Generation of the 2-Torsion Part of the Brauer Group of a Local Quintic by Quaternion Algebras

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1 Introduction

Let X be a smooth projective connected variety over field k of characteristic zero, and $K = k(X)$ be its function field. It is known (see e.g. [6]) that there is a natural inclusion of $Br\,X$ in $Br(K)$ and this inclusion identifies $Br\,X$ with unramified Brauer group $Br_{nr}(K|k)$. Below we shall write $Br\,X$ instead of $Br_{nr}(K|k)$ keeping in mind this identification.

The group $Br\,X$ is of great importance for many problems. It is enough to mention the problems of rationality of varieties, the problem of Brauer-Manin obstructions to various Hasse principles and, finally, the classical problem of the description of finite-dimensional division algebras over function fields. (see [6],[10], [11].)

In this paper we are interested in the problem of description of $Br\,X$ for a smooth geometrically connected projective curves over local fields, more precisely, for finite extensions k of $\boldsymbol{Q}_p$. The classical class field theory gives us a complete description of Brauer group $Br(k)$ of k (and also of finite dimensional central division algebras over k). But if $k(X)$ is a function field of smooth geometrically connected projective variety over k, the situation is much more complicated. In spite of many results obtained in connection with the developing of multidimensional class field theory ([8],[9]), unfortunately, we do not have a complete description as in the classical situation even in case of curves X (a "complete" description means here a presentation of each element of $Br(k(X))$ by the corresponding central division algebra over $k(X)$, in other words, it means a description of all central division algebras over $k(X)$). Traditions of local-to-global principle prescribe us before the investigation of a global situation to describe all local ones. In case of curves X over k this requires to investigate the groups $Br(k_v(X_v))$ for all v, where k_v is a completion of k at absolute value v on k and $X_v = X \times_k k_v$. By Tsen's theorem $Br(k_v(X_v))$ is trivial in case of complex v and its structure is well-known for real v according to results in [1], [2], [3] and [7].

Thus, in this situation the case of non-Archimedean v is the main unknown one. In the following, by a local field we shall understand a finite extension of the field of p-adic numbers $\boldsymbol{Q}_p$. Let now X be a smooth geometrically connected projective curve over a local field k. Then it follows from [4] that there exists the exact sequence

$$0 \longrightarrow Br\,X \longrightarrow Br(k(X)) \xrightarrow{\phi} \coprod_{P\in X} \chi(G_P) \xrightarrow{\psi} \chi(G) \longrightarrow 0.$$

Here G denotes the Galois group of the separable closure of k, $G_P = Gal((k_P)_s|k_P)$, k_P is the residue field at $P \in X$ and $\chi(G_P)$, $\chi(G)$ are the groups of continuous characters of G_P and G respectively (for definitions of ϕ and ψ see [4]). In view of this sequence the group $Im\,\phi$ is known, so the group $Br\,X$ is of main interest for the description of $Br(k(X))$. The structure of $Br\,X$ is well-known in case of curves of genus zero, but even in case of elliptic curves the complete structure of $Br\,X$ is unknown. More generally, one can consider a smooth connected projective model X of an affine curve defined by the equation $y^2 = f(x)$, where $f(x)$ is a polynomial with coefficients in k without multiple roots and one can ask for the structure of $Br\,X$. One of the aims of our paper is to answer some questions related to this. Since $Br\,X$ is an abelian periodic group, the main problem of its description is to find the m-torsion part of it as an abstract abelian group for any m. Among the groups ${}_mBr\,X$ the group ${}_2Br\,X$ is of particular importance, because its elements are represented by central division algebras with involutions, and this group is, according to the result of Merkurjev [12], related to the theory of quadratic forms by the isomorphism ${}_2Br(k(x)) \cong I^2/I^3$, where I is the fundamental ideal of Witt ring of $k(X)$. Another important problem is to find an explicit presentation of all elements of ${}_m Br\, X$ (or at least to find such presentation for generators of ${}_m Br\, X$) by central division algebras.

Of course, solutions of these two problems strongly depend on the curve X. Both problems have trivial solutions in case of curves of genus zero. For genus one curves one has more complicated situation. If X is an elliptic curves both these problems were solved completely in [13] for non-dyadic fields k. There are also a few preliminary results in case of dyadic fields and in case of principal homogeneous spaces which will be published elsewhere. For hyperelliptic curves only the case of non-dyadic local curves with good reduction is considered (see [14]).

Our main object of consideration is the group ${}_2 Br\, X$ for X being a smooth projective model of the affine curve defined by the equation $y^2 = f(x)$, where $f(x)$ is a polynomial of the fifth degree with coefficients in k without multiple roots. The presentation of generators of ${}_2 Br\, X$ by quaternion algebras depends on the irreducible factors of $f(x)$ over k. We will consider all possible decomposition cases of $f(x)$ step by step. The main result is the complete list of quaternion algebras representing generators of the group ${}_2 Br\, X$.

2 Splitting type (1,1,1,1,1).

Preliminary results. Let k be a local field of characteristic 0 with residue field $\bar{k}$. Let $\pi \in k$ denote a prime element, and let O_k, O_k^*, πO_k be its valuation ring, the group of units and the maximal ideal of the valuation ring. For $u \in O_k^*$ let $\bar{u}$ denote its class *modulo* πO_k. We will consider also a hyperelliptic curve C defined over k by the equation $y^2 = f(x)$, where $f(x) \in O_k[x]$ is a monic polynomial without multiple roots and $deg\, f = 5$. Let us assume also that $f(x) = \prod_{i=1}^m f_i(x)$, $f_i(x) \in O_k[x]$ are monic irreducible polynomials. In what follows let α be a fixed unit of k which is not a square in k, $\bar{g}(x)$ will denote the reduction of a polynomial $g(x) \in O_k[x]$. For $a, b \in k^*$ we write $a \sim b$ if a and b belong to the same class *modulo* $(k^*)^2$.

In our further considerations we will need the following lemmas.

Lemma 1

$$|\, {}_2 Br\, C| = 2^m,$$

particularly, in this section

$$|\, {}_2 Br\, C| = 32.$$

For the proof see theorem 1 and §4 in [14].

Lemma 2 *Let $g(x) \in k[x]$ be a monic divisor of $f(x)$, $g(x) \notin k$ and let either n be odd or $deg\, g(x)$ be even. Then for any $a \in k^*$ the quaternion algebra*

$$\left(\frac{a\ ,\ g(x)}{k(C)}\right)$$

is unramified and not isomorphic to the scalar algebra

$$\left(\frac{\pi\ ,\ \alpha}{k(C)}\right).$$

The proof is similar to the proof of lemma 7 in [14].

Lemma 3 *Let $g(x) \in O_k[x]$ be a monic divisor of $f(x)$. If $\bar{g} \notin \bar{k}[x]^2$ and $\overline{f(x)/g(x)} \notin \bar{k}[x]^2$, then the quaternion algebra*

$$\left(\frac{\pi\ ,\ g(x)}{k(C)}\right)$$

is non-trivial.

For the proof see [14], proposition 4.

Lemma 4 *Let $g(x) \in O_k[x]$ be a monic divisor of $f(x)$. If the quaternion algebra*

$$\bar{A} = \left(\frac{\bar{\alpha}\ ,\ \bar{g}(x)}{\bar{k}(x, \sqrt{\bar{f}(x)})} \right)$$

is non-trivial one, then the algebra

$$A = \left(\frac{\alpha\ ,\ g(x)}{k(C)} \right)$$

is also non-trivial.

Proof. If $A \sim 1$, then $k(x)(\sqrt{f(x)})$ is a splitting field for the algebra

$$A_0 = \left(\frac{\alpha\ ,\ g(x)}{k(x)} \right),$$

therefore this field is isomorphic to a maximal subfield of A_0. Then

$$f = \varepsilon_1^2\alpha + \varepsilon_2^2 g - \varepsilon_3^2 \alpha g,\ \varepsilon_i \in k(x).$$

If $\varepsilon_i = p_i/q_i,\ p_i \in k[x],\ q_i \in k[x]$ let $s_i = cont_\pi p_i - cont_\pi q_i$. We have

$$f = \pi^{2s_1}\frac{{p'_1}^2}{{q'_1}^2}\alpha + \pi^{2s_2}\frac{{p'_2}^2}{{q'_2}^2}g - \pi^{2s_3}\frac{{p'_3}^2}{{q'_3}^2}\alpha g,$$

where p'_i and q'_i are primitive polynomials. If h is the greatest common divisor of $q_1,\ q_2,\ q_3$ and h is primitive polynomial, then

$$h^2 f = \pi^{2s_1}\mu_1^2\alpha + \pi^{2s_2}\mu_2^2 g - \pi^{2s_3}\mu_3^2\alpha g \quad (*)$$

and all the polynomials μ_i, h are primitive. In particular, $\mu_i, h \in O_k[x]$. By comparing the π-contents of both sides of the previous equality we obtain that all s_i cannot be positive. If all of them are equal to 0, then we have

$$\bar{f} = \bar{\varepsilon}_1^2\bar{\alpha} + \bar{\varepsilon}_2^2\bar{g} - \bar{\varepsilon}_3^2\bar{\alpha}\bar{g}$$

and $\bar{A} \sim 1$. If $s_{i_0} = \min_{1 \le i \le 3}\{s_i\} < 0$, then after comparing the π-contents of both sides of (*) we obtain that there exists $j_0 \neq i_0$ such that $s_{i_0} = s_{j_0}$. If $s_1 = s_2 < s_3$, then after taking the reduction $\bar{\mu}_1^2\bar{\alpha} + \bar{\mu}_2^2\bar{g} = 0$. If $s_1 = s_3 < s_2$, then $\bar{\mu}_1^2\bar{\alpha} - \bar{\mu}_3^2\bar{\alpha}\bar{g} = 0$. If $s_2 = s_3 < s_1$, then $\bar{\mu}_2^2\bar{g} - \bar{\mu}_3^2\bar{\alpha}\bar{g} = 0$. Finally, let $s_1 = s_2 = s_3$, so $\bar{\mu}_1^2\bar{\alpha} + \bar{\mu}_2^2\bar{g} - \bar{\mu}_3^2\bar{\alpha}\bar{g} = 0$. In all these cases we have that there exists $\theta \in \bar{A}$ such that $Nrd(\theta) = 0$ and this is a contradiction with $\bar{A} \not\sim 1$.

Lemma 5 *Let V be a variety defined over a finite field $\boldsymbol{F}_q$. Then for any integral positive number N there exists an integral positive n_0 such that for any $n > n_0$ the subset $V(\boldsymbol{F}_{q^n})$ of all $\boldsymbol{F}_{q^n}$-rational points of V consists of more then N elements.*

Proof is straightforward in view of the Lang-Weil theorem.

Lemma 6 *Let K be an algebraically closed field and $f(x), g(x) \in K[x]$. Then the system*

$$\begin{cases} y^2 = f(x) \\ z^2 = g(x) \end{cases}$$

defines a variety V in $\boldsymbol{P}^3(K)$ if and only if $f(x) \notin K[x]^2$, $g(x) \notin K[x]^2$ and $f(x)g(x) \notin K[x]^2$.

Proof. It is sufficient to prove that the K-algebra $K[V] = K[x,y,z]/(y^2 - f(x), z^2 - g(x))$ has no zero divisors. Let us consider the 4-dimensional commutative algebra A over the field $K(x)$ with the basis $1, i, j, k$ and with the multiplication rule $i^2 = f, j^2 = g, ij = ji = k$. The associativity of the multiplication of the basis elements is evident. Let us consider the map

$$\phi : K[V] \longrightarrow A,\ \ \phi([p(x,y,z)]) = p(x,i,j),\ \ p \in K[x,y,z].$$

Because of $\phi((y^2-f(x), z^2-g(x)))=0$, ϕ is correctly defined. Furthermore, ϕ is a ring homomorphism. Let us show that ϕ is an embedding. Indeed, let $p\in K[x,y,z]$. We have $p(x,y,z)=p_0(x)+p_1(x)y+p_2(x)z+p_3(x)yz+q(x,y,z)(y^2-f(x))+r(x,y,z)(z^2-g(x))$. Then $\phi([p])=p_0+p_1i+p_2j+p_3k$. If $\phi([p])=0$, then $p_0=p_1=p_2=p_3=0$ and $p\in(y^2-f(x), z^2-g(x))$, so $[p]=0$. Since $K[V]\in A$ it is enough to prove that A has no zero divisors. Let $I=\{a\in A, ab=0 \text{ for some } b\in A, b\neq 0\}$ and $a\in I, a\neq 0$. If $a=a_0+a_1i+a_2j+a_3k$, then

$$a(\alpha+\beta k)=(\alpha a_0+\beta a_3fg)+(\alpha a_1+\beta a_2 g)i+(\alpha a_2+\beta a_1 f)j+(\alpha a_3+\beta a_0)k\in I.$$

$$ai=a_1f+a_0i+a_3fj+a_2k\in I.$$

$$aj=a_2g+a_3gi+a_0j+a_1k\in I.$$

$$ak=a_3fg+a_2gi+a_1fj+a_0k\in I.$$

Since $a\neq 0$, then we have that among of a_l there exists a non-zero coefficient. Then after replacement of a by ai, aj, or ak if it is necessary we can assume that $a_0\neq 0$. If $a_3\neq 0$ let us replace a by $a'=a(\alpha+\beta k)$, where $\beta=-\alpha a_3/a_0$. After the replacement we have $a'_3=0$. If $a'_0=0$, then $a_0^2=a_3^2fg$. Hence $fg\in K[x]^2$ and we have a contradiction. Thus $a'_0\neq 0$ and one can assume from the very beginning that $a=1+a_1i+a_2j$. If $ab=0, b\in I$, then in view of the previous arguments we can suppose that $b=1+x_1i+x_2j$. Therefore,

$$(1+a_1i+a_2j)(1+x_1i+x_2j)=(1+a_1x_1f+a_2x_2g)+(a_1+x_1)i+(a_2+x_2)j+(a_2x_1+a_1x_2)k=0,$$

so

$$\begin{cases} x_1=-a_1,\\ x_2=-a_2,\\ a_1a_2=0,\\ a_1^2f+a_2^2g=1.\end{cases}$$

If $a_1=0$, then $g\in K[x]^2$ and if $a_2=0$, then $f\in K[x]^2$. The lemma is proved.

The reduction theorem. From now on we will assume that $f(x)\in k[x]$ completely splits over k. Let $k(C)$ be the function field of C. If f is of odd degree, we may assume without loss of generality that $f(x)\in O_k[x]$ and $f(x)$ is monic. We also may assume that $f(x)=x(x-\pi^\varepsilon u)g(x)$, where $0\leq\varepsilon\leq 1$, $u\in O_k^*$, and $g(x)\in O_k[x]$.

Proof: By a linear transformation we obtain that $f(x)$ is divisible by x, hence

$$f(x)=x(x-\pi^l u)\prod_{i=1}^{n-2}(x-\pi^{k_i}u_i),$$

where $0\leq l\leq k_1\leq k_n$ and $u,u_1,\ldots,u_n\in O_k^*$. Let $l=2m+\varepsilon$ with $\varepsilon\in\{0,1\}$ and $x=\pi^{2m}x'$. Then

$$y^2=\pi^{10m}x'(x'-\pi^\varepsilon u)\prod_{i=1}^{n-2}(x'-\pi^{k_i-2m}u_i),$$

and the transformation $y=y'\pi^{5m}$ will lead us to the equation of the desired type.

Let now f be of degree 5. The aim of this section is to prove that by means of some special transformation of variables we may restrict our consideration to some special forms of $f(x)$. More precisely, we prove the

Theorem 1 *Let $f(x)$ be as above, such that, in addition $\bar f(x)$ has multiple roots over $\bar k$ (bad reduction). Then $f(x)$ can be transformed by an appropriate transformation $x\mapsto x+m$, where $m\in\pi O_k$ into one of the following forms.*

I. $f(x)=x(x-\pi u)(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)$,

where $u,u_i\in O_k^*$, $1\leq k_1\leq k_2\leq k_3$. *If* $k_i=1$, *then* $\bar u_i\neq\bar u$.

If $k_i=k_j$, $i\neq j$, *then* $\bar u_i\neq\bar u_j$.

I'_*. $f(x)=x(x-\pi u)(x-\pi^{k_1}u_1)(x-\pi^{k_1}(u_1+\pi^s v))(x-\pi^{k_3}u_3)$,
where $1<k_1<k_3$, $u,v,u_1,u_3\in O_k^*$ *and* $s>0$.

I'_{**}. $f(x)=x(x-\pi u)(x-\pi(u+\pi^s v))(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)$,
where $1\le k_2\le k_3$, $1<k_3$, $u,v,u_2,u_3\in O_k^*$.
If $k_2=1$, *then* $\bar u\ne\bar u_2$, *if* $k_2=k_3$, *then* $\bar u_2\ne\bar u_3$, *and* $s>0$.

II. $f(x)=x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)(x+e)$,
where $1\le k_1\le k_2\le k_3$, $u_i,e\in O_k^*$, *and if* $k_i=k_j$, $i\ne j$, *then* $\bar u_i\ne\bar u_j$.

II'. $f(x)=x(x-\pi^{k_1}u_1)(x-\pi^{k_1}(u_1+\pi^s v))(x-\pi^{k_3}u_3)(x+e)$,
where $1\le k_1<k_3$, $u_i,v\in O_k^*$, $s>0$.

III. $f(x)=x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x+e)(x+e-\pi^l v)$,
where $1\le k_1\le k_2$, $u_i,e\in O_k^*$, $l>0$, *and if* $k_1=k_2$, *then* $\bar u_1\ne\bar u_2$.

IV. $f(x)=x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x+e_1)(x+e_2)$,
where $1\le k_1\le k_2$, $u_i,e_i\in O_k^*$, $\bar e_1\ne\bar e_2$, *and if* $k_1=k_2$, *then* $\bar u_1\ne\bar u_2$.

V. $f(x)=x(x-\pi^{k_1}v_1)(x+u)(x+u-\pi^{k_2}v_2)(x+e)$,
where $k_1,k_2\ge 1$, $u,e,v_i\in O_k^*$, $\bar u\ne\bar e$.

VI. $f(x)=x(x-\pi^k u)(x+e_1)(x+e_2)(x+e_3)$,
where $k\ge 1$, $u,e_i\in O_k^*$, $\bar e_i\ne\bar e_j$ *if* $i\ne j$.

The proof of the theorem is based on the following lemmas.

Lemma 7 *Let* $\bar f(x)=x^5$. *Then by an appropriate transformation* $x\mapsto x+n, n\in\pi O_k$ *the polynomial* $f(x)$ *can be transformed to one of the forms* I,I'_*,I'_{**}.

Proof. Assume that $f(x)$ cannot be transformed to the form I. Then by an appropriate transformation it can be transformed to a polynomial of the form

$$x(x-\pi u)(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)\qquad(**)$$

where $u,u_i\in O_k^*$, $0<k_1\le k_2\le k_3>1$, all elements $\bar u_i$ such that $k_i=k_3$ are pairwise unequal and in case $k_1=k_2=1$ $\bar u_1\ne\bar u_2$. Indeed, by above considerations

$$f(x)=x(x-\pi v)(x-\pi^{k_1}v_1)(x-\pi^{k_2}v_2)(x-\pi^{k_3}v_3),$$

where $v,v_i\in O_k^*$ and $0<k_1\le k_2\le k_3$. If $k_3=1$, then the change of variables $x\mapsto x+\pi v_3$ transforms $f(x)$ to the polynomial

$$x(x-\pi(v-v_3))(x-\pi(v_1-v_3))(x-\pi(v_2-v_3))(x-\pi(-v_3)).$$

By assumption this polynomial is not of the form I, so at least one of the elements $\overline{v-v_3}$, $\overline{v_1-v_3}$, $\overline{v_2-v_3}$ is zero and this means that for the transformed polynomial after renumeration the exponent k_3 is greater than 1. Thus we may assume from the beginning that $k_3>1$.

Let now $k_2=k_3$. Then the transformation $x\mapsto x+\pi^{k_3}v_3$ leads us to the polynomial

$$x(x-\pi(v-\pi^{k_3-1}v_3))(x-\pi^{k_1}(v_1-\pi^{k_3-k_1}v_3))(x-\pi^{k_3}(-v_3))(x-\pi^{k_3}(v_2-v_3)).$$

If $k_1\ne k_3$, then this polynomial is of the form $(**)$. If $k_1=k_3$, then we have the polynomial

$$x(x-\pi(v-\pi^{k_3-1}v_3))(x-\pi^{k_3}(v_1-v_3))(x-\pi^{k_3}(-v_3))(x-\pi^{k_3}(v_2-v_3)).$$

If $\bar v_1-\bar v_3\ne\bar v_2-\bar v_3$, then this polynomial is also of the form $(**)$. If $\bar v_1-\bar v_3=\bar v_2-\bar v_3=0$ then we have either the above case $k_1\ne k_2=k_3$ or case $k_1<k_2<k_3$ and in the last case our polynomial is

again of the form $(**)$. Thus $\bar{v}_1 - \bar{v}_3 = \bar{v}_2 - \bar{v}_3 \neq 0$. In this case it is enough to use the transformation $x \mapsto x + \pi^{k_3}(v_1 - v_3)$. Hence in all possible cases $f(x)$ can be transformed to a polynomial of the form $(**)$. Thus we have: in case $k_1 = k_2 = 1$ $f(x)$ can be transformed to

$$x(x-\pi u)(x-\pi(u+\pi^s v))(x-\pi u_2)(x-\pi^{k_3}u_3), \tag{1}$$

$s > 0,\ k_3 > 1,\ u, u_i, v \in O_k^*,\ \bar{u} \neq \bar{u}_2$.

In case $k_2 > 1,\ k_1 = 1$ $f(x)$ can be transformed to

$$x(x-\pi u)(x-\pi(u+\pi^s v))(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3), \tag{2}$$

(if $k_2 = k_3,\ \bar{u}_2 \neq \bar{u}_3,\ s > 0$).

The remaining case is

$$x(x-\pi u)(x-\pi^{k_1}u_1)(x-\pi^{k_1}(u_1+\pi^s v))(x-\pi^{k_3}u_3), \tag{3}$$

where $1 < k_1 < k_3,\ s > 0$. Combining together cases 1 and 2 we will obtain case of the form I'_{**} and case 3 is the case of the form I'_{*}.

Lemma 8 *$\bar{f}(x) = x^4(x+\bar{e})$, where $\bar{e} \neq 0$. Then by an appropriate transformation $x \mapsto x + m$, $m \in \pi O_k$ $f(x)$ can be transformed to one of forms II, II'.*

Proof. Let

$$f(x) = x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)(x+e),$$

where $u_i, e \in O_k^*$ and $0 < k_1 \leq k_2 \leq k_3$.

Firstly, let $k_3 = 1$ and all $\bar{u}_i$ be pairwise unequal. Then $f(x)$ is of the form II. Otherwise there exist $i \neq j$ such that $u_i \neq u_j$ but $\bar{u}_i = \bar{u}_j$. Then the transformation $x \mapsto x + \pi u_i$ leads us to the case where $k_3 > 1$. Let now $k_3 > 1$. Then there are the following possibilities for k_1, k_2, k_3.

(i) $k_1 = k_2 = 1$,

(ii) $1 = k_1 < k_2$,

(iii) $1 < k_1 = k_2 = k_3$,

(iv) $1 < k_1 < k_2$,

(v) $1 < k_1 = k_2 < k_3$.

Let us consider all possibilities step by step.

In case i) if $\bar{u}_1 \neq \bar{u}_2$, then $f(x)$ is of the form II. Otherwise after the transformation $x \mapsto x + \pi u_1$ we will have a polynomial of the form II'.

In case ii) if $k_2 = k_3$ and $\bar{u}_2 = \bar{u}_3$, then the transformation $x \mapsto x + \pi^{k_3}u_3$ leads us to a polynomial of the form II. In case ii) if $k_2 \neq k_3$ or $k_2 = k_3$ but $\bar{u}_2 \neq \bar{u}_3$, then $f(x)$ is of the form II.

In case iii) if u_i $(i = 1, 2, 3)$ are pairwise unequal, then $f(x)$ is of the form II. Otherwise, without loss of generality let us suppose $\bar{u}_1 = \bar{u}_3$. Then the transformation $x \mapsto x + \pi^{k_3}u_3$ leads us to the polynomial

$$x(x-\pi^{k_3}(-u_3))(x-\pi^{k_3}(u_1-u_3))(x-\pi^{k_3}(u_2-u_3))(x+e+\pi^{k_3}u_3).$$

This polynomial either is of the form II or can be transformed to one of cases iv), v).

In case iv) if either $k_2 \neq k_3$ or $k_2 = k_3$ but $\bar{u}_2 \neq \bar{u}_3$ $f(x)$ is of the form II. Otherwise, the transformation $x \mapsto x + \pi^{k_3}u_3$ leads us to case iv) where $k_2 \neq k_3$.

In case v) it is clear that $f(x)$ is either of the form II or II'.

Lemma 9 *By means of an appropriate transformation $x \mapsto x + m$, $m \in \pi O_k$ $f(x)$ can be transformed to a polynomial of the form:*

III. $x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x+e)(x+e-\pi^l v)$,

where $u_i, e \in O_k^*$, $l > 0$, $1 \le k_1 \le k_2$

and if $k_1 = k_2$, *then* $\bar{u}_1 \neq \bar{u}_2$

(case $\bar{f}(x) = x^3(x+\bar{e})^2$, $\bar{e} \neq 0$*).*

IV. $x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x+e_1)(x+e_2)$,

where $u_i, e_i \in O_k^*$, $\bar{e}_1 \neq \bar{e}_2$, $1 \le k_1 \le k_2$ *and if* $k_1 = k_2$, *then* $\bar{u}_1 \neq \bar{u}_2$

(case $\bar{f}(x) = x^3(x+\bar{e}_1)(x+\bar{e}_2)$, $\bar{e}_i \neq 0$, $\bar{e}_1 \neq \bar{e}_2$*).*

V. $x(x-\pi^{k_1}v_1)(x+u)(x+u-\pi^{k_2}v_2)(x+e)$,

where $u, e, v_i \in O_k^*$, $k_1, k_2 \ge 1$, $\bar{u} \neq \bar{e}$

(case $\bar{f}(x) = x^2(x+\bar{u})^2(x+\bar{e})$, $\bar{e}\bar{u} \neq 0$, $\bar{u} \neq \bar{e}$*).*

VI. $x(x-\pi^k u)(x+e_1)(x+e_2)(x+e_3)$,

where $u, e_i \in O_k^*$, $\bar{e}_i \neq \bar{e}_j$ *if* $i \neq j$

(case $\bar{f}(x) = x^2(x+\bar{e}_1)(x+\bar{e}_2)(x+\bar{e}_3)$, $\bar{e}_1\bar{e}_2\bar{e}_3 \neq 0$, $\bar{e}_i \neq \bar{e}_j$ *if* $i \neq j$*).*

Proof. Let $\bar{f}(x) = x^3(x+\bar{e})^2$. If either $k_1 \neq k_2$ or $k_1 = k_2$ but $\bar{u}_1 \neq \bar{u}_2$, then $f(x)$ is of the form III. Otherwise the transformation $x \mapsto x + \pi^{k_2}u_2$ leads us to case $k_1 \neq k_2$, so we have transformed $f(x)$ to the form III.

Let $\bar{f}(x) = x^3(x+\bar{e}_1)(x+\bar{e}_2)$. Then similarly to the previous case either $f(x)$ is of the form III or one may use the transformation $x \mapsto x + \pi^{k_2}u_2$.

The two last splitting cases of $\bar{f}(x)$ are obvious.

Now our theorem follows from lemmas 7–9 because all possible cases of splitting of $\bar{f}(x)$ are considered.

It follows from the theorem that in the case of bad reduction of C given by equation $y^2 = f(x)$ to describe quaternion generation of ${}_2Br\,C$ we may restrict our attention to $f(x)$ given by forms I–VI, I'_*, I'_{**}, II'. For the further considerations we will need the following list of subcases.

The cases of consideration.

I. $f(x) = x(x-\pi u)(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)$,

where $u, u_i \in O_k^*$, $1 \le k_1 \le k_2 \le k_3$. If $k_i = 1$, then $\bar{u}_i \neq \bar{u}$.

If $k_i = k_j$, $i \neq j$, then $\bar{u}_i \neq \bar{u}_j$.

*I*1. $k_1 \equiv k_2(2)$.

*I*2. *I*2$'$. $k_1 \not\equiv k_2(2)$, $k_1 \equiv k_3(2)$.

*I*2$''$. $k_1 \not\equiv k_2(2)$, $k_1 \not\equiv k_3(2)$, $k_2 \neq k_3$, $-uu_1u_2 \sim 1$.

*I*3. $k_1 \not\equiv k_2(2)$, $k_1 \not\equiv k_3(2)$ and either $k_2 = k_3$ or $-uu_1u_2 \not\sim 1$.

I'_*. $f(x) = x(x-\pi u)(x-\pi^{k_1}u_1)(x-\pi^{k_1}(u_1+\pi^s v))(x-\pi^{k_3}u_3)$,

where $u, v, u_i \in O_k^*$, $1 < k_1 < k_3$ and $s > 0$.

I'_{**}. $f(x) = x(x-\pi u)(x-\pi(u+\pi^s v))(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)$,

where $u, v, u_i \in O_k^*$, $1 \le k_2 \le k_3$, $1 < k_3$ and $s > 0$.

If $k_2 = 1$, then $\bar{u} \neq \bar{u}_2$, and if $k_2 = k_3$, then $\bar{u}_2 \neq \bar{u}_3$.

$I'_{**}1$. $k_2 \equiv 1(2)$.

$I'_{**}2$. $k_2 \equiv 0(2)$ and either $k_3 \equiv 1(2)$ or ($k_3 \equiv 0(2)$ but $k_2 \neq k_3$ and $-u_2 \sim 1$).

$I'_{**}3$. $k_2 \equiv k_3 \equiv 0(2)$ and either $k_2 = k_3$ or $-u_2 \not\sim 1$.

II. $f(x) = x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)(x+e)$,

where $u_i, e \in O_k^*$, $1 \le k_1 \le k_2 \le k_3$, if $k_i = k_j$, $i \neq j$, then $\bar{u}_i \neq \bar{u}_j$.

$II1.$ $e \sim 1$.

$II1,1.$ $k_1 \not\equiv k_2(2)$.

$II1,2.$ Either $k_1 \equiv k_2(2)$, $k_1 \not\equiv k_3(2)$ or $k_1 \equiv k_2 \equiv k_3(2)$ but $k_2 \neq k_3$ and $u_1u_2 \sim 1$.

$II1,3.$ $k_1 \equiv k_2 \equiv k_3(2)$ and either $k_2 = k_3$ or $u_1u_2 \not\sim 1$.

$II2.$ $e \not\sim 1$.

$II2,1.$ $k_1 \not\equiv k_2(2)$.

$II2,2.$ $k_1 \equiv k_2(2)$, $k_1 \not\equiv k_3(2)$ or $k_1 \equiv k_2 \equiv k_3(2)$ but $k_2 \neq k_3$ and $u_1u_2 \not\sim 1$.

$II2,3.$ $k_1 \equiv k_2 \equiv k_3(2)$ and either $k_2 = k_3$ or $u_1u_2 \sim 1$.

$II'.$ $f(x) = x(x - \pi^{k_1}u_1)(x - \pi^{k_1}(u_1 + \pi^s v))(x - \pi^{k_3}u_3)(x + e)$,
where $u_i, e, v \in O_k^*$, $1 \leq k_1 < k_3$, $s > 0$.

$II'1.$ $e \sim 1$.

$II'2,1.$ $e \not\sim 1$, $k_1 \not\equiv k_3(2)$.

$II'2,2.$ $e \not\sim 1$, $k_1 \equiv k_3(2)$.

$III.$ $f(x) = x(x - \pi^{k_1}u_1)(x - \pi^{k_2}u_2)(x + e)(x + e - \pi^l v)$,
where $u_i, e, v \in O_k^*$, $1 \leq k_1 \leq k_2$, $l > 0$ and if $k_1 = k_2$, then $\bar{u}_1 \neq \bar{u}_2$.

$III1.$ $-e \not\sim 1$, $k_1 \not\equiv 0(2)$.

$III2.$ $-e \sim 1$, $k_1 \not\equiv 0(2)$.

$III3.$ $-e \not\sim 1$ and either $(k_1 \equiv 0(2),\ k_2 \not\equiv 0(2))$ or
$(k_1 \equiv k_2 \equiv 0(2)$ and $k_1 < k_2,\ -u_1 \sim 1)$.

$III4.$ $-e \sim 1$ and either $(k_1 \equiv 0(2),\ k_2 \not\equiv 0(2))$ or
$(k_1 \equiv k_2 \equiv 0(2)$ and $k_1 < k_2,\ -u_1 \sim 1)$.

$III5.$ $-e \not\sim 1$, $k_1 \equiv k_2 \equiv 0(2)$ and either $k_1 = k_2$ or $-u_1 \not\sim 1$.

$III6.$ $-e \sim 1$, $k_1 \equiv k_2 \equiv 0(2)$ and either $k_1 = k_2$ or $-u_1 \not\sim 1$.

$IV.$ $f(x) = x(x - \pi^{k_1}u_1)(x - \pi^{k_2}u_2)(x + e_1)(x + e_2)$,
where $u_i, e_i \in O_k^*$, $\bar{e}_1 \neq \bar{e}_2$, $1 \leq k_1 \leq k_2$, and if $k_1 = k_2$, then $\bar{u}_1 \neq \bar{u}_2$.

$IV1.$ $k_1 \equiv k_2 \equiv 0(2)$ and either $k_1 = k_2$ or $-e_1e_2u_1 \not\sim 1$.

$IV2.$ The other cases.

$V.$ $f(x) = x(x - \pi^{k_1}u_1)(x + u)(x + u - \pi^{k_2}u_2)(x + e)$,
where $u, e, v_i \in O_k^*$, $\bar{u} \neq \bar{e}$, $k_1, k_2 > 0$.

$V1.$ $e \not\sim 1$, $e - u \not\sim 1$.

$V2.$ $e \not\sim 1$, $e - u \sim 1$.

$V3.$ $e \sim 1$, $e - u \not\sim 1$.

$V4.$ $e \sim 1$, $e - u \sim 1$.

$VI.$ $f(x) = x(x - \pi^k u)(x + e_1)(x + e_2)(x + e_3)$,
where $u, e_i \in O_k^*$, $\bar{e}_i \neq \bar{e}_j$ if $i \neq j$.

$VI1.$ $e_1e_2e_3 \not\sim 1$.

$VI2.$ $e_1e_2e_3 \sim 1$.

Presentation by quaternion algebras.

Theorem 2 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of form I. Then the class of (π, α) and the classes of the following quaternion algebras generate ${}_2Br\,C$.*

I1. $\{(\alpha, l_i)\}$, *where l_i is a monic linear divisor of $f(x)$.*

I2. (α, x), $(\alpha, x - \pi u)$, $(\alpha, x - \pi^{k_1} u_1)$, (π, x).

I3. (α, x), $(\alpha, x - \pi u)$, $(\pi, x - \pi^{k_3} u_3)$, (π, x).

Proof. By lemma 1 the order of ${}_2Br\,C$ is 32, so to prove the theorem it is enough to prove that in each case $I1, I2, I3$ the classes of the corresponding algebras generate the subgroup H of ${}_2Br\,C$ of order 16 such that $[(\pi, \alpha)] \notin H$.

Observe firstly that in order to prove that some algebra is non-trivial it is enough to prove that some of its completions is non-trivial. We have

$$(\alpha, x)_{x-\pi u} \sim (\alpha, \pi u) \not\sim 1,$$

$$(\alpha, x - \pi^{k_i} u_i)_{x-\pi u} \sim (\alpha, \pi u - \pi^{k_i} u_i) \sim (\alpha, \pi) \not\sim 1 \; (i = 1, 2, 3),$$

$$(\alpha, x - \pi u)_x \sim (\alpha, \pi u) \not\sim 1.$$

Let now $\{i, j, t, m, n\} = \{1, 2, 3, 4, 5\}$. Then $(\alpha, l_i l_j l_t l_m l_n) \sim 1$, so that $(\alpha, l_i l_j l_t l_m) \sim (\alpha, l_n) \not\sim 1$ and $(\alpha, l_i l_j l_t) \sim (\alpha, l_m l_n)$. It follows from the above consideration that to prove that the order of the group H is 16 it is enough to prove that all the algebras $(\alpha, l_i l_j)$ are non-trivial. If $l_i = x - \pi u$, then

$$(\alpha, (x - \pi u) l_j)_{x-\pi u} \sim (\alpha, l_t l_m l_n)_{x-\pi u} \sim (\alpha, \pi^3) \not\sim 1.$$

Let us consider the remaining cases, where $l_i, l_j \neq x - \pi u$.

$$(\alpha, x(x - \pi^{k_1} u_1))_{x-\pi^{k_1} u_1} \sim (\alpha, (x - \pi u)(x - \pi^{k_2} u_2)(x - \pi^{k_3} u_3))_{x-\pi^{k_1} u_1} \sim (\alpha, \pi^{2k_1+1}) \not\sim 1.$$

$$(\alpha, x(x - \pi^{k_3} u_3))_{x-\pi^{k_3} u_3} \sim (\alpha, \pi^{k_1+k_2+1}) \sim (\alpha, \pi) \not\sim 1.$$

$$(\alpha, (x - \pi^{k_1} u_1)(x - \pi^{k_2} u_2))_{x-\pi^{k_1} u_1} \sim (\alpha, \pi^{k_1} u_1 (\pi^{k_1} u_1 - \pi u)(\pi^{k_1} u_1 - \pi^{k_3} u_3)) \sim (\alpha, \pi) \not\sim 1.$$

$$(\alpha, (x - \pi^{k_1} u_1)(x - \pi^{k_3} u_3))_{x-\pi^{k_1} u_1} \sim (\alpha, \pi^{k_1} u_1 (\pi^{k_1} u_1 - \pi u)(\pi^{k_1} u_1 - \pi^{k_2} u_2)) \sim (\alpha, \pi) \not\sim 1.$$

$$(\alpha, x(x - \pi^{k_2} u_2))_{x-\pi^{k_2} u_2} \sim (\alpha, \pi^{k_1+k_2+1}) \sim (\alpha, \pi) \not\sim 1.$$

$$(\alpha, (x - \pi^{k_2} u_2)(x - \pi^{k_3} u_3))_{x-\pi^{k_2} u_2} \sim (\alpha, \pi^{k_2} u_2 (\pi^{k_2} u_2 - \pi u)(\pi^{k_2} u_2 - \pi^{k_1} u_1)) \sim (\alpha, \pi) \not\sim 1.$$

To finish the consideration of case $I1$ one needs to observe that by the lemma 2 all the algebras from the list in case $I1$ are unramified and non-isomorphic to (π, α).

In case $I2$ let H be the subgroup of ${}_2Br\,C$ generated by the classes of algebras (α, x), $(\alpha, x - \pi u)$, $(\alpha, x - \pi^{k_1} u_1)$, (π, x). All these algebras are unramified over $k(C)$. Let us prove that H is of order 16. Just by the same way as in the previous case one can prove that the algebras $(\alpha, x - \pi u)$, $(\alpha, x(x - \pi^{k_1} u_1))$, (α, x), $(\alpha, x - \pi^{k_1} u_1)$, $(\alpha, x(x - \pi^{k_1} u_1))$, $(\alpha, x(x - \pi u))$, $(\alpha, (x - \pi u)(x - \pi^{k_1} u_1))$ are non-trivial. To prove that $[(\alpha, x)]$, $[(\alpha, x - \pi u)]$, $[(\alpha, x - \pi^{k_1} u_1)]$ generate a group of order 8 one needs only to prove that $(\alpha, x(x - \pi u)(x - \pi^{k_1} u_1)) \sim (\alpha, (x - \pi^{k_2} u_2)(x - \pi^{k_3} u_3)) \not\sim 1$.

In case $I2'$ we have $(\alpha, (x - \pi^{k_2} u_2)(x - \pi^{k_3} u_3))_{x-\pi^{k_3} u_3} \sim (\alpha, \pi^{k_1+k_3+1}) \not\sim 1$.

In case $I2''$ $k_2 + 1 < k_3$ and $(\alpha, (x - \pi^{k_2} u_2)(x - \pi^{k_3} u_3))_{x-\pi^{k_2+1}} \sim (\alpha, (\pi^{k_2+1} - \pi^{k_2} u_2)(\pi^{k_2+1} - \pi^{k_3} u_3)) \sim (\alpha, \pi) \not\sim 1$ since $f(\pi^{k_2+1}) \sim -\pi^{k_1+k_2+1} u u_1 u_2 \sim -u u_1 u_2 \sim 1$.

To prove that $(\pi, x) \not\sim 1$ let us consider a curve $(-1)^{k_2} \bar{\alpha} \bar{u} \bar{u}_1 x^2 - \bar{u} \bar{u}_1 \bar{u}_2 = y^2$ defined over the residue field $\bar{k}$ of k. This curve has a smooth point $(\tilde{a}, \tilde{b})$ such that $\tilde{a}, \tilde{b} \in \bar{k}(\tilde{a})$, where $\bar{k}(\tilde{a})$ is an extension of $\bar{k}$ of some prime odd degree. Then for the unramified extension N over k with residue field $\bar{k}(\tilde{a})$ there is a lift

$(a,b) \in N^2$ of $(\tilde a, \tilde b)$ to a point of a curve $(-1)^{k_2}\alpha u u_1 x^2 - u u_1 u_2 = y^2$. Set $w = (-1)^{k_2}\alpha a^2$ and $\theta = \pi^{k_2} w$. It is easy to see that $f(\theta) \sim (-1)^{k_2}\alpha u u_1 a^2 - u u_1 u_2 \sim 1$. Then it follows that

$$\left(\frac{\pi\ ,\ x}{k(x, \sqrt{f(x)})}\right)_{x-\theta} \sim \left(\frac{\pi\ ,\ \theta}{N(\sqrt{f(\theta)})\langle x-\theta\rangle}\right) \sim \left(\frac{\pi\ ,\ (-1)^{k_2}w}{N\langle x-\theta\rangle}\right) \sim \left(\frac{\pi\ ,\ \alpha}{N\langle x-\theta\rangle}\right) \not\sim 1.$$

Hence $(\pi, x) \not\sim 1$. Since the algebra

$$\left(\frac{\pi\ ,\ x}{k(\sqrt{\alpha})(x, \sqrt{f(x)})}\right)$$

belongs again to case $I2$ (with constant field $k(\sqrt{\alpha})$, then $k(\sqrt{\alpha})(\sqrt{f(x)})$ does not split the algebra (π, x) but it splits all the algebras (α, x), $(\alpha, x-\pi u)$, $(\alpha, x-\pi^{k_1}u_1)$, so it follows that elements $[(\alpha, x)]$, $[(\alpha, x-\pi u)]$, $[(\alpha, x-\pi^{k_1}u_1)]$ and $[(\pi, x)]$ generate a group H of order 16. In view of lemma 2 $[(\pi,\alpha)] \notin H$. This finishes the consideration of case $I2$.

In case $I3$ one can prove similar to case $I2$ that $[(\alpha, x)]$, $[(\alpha, x-\pi u)]$ generate a group of order 4. As to the non-triviality of (π, x) one can prove this just in the same way as in the previous case if $k_2 < k_3$. If $k_2 = k_3$ consider the curve over $\bar k$ defined by the equation

$$(-1)^{k_2}\bar\alpha \bar u \bar u_1 \left(x^2 - \frac{(-1)^{k_2}\bar u_2}{\bar\alpha}\right)\left(x^2 - \frac{(-1)^{k_2}\bar u_3}{\bar\alpha}\right) = y^2.$$

Then there exists an extension $\bar k(\tilde a)|\bar k$ of odd prime degree with a smooth point $(\tilde a, \tilde b)$ such that $\tilde b \in \bar k(\tilde a)$. Let $(a,b) \in N^2$ be a lifted point on the curve defined by the equation

$$(-1)^{k_2}\alpha u u_1 \left(x^2 - \frac{(-1)^{k_2}u_2}{\alpha}\right)\left(x^2 - \frac{(-1)^{k_2}u_3}{\alpha}\right) = y^2,$$

where $N|k$ is the unramified extension with residue field $\bar k(\tilde a)$, $w = (-1)^{k_2}\alpha a^2$. Set $\theta = \pi^{k_2}w$. Then $f(\theta) \sim w(w-u_2)(w-u_3)u u_1 \sim 1$ and consequently

$$\left(\frac{\pi\ ,\ x}{k(x, \sqrt{f(x)})}\right)_{x-\theta} \sim \left(\frac{\pi\ ,\ \theta}{N(\sqrt{f(\theta)})\langle x-\theta\rangle}\right) \sim \left(\frac{\pi\ ,\ \alpha}{N\langle x-\theta\rangle}\right) \not\sim 1.$$

Thus $(\pi, x) \not\sim 1$.

Let N be an extension of k as above for a curve defined by the equation $u u_1((-1)^{k_2}\alpha x^2 - u_2) = y^2$. Then such a curve has a point (a,b) over N. In case $k_2 < k_3$ for $\theta = \pi^{k_2}(-1)^{k_2}\alpha a^2$ we have $f(\theta) \sim 1$ and

$$\left(\frac{\pi\ ,\ x-\pi^{k_3}u_3}{k(x, \sqrt{f(x)})}\right)_{x-\theta} \sim \left(\frac{\pi\ ,\ \theta}{N(\sqrt{f(\theta)})\langle x-\theta\rangle}\right) \sim \left(\frac{\pi\ ,\ \alpha}{N\langle x-\theta\rangle}\right) \not\sim 1.$$

In case $k_2 = k_3$ one need to consider instead of the latter curve, a curve defined by the equation

$$y^2 = (-1)^{k_2}\alpha u u_1((-1)^{k_2}\alpha x^2 - u_2)((-1)^{k_2}\alpha x^2 - u_3)$$

and proceed by analogy as in case $k_2 < k_3$.

One can prove in a similar way that $(\pi, x(x-\pi^{k_3}u_3)) \not\sim 1$. If $k_2 < k_3$ consider a curve $y^2 = \alpha(x^2 + u_2/(u u_1))(x^2 + (u_2 - u_3)/(u u_1))$ and in case $k_2 = k_3$ a curve $y^2 = \alpha x^2 - u_3$ and proceed by analogy as in the previous cases.

To finish the consideration of case $I3$ let us prove that groups $\langle [(\alpha, x)],\ [(\alpha, x-\pi u)]\rangle$ and $\langle [(\pi, x)],\ [(\pi, x-\pi^{k_3}u_3)]\rangle$ have trivial intersection.

Since the extension of k by $\sqrt{\alpha}$ leaves us either in case $I2$ or in case $I3$, the algebra (π, x) does not split by $k(\sqrt{\alpha})(x, \sqrt{f(x)})$ and consequently $[(\pi, x)] \notin \langle [(\alpha, x)],\ [(\alpha, x-\pi u)]\rangle$. The transformation $x \mapsto x + \pi^{k_3}u_3$ leaves us again either in case $I2$ or in case $I3$, so that $[(\pi, x-\pi^{k_3}u_3)] \notin \langle [(\alpha, x)],\ [(\alpha, x-\pi u)]\rangle$. Let us consider the algebra $(\pi, x(x-\pi^{k_3}u_3))$. If $k_2 = k_3$, then the extension of constants by $\sqrt{\alpha}$ shows us that $[(\pi, x(x-\pi^{k_3}u_3))] \notin \langle [(\alpha,x)],\ [(\alpha, x-\pi u)]\rangle$. Let now $k_2 \neq k_3$. Comparing completions of $(\pi, x(x-\pi^{k_3}u_3))$ and $(\alpha, x-\pi u)$:

$$(\pi, x(x-\pi^{k_3}u_3))_{x-\pi^{k_2}u_2} \sim (\pi, \pi^{k_2}u_2(\pi^{k_2}u_2-\pi^{k_3}u_3)) \sim 1,$$

$$(\alpha, x-\pi u)_{x-\pi^{k_2}u_2} \sim (\alpha, \pi^{k_2}u_2-\pi u) \sim (\alpha,\pi) \not\sim 1,$$

so one can see that $(\pi, x(x-\pi^{k_3}u_3))$ and $(\alpha, x-\pi u)$ are non-isomorphic. Comparing completions of $(\pi, x(x-\pi^{k_3}u_3))$ and (α, x) in $x-\pi u$ one can see that these algebras are also non-isomorphic. And finally we have for $i\in\{1,2\},\ k_i \equiv 0(2)$ that

$$(\alpha, x(x-\pi u))_{x-\pi^{k_i}u_i} \sim (\alpha, \pi^{k_i+1}) \not\sim 1 \text{ and } (\pi, x(x-\pi^{k_3}u_3))_{x-\pi^{k_i}u_i} \sim (\pi, (\pi^{k_i}u_i)^2) \sim 1.$$

Hence $(\alpha, x(x-\pi u)) \not\sim (\pi, x(x-\pi^{k_3}u_3))$. This completes the proof.

In cases I'_* and I'_{**} we have

Theorem 3 *Let C be given by the equation $y^2=f(x)$, where $f(x)$ is of form I'. Then the class of (π,α) and the classes of the following quaternion algebras generate ${}_2Br\,C$.*

I'_*. *$\{(\alpha, l_i)\}$, where l_i is a monic linear divisor of f.*

$I'_{**}1$. *$\{(\alpha, l_i)\}$, where l_i is a monic linear divisor of f.*

$I'_{**}2$. *$(\alpha,x),\ (\alpha,x-\pi u),\ (\alpha, x-\pi u'),\ (\pi,x),\ u'=u+\pi^s v$.*

$I'_{**}3$. *$(\alpha,x),\ (\alpha,x-\pi u),\ (\pi, x-\pi^{k_3}u_3),\ (\pi,x)$.*

Proof. Similarly to case I one can prove that $(\alpha, l_i)\not\sim 1$ for a monic linear divisor l_i of $f(x)$. In case I'_* if $l_i = x-\pi u$ and $i\neq j$ we have similarly to case I $(\alpha, l_il_j)\not\sim 1$. Furthermore,

$$(\alpha, x(x-\pi^{k_1}u_1))_x \sim (\alpha, \pi^{k_1+k_3+1}), (\alpha, x(x-\pi^{k_1}u_1))_{x-\pi^{k_3}u_3} \sim (\alpha, \pi^{k_1+k_3}).$$

One of the algebras $(\alpha,\pi^{k_1+k_3})$ and $(\alpha,\pi^{k_1+k_3+1})$ is non-trivial, therefore $(\alpha, x(x-\pi^{k_1}u_1))\not\sim 1$. Similarly one can prove that $(\alpha, x(x-\pi^{k_1}(u_1+\pi^s v)))\not\sim 1$. We have also

$$(\alpha, x(x-\pi^{k_3}u_3))_x \sim (\alpha, \pi^{2k_1+1}) \not\sim 1$$

and

$$(\alpha, (x-\pi^{k_1}u_1)(x-\pi^{k_1}u_1'))_{x-\pi^{k_1}u_1} \sim (\alpha, x(x-\pi u)(x-\pi^{k_3}u_3))_{x-\pi^{k_1}u_1} \not\sim 1.$$

For the algebra $(\alpha, (x-\pi^{k_1}u_1)(x-\pi^{k_3}u_3))$ we have

$$(\alpha, (x-\pi^{k_1}u_1)(x-\pi^{k_3}u_3))_{x-\pi^{k_1}u_1} \sim (\alpha, x(x-\pi^{k_1}u_1')(x-\pi u))_{x-\pi^{k_1}u_1} \sim (\alpha,\pi^{s+1}),$$

$$(\alpha, (x-\pi^{k_1}u_1)(x-\pi^{k_3}u_3))_{x-\pi^{k_1}u_1'} \sim (\alpha, \pi^{2k_1+s}) \sim (\alpha,\pi^s).$$

One of the algebras $(\alpha,\pi^s),\ (\alpha,\pi^{s+1})$ is non-trivial, so that $(\alpha, (x-\pi^{k_1}u_1)(x-\pi^{k_3}u_3))\not\sim 1$.

In a similar way one proves that $(\alpha, (x-\pi^{k_1}u_1')(x-\pi^{k_3}u_3))\not\sim 1$. Now by analogy to case I we complete the proof.

Let us consider case I'_{**}. We have

$$(\alpha, (x-\pi u)(x-\pi u'))_{x-\pi u} \sim (\alpha, x(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3))_{x-\pi u} \sim (\alpha, \pi^3) \not\sim 1.$$

We have also $(\alpha, x(x-\pi u))\not\sim 1$, since

$$(\alpha, x(x-\pi u))_{x-\pi u} \sim (\alpha, \pi^{s+1}) \text{ and } (\alpha, x(x-\pi u))_{x-\pi u'} \sim (\alpha,\pi^s).$$

Similarly $(\alpha, x(x-\pi u'))\not\sim 1$.

Consider now the algebra $(\alpha, (x-\pi u)(x-\pi^{k_2}u_2))$. We have $(\alpha, (x-\pi u)(x-\pi^{k_2}u_2))\not\sim 1$, since

$$(\alpha, (x-\pi u)(x-\pi^{k_2}u_2))_{x-\pi u} \sim (\alpha,\pi^{s+1}) \text{ and } (\alpha, (x-\pi u)(x-\pi^{k_2}u_2))_{x-\pi u'} \sim (\alpha,\pi^s).$$

Similarly $(\alpha,(x-\pi u')(x-\pi^{k_2}u_2)) \not\sim 1$.

Furthermore, $(\alpha,(x-\pi u)(x-\pi^{k_3}u_3)) \not\sim 1$, since

$$(\alpha,(x-\pi u)(x-\pi^{k_3}u_3))_{x-\pi u} \sim (\alpha,\pi^{s+1}) \text{ and } (\alpha,(x-\pi u)(x-\pi^{k_3}u_3))_{x-\pi u'} \sim (\alpha,\pi^{s}).$$

Similarly $(\alpha,(x-\pi u')(x-\pi^{k_3}u_3)) \not\sim 1$.

We have also $(\alpha,x(x-\pi^{k_2}u_2)) \not\sim 1$, since

$$(\alpha,x(x-\pi^{k_2}u_2))_{x-\pi^{k_2}u_2} \sim (\alpha,(x-\pi u)(x-\pi u')(x-\pi^{k_3}u_3))_{x-\pi^{k_2}u_2} \sim (\alpha,\pi^{k_2})$$

and k_2 is odd.

Then

$$(\alpha,x(x-\pi^{k_3}u_3))_x \sim (\alpha,(x-\pi u)(x-\pi u')(x-\pi^{k_2}u_2))_x \sim (\alpha,\pi^{k_2})$$

and k_2 is odd.

Finally, $(\alpha,(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)) \not\sim 1$, since

$$(\alpha,(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3))_{x-\pi^{k_2}u_2} \sim (\alpha,x(x-\pi u)(x-\pi u'))_{x-\pi^{k_2}u_2} \sim (\alpha,\pi^{k_2}) \not\sim 1.$$

It is clear from the above considerations that similarly to case I one can complete the proof in case I'_{**}.

Let us consider the next case. We can conclude that the group $\langle[(\alpha,x)],\ [(\alpha,x-\pi u)],\ [(\alpha,x-\pi u')]\rangle$ is of order 8 in case $I'_{**}2$ and $\langle[(\alpha,x)],\ [(\alpha,x-\pi u)]\rangle$ is of order 4 in case $I'_{**}3$. The non-triviality of the algebra (π,x) in both cases can be established by considerations similar to the ones in the proof of the theorem 2. As to the algebra $(\pi,x-\pi^{k_3}u_3)$ in case $I'_{**}3$ one can use the replacement $x'=x-\pi^{k_3}u_3$ and reduces this case to the case of the algebra (π,x). The non-triviality of the algebra $(\pi,x(x-\pi^{k_3}u_3))$ in case $I'_{**}3$ can be obtained by using arguments similar to the ones in case I. To complete the proof in case $I'_{**}2$ let us observe that the extension of constants by $\sqrt{\alpha}$ does not split (π,x) but splits any algebra of the form $(\alpha,g(x))$.

To complete the proof of the theorem in case $I'_{**}3$ we need to prove that the groups $\langle[(\alpha,x)],\ [(\alpha,x-\pi u)]\rangle$ and $\langle[(\pi,x)],\ [(\pi,x-\pi^{k_3}u_3)]\rangle$ have a trivial intersection. But in case $I'_{**}3,\ k_2=k_3$ the extension of constants by $\sqrt{\alpha}$ leaves us in case $I'_{**}3$ and this completes the proof in case $k_2=k_3$. If we are in case $I'_{**}3,\ k_2\neq k_3$ then the similar extension of constants leads us to case $I'_{**}2$ so that $[(\pi,x)]\notin\langle[(\alpha,x)],\ [(\alpha,x-\pi u)]\rangle$. After the replacement $x'=x-\pi^{k_3}u_3$ we are left again in the case $I'_{**}3$, then $[(\pi,x-\pi^{k_3}u_3)]\notin\langle[(\alpha,x)],\ [(\alpha,x-\pi u)]\rangle$. The proof of the fact that $[(\pi,x(x-\pi^{k_3}u_3))]\notin\langle[(\alpha,x)],\ [(\alpha,x-\pi u)]\rangle$ is just the same as the corresponding part of the proof of the theorem 2. This completes the proof of the theorem.

In case II we have

Theorem 4 *Let C be given by the equation $y^2=f(x)$, where $f(x)$ is of form II. Then the class of (π,α) and the classes of the following algebras generate ${}_2Br\,C$.*

$II1,1.$ $(\alpha,x),\ (\alpha,x-\pi^{k_1}u_1),\ (\alpha,x-\pi^{k_2}u_2),\ (\pi,x)$.

$II1,2.$ $(\alpha,x),\ (\alpha,x-\pi^{k_1}u_1),\ (\pi,x-\pi^{k_1}u_1),\ (\pi,x)$.

$II1,3.$ $(\alpha,x),\ (\pi,x-\pi^{k_1}u_1),\ (\pi,x-\pi^{k_2}u_2),\ (\pi,x)$.

$II2,1.$ $(\alpha,x),\ (\alpha,x-\pi^{k_3}u_3),\ (\pi,x+e),\ (\pi,x)$.

$II2,2.$ $(\alpha,x),\ (\pi,x-\pi^{k_1}u_1),\ (\pi,x+e),\ (\pi,x)$.

$II2,3.$ $(\pi,x-\pi^{k_1}u_1),\ (\pi,x-\pi^{k_2}u_2),\ (\pi,x-\pi^{k_3}u_3),\ (\pi,x)$.

Proof. Observe first of all that in case II $(\alpha,x+e)\sim 1$ for any $e\in O_k^*$ and $(\pi,x+e)\not\sim 1$ if and only if $e\in O_k^*\setminus(O_k^*)^2$. Indeed, one needs only to prove that for any irreducible polynomial $p(x)\in k[x]$ we have $(\alpha,x+e)_{p(x)}\sim 1$, and for any irreducible $p(x)$ we have $(\pi,x+e)_{p(x)}\sim 1$ if $e\in(O_k^*)^2$ but there exists $p(x)$ such that $(\pi,x+e)_{p(x)}\not\sim 1$ if $e\notin(O_k^*)^2$.

Assume firstly that θ is a root of $p(x)$. Then

$$(\alpha,x+e)_{p(x)} \sim (\alpha,\theta+e) \sim (\alpha,\theta(\theta-\pi^{k_1}u_1)(\theta-\pi^{k_2}u_2)(\theta-\pi^{k_3}u_3))$$

and if $\theta \notin \pi O_k$, then $(\alpha, x+e)_{p(x)} \sim (\alpha, \theta^4) \sim 1$. Let $\theta \in \pi O_k$, then

$$(\alpha, x+e)_{p(x)} \sim (\alpha, \theta + e) \sim (\alpha, e) \sim 1.$$

Assume that $e \in (O_k^*)^2$, then similar arguments show that $(\pi, x+e) \sim 1$. In case $e \notin (O_k^*)^2$ we have $(\pi, x+e)_x \sim (\pi, e) \not\sim 1$.

Consider case $II1$. Let $l_0 = x$, $l_j = x - \pi^{k_j} u_j$, $j = 1,2,3$. Since $\bar{f}(x)/\bar{l}_j(x)$, $\bar{l}_j(x) \notin \bar{k}[x]^2$, $j = 0,1,2,3$, then by lemma 3 $(\pi, l_j) \not\sim 1$. Furthermore,

$$\left(\frac{\bar{\alpha}\ ,\ \bar{l}_i(x)}{\bar{k}(x)(\sqrt{\bar{f}(x)})}\right) = \left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}(x)(\sqrt{x+\bar{e}})}\right).$$

Since in case $II1$ $e \sim 1$ we have

$$\left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}(x)(\sqrt{x+\bar{e}})}\right)_x \sim \left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}\langle x\rangle}\right) \not\sim 1.$$

Hence

$$\left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}(x)(\sqrt{x+\bar{e}})}\right) \not\sim 1$$

and in view of lemma 4 $(\alpha, l_i) \not\sim 1$, $(i = 0,1,2,3)$. Thus all the algebras listed in cases $II1,1$, $II1,2$, $II1,3$ are non-trivial. Since $(\alpha, x+e) \sim 1$, then $(\alpha, l_0 l_1 l_2) \sim (\alpha, l_3) \not\sim 1$ and hence in order to prove that $\langle[(\alpha, l_0)],\ [(\alpha, l_1)],\ [(\alpha, l_2)]\rangle$ is a group of order 8 it is enough to prove that $(\alpha, l_0 l_1) \not\sim 1$, $(\alpha, l_0 l_2) \not\sim 1$, $(\alpha, l_1 l_2) \not\sim 1$. We have

$$(\alpha, l_0 l_1)_{x - \pi^{k_2} u_2} \sim (\alpha, \pi^{k_2} \pi^{k_1}) \sim (\alpha, \pi) \not\sim 1,$$

$$(\alpha, l_0 l_2)_{x - \pi^{k_2} u_2} \sim (\alpha, l_1 l_3)_{x - \pi^{k_2} u_2} \sim (\alpha, \pi^{k_1} \pi^{k_2}) \not\sim 1,\ (\alpha, l_1 l_2)_x \sim (\alpha, \pi^{k_1} \pi^{k_2}) \not\sim 1.$$

It is clear that the extension of constants by $\sqrt{\alpha}$ leaves us in case $II1$ and this implies that $[(\pi, x)] \notin \langle[(\alpha, l_0)],\ [(\alpha, l_1)],\ [(\alpha, l_2)]\rangle$, so that case $II1,1$ is considered.

In case $II1,2$ let us show firstly that $(\alpha, x(x - \pi^{k_1} u_1)) \not\sim 1$ and $(\pi, x(x - \pi^{k_1} u_1)) \not\sim 1$. If $k_2 \not\equiv k_3(2)$ we have

$$(\alpha, x(x - \pi^{k_1} u_1))_x \sim (\alpha, (x - \pi^{k_2} u_2)(x - \pi^{k_3} u_3))_x \sim (\alpha, \pi^{k_2 + k_3}) \not\sim 1.$$

If $k_2 \equiv k_3(2)$, then

$$f(\pi^{k_2+1}) \sim \pi^{k_2+1}(-\pi^{k_1} u_1)(-\pi^{k_2} u_2)\pi^{k_2+1} \sim u_1 u_2 \sim 1$$

and this implies

$$(\alpha, x(x - \pi^{k_1} u_1))_{x - \pi^{k_2+1}} \sim (\alpha, \pi^{k_1 + k_2 + 1}) \not\sim 1,$$

so that $(\alpha, x(x - \pi^{k_1} u_1)) \not\sim 1$.

To prove $(\pi, x(x - \pi^{k_1} u_1)) \not\sim 1$ let us find a polynomial $p(x)$ such that $(\pi, x(x - \pi^{k_1} u_1))_p \not\sim 1$. Let $p(x)$ be a minimal polynomial of element $\theta = \pi^{k_3} w$, where w is a unit from the algebraic closure of k. Then

$$f(\theta) \sim \pi^{k_3} w(-\pi^{k_1} u_1)(-\pi^{k_2} u_2)\pi^{k_3}(w - u_3) \sim u_1 u_2 w(w - u_3).$$

Assume now that $k(w)|k$ is an unramified prime odd degree extension (or its degree is 1). Then

$$(\pi, x(x - \pi^{k_1} u_1))_{p(x)} \sim \left(\frac{\pi\ ,\ (-1)^{k_1 + k_3 + 1} u_1 w}{k(w)\langle x - \theta\rangle \sqrt{u_1 u_2 w(w - u_3)}}\right).$$

And if $u_1 u_2 w(w - u_3) \sim 1$ and $(-1)^{k_1 + k_3 + 1} u_1 w \sim \alpha$ then $(\pi, x(x - \pi^{k_1} u_1))_{p(x)} \not\sim 1$. To find such w let us consider a curve over $\bar{k}$ given by the equation $y^2 = \bar{u}_1 \bar{u}_2 x^2 + (-1)^{k_1 + k_3} \bar{u}_2 \bar{u}_3 \bar{\alpha}$.

There exists a finite odd degree extension of $\bar{k}$, where our curve has a smooth point $(\tilde{a}, \tilde{b})$ which can be lifted to a point (a, b) on the curve $y^2 = u_1 u_2 x^2 + (-1)^{k_1 + k_3} u_2 u_3 \alpha$. And we will be done if put $w = (-1)^{k_1 + k_3 + 1} u_1 \alpha a^2$. It follows immediately from the above considerations that $\langle[(\alpha, x)],\ [(\alpha, x - \pi^{k_1} u_1)]\rangle$ and $\langle[(\pi, x)],\ [(\pi, x - \pi^{k_1} u_1)]\rangle$ are groups of order 4. Since the extension of constants by $\sqrt{\alpha}$ leaves us in case $II1,2$ these two groups have the trivial intersection and this completes the consideration of case $II1,2$.

In case $II1,3$ in order to prove that $\langle[(\pi,x)],\ [(\pi,x-\pi^{k_1}u_1)]\ [(\pi,x-\pi^{k_2}u_2)]\rangle$ is a group of order 8 in view of $(\pi,x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2))\sim(\pi,x-\pi^{k_3}u_3)\not\sim 1$ it is enough to establish that the algebras $(\pi,x(x-\pi^{k_1}u_1))$, $(\pi,x(x-\pi^{k_2}u_2))$, $(\pi,(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2))$ are non-trivial. If $k_2<k_3$, then the non-triviality of the first algebra can be checked similar to the previous case. Let now $k_2=k_3$ and assume that w is a unit from the algebraic closure of k with the above mentioned properties, $\theta=\pi^{k_3}w$, and $p(x)$ is the minimal polynomial of θ. Then we have $f(\theta)\sim w(\pi^{k_3-k_1}w-u_1)(w-u_2)(w-u_3)$ and

$$(\pi,x(x-\pi^{k_1}u_1))_{p(x)}\sim\left(\frac{\pi\ ,\ w(w\pi^{k_3-k_1}-u_1)}{k(w)\langle x-\theta\rangle(\sqrt{w(\pi^{k_3-k_1}w-u_1)(w-u_2)(w-u_3)})}\right)$$

and if

$$\begin{cases} w(\pi^{k_3-k_1}w-u_1)\sim\alpha\\ (w-u_2)(w-u_3)\sim\alpha,\end{cases}$$

then $(\pi,x(x-\pi^{k_1}u_1))_{p(x)}\not\sim 1$.

Assume that $k_1<k_3$, then one can take $w=-\alpha u_1a^2$, where (a,b) is a point on a curve $y^2=\alpha(x^2+u_2/(\alpha u_1))(x^2+u_3/(\alpha u_1))$.

If $k_1=k_2=k_3$, then we have a system

$$\begin{cases} w(w-u_1)\sim\alpha\\ (w-u_2)(w-u_3)\sim\alpha\end{cases}$$

The existence of w in this case can be established by inspection of points on the curve defined by the following system in view of lemmas 5 and 6.

$$\begin{cases} y^2=\bar{\alpha}x(x-\bar{u}_1)\\ z^2=\bar{\alpha}(x-\bar{u}_2)(w-\bar{u}_3).\end{cases}$$

If $k_2\neq k_3$, then the non-triviality of the algebra $(\pi,x(x-\pi^{k_2}u_2))$ can be checked by analogy as in the previous cases.

Let $k_2=k_3$ and $k_1<k_2$. If as above $\theta=w\pi^{k_3}$, then we have $f(\theta)\sim\pi^{k_3}w(-\pi^{k_1}u_1)\pi^{k_3}(w-u_2)\pi^{k_3}(w-u_3)\sim-wu_1(w-u_2)(w-u_3)$ and for minimal polynomial $p(x)$ of θ over k

$$(\pi,x(x-\pi^{k_2}u_2))_{p(x)}\sim\left(\frac{\pi\ ,\ w(w-u_2)}{k(w)\langle p\rangle(\sqrt{f(\theta)})}\right).$$

The non-triviality of the algebra $(\pi,x(x-\pi^{k_2}u_2))_{p(x)}$ will be follow from the existence of solution of the system

$$\begin{cases} w(w-u_2)\sim\alpha\\ -u_1(w-u_3)\sim\alpha\end{cases}$$

and $k(w)|k$ is of odd prime degree or its degree is 1. The existence of such w as usual can be deduced by the inspection of points of the reduced curve.

Let us show that $(\pi,(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2))\not\sim 1$. If $k_2\neq k_3$, then

$$(\pi,(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2))_x\sim(\pi,\pi^{k_1+k_2}u_1u_2)\sim(\pi,u_1u_2)\not\sim 1.$$

Let $k_2=k_3$ and $k_1<k_3$ (if $k_1=k_2$, then one can prove everything as in one of the previous cases since $(\pi,(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2))\sim(\pi,x(x-\pi^{k_3}u_3))$).

By analogy