu (Eds.), Geometric topology: recent developments (pp. 1–38). LNM 1504, Springer.
Cheeger, J. & Colding, T. (1996). Lower bounds on Ricci curvature and the almost rigidity of warped products. Annals of math. 144, 189–237.
Cheeger, J. & Colding, T. (1997a). On the structure of spaces with Ricci curvature bounded below: I. J. of differential geometry, 46, 406–480.
Cheeger, J. & Colding, T. (1997b). On the structure of spaces with Ricci curvature bounded below, II.
Cheeger, J. & Colding, T. (1998). On the structure of spaces with Ricci curvature bounded below, III.
Cheeger, J. & Ebin, D. (1975). Comparison theorems in Riemannian Geometry. North-Holland.
Cheeger, J. & Gromoll, D. (1971). The splitting theorem for manifolds of nonnegative Ricci curvature. J. Differential Geometry, 6, 119–128.
Cheeger, J. & Gromoll, D. (1972). On the structure of complete manifolds of nonnegative curvature. Annals of math., 96, 413–443.

Cheeger, J. & Gromoll, D. (1980). On the lower bound for the injectivity radius of 1/4 – pinched Riemannian manifolds. J. differential geometry, 26, 437–442.
Cheeger, J. & Gromov, M. (1985). On the characteristic numbers of complete manifolds of bounded curvature and finite volume. In I. Chavel & H. Farkas (Eds.), Differential geometry and complex analysis (pp. 115–154). Springer.
Cheeger, J. & Gromov, M. (1986–1990). Collapsing Riemannian manifolds while keeping their curvature bounded. J. differential geometry, 23–32, 309–346, 269–298.
Cheeger, J. & Gromov, M. (1989). Chopping Riemannian manifolds. In B. Lawson & K. Teneblat (Ed.), Differential geometry: a symposium in honour of Manfredo do Carmo, (pp. 85–94). Rio de Janeiro: Longman scientific.
Cheeger, J. & Rong, X. (1996). Existence of polarized F-structures on collapsed manifolds with bounded curvature and diameter. Geometric and functional analysis, 6, 411–429.
Cheeger, J. & Simons, J. (1985). Differential characters and geometric invariants. In Geometry and topology, LNM 1167, Springer.
Cheeger, J. & Taylor, M. (1982). On the diffraction of waves by conical singularities, I and II. Communications in pure and appl. math., 35, 275–231 and 487–529.
Cheeger, J. & Tian, G. (1994). On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay. Inventiones math., 118, 493–571.
Cheeger, J., Colding, T. & Tian, G. (1997). Constraints on singularities under Ricci curvature bounds. C. R. Acad. Sc. Paris, 324, 645–649.
Cheeger, J., Fukaya, K. & Gromov, M. (1992). Nilpotent structures and invariant metrics on collapsed manifolds. J. of the American math. society, 5, 327–372.
Cheeger, J., Müller, W., & Schrader, R. (1984). On the curvature of piecewise flat spaces. Communic. Math. Phys., 92, 405–454.
Cheeger, J., Müller, W., & Schrader, R. (1986). Kinematic and tube formulas for piecewise linear spaces. Indiana Univ. math. J., 35, 737–753.
Chen, H. (1990). Manifolds with 2-nonnegative curvature operator. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54, part 3 (pp. 129–133). Los Angeles: American math. soc. 1993.
Cheng, S. (1975). Eigenvalue comparison theorems and its geometric applications. Math. Zeitschrift, 143, 289–297.
Chern, S.-S. (1944). A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds. Annals of math., 45, 747–752.
Chern, S.-S. (1946). Characteristic classes of Hermitian manifolds. Annals of math., 47, 87–121.
Chern, S.-S. (1955). On curvature and characteristic classes of a Riemannian manifold. Abh. math. sem. Hamburg, 20, 117–126.
Chern, S.-S. & Simons, J. (1974). Characteristic forms and geometric invariants. Annals of math., 99, 48–69.
Chevalley, C. (1997). Collected work, volume 2, Springer.
Chow, B. (1991). The Ricci-flow on the 2-sphere. J. of diff. geometry, 33, 325–334.
Christodoulou, D. & Kleinerman, S. (1993). The global linear stability of Minkowski space. Princeton University Press.
Cohn-Vossen, S. (1935). Kürzeste Wege und Totalkrümmung auf Flächen. Compositio math., 2, 63–133.
Cohn-Vossen, S. (1936). Totalkrümmung und geodätische Linien auf einfach zusammenhängenden vollständingen Flächenstücken. Recueil math. Moscou, 43, 139–163.
Colding, T. (1996a). Large manifolds with positive Ricci curvature. Inventiones math., 124, 193–214.
Colding, T. (1996b). Ricci curvature and volume convergence. Annals of math.
Colding, T. (1996c). Shape of manifolds with positive Ricci curvature. Inventiones math., 124, 175–191.
Colding, T. & Minicozzi II, W. (1997). Weyl type bounds for harmonic functions.
Colin de Verdière, Y. (1973). Spectre du laplacien et longueurs des géodésiques périodiques I et II. Compositio math., 27, 83–106 et 159–184.
Colin de Verdière, Y. (1977). Quasi-modes sur les variétés Riemanniennes. Inventiones Math., 3, 15–52.

Colin de Verdière, Y. (1979). Sur le spectre des opérateurs elliptiques à bicaractéristiques toute périodiques. Commentarii math. Helvetici, 54, 508–522.
Colin de Verdière, Y. (1985). Ergodicité et fonctions propres du laplacien. Communic. math. physics, 102, 497–502.
Colin de Verdière, Y. (1987). Construction de laplaciens dont une partie finie du spectre est donnée. Annales scient. Ecol. norm. sup., 20, 599–615.
Colin de Verdière, Y. (1992). Spectre du laplacien: survol partiel depuis le BGM et problèmes. In A. Besse (Ed.), Table ronde de géométrie différentielle, Congrès et séminaires 1 (pp. 233–252). Luminy: Soc. math. France 1996.
Connes, A. (1994). Noncommutative geometry. Academic Press.
Connes, A. (1995–6). Brisure de symétrie spontanée et géométrie du point de vue spectral. In Séminaire Bourbaki, Paris, Astérisque, Société math. France.
Conway, J. (1995). Sphere packings, lattices, codes and greed. In Proceedings of the International Congress of Mathematicians Birkhäuser.
Conway, J. H. & Sloane, N. J. A. (1993). Shere Packings, Lattices and Groups. Springer
Corlette, K. (1992). Archimedean superrigidity and hyperbolic geometry. Annals of math., 135, 165–182.
Croke, C. (1980). Some isoperimetric inequalities and eigenvalue estimates. Ann. sci. Ec. norm. sup. Paris, 13, 419–435.
Croke, C. (1982). An eigenvalue pinching theorem. Inventiones math., 68, 253–256.
Croke, C. (1984). A sharp four dimensional isoperimetric inequality. Commentarii math. helvetici, 59, 187–192.
Croke, C. (1988a). An isoembolic pinching theorem. Inventiones math., 92, 385–387.
Croke, C. (1988b). Area and the length of the shortest closed geodesic. J. of differential geometry, 27, 1–21.
Croke, C. (1991). Rigidity and the distance between boundary points. J. differential geometry, 33, 445–464.
Croke, C. (1992). Volume of balls in manifolds without conjugate points. International journal of math., 3, 455–467.
d'Ambra, G. (1988). Isometry groups of Lorentz manifolds. Inventiones math., 92, 555–565.
d'Ambra, G. & Gromov, M. (1991). Lectures on transformations groups: geometry and dynamics. Surveys in differential geometry: supplement to the J. of differential geometry, 1, 19–111.
Dai, X. & Wei, G. (1995). A comparison-estimate of Toponogov type for Ricci curvature. Math. Annalen, 303, 297–306.
Darboux, G. (1887, 1889, 1894, 1896). Leçons sur la théorie générale des surfaces (4 vol.). Gauthier-Villars.
Darboux, G. (1972). Lecons sur la théorie générale des surfaces. Chelsea.
de Rham, G. (1931). Sur l'analysis situs des variétés à n dimensions. J. math. pures et appliquées, 10, 115–200.
de Rham, G. (1952). Sur la réductibilité d'un espace de Riemann. Commentarii math. helvetici, 26, 328–344.
de Rham, G. (1954). Variétés différentiables. Hermann.
Deligne, P. & Mostow, G. (1986). Monodromy of hypergeometric functions and no-lattice integral monodromy. Publications math. IHES, 63, 5–89.
Deligne, P., Griffiths, P., Morgan, J. & Sullivan, D. (1975). Real homotopy of Kähler manifolds. Inventiones math., 29, 245–274.
Diaconis, P., Elworthy, D., Nelson, E., Papanicolaou, G. & Varadhan, S. (Ed.) (1985–87). Ecole d'été de probabilités de Saint-Flour. LNM 1362, Springer.
Dierkes, U., Hildebrandt, S., Küster, A. & Wohlrab, O. (1992). Minimal surfaces I and II. Springer.
Dieudonné, J. (1972). Treaty of Analysis. Academic Press.
Dillen, F. & Verstraelen, L. (Ed.) (1992). Geometry and topology of submanifolds. World scientific.
do Carmo, M. (1976). Differential geometry of curves and surfaces. Prentice-Hall.
do Carmo, M. (1992). Riemannian geometry. Birkhäuser.

Dombrowski, P. (1979). 150 years after Gauss. Astérisque 62, Société mathématique de France.

Donaldson, S. (1996). The Seiberg-Witten equations and 4-manifolds topology. Bull. American math. soc., 33, 45–70.

Donaldson, S. & Kronheimer, P. (1996). The geometry of four-manifolds. Oxford University Press.

Donnay, V. (1988). Geodesics flow on the two-sphere II: ergodicity. In Lecture notes in math. n° 1342: Dynamical systems (pp. 112–153). Springer.

Donnelly, H. & Fefferman, C. (1988). Nodal sets of eigenfunctions on Riemannian manifolds. Inventiones math., 93, 161–183.

Donnelly, H. & Fefferman, C. (1990). Growth and geometry of eigenfunctions of the Laplacian. In C. Sadosky (Eds.), (pp. 635–655). Dekker.

Duistermaat, J. & Guillemin, V. (1975). The spectrum of positive elliptic operators and periodic bicharacteristics. Inventiones math., 29, 39–79.

Dyck (1888). Beiträge zur Analysis Situs I. Math. Annalen, 32.

Eberlein, P. & O'Neill, B. (1973). Visibility manifolds. Pacific J. of math., 46, 45–109.

Eberlein, P., Hamenstädt, U. & Schroeder, V. (1990). Manifolds of nonpositive curvature. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54, part 3 (pp. 179–227). Los Angeles: American math. society 1993.

Ebin, D. (1970). The space of Riemannian metrics. In Global analysis, 15 (pp. 11–40). Berkeley: American math. soc.

Ebin, D. (1972). Espace des métriques riemanniennes et mouvement des fluides via les variétés d'applications. Ecole polytechnique.

Ebin, D. & Marsden, J. (1970). Groups of diffeomorphisms and the motion of an incompressible fluid. Annals of math., 92, 102–163.

Eells, J. & Lemaire, L. (1988). Another report on harmonic maps. Bull. London Math. Soc., 20, 385–524.

Eells, J. & Rato, A. (1993). Harmonic maps and minimal immersions with symmetries. Princeton University Press.

Eells, J. & Sampson, J. (1964). Harmonic mappings of Riemannian manifolds. American J. of math., 86, 109–160.

Ehresmann, C. (1950). Les connexions infinitésimales dans un espace fibré différentiable. In Colloque de topologie, (pp. 29–550. Bruxelles.

Einstein, A. (1916). Näherungsweise Integration der Feldgleichungen der Gravitation. S. B. Preuss. Akad. Wiss., 688–696.

El Alaoui, E.-L., Gauthier, J.-P. & Kupka, I. (1996). Small sub-Riemannian balls on R3, J. of dynamical control systems, 2, 359–421

Elworthy, K. (1988). Geometric aspects of diffusions on manifolds. LNM 1362, Springer.

Epstein, D. (1975). Natural tensors on Riemannian manifolds. J. of differential geometry, 10, 631–645.

Eschenburg, J.-H. (1982). New examples of Riemannian manifolds of strictly positive curvature. Inventiones math., 66, 469–480.

Eschenburg, J-H. (1986). Local convexity and nonnegative curvature: Gromov's proof of the shere theorem. Inventiones math., 84, 507–522.

Eschenburg, J-H. (1992). Inhomogeneous spaces of positive curvature. Differential geometry and its applications, 2, 123–132.

Eschenburg, J.-H. (1994). Comparison theorems in Riemannian geometry. Trento: Universita di Trento.

Eschenburg, J-H. & Heintze, E. (1984). An elementary proof of Cheeger-Gromoll splitting theorem. Annals of global analysis and geometry, 2, 141–151.

Eschenburg, J.-H. & Wang, M. (1997). The ODE system arising from cohomogeneity one Einstein metrics, in Andersen et al. (ed.), Geometry Physics, Aarhus, Proceedings, Lect. Notes in Pure & Appl. Math. 184, 157–165, Marcel Dekker.

Faber, G. (1923). Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt. Sitzungberichte der Bayerischen Akad. Wissen., 169–172.

Farber, M., Katz, G. & Levine, J. (1996). Morse theory for harmonic forms No. M/96/34). I H E S.
Farrell, F. (1996). Surgical methods in rigidity. Tata Insitute lectures, Springer.
Farrell, F. & Jones, L. (1989a). Negatively curved manifolds with exotic smooth structures. J. of Amer. math. Soc., 2, 899–908.
Farrell, T. & Jones, L. (1989b). A topological analogue of Mostow's rigidity theorem. J. American math. soc., 2, 257–370.
Farrell, F. & Jones, L. (1993). Non-uniform lattices and exotic smooth structures. J. of differential geom., 38, 235–261.
Farrell, F. & Jones, L. (1994). Complex hyperbolic manifolds and exotic smooth structures. Inventiones math., 117, 57–74.
Federer, H. (1969). Geometric measure theory. Springer.
Federer, H. & Fleming, W. (1960). Normal and integral currents. Annals of math., 72, 458–520.
Felix, Y. & Halperin, S. (1982). Rational Lusternik-Schnirelman category and its applications. Transactions of the AMS, 273, 1–37.
Flach, N. (1994). Diamètre des variétés à courbure sectionnelle positive. C. R. Acad. sciences Paris, 318, 827–830.
Foulon, P. (1997). Locally symmetric Finsler spaces in negative curvature. C. R. Acad. Sciences Paris, 324, 1127–1132.
Franks, J. (1992). Geodesics on S^2 and periodic points of annulus diffeomorphisms. Inventiones math., 108, 403–418.
Freed, D. & Uhlenbeck, K. (1984). Instantons and four-manifolds. Springer.
Fröhlich, J., Granjean, O. & Recknagel, A. (1997). Supersymmetric quantum theory. ETH-Hönggerberg, Zürich.
Fubini, G. (1903). Sulle metriche definite da una forma Hermitiana. Atti Inst. Veneto, 6, 501–513.
Fukaya, K. (1984). A finiteness theorem for negatively curved manifolds. J. of differential geometry, 20, 497–521.
Fukaya, K. (1987a). Collapsing Riemannian manifolds to ones of lower dimension. J. of differential geometry, 25, 139–156.
Fukaya, K. (1987b). Collapsing Riemannian manifolds and eigenvalues of the Laplace operator. Inventiones math., 87, 517–547.
Fukaya, K. (1988). A boundary of the set of the Riemannian manifolds with bounded curvature and diameters. J. of differential geometry, 28, 1–21.
Fukaya, K. (1989). Collapsing Riemannian manifolds to ones of lower dimension II. J. math. society of Japan, 41, 333–356.
Fukaya, K. (1990). Hausdorff convergence of Riemannian manifolds and its applications. In Recent topics in differential and analytic geometry (pp. 143–238). Tokyo: North Holland and Kinokuniya.
Fukaya, K. & Yamaguchi, T. (1991). Almost nonpositively curved manifolds. J. differential geometry, 33, 67–90.
Fukaya, K. & Yamaguchi, T. (1992). The fundamental group of almost nonnegatively curved manifolds. Annals of math., 136, 253–333.
Funar, L. & Grimaldi, R. (1977). La topologie à l'infini des variétés géométrie bornée et à croissance linéaire. J. math. pures et appl.
Futaki, A. (1983). An obstruction to the existence of Einstein-Kähler metrics. Inventiones math., 73, 437–443.
Gaffney, M. (1958). Asymptotic distributions associated with the Laplacian for forms. Communic. pure and appl. math., XI, 535–545.
Gage, M. & Hamilton, R. (1986). The heat equation shrinking plane curves. J. of differential geometry, 23, 69–96.
Galicki, K. & Laswon, B. (1988). Quaternionic reduction and quaternionic orbifolds. Math. annalen, 282, 1–21.
Gallot, S. (1981). A Sobolev inequality and some geometric applications. In M. Berger, S. Murakami & T. Ochiai (Ed.), Spectra of Riemannian manifolds, (pp. 45–55). Kyoto: Kaigai.

Gallot, S. (1986). Inégalités isopérimétriques et analytiques sur les variétés riemanniennes. In Sulle geometria delle varietà differenziabili, 163–164 (pp. 31–92). Rome. Astérisque 163–164, Société mathématique de France.

Gallot, S. (1987). Isoperimetric inequalities based on integral norms of Ricci curvature. In Colloque Paul Lévy sur les processus stochastiques (pp. 191–216). Ecole polytechnique. Astérisque 157–158, Soc. math. France.

Gallot, S. (1998). Volumes, courbure de Ricci et convergence des variétés (d'après T. Colding et Cheeger-Colding): séminaire Bourbaki 1997–98, novembre 1997. Astérisque. Société mathématique de France.

Gallot, S., Hulin, D. & Lafontaine, J. (1990). Riemannian geometry. Springer.

Gallot, S. & Meyer, D. (1975). Opérateur de courbure et laplacien des formes différentielles d'une variété riemannienne. J. math. pures et appliquées, 54, 259–284.

Gao, L. (1990a). Convergence of Riemannian manifolds; Ricci and $L^{n/2}$ curvature pinching. J. differential geometry, 32, 349–381.

Gao, L. (1990b). $L^{n/2}$ curvature pinching. J. differential geometry, 32, 713–774.

Gasqui, J. & Goldschmidt, H. (1994). The infinitesimal rigidity of the complex quadrics of dimension four. American J. of Math., 116, 501–539.

Gasqui, J. & Goldschmidt, H. (1997). Radon transforms and spectral rigidity on the complex quadrics and the real Grassmannians of rank two. J. reine angew. Math., 480, 1–69.

Gauduchon, P. (1992/93). Variétés riemanniennes autoduales. In Séminaire Bourbaki, Astérisque 216 (pp. 151–186). Paris: Société math. de France.

Gentili, G., Marchiafava, S. & Pontecorvo, M. (Ed.). (1994). Quaternionic structures in mathematics and physics. Trieste: SISSA, Trieste.

Ghys, E. & de la Harpe, P. (Ed.). (1990). Sur les groupes hyperboliques d'après Mikhael Gromov. Birkhäuser.

Gibbon, G. & Hawking, S. (1978). Gravitational multi-instantons. Physics letters B, 78, 430–432.

Gil-Medrano, O. & Michor, P. (1991). The Riemannian manifold of all Riemannian metrics. Quaterly j. of math. Oxford, 42, 183–202.

Gilkey, P. (1973a). The spectral geometry of real and complex manifolds. In Differential geometry, (pp. 265–280). Stanford: American math. soc. 1975

Gilkey, P. (1973b). Curvature and the eigenvalues of the Laplacian for elliptic complexes. Adv. in math., 10, 344–382.

Gilkey, P. (1974). The index theorem and the heat equation. Publish or Perish, Boston.

Gilkey, P. (1995). Invariance theory, the heat equation, and the Atiyah-Singer index theorem (second edition). Boca Raton: CRC Press.

Gillet, H. & Soulé, C. (1992). An arithmetic Riemann-Roch theorem. Inventiones math., 110, 473–543.

Gluck, H. & Singer, D. (1978). Scattering of geodesic fields, II. Annals of math., 110, 205–225.

Godbillon, C. (1991). Feuilletages, Etudes géométriques. Birkhäuser.

Goncharov, A. (1997). Volumes of hyperbolic manifolds, and mixed Tate motives.

Gordon, C. (1994). Isospectral closed Riemannian manifolds which are not locally isometric, part II. Contemporary math., 121–131.

Gordon, C. & Mao, Y. (1994). Comparisons of Laplace spectra, length spectra and geodesic flows of some Riemannian manifolds. Mathematical research letters, 1, 677–688.

Gornet, R. (1996). The marked length spectrum on forms on Riemannian nilmanifolds. Commentarii math. helv., 71, 297–329.

Gornet, R. (1998). Continuous families of Riemannian manifolds isospectral on functions but not on 1-forms.

Grayson, M. (1989). Shortening imbedded curves. Annals of math., 129, 71–111.

Green, L. (1963). Auf Wiedersehenflächen. Annals of math.,78, 289–299.

Greene, R. (1978). Complete metrics of bounded curvature on noncompact manifolds. Arch. Math., 31, 89–95.

Greene, R. (1990). Some concepts and methods in Riemannian geometry. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 3 (pp. 1–22). Los Angeles: American math. soc. 1993

Greene, R. (1994). Homotopy finiteness theorems and Helly's theorem. The journal of Geometric Analysis, 4, 317–325.

Greene, R. (1997). A genealogy of noncompact manifolds of nonnegative curvature: history and logic. In K. Grove & P. Petersen (Eds.), Comparison geometry (pp. 99–134). Cambridge U. Press.

Greene, R. & Petersen, P. (1992). Little topology, big volume. Duke math. J., 67, 173–290.

Greene, R. & Shiohama, K. (1981a). Convex functions on complete noncompact manifolds: differentiable structure. Annales scientifiques Ecol. norm. sup., 14, 357–367.

Greene, R. & Shiohama, K. (1981b). Convex functions on complete noncompact Riemannian manifolds. Inventiones math., 63, 129–157.

Greene, R. & Wu, H. (1988). Lipschitz convergence of Riemannian manifolds. Pacific J. of math., 131-1, 119–141.

Greene, R. & Wu, H. (1990). Nonnegatively curved manifolds which are flat outside a compact set. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54, 3 (pp. 327–335). Los Angeles: American math. soc. 1993

Greene, R. & Yau, S.-T. (Ed.). (1993). Differential geometry, Los Angeles 1990, Proceedings of symposia in pure math. American math. soc.

Greene, R. & Yau, S.-T. (Ed.). (1997). Mirror symmetry II. American math. Soc., International Press.

Griffiths, P. & Harris, J. (1978). Principle of algebraic geometry. John Wiley.

Gromoll, D. (1966). Differenzierbare Strukturen und Metriken positiver Krümmung auf Sphären. Math. annalen, 164, 353–371.

Gromoll, D. (1990). Spaces of nonnegative curvature. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54, part 3 (pp. 337–356). Los Angeles: American math. society. 1993

Gromoll, D. & Grove, K. (1981). On metrics on S^2 all of whose geodesics are closed. Invent. math., 65, 175–177.

Gromoll, D. & Grove, K. (1987). A generalization of Berger's rigidity theorem for positively curved manifolds. Ann. sci. Ecole normale sup. Paris, 20, 227–239.

Gromoll, D. & Meyer, W. (1969a). Periodic geodesics on compact Riemannian manifolds. J. differential geom., 3, 493–510.

Gromoll, D. & Meyer, W. (1969b). On complete manifolds of positive curvature. Annals of math., 90, 75–90.

Gromoll, D. & Meyer, W. (1974). An exotic sphere with nonnegative curvature. Annals of math., 100, 401–406.

Gromoll, D. & Wolf, J. (1971). Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of nonpositive curvature. Bull. American math. soc., 77, 545–552.

Gromov, M. (1978a). Almost flat manifolds. J. of differential geometry, 13, 231–241.

Gromov, M. (1978b). Manifolds of negative curvature. J. of differential geometry, 13, 223–230.

Gromov, M. (1978c). Homotopical effects of dilatation. J. of differential geometry, 13, 303–310.

Gromov, M. (1980). Paul Levy's isoperimetric inequality No. M/80/320). Institut des Hautes Etudes Scientifiques. Reprinted as Appendix C of (Gromov, 1998).

Gromov, M. (1981a). Hyperbolic manifolds according to Thurston and Jorgensen. In Lecture Notes in Math. 842 (pp. 40–53). Springer.

Gromov, M. (1981b). Groups of polynomial growth and expanding maps. Publications math. I.H.E.S., 53, 53–73.

Gromov, M. (1981c). Curvature, diameter and Betti numbers. Commentarii math. helvetici, 56, 179–195.

Gromov, M. (1983a). Filling Riemannian manifolds. J. of Diff. Geometry, 18, 1–147.

Gromov, M. (1983b). Volume and bounded cohomology. Public. math. IHES, 56, 213–37.

Gromov, M. (1985). Pseudo-holomorphic curves in symplectic manifolds. Inventiones math., 82, 307–347.

Gromov, M. (1986). Partial Differential Relations. Springer.

Gromov, M. (Ed.) (1987a). Hyperbolic groups. in Essays in group theory, S. Gersten, ed. MSRI publications 8, Springer.

Gromov, M. (1987b). Entropy, homology and semi-algebraic geometry. In Séminaire Bourbaki, Astérisque 145–146 (pp. 225–240), Paris: Société mathématique de France.

Gromov, M. (1988). Widths and related invariants of Riemannian manifolds. Astérisque, 163–164, 93–109.
Gromov, M. (1990). Stability and pinching. In Giornate di Topologica e geometria delle varieta. Bologna: Universita degli studi di Bologna.
Gromov, M. (1991a). Kähler hyperbolicity and L^2 Hodge theory. Journal of Differ. geom., 33, 263–292.
Gromov, M. (1991b). Sign and geometric meaning of curvature. Rendiconti Seminario mat. e Fisic. di Milano, 56, 10–123.
Gromov, M. (1991c). Foliated Plateau problem I et II. Geometric and functional analysis, 1, 14–79 and 253–320.
Gromov, M. (1992a). Systoles and intersystolic inequalities. In A. Besse (Ed.), Table ronde de géométrie différentielle, 1 (pp. 291–362). Luminy, Séminaires et congrès 1, Soc. math. France 1996.
Gromov, M. (1992b). Metric invariants of Kähler manifolds. In R. Caddeo & F. Tricerri (Ed.), Differential geometry and topology, (pp. 90–116). Alghero: World scientific.
Gromov, M. (1993). Asymptotic invariants of infinite groups. In G. Noble & M. Roller (Eds.), Geometric group theory (pp. 75–263). London math. soc.
Gromov, M. (1996). Positive curvature, macroscopic dimension, spectral gaps and higher signatures. In S. Gindikin, J. Lepowski & R. Wilson (Eds.), Functional analysis in the eve of the 21st century (pp. 1–213). Birkhäuser.
Gromov, M. (1998). Metric structures for Riemannian and non Riemannian manifolds, Birkhäuser.
Gromov, M. & Lawson, B. (1980a). The classification of simply connected manifolds of positive curvature. Annals of math., 111, 423–434.
Gromov, M. & Lawson, B. (1980b). Spin and scalar curvature in the presence of a fundamental group I. Annals of math., 111, 209–230.
Gromov, M. & Lawson, B. (1988). Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. Publications math. I H E S, 58, 83–196.
Gromov, M. & Pansu, P. (1990). Rigidity of lattices: an introduction. In P. de Bartolomeis & F. Tricerri (Ed.), Geometric topology; recent developments, LNM, n° 1504. Montecatini: Springer.
Gromov, M. & Piatetski-Shapiro, I. (1988). Non-arithmetic groups in Lobachevskij spaces. Public. Math. Inst. Hautes Etud. Sci., 66, 93–103.
Gromov, M. & Schoen, R. (1992). Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. Publications math. IHES, 76, 165–246.
Gromov, M. & Thurston, W. (1987). Pinching constants for hyperbolic manifolds. Inventiones math., 89, 1–12.
Gromov, M., Lafontaine, J. & Pansu, P. (1981). Structures métriques pour les variétés riemanniennes. Cedic-Nathan.
Grove, K. (1985). Metric differential geometry. In V. L. Hausen (Ed.), Nordic summer school, 1263 (pp. 171–227). Lingby 1985: Springer.
Grove, K. (1990). Critical point theory for distance functions. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 3 (pp. 357–385). Los Angeles: Amer. math. soc. 1993
Grove, K. (1992). Ramifications of the classical sphere theorem. In A.-L. Besse (Ed.), Table ronde de géométrie differentielle, 1 (pp. 363–376). Luminy, Séminaires et congrès 1, Société math. France 1996.
Grove, K. & Halperin, S. (1983). Contributions of rational homotopy theory to global problems in geometry. Publications math. IHES, 56, 379–385.
Grove, K. & Petersen, P. (1988). Bounding homotopy types by geometry. Annals of math., 128, 195–206.
Grove, K. & Petersen, P. (1993). A radius sphere theorem. Inventiones math., 112, 577–583.
Grove, K. & Shiohama, K. (1977). A generalized sphere theorem. Annals of math., 106, 201–211.
Grove, K. & Wilhelm, F. (1995). Hard and soft packing radius theorems. Annals of math., 142, 213–237.
Grove, K. & Wilhelm, F. (1997). Geometric constraints on exotic spheres via Alexandrov geometry. J. für reine und ang. Mathematik, 487, 201–217.

Grove, K., Petersen, P. & Wu, J.-Y. (1990–1991). Geometric finiteness theorems via controlled topology. Inventiones math., 99 and 104, 205–213 and 221–222.
Guillemin, V. (1976). The Radon transform on Zoll surfaces. Advances in math., 22, 85–119.
Guillemin, V. (1993). Wave-trace invariants and a theorem of Zelditch. International math. research notices, 12, 303–308.
Guillemin, V. (1996). Wave trace invariants. Duke math. j., 83, 287–352.
Guillemin, V. & Kazhdan, D. (1980). Some inverse spectral results for negatively-curved 2-manifolds. Topology, 19, 301–312.
Hadamard, J. (1898). Les surfaces à courbure opposées et leurs lignes géodésiques. J. Math. pures et appl., 4, 27–73.
Hadamard, J. (1901). Sur l'itération et les solutions asymptotiques des équations différentielles. Bulletin de la Soc. math. France, 24, 224–228.
Haefliger, A. (1956). Sur l'extension du groupe structural d'un espace fibré. Compres rendus Acad. sciences Paris, 253, 558–560.
Hamilton, R. (1982). Three-manifolds with positive Ricci curvature. J. of differential geometry, 17, 255–306.
Hamilton, R. (1986). Four-manifolds with positive curvature operator. J. of differential geometry, 37, 225–243.
Harvey, R. & Lawson, B. (1982). Calibrated geometries. Acta mathematica, 148, 48–157.
Hawking, S. & Ellis, G. (1973). The large structure of space time. Cambridge University Press.
Hawley, N. (1953). Constant holomorphic sectional curvature. Canadian j. of math., 5, 53–56.
Hebda, J. (1994). Metric structure of cut-loci in surfaces and Ambrose's problem. J. differential geometry, 40, 621–642.
Hebda, J. (1995). Cut Loci of submanifolds in space forms and in the geometry of Möbius and Lie. Geometriae Dedicata, 55, 75–93.
Heber, J. (1997) Geometric and algebraic structure of noncompact homogeneous Einstein spaces. Augsburg.
Hebey, E. (1993). Courbure scalaire et géométrie conforme. J. of geometry and physics, 10, 345–380.
Hebey, E. (1996). Optimal Sobolev inequalities on complete Riemannian manifolds with Ricci curvature bounded below and positive injectivity radius. American Journal of Math. 118, 291–300.
Hebey, E. (1996). Sobolev spaces on Riemannian manifolds. LNM 1635, Springer.
Hebey, E. (1997). Introduction à l'analyse non linéaire sur les variétés. Diderot.
Hebey, E. & Herzlich, M. (1998). Convergence of Riemannian manifolds: a status report. Rendiconti di Matematica e delle sue applicazione. Roma.
Hehjal, D. (1976). The Selberg trace formula and the Riemann zeta-function. Duke math. J., 43, 441–481.
Heintze, E. (1976) Mannigfaltigkeiten negativer Krümmung. Habilitationsschrift, Bonn.
Heintze, E. & Liu, X. (1997). Homogeneity of infinite dimensional isoparamterci submanifolds.
Heintze, E. & Karcher, H. (1978). A general comparison theorem with applications to volume estimates for submanifolds. Ann. Sci. Ec. Norm. Sup. Paris, 11, 451–470.
Helein, F. (1996). Applications harmoniques, lois de conservation et repères mobiles. Diderot.
Hernandez, L. (1991). Kähler manifolds and 1/4-pinching. Duke math. J., 62, 601–611.
Hersch, J. (1970). Quatre propriétés des membranes sphériques. C.R. Acad. Sc. Paris, 270, 1645–1648.
Hijazi, O. (1995). Lower bounds for the eigenvalues of the Dirac operator. J. of geometry and physics, 16, 27–38.
Hilbert, D. (1915). Die Grundlagen der Physik. Nach. Ges. Wiss. Göttingen, 461–472.
Hingston, N. (1993). On the growth of the number of closed geodesics on the two-sphere. International math. research notices, 9, 253–262.
Hirsch, M. & Mazur, B. (1974). Smoothing of piecewise linear manifolds. Princeton university press.
Hirzebruch, F. & Kodaira, K. (1957). On the complex projective spaces. J. de math. pures et appl., 36, 201–216.

Hirzebruch, F. (1966). Topological methods in Algebraic geometry. Springer.
Hitchin, N. (1974). Harmonic spinors. Advances in math., 14, 1–55.
Hitchin, N. (1991/92). The hyperkähler manifolds. In Séminaire Bourbaki, (pp. 137–166). Paris, Astérisque 206, Société math. de France.
Hitchin, N. (1995). Twistor spaces, Einstein metrics and isomonodromic deformations. J. of differential geometry, 42, 30–112.
Hodge, W. (1941). The theory and applications of harmonic integrals (second edition 1952). Cambridge University Press.
Hofer, H. & Zehnder, E. (1994). Symplectic capacities. Birkhäuser.
Hofer, H., Taubes, C., Weinstein, A. & Zehnder, E. (Ed.). (1995). The Floer memorial volume. Birkhäuser.
Hopf, E. (1939). Statistik des geodätischen Linien in Mannigfaltigkeiten negativer Krümmung. Ber. Vehr. Sächs. Akad. Wiss. Leipzig, 91, 261–304.
Hopf, E. (1948). Closed surfaces without conjugate points. Proc. Nat. Acad. Sciences, 34, 47–51.
Hopf, H. (1925a). Die Curvatura integra Clifford-Kleinischer Raumformen. Nach. Ges. d. Wiss. Göttingen, Math. Physik Klasse.
Hopf, H. (1925b). Über die Curvatura integra geschlossener Hyperflächen. Math. Annalen 95, p. 340–367.
Hopf, H. (1932). Differential Geometrie and topologische Gestalt. Jahresbericht der DMV, 41, 209–229.
Hopf, H. (1951). Über Flächen mit einer Relation zwischen den Hauptkrümmungen. Math. Nachrichten, 4, 232–249.
Hopf, H. (1983). Differential geometry in the large. LNM 1000, Springer.
Hopf, H. & Rinow, W. (1931). Über den Begriff der vollständigen differentialgeometrischen Flächen. Commentarii math. Helv., 3, 209–225.
Hörmander, L. (1968). The spectral function of an elliptic operator. Acta mathematica, 121, 193–218.
Hörmander, L. (1985). The analysis of linear partial differential operators: I, II, III, IV. Springer.
Huang, H.-M. (1981). Some remarks on the pinching problem. Bull. Inst. math. Sinica, 9, 321–340.
Huber, H. (1961). Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen II. Math. annalen, 142, 143, 385–398, 463–464.
Hulin, D. (1995). Sous-variétés complexes d'Einstein de l'espace projectif. Bull. soc. math. France, 124, 277–298.
Hummel, C. & Schroeder, V. (1997). Tits geometry associated with closed real-analytic manifolds of non positive curvature and dimension 4. Univ. Pennsylvania and Zürich Univ.
Husemoller, D. (1975). Fibre bundles. Springer.
Igusa, J. (1954). On the structure of a certain class of Kähler manifolds. American j. of math., 76, 669–678. Im Hof, H.-C. & Ruh, E. (1975). An equivariant pinching theorem. Commentarii math. helvetici, 50, 389–401.
Im Hof, H.-C. & Ruh, E. (1975) An equivariant pinching theorem. Commentarii math. Helvetici, 50, 389–401.
Itoh, J. (1996). The length of the cut-loci on a surface and Ambrose's problem. J. of differential geometry, 43, 642–651.
Itoh, J.-I. & Tanaka, M. (1997). On the Hausdorff dimension of a cut-locus on a smooth Riemannian manifold.
Iwaniec (1975). Introduction to the spectral theory of automorphic forms. Biblioteca de la revista matematica iberoamericana.
Jensen, G. (1973). Einstein metrics on principal fibre bundles. J. of diff. geometry, 8, 599–614.
Jorgensen, T. (1977). Compact 3-manifolds of constant negative curvature fibering over the circle. Annals of math., 106, 61–72.
Jost, J. (1991). Two-dimensional geometric variational problems. Wiley-Interscience.
Jost, J. (1995). Riemannian geometry and geometric analysis. Springer.
Jost, J. (1996). Compact Riemann surfaces. Springer.

Jost, J. & Karcher, H. (1982). Geometrische Methoden zur Gewinnung von a-priori Schranken für harmonische Abbildungen. Manuscripta mathematica, 40, 27–77.

Jost, J. & Yau, S.-T. (1990). Harmonic maps and superrigidity. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54 (pp. 245–280). Los Angeles: American math. Soc.

Joyce, D. (1996a). Compact 8-manifolds with holonomy Spin(7). Inventiones math., 123, 507–552.

Joyce, D. (1996b). Compact manifolds with holonomy G_2: I. II J. of differential geometry, 43, 291–328, 329–375.

Joyce, D. (1997). A theory of quaternionic algebra, with applications to hypercomplex geometry.

Kamberov, G. (1997). Prescribing mean curvature: existence and uniqueness problems.

Kapovich, M. & Leeb, B. (1996). 3-manifolds groups and nonpositive curvature.

Karcher, H. (1988). A geometric classification of positively curved symmetric spaces and the isoparametric construction of the Cayley plane. In: On the geometry of differentiable manifolds (pp. 111–136). Astérisque 163–164, Soc. Math. France.

Karcher, H. (1989). Riemann comparison constructions. In S. S. Chern (Eds.), Global differential geometry (pp. 170–222). The mathematical association of America.

Kasue, A. (1989). A convergence theorem for Riemannian manifolds and some applications. Nagoya math. j., 114, 21–51.

Katok, A. (1982). Entropy and closed geodesics. Ergodic theory and dynamical systems, 2, 339–365.

Katok, A. (1988). Four applications of conformal equivalence to geometry and dynamics. Ergodic Th. and Dynamical Syst., 8, 139–152.

Katok, A. & Hasselblatt, B. (1995). Introduction to the modern theory of dynamical systems. Cambridge Univeristy Press.

Katok, S. (1996). Coding of closed geodesics after Gauss and Morse. Geometriae dedicata, 63, 123–145.

Katsuda, A. (1985). Gromov's convergence theorem and its applications. Nagoya math. J., 100, 11–48.

Katz, M. (1983). The filling radius of two-point homogeneous spaces. J. of differential geom., 18, 503–511.

Katz, M. (1991). The rational filling radius of complex projective space. Topology and its applications, 42, 201–215.

Katz, N. (1998). Filling volume of the circle.

Kazdan, J. (1981/82). Positive energy in general relativity. In Séminaire Bourbaki, Astérisque 92–93 (pp. 315–330). Paris: Société math. France.

Kazdan, J. & Warner, F. (1975a). Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature. Ann. of math., 101, 317–331.

Kazdan, J. & Warner, F. (1975b). Scalar curvature and conformal deformation of Riemannian structure. J. of differential geom., 10, 113–134.

Killing, W. (1885). Nicht euklidische Raumformen. Leipzig.

Kirby, R. & Siebenmann, L. (1977). Foundational essays on topological manifolds, smoothings, and triangulations. Annals of math. studies 88, Princeton University Press.

Kiyohara, K. (1984). On infinitesimal C_{2p}-deformations of standard metrics on spheres. Hokkaido math. Journal, 13, 151–231.

Kiyohara, K. (1987). Riemannian metrics with periodic geodesic flow on projective spaces. Japan J. math., 13, 209–234.

Kleiner, B. (1992). An isoperimetric comparison theorem. Inventiones math., 108, 37–47.

Kleiner, B. & Leeb, B. (1997). Rigidity of quasi-isometries for symmetric spaces and euclidean buildings.

Klingenberg, W. (1959). Contributions to Riemannian geometry in the large. Annals of math., 69, 654–666.

Klingenberg, W. (1961). Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung. Commentarii math. Helvetici, 35, 47–54.

Klingenberg, W. (1978a). Lectures on closed geodesics. Springer.

Klingenberg, W. (1978b). A course in differential geometry. Springer.

Klingenberg, W. (1982). Riemannian geometry Second edition in 1995. de Gruyter.
Klingenberg, W. & Pinl, M. (1981). Die Bedeutung von Christoffel für die Geometrie, E. B. Christoffel Weg zum absoluten Differentialkalkül und sein Betrag zur Theorie des Krümmungstensors. In P. Butzer & F. Fehér (Eds.), E. B. Christoffel (pp. 462–473)
Klingenberg, W. & Sakai, T. (1980). Injectivity radius estimate for (1/4)-pinched manifolds. Arch. math., 34, 371–376.
Klingenberg, W. & Takens, F. (1972). Generic properties of geodesic flow. Math. annalen, 197, 323–334.
Klotz Milnor, T. (1972). Efimov's theorem about complete immersed surfaces of negative curvature. Adavances in math., 8, 474–543.
Knieper, G. & Weiss, H. (1994). A surface with positive curvature and positive topological entropy. J. Differential geometry, 39, 229–249.
Kobayashi, S. (1989). On conjugate and cut-loci. In S. S. Chern (Eds.), Global differential geometry (pp. 140–169). The math. Assoc. of America.
Kobayashi, S. & Nomizu, K. (1963–1969). Foundations of differential geometry I and II. Interscience.
Koszul, J.-L. (1960). Lectures on fibre bundles and differential geometry. Bombay: Tata institute of fundamental research.
Kowalski, O. & Vlasek, Z. (1993). Homogeneous Einstein metrics on Aloff-Wallach spaces. Differential geometry and its applic., 3, 157–167.
Krahn, E. (1924). Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises. Math. Annalen, 94, 97–100.
Kreck, M. & Stolz, S. (1993). Nonconnected moduli spaces of positive sectional curvature metrics. Journal of the Amer. math. Soc., 6, 825–849.
Kronecker, L. (1869). Über Systeme von Functionen mehererer Variablen. Monatsber. d. Kgl. Peuss. Akad. d. Wiss. zu Berlin, 2. Abhdl.
Kühnel, W. & Rademacher, H.-B. (1995). Essential conformal fields in pseudo-riemannian geometry. J. math. pures et appl., 74, 453–481.
Kuiper, N. (1955). On C^1-isometric imbeddings I and II. Indigationes math., 17, 545–566 and 683–689.
Kulkarni, R. (1970). Curvature and metric. Annals of math., 91, 311–331.
Kuranishi, M. (1990). On some metric on $S^2 \times S^2$. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54 part 3 (pp. 439–450). Los Angeles: American math. soc. 1993
Lafontaine, J. (1985/86). Mesures de courbure des variétés riemanniennes et des polyèdres (d'après Cheeger, Müller et Schrader). In Séminaire Bourbaki, 145–146 (pp. 241–256). Paris: Société mathématique de France.
Lanczos, C. (1922). Ein vereinfachendes Koordinatensystem für die Einsteinschen Gravitationsgleichungen. Phys. Z., 23, 537–539.
Lang, S. (1972). Differentiable manifolds. Addison-Wesley.
Lang, S. (1996). Differentiable manifolds and Riemannian geometry. Springer
Langevin, R. (1997). Introduction to integral geometry. In 21° Coloquio Brasileiro de matematica. Rio de Janeiro: IMPA.
Lanzendorf, M. (1997). Einstein metrics with nonpositive sectional curvature on extensions of Lie algebras of Heisenberg type. Geometriae dedicata, 66, 187–202.
Lawson, B. & Michelsohn, M.-L. (1984). Embedding and surrounding with positive mean curvature. Inventiones math., 77, 399–419.
Lawson, B. & Michelsohn, L. (1989). Spin geometry. Princeton University Press.
Lawson, B. & Yau, S.-T. (1972). Compact manifolds of nonpositive curvature. J. differential geometry, 7, 211–228.
Lawson, B. & Yau, S.-T. (1974). Scalar curvature, non-abelian group actions, and the degree of symmetry of exotic spheres. Commentarii math. helvetici, 49, 232–244.
Lazutkin, V. (1993). KAM theory and semiclassical approximations to eigenfunctions. Springer.
LeBrun, C. (1995). Einstein metrics and Mostow rigidity. Mathematical reseach letters(2), 1–8.
LeBrun, C. (1996). Four-manifolds without Einstein metrics. Math. research letters, 3, 1–15.

LeBrun, C. & Salamon, S. (1994). Strong rigidity of positive quaternionic manifolds. Inventiones math., 118, 109–132.
Ledoux, M. (1992/93). Inégalités isopérimétriques en analyse et probabilités. In Séminaire Bourbaki, Astérisque 216 (pp. 343–376). Paris: Société math. de France.
Levi-Civita, T. (1917). Nozione di parallelismo in una varieta qualunque e consequente specificazione geometrica della curvatura Riemanniana. Rendiconti di cir. Palermo, 42, 73–205.
Levi-Civita, T. & Ricci, G. (1901). Méthodes de calcul différential absolu et leurs applications. Math. Annalen, 54, 125–201.
Lewin, L. (Ed.). (1991). Structural properties of polylogarithms. American math. soc.
Lewy, H. (1938). On differential geometry in the large I. Transactions of the A M S, 43, 258–270.
Lichnerowicz, A. (1950). Courbure, nombres de Betti, et espaces symétriques. In International congress of math., (pp. 216–223). Harvard.
Lichnerowicz, A. (1955). Théorie globale des connexions et des groupes d'holonomie. Ed. Cremonese, Rome.
Lichnerowicz, A. (1958). Géométrie des groupes de transformations. Dunod.
Lichnerowicz, A. (1961). Propagateurs et commutateurs en relativité générale. Publications math. IHES, 10, 293–344.
Lichnerowicz, A. (1963). Spineurs harmoniques. C. R. Acad. Sciences Paris, 257, 7–9.
Llarull, M. (1995). Scalar curvature and volume of a Riemannian manifold. Geometriae dedicata, 56, 1–4.
Llarull, M. (1996). Scalar curvature estimates for $(4n + k)$-dimensional spin manifolds. Differential geometry and applic, 6, 321–326.
Löbell, F. (1931). Beispiele geschlossener dreidimensionaler Clifford-Kleinscher Räume negativer Krümmung. Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math-Phys. Kl., 83, 167–174.
Lohkamp, J. (1992). Ricci curvature modulo homotopy. In A. Besse (Ed.), Table ronde de géométrie différentielle, (pp. 437–451). Luminy, Séminaires et congrès 1, Soc. math. France.
Lohkamp, J. (1994). Manifolds of negative Ricci curvature. Annals of math., 140, 655–683.
Lohkamp, J. (1996a). Global and local curvatures. In M. Lovric, M. Min-Oo & M. Wang (Eds.), Riemannian geometry (pp. 23–52). American math. society.
Lohkamp, J. (1996b). Discontinuity of geometric expansions. Commentarii math. helv., 71, 213–228.
Lohkamp, J. (1998). To appear.
Lück, W. (1996). L^2 invariants of regular coverings of compact manifolds and CW-complexes (survey). Münster Universität.
Luo, W. & Sarnak, P. (1995). Quantum ergodicity of eigenfunctions on PSL2(Z)\2. Publications math. I . H. E. S., 81, 207–237.
Luo, W. & Sarnak, P. (1996). Number variance for arithmetic hyperbolic surfaces.
Luo, W., Rudnick, Z. & Sarnak, P. (1995). On Selberg's eigenvalue conjecture. Geometric and functional analysis, 5, 387–401.
Lusternik, L. & Fet, A. (1951). Variational problems in closed manifolds. Doklady Akad. Nauk SSR, 81, 17–18.
Lusternik, L. & Schnirelman, L. (1929). Sur le problème de trois géodésiques fermées sur les surfaces de genre 0. C. R. Acad. Sciences Paris, 189, 269–271.
Malliavin, P. & Stroock, D. (1996). Short time behavior of the heat kernel and its logarithmic derivatives. J. of differential geom., 44, 550–570.
Mané, R. (1987). Ergodic theory and differentiable dynamics. Springer.
Mané, R. (1997). On the topological entropy of geodesic flows. J. of differential geometry, 45, 74–93.
Manning, A. (1979). Topological entropy for geodesic flows. Annals of math., 110, 567–573.
Margerin, C. (1984a). Pointwise-pinched manifolds are space forms. In AMS summer institute, (pp. 327–328). Arcata: American math. soc. 1986.
Margerin, C. (1984b). Un théorème optimal pour le pincement faible en dimension 4. C.R. Acad. sci. Paris, 303, 713–716.
Margerin, C. (1991, 1993, 1994). Théorie de la déformation en géométrie métrique No. M1005.0890, M1100.1293, M1073.0593). Ecole polytechnique.

Margerin, C. (1994). Un caractérisation optimale de la structure différentiable de la sphère en terme de coubure pour preque toutes les dimensions. C.R. Acad. sci. Paris, 319, 605–607.
Margulis, G. (1969). Applications of ergodic theory to the investigation of manifolds of negative curvature. Functional analysis and applic., 3, 335–336.
Margulis, G. (1977). Discrete groups of motions of manifolds of non-positive curvature. A M S Translations, 39, 33–45.
McCleary, J. & Ziller, W. (1987). On the free loop space of homogeneous spaces. American j. of math., 109, 765–782.
McDuff, D. & Salamon, D. (1995). Introduction to symplectic topology. Oxford: Clarendon Press.
McKean, H. & Singer, I. (1967). Curvature and the eigenvalues of the Laplacian. J. Differential Geometry, 1, 43–69.
Mentges, H. (1987). Über geodätische Schleifen. Bonn.
Meyer, D. (1971). Sur les variétés riemanniennes à opérateur de courbure positif. C. R. Acad. Sciences Paris, 272, 482–485.
Meyer, W. (1989). Toponogov's theorem and its applications. In College on differential geometry. Trieste: ITP Trieste.
Micallef, M. & Moore, J. (1988). Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes. Annals of math., 127, 199–227.
Michel, R. (1994). Restriction de la distance géodésique à un arc et rigidité. Bull. soc. math. France, 122, 435–442.
Milnor, J. (1963). Morse theory. Princeton Univ. Press.
Milnor, J. (1964). Eigenvalues of the Laplace operator on certain manifolds. Proc. nat. Acad. Sci. USA, 51, 542.
Milnor, J. (1966). Whitehead torsion. Bulletin of the A.M.S., 72, 351–426.
Milnor, J. (1968). A note on curvature and the fundamental group. J. differential geometry, 2, 1–7.
Milnor, J. & Stasheff, J. (1974). Characteristic classes. Annals of math. studies 76, Princeton University Press.
Min-Oo, & Ruh, E. (1979). Comparison theorems for compact symmetric spaces. Annales scientif. Ecol. norm. sup., 12, 335–353.
Min-Oo, & Ruh, E. (1981). Vanishing theorems and almost symmetric spaces of noncompact type. Math. Annalen, 257, 419–433.
Minakshisundaram, S. (1953). Eigenfunctions on Riemanian manifolds. J. indian math. Soc., 17, 118–165.
Minakshisundaram, S. & Pleijel, A. (1949). Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds. Canadian J. of math., 1, 242–256.
Misner, C., Thorne, K. & Wheeler, J. (1973). Gravitation. Freeman.
Mok, N. (1988). The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature. J. of differential geometry, 27, 179–214.
Morgan, F. (1988). Geometric measure theory: a beginner's guide. Academic Press.
Morgan, J. (1996). The Seiberg-Witten equations and applications to the topology of smooth 4-manifolds. Math. notes 44, Princeton university press.
Mori, S. (1979). Projective manifolds with ample tangent bundles. Annals of math., 110, 593–606.
Morse, M. (1921a). A one-to-one representation of geodesics on a surface of negative curvature. American J. of math., 43, 33–51.
Morse, M. (1921b). Recurrent geodesics on a surface of negative curvature. Transactions of the A. M. S., 22, 84–100.
Morse, M. (1934). Calculus of variations in the large. Providence: American math. soc.
Mostow, G. (1968). Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms. Public. math. I H E S, 34, 53–104.
Mostow, G. D. (1980). A remarkable class of polyhedra in complex hyperbolic space. Pacific J. of Math., 86, 171–276.
Mostow, G. & Siu, Y.-T. (1980). A compact Kähler surface of negative curvature non covered by the ball. Annals of math., 112, 321–360.

Müller, W. (1978). Analytic torsion and R-torsion of Riemannian manifolds. Advances in math., 28, 233–305.

Munoz, M., J. & Valdés, A. (1996). Génération d'anneaux d'invariants différentiels des métriques riemanniennes. C. R. Acad. Sc. Paris, 323, 643–646.

Myers, S. (1935a). Riemannian manifolds in the large. Duke math. J., 1, 39–49.

Myers, S. (1935b). Connections between differential geometry and toplogy, I. Duke math. J., 35, 376–391.

Myers, S. (1936). Connections between differential geometry and topology, II. Duke math. J., 2, 95–102.

Myers, S. (1941). Riemannian manifolds with positive mean curvature. Duke math. J., 8, 401–404.

Nabonnand, P. (1995). Contribution à l'histoire de la théorie des géodésiques au XIXème siècle. Revue d'histoire des math., 1, 159–200.

Nabutovsky, A. (1996a). Fundamental group and contractible closed geodesics. Communications in pure and appl. math., 49, 1257–1270.

Nabutovsky, A. (1996b). Disconnectedness of sublevel sets of some Riemannian functionals. Geometric and functional analysis, 6, 702–725.

Nabutovsky, A. (1996c). Geometry of the space of triangulations of compact manifolds. Communications in math. physics, 181, 303–330.

Nabutovsky, A. & Weinberger, S. (1998). Variational problems for Riemannian functionals and arithmetic groups.

Nadirashvili, N. (1996). Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces. Inventiones math., 126, 457–466.

Nash, J. (1954). C^1-isometric imbeddings. Annals of math., 60, 383–396.

Nash, J. (1956). The embedding problem for Riemannian manifolds. Annals of math., 63, 20–63.

Nirenberg, L. (1953). The Weyl and Minkowski problems in differential geometry in the large. Communications in pure and applied math., VI, 337–394.

Nitsche, J. (1996). Minimal surfaces. Springer.

Nomizu, K. (1954). Affine connections on homogeneous spaces. American j. of math., 76, 33–65.

Obata, M. (1962). Certain conditions for a Riemannian manifold to be isometric with a sphere. J. math. Soc. Japan, 14, 333–340.

Oesterlé, J. (1992/93). Polylogarithmes. In Séminaire Bourbaki, Astérisque 216 (pp. 49–68). Paris: Société math. de France.

Omori, H. (1983). Construction problems of Riemannian manifolds. In M. Berger, S. Murakami & T. Ochiai (Eds.), Spectra of Riemannian manifolds (pp. 79–90). Kaigai.

O'Neill, B. (1966). The fundamental equations of a submersion. Michigan math. journal, 13, 459–469.

O'Neill, B. (1983). Semi-Riemannian geometry. Academic Press.

Osgood, B., Phillips, R. & Sarnak, P. (1988). Extremals of determinants of Laplacians. J. of functional analysis, 80, 148–211.

Osgood, O., Phillips, R. & Sarnak, P. (1989). Moduli space, heights and isospectral sets of plane domains. Annals of Math., 129, 293–362.

Osserman, R. (1996). Minimal surfaces. Encyclopedia of Mathematical Sciences, volume 90, Geometry V. Springer.

Otal, J.-P. (1990a). Le spectre marqué des surfaces à courbure négative. Annals of math., 131, 151–162.

Otal, J.-P. (1990b). Sur les longueurs des géodésiques d'une métrique à courbure négative dans le disque. Commentarii math. helvetici, 65, 334–347.

Otsu, Y. (1991). On manifolds of positive Ricci curvature and large diameter. Math. Zeitschrift, 2, 255–264.

Otsu, Y. (1993). On manifolds of small excess. American J. of math., 115, 1229–1280.

Otsu, Y. & Shioya, T. (1994). The Riemannian structure of Alexandrov spaces, Journal of differential geometry 39, 629–658.

Otsu, Y., Shiohama, K. & Yamaguchi, T. (1989). A new version of the differentiable sphere theorem. Inventiones math., 98, 219–228.

Page, D. (1979). A compact gravitational instanton. Phys. letters, 79 B, 235–238.
Palais, R. & Smale, S. (1964). A generalized Morse theory. Bulletin of the Amer. math. soc., 70, 165–172.
Palais, R. & Terng, C.-L. (1988) Critical point theory and submanifold geometry. LNM 1353, Springer.
Pandey, A., Bohigas, O. & Giannoni, M. (1984). Characterization of chaotic quantum spectra and universality of level fluctuations laws. Physical review letters, 52, 1, 1–4.
Pansu, P. (1983–1984). Effondrement des variétés riemanniennes. In Séminaire Bourbaki, 618 (pp. 63–82). Paris: Société math. France.
Pansu, P. (1989). Métriques de Carnot-Carathéodory et quasi-isométries des espaces symétriques de rang 1. Annals of math., 129, 1–60.
Pansu, P. (1990/91). Le flot géodésique des variétés riemanniennes à courbure négative. In Séminaire Bourbaki, Astérisque 201–202-203 (pp. 269–298). Paris: Société math. France.
Pansu, P. (1993–94). Sous-groupes discrets des groupes de Lie: rigidité, arithméticité. In Séminaire Bourbaki, 227 (pp. 69–105). Paris, Astérsique 618, Société math. France.
Pansu, P. (1996). Introduction to L^2 Betti numbers. In M. Lovric, M. Min-Oo & M. Wang (Eds.), Riemannan geometry (pp. 53–86). Fields Institute monographs, American math. soc.
Pansu, P. (1996/97). Volume, courbure et entropie (d'après G. Besson, G. Courtois et S. Gallot). In Séminaire Bourbaki. Paris, Astérisque, Société math. France.
Pansu, P. (1997a). Differentiability of volume growth. Rend. circolo mat. Palermo, 49 (suppl.), 159–176.
Pansu, P. (1997b). Sur la régularité du profil isopérimétrique des surfaces riemanniennes compactes. Annales de l'Institut Fourier.
Parry, W. & Pollicott, M. (1990). Zeta functions and the periodic orbit structure of hyperbolic dynamics. Astérisque 187–188, Société math. France.
Paternain, G. (1992). On the topology of manifolds with completely integrable geodesic flow. Ergodic theory and dynamical systems, 12, 109–121.
Paternain, G. (1993). Real analytic convex surfaces with positive topological entropy and rigid body dynamics. Manuscripta math., 78, 397–402.
Paternain, G. & Paternain, M. (1994). Topological entropy versus geodesic entropy. International J. of mathematics, 5, 213–218.
Patodi, V. (1971). Curvature and the eigenforms of the Laplace operator. J. of differential geom., 5, 233–249.
Paun, M. (1997). Sur le groupe fondamental des variétés kählériennes compactes à classe de Ricci numériquement effective. Comptes rendus Acad. Sc. Paris, 324, 1249–1254.
Pelletier, F., & Valère Bouche, L. (1992). The problem of geodesics, intrinsic derivation and the use of control theory in singular sub-riemannian geometry. In A. Besse (Ed.), Table ronde de géométrie différentielle, 1 (pp. 453–512). Luminy, Séminaires et congrès 1, Société math. France.
Perelman, G. (1994a). Manifolds of positive Ricci curvature with almost maximal volume. Journal of the AMS, 7, 299–305.
Perelman, G. (1994b). Proof of the soul conjecture of Cheeger and Gromoll. J. of differential geom., 40, 209–212.
Perelman, G. (1994c). Spaces with curvature bounded below. In International Congress of mathematicians, (pp. 517–525). Zürich: Birkhäuser.
Perelman, G. (1995a). Widths of nonnegatively curved spaces. Geometric and functional analysis, 5, 445–463.
Perelman, G. (1995b). A diameter sphere theorem for manifolds of positive Ricci curvature. Math. Zeitschrift, 218, 595–595.
Perelman, G. (1997). Constructions of manifolds of positive Ricci curvature with big volume and large Betti numbers.
Pesce, H. (1992). Nilvariétés isospectrales. In A.-L. Besse (Ed.), Table ronde de géométrie différentielle, 1 (pp. 513–530). Luminy, Sémianires et congrès 1, Société math. France 1996.
Peters, S. (1984). Cheeger's finiteness theorem for diffeomorphism classes of manifolds. J. für reine und angew. Math., 349, 77–82.
Peters, S. (1987). Convergence of Riemannian manifolds. Compositio math., 62, 3–16.

Petersen, P. (1990). Gromov-Hausdorff convergence of metric spaces. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54, part 3 (pp. 489–504). Los Angeles: American math. society.

Petersen, P. (1996). Comparison geometry problem list. In M. Lovric, M. Min-Oo & M. Wang (Eds.), Riemannian geometry (pp. 87–115). Fields Institute monographs, American math. soc.

Petersen, P. (1997). Convergence theorems in Riemannian geometry. In Comparison geometry MSRI publications, Cambridge University Press.

Petersen, P. (1998a). Riemannian geometry. Springer.

Petersen, P. (1998b). Global Riemannian geometry : an historical account.

Petersen, P. & Wei, G. (1996a). Relative volume comparison with integral curvature bounds.

Petersen, P., & Wei, G. (1996b). Analysis and geometry on manifolds with integral Ricci curvature bounds.

Petersen, P., Shteingold, S. & Wei, G.-F. (1996). Comparison geometry with integral curvature bounds.

Petersen, P., Wei, G. & Ye, R. (1997). Control geometry via smoothing.

Petersen, P., Wilhelm, F. & Zhu, S.-H. (1995). Spaces on and beyond the boundary of existence. J. of geometric analysis, 5, 419–426.

Petroll, D. (1996) Existenz und Transversalität von homoklinen und heteroklinen Orbits beim geodätischen Fluss. Thesis, Universität Freiburg.

Pinkall, U. & Sterling, I. (1989). On the classification of constant mean curvature tori. Annals of math., 130, 407–451.

Pinsky, M. (1989). Inverse questions in stochastic differential geometry. In Probabilities. Singapour, de Gruyter.

Pinsky, M. (1990). Feeling the shape of a manifold with Brownian motion. In Barlow & Bingham (Eds.), Stochastic analysis, Cambridge University press.

Pogorelov, A. (1973). Extrinsic geometry of convex surfaces. American mathematical society.

Poincaré, H. (1905). Sur les lignes géodésiques des surfaces convexes. Trans. Amer. Math. Soc., 6, 237–274.

Pollicott, M. & Rocha, A. (1997). A remarkable formula for the determinant of the Laplacian. Inventiones math., 130, 399–414.

Prasad, G. (1989). Volumes of S-arithmetic quotients of semi-simple Lie groups. Publications math. IHES, 69, 91–117.

Preissmann, A. (1942–43). Quelques propriétés globales des espaces de Riemann. Comm. math. Helvetici, 15, 175–216.

Prüfer, F., Tricerri, F. & Vanhecke, L. (1996). Curvature invariants, differential operators and local homogeneity. Transactions of the A M S., 348, 4643–4651.

Przytycki, F. (1980). An upper estimation of the topological entropy of diffeomorhisms. Inventiones math., 59, 205–213.

Pu, P. (1952). Some inequalities in certain nonorientable manifolds. Pacific J. Math., 2, 55–71.

Püttmann, T. (1997). Improved pinching constants of odd-dimensional homogeneous spaces, Bochum University.

Rademacher, H. (1994a). On a generic property of geodesic flows. Math. Annalen, 298, 101–116.

Rademacher, H. (1994b). The Fadell-Rabinowitz index and closed geodesics. J. London math. soc., 50, 609–624.

Ratcliffe, J. (1994). Foundations of hyperbolic manifolds. Springer.

Rauch, H. (1951). A contribution to differential geometry in the large. Annals of math., 54, 38–55.

Ray, D. & Singer, I. (1971). R-torsion and the Laplacian on Riemannian manifolds. Advances in math., 7, 145–210.

Real, C. (1996). Métriques d'Einstein-Kähler sur les variétés à première classe de Chern positive. C. R. Acad. Sc. Paris, 322, 461–464.

Reshetnyak, Y. (1993). Geometry IV. Encyclopedia of math. sciences 70, Springer.

Ritoré, M. & Ros, A. (1992). Stable constant mean curvature tori and the isoperimetric problem in three space form. Commentarii math. helvetici, 67, 293–305.

Robert, G. (1994) Invariants topologiques et géométriques reliés aux longueurs des géodésiques et aux sections harmoniques de fibrés. Grenoble.
Rong, X. (1993). The limiting eta-invariant of collapsed 3-manifolds. J. of differential geometry, 37, 535–568.
Rong, X. (1995). Rationality of geometric signatures of complete 4-manifolds. Inventiones math., 120, 513–554.
Rong, X. (1996a). On the fundamental group of manifolds of positive sectional curvature. Annals of math., 143, 397–411.
Rong, X. (1996b). The almost cyclicity of the fundamental group of positively curved manifolds. Inventiones math., 126, 47–64.
Rong, X. (1997a). Positive curvature, local and global symmetry, and fundamental group.
Rong, X. (1997b). A Bochner theorem and its applications. Duke math. j.
Rosenberg, J., & Stolz, S. (1994). Manifolds of positive scalar curvature. In G. Carlsson, R. Cohen & W.-C. Hsiang J. (Eds.), Algebraic topology and its applications (pp. 241–267). Springer.
Rotman, R. (1997). Upper bounds on the length of the shortest geodesic on simply connected manifolds.
Rudnick, Z. & Sarnak, P. (1996). Zeros of principal zeta-functions and random matrix theory, Duke math. J. 81, 269–322.
Ruh, B. (1985). Krümmungstreue Diffeomorphismen Riemannscher und pseudo-Riemannscher Mannigfaltigkeiten. Math. Zeitschrift, 189, 371–391.
Ruh, E. (1971). Curvature and differentiable structures on spheres. Commentarii math. helvetici, 46, 127–136.
Ruh, E. (1982a). Almost flat manifolds. J. of differential geom., 17, 1–14.
Ruh, E. (1982b). Riemannian manifolds with bounded curvature ratios. J. of differential geometry, 17, 643–653.
Sachs, R. & Wu, H. (1977). General relativity for mathematicians. Springer.
Sakai, T. (1978). Cut-loci of compact symmetric spaces. In Minimal submanifolds and geodesics, (pp. 193–208). Tokyo: Kaigai publications, Tokyo.
Sakai, T. (1996). Riemannian geometry. American math. soc.
Salamon, S. (1982). Quaternionic manifolds. Symposia math., XXVI, 139–151.
Salamon, S. (1989). Riemannian geometry and holonomy groups. Longman scientific and technical.
Salamon, S. (1996). On the cohomology of Kähler and Hyper-Kähler manifolds. Topology, 35, 137–155.
Sambusetti, A. (1996). An obstruction to the existence of Einstein metrics on 4-manifolds. C. R. Acad. sciences Paris, 322, 1213–1218.
Samiou, E. (1997). The symmetry of 2-flat homogeneous spaces. Geometriae dedicata, 65, 161–165.
Sampson, J. (1986). Some applications of harmonic maps to Kähler geometry. Contemporary math., 49. 125–134.
Santalo, L. (1976). Integral geometry and geometric probability. Addison-Wesley.
Sarnak, P. (1995). Arithmetic Quantum Chaos. In Isr. math. conf. (pp. 183–236).
Sarnak, P. (1997a). Extremal geometries. Contemporary math., 201, 1–7.
Sarnak, P. (1997b). Values at integers of binary quadratic forms, Princeton University.
Sarnak, P. (1997c). Spectra and eigenfunctions of the Laplacian, Princeton University.
Satake (1956). On a generalization of the notion of manifold. Proc. nat. acad. sc. USA, 42, 359–363.
Savage (1982). The space of positive definite matrices and Gromov's invariant. Transactions of the AMS, 274, 239–263.
Savo, A. (1996). Une méthode de symétrisation et quelques applications, Comptes rendus Acad. Sc. Paris, 322, 861–864
Savo, A. (1998). A mean-value lemma and applications. Geometriae dedicata.
Schmidt, E. (1948–1949). Die Brunn-Minkowskischen Ungleichungen und ihr Spielgebild sowie die Isoperimetrische der Kugel in der euklidischen und nichteuklidischen Geometrie I, II. Math. Nachrichten, 1, 2, 81–157, 171–244.

Schneider, R. & Wieacker, J. (1993). Integral geometry. In Handbook of convexity (pp. 1349–1390). North-Holland.

Schnirelman, A. (1974). Ergodic properties of eigenfunctions (in Russian), Usp. math. Nauk, 29, 181–182.

Schoen, R. (1991). Recent progress on non-linear problems in geometry. J. of differential geometry, surveys.

Schoen, R. & Yau, S.-T. (1979). On the structure of manifolds with positive scalar curvature. Manuscripta math., 28, 159–183.

Schoen, R. & Yau, S.-T. (1981). Proof of the positive mass theorem. Communications in math. physics, 79, 231–260.

Schoenberg, J. (1932). Some applications of the calculus of variations to Riemannian geometry. Annals of math., 33, 485–495.

Schouten, J. (1954). Ricci-calculus. Springer.

Schwartz, R. (1995). The quasi-isometry classification of rank one lattices. Publications math. IHES, 82, 133–168.

Scott, P. (1983). The geometries of 3-manifolds. Bulletin of the AMS, 15, 401–487.

Seiberg, N. & Witten, E. (1994). Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD. Nuclear physics B, 341, 581–640.

Semmes, S. (1994). Finding structures in sets with little smoothness. In Intern. congress of math., (pp. 875–885). Zürich: Birkhäuser.

Semmes, S. (1996a). Where the buffalo roam: infinite process and infinite complexity. M/96/77). I H E S.

Semmes, S. (1996b). Good metric spaces without good parametrizations. Revista math. iberoamericana, 12, 187–275.

Serre, J.-P. (1951). Homologie singulière des espaces fibrés. Annals of Math., 54, 425–505.

Sha, J. & Yang, D. (1991). Positive Ricci curvature on the connected sums of $S^n \times S^m$. J. of differential geometry, 33, 127–137.

Sha, J.-P. & Yang, D. (1993). Positive Ricci curvature on compact simply connected 4-manifolds. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54, part 3 (pp. 529–538). Los Angeles: American math. soc. 1993

Shankar, K. (1998). On fundamental groups of positively curved manifolds.

Shen, Z. (1996). Complete manifolds with nonnegative Ricci curvature and large volume growth. Inventiones math., 125, 393–404.

Shen, Z. (1998). Curvature, distance and volume in Finsler geometry.

Shi, W.-X. (1989). Deforming the metric in complete manifolds. J. of differential geometry, 30, 225–301.

Shikata, Y. (1967). On the differentiable pinching problem. Osaka math. J., 4, 279–287.

Shimizu, Y. & Shudo, A. (1995). Polygonal billiards: correspondence between classical trajectories and quantum eigenstates. Chaos, solitons and fractals, 5, 1337–1362.

Shiohama, K. (1983). A sphere theorem for manifolds of positive Ricci curvature. Transactions of A M S, 257, 811–819.

Shiohama, K. (1990). Recent developments in sphere theorems. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54 (pp. 551–576). Los Angeles: American math. soc. 1993

Shiohama, K. & Tanaka, M. (1992). Cut-loci and distance spheres in Alexandrov surfaces. In A. Besse (Ed.), Table ronde de géométrie différentielle, 1 (pp. 531–560). Luminy, Séminaires et congrès 1, Société math. France.

Shioya, T. (1992). Geometry of total curvature. In A. Besse (Ed.), Table ronde de géométrie différentielle, 1 (pp. 561–600). Luminy, Séminaires et congrès 1, Société math. France.

Simon, L. (1983). Lectures on geometric measure theory. Canberra: Australian national university.

Simons, J. (1962). On the transitivity of holonomy systems. Annals of math.76, 213–234.

Sinai, Y. (1976). Introduction to ergodic theory. Math. notes 18, Princeton Univ. Press.

Singer, I. (1960). Infinitesimally homogeneous spaces. Communications in pure and applied math., 13, 685–697.

Singer, I. & Thorpe, J. (1969). The curvature of 4-dimensional Einstein spaces. In Global analysis, papers in honour of K. Kodaira (pp. 355–365). Princeton University Press.

Siu, Y.-T. (1980). The complex analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds. Annals of math., 112, 73–111.

Siu, Y.-T. & Yau, S.-T. (1980). Compact Kähler manifolds of positive bisectional curvature. Inventiones math., 59, 189–204.

Skorniakov, L. (1955). Metrization of the projective plane in connection with a given system of curves. Izvestiya Akad. Nauk SSSR, 19, 471–482.

Sommese (1975). Quaternionic manifolds, Math. Annalen, 212, 191–214

Spatzier, R. (1990). Riemannian manifolds with completely integrable geodesic flows. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54, 3 (pp. 599–608). Los Angeles: American math. soc. 1993

Spivak, M. (1970). A comprehensive introduction to differential geometry. Publish or Perish.

Steenrod, N. (1951). The topology of fiber bundles. Princeton university press.

Stolz, S. (1992). Simply connected manifolds of positive scalar curvature. Annals of math., 136, 511–540.

Stolz, S. (1996). A conjecture concerning positive Ricci curvature and the Witten genus. Math. Annalen, 304, 785–800.

Stroock, D. (1996). Some thoughts about Riemannian structures on path space. Gazette Soc. math. France, 68, 31–45.

Struwe, M. (1988). Plateau's problem and the calculus of variations. Princeton University Press.

Study, E. (1905). Kürzeste Wege im komplexen Gebiete. Math. annalen, 60, 321–377.

Sullivan, D. (1977). Infinitesimal computations in topology. Publications math. IHES, 47, 269–332.

Sunada, T. (1985). Riemannian coverings and isospectral manifolds. Annals of math., 121, 169–186.

Suyama, Y. (1995). Differentiable sphere theorem by curvature pinching. Tohoku math. J., 47, 15–29.

Svarc (1955). Doklady Akad Nauk SSSR, 105, 32–34.

Synge, J. (1925). The first and second variation of the length integral in Riemann space. Proc. London math. soc., 25, 247–264.

Synge, J. (1936). On the connectivity of spaces of positive curvature. Quarterly J. math. Oxford, 7, 316–320.

Szabo, Z. I. (1991). A short topological proof for the symmetry of 2-point homogeneous spaces. Inventiones math., 106, 61–64.

Taimanov, I. (1993). On the existence of three non self-intersecting closed geodesics on manifolds homeomorphic to the 2-sphere. Russian Acad. Sci. Izv. Math., 40, 565–590.

Taimanov, I. (1997). On totally geodesic embeddings of 7-dimensional manifolds into 13-dimensional positively curved manifolds. Math. sbornik.

Takeuchi, M. (1979). On the conjugate loci and cut-loci of compact symmetric spaces II. Tsukuba Journal of math., 3, 1–29.

Tanno, S. (1980). A characterization of canonical sphere by the spectrum. Math. Zeitschrift, 175, 267–274.

Taubes, C. (1992). The existence of anti-self dual conformal structures. J. of differential geometry, 36, 163–253.

Terng, C.-L. (1990). Recent progress in submanifold geometry. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54, part 1 (pp. 439–484). Los Angeles: American math. soc. 1993.

Terng, C.-L. & Thorbergsson, G. (1995). Submanifolds geometry in symmetric spaces. J. differential geometry, 42, 665–718.

Thomas, T. (1935). Riemannian spaces and their characterizations. Acta math., 67, 169–211.

Thorpe, J. (1969). Some remarks on the Gauss-Bonnet integral. J. math. mech., 18, 779–786.

Thurston, W. (1978). Geometry and topology of 3-manifolds. Princeton University.

Tian, G. (1990). On Calabi's conjecture for complex surfaces. Inventiones math., 101, 101–172.

Tian, G. (1991). On one of Calabi's problem. In Proceedings of symposia in pure math. 52, Bedford, E., d'Angelo, J., Greene, R. & Krantz, S. (Ed.) (volume 2, pp. 543–556). American math. soc. 1991

Tian, G. (1997). Kähler-Einstein metrics with positive scalar curvature. Inventiones math., 130, 1–38.

Tits, J. (1955). Sur certaines classes d'espaces homogènes de groupes de Lie. Mémoires de l'Acadmie des Sciences, XXIX, Bruxelles

Toledo, D. (1997). Rigidity theorems in Kähler geometry and fundamental groups of varieties.

Tondeur, P. (1988). Foliations on Riemannian manifolds. Springer.

Toponogov, V. (1959). Riemann spaces with curvature bounded below. Uspehi Mat. Nauk, 14.

Toponogov, V. (1964a). Riemannian spaces having their curvature bounded below by a positive number. Translations of the AMS, 37, 291–337.

Toponogov, V. (1964b). Riemannian spaces which contain straight lines. Americ. math. soc. translations, 37, 287–290.

Tricerri, F. & Vanhecke, L. (1983). Homogeneous structures on Riemannian manifolds. Cambridge university press.

Tricerri, F. & Vanhecke, L. (1986). Variétés riemanniennes dont le tenseur de courbure est celui d'un espace symétrique riemannien irréductible. C.R. Acad. Sc. Paris, 302, 233–235.

Tsukamoto, C. (1981). Infinitesimal Blaschke conjecture on projective spaces. Ann. scient. Ec. norm. sup., 14, 339–356.

Tuschmann, W. (1997). On the structure of compact simply connected manifolds of positive sectional curvature. Geometriae dedicata, 67, 107–116.

Unnebrink, S. (1997). Über die Gestalt der Enden asymptotisch flacher Riemannscher Mannigfaltigkeiten.

van Mangoldt, H. (1881). Über diejenigen Punkte auf positiv gekrümmten Flächen, welche die Eigenschaft haben, dass die von ihnen ausgehenden geodätischen Linien nie aufhören, kürzeste Linien zu sein. J. Reine Angew. Math., 91, 23–52.

Varadhan, S. (1967). Diffusion processes in small time. Communications in pure and appl. math., 20, 659–685.

Vérovic, P. (1996) Entropie et métriques de Finsler. Joseph Fourier, Grenoble I.

Vignéras, M.-F. (1980a). Arithmétique des algèbres de quaternions. LNM 800, Springer.

Vignéras, M.-F. (1980b). Variétés riemanniennes isospectrales et non isométriques. Annals of math., 112, 21–32.

Ville, M. (1985). On (1/4)-pinched 4-dimensional manifolds of negative curvature. Annals of global analysis and geometry, 3, 329–336.

Ville, M. (1987). Sur le volume des variétés riemanniennes pincées. Bull. soc. math. France, 115, 127–139.

Vinberg, E. B. (1984). The absence of crystallographic groups of reflections in Lobachevskii spaces of large dimension. Trans. Mosc. Math. Soc., 1985, 75–112.

Vinberg, E. B. (1993). Geometry II. Encyclopedia of math. sciences 22, Springer.

Voisin, C. (1996). Symétrie miroir. Société math. France.

Wall, C. T. C. (1977). Geometric properties of generic differentiable manifolds. In P. a. d. Carmo (Eds.), Geometry and topology (pp. 707–774). Springer.

Walters, P. (1982). An introduction to ergodic theory. Springer.

Wang, H. (1952). Two-point homogeneous spaces. Annals of math., 55, 177–191.

Wang, H.-C. (1972). Topics on totally discontinuous groups. In W. Boothby & G. Weiss (Eds.), Symmetric spaces (pp. 459–487). Marcel Dekker.

Wang, M. (1982). Some examples of homogeneous Einstein manifolds in dimension seven. Duke math. j., 49, 23–28.

Wang, M. (1992). Einstein metrics and quaternionic Kähler manifolds. Math. Zeitschrift, 210, 305–326.

Wang, M. & Ziller, W. (1986). Einstein metrics with positive scalar curvature. In Curvature and topology of Riemannian manifolds. Springer.

Warner, F. (1965). The conjugate locus of a Riemannian manifold. American j. of math., 87, 575–604.

Wei, G. (1988). Examples of complete manifolds of positive Ricci curvature with nilpotent isometry groups. Bulletin of the AMS, 19, 313–313.

Wei, G. (1998). Ricci curvature and Betti numbers. J. of geometric analysis.

Weil, A. (1926). Sur les surfaces à courbure négative. C. R. Acad. sc. Paris, 182, 1069–1071.
Weil, A. (1958). Variétés kählériennes. Hermann.
Weinstein, A. (1967). On the homotopy type of positively pinched manifolds. Archiv der math., 18, 523–524.
Weinstein, A. (1970). Sur la non densité des géodésiques fermées. C. R. Acad. Sciences Paris, 271, 504.
Weinstein, T. (1996). An introduction ot Lorentz surfaces. Walter de Gruyter.
Weiss, M. (1993). Pinching and cooncordance theory. J. of diffential geometry, 38, 387–416.
Weitzenböck, R. (1923). Invariantentheorie. Groningen: Noordhoff.
Wells, R. (1980). Differential analysis on complex manifolds. Springer.
Wente, H. (1986). Counterexample to a conjecture of H. Hopf. Pacific j. of math., 121, 193–243.
Weyl, H. (1911). Über die asymptotische Verteilung der Eigenwerte. Nach. d. Königl. Ges. d. Wiss. zu Göttingen, 110–117.
Whitehead, H. (1935). On the covering of a complete space by geodesics. Annals of math., 36, 679–704.
Whitney, H. (1936). Differentiable manifolds. Annals of math., 37, 645–680.
Wilhelm, F. (1996). The radius rigidity theorem for manifolds of positive curvature. J. of differential geometry, 44, 634–665.
Wilhelm, F. (1997). On intermediate Ricci curvature and fundamental groups. Illinois J. of math., 41, 1–7.
Wilking, B. (1997). Dissertation, Münster, Germany.
Wilking, B. (1998). The fundamental groups of complete manifolds of nonnegative curvature.
Willmore, T. (1993). Riemannian geometry. Clarendon press Oxford.
Wirtinger, W. (1936). Eine Determinantenidentität und ihre Anwendung auf analytische Gebilde und Hermitesche Massbestimmung. Monatshefte für Math. und Physik, 44, 343–365.
Wolf, J. (1972). Spaces of constant curvature (second ed.). Berkeley: J. Wolf.
Wu, H. (1987). The Bochner technique in differential geometry. Math. reports, London.
Wu, J.-Y. (1990). Hausdorff convergence and sphere theorem. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54, part 3 (pp. 685–692). Los Angeles: American math. soc.
Yamabe, H. (1960). On a deformation of Riemannian structures on compact manifolds. Osaka math. J., 12, 21–37.
Yamaguchi, T. (1988). Homotopy type finiteness theorems for certain precompact families of Riemannian manifolds. Proc. American math. soc., 102, 660–666.
Yamaguchi, T. (1991). Collapsing and pinching under a lower curvature bound. Annals of math., 133, 317–357.
Yamaguchi, T. (1992). A convergence theorem in the geometry of Alexandrov spaces. In A.-L. Besse (Ed.), Table ronde de géométrie différentielle, (pp. 601–642). Luminy, Séminaires et congrès 1, Société math. France.
Yang, C. (1980). Fibre bundles and the physics of the magnetic monopoles. In W. Hsiang (Eds.), The Chern symposium Berkeley: Springer.
Yang, D. (1992). Convergence of Riemannian manifolds with integral bounds on curvature I and II. Ann. Scientifiques E N S, 25, 75–105 and 179–199.
Yau, S.-T. (1975). Harmonic functions on complete Riemannian manifolds. Comm. pure and appl. math., 28, 201–228.
Yau, S.-T. (1977). On Calabi's conjecture and some new results in algebraic geometry. Proceed. nat. acad. sciences USA, 74, 1798–1799.
Yau, S.-T. (1978). On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I. Communic. pure and applied math., XXXI, 339–411.
Yau, S.-T. (1982). Problem section. In S.-T. Yau (Eds.), Seminar on differential geometry (pp. 669–706). Annals of math. studies 102, Princeton Univ. Press.
Yau, S.-T. (1987). Nonlinear analysis in geometry. L'enseignement math., 33, 109–158.
Yau, S.-T. (1990). Open problems in geometry. In R. Greene & S.-T. Yau (Ed.), Differential geometry, 54, part 3 (pp. 1–28). Los Angeles: American math. soc. 1993.
Yau, S.-T. & Zheng, F. (1991). Negatively (1/4)-pinched Riemannian metric on a compact Kähler manifold. Inventiones math., 103, 527–535.

Yomdin, Y. (1987). Volume growth and entropy. Israël J. of Math., 57, 285–300.
Yu, G. (1997). Zero-in-the-spectrum conjecture, positive scalar curvature and asymptotic dimension. Inventiones math., 127, 96–126.
Zamfirescu, T. (1996). Conjugate points and closed geodesic arcs on convex surfaces, Geom. dedicata, 62, 99–105.
Zelditch, S. (1987). Uniform distribution of eigenfunctions on compact hyperbolic surfaces. Duke math. J., 55, 919–941.
Zelditch, S. (1992). Selberg trace formula and equidistribution theorems. Memoirs of the AMS, 96, vol. 465, Providence: American math. Soc.
Ziller, W. (1977). The free loop space of globally symmetric spaces. Inventiones math., 41, 1–22.
Ziller, W. (1982). Geometry of the Katok example. Ergodic theory and dyn. systems, 3, 135–157.
Zoll, O. (1903). Über Flächen mit Scharen geschlossener geodätischer Linien. Math. Annalen, 57, 108–133.

Marcel Berger
IHES
35, Rte de Chartres
F-91440 Bures-Sur-Yvette
France
e-mail: berger@ihes.fr

*(Eingegangen 4. 8. 1997;
in revidierter Form 23. 3. 1998)*

Hermann Weyl
Die Idee der Riemannschen Fläche

Herausgegeben von
Prof. Dr. **Reinhold Remmert**
Münster

Mit Beiträgen von
Prof. Dr. **Reinhold Remmert**
Münster
Prof. Dr. **Michael Schneider**
Bayreuth
Prof. Dr. **Stefan Hildebrandt**
Bonn
Prof. Dr. **Klaus Hulek**
Hannover
Prof. Dr. **Samuel Patterson**
Göttingen

1997. XIV, X, 240 Seiten
mit 7 Bildern.
16,2 x 23,5 cm.
Geb. DM 44,–
ÖS 321,– / SFr 40,–
ISBN 3-8154-2096-2

(TEUBNER-ARCHIV zur Mathematik, Suppl. 5)

Hermann Weyls im Verlag B.G. Teubner in Leipzig erschienenes Jugendwerk »Die Idee der Riemannschen Fläche« war bahnbrechend für die Entwicklung der Mathematik des 20. Jahrhunderts. Es ist aus Weyls Vorlesung vom Wintersemester 1911/12 an der Universität Göttingen hervorgegangen.

Diese von Reinhold Remmert herausgegebene Edition des mathematischen Kultbuches aus dem Jahre 1913 wird komplettiert durch aktuelle Essays führender Wissenschaftler und durch biographische Materialien.

Der Herausgeber schreibt in seinem Proömium: »Weyl war 1912 seiner Zeit voraus… . An die Stelle der erfindungsreichen schöpferischen Phantasie eines Riemann und Klein, die ein verheißenes Land suchte, tritt der systematische Anbau auf sicher erworbenem Boden. Das Werk mit seinem Reichtum an Gedanken trägt den Keim zukünftiger Entwicklung in sich; es bereitete den Weg zur Theorie der topologischen, differenzierbaren und komplexen Mannigfaltigkeiten.«

Preisänderungen vorbehalten.

B. G. Teubner Stuttgart · Leipzig
Postfach 80 10 69 · 70510 Stuttgart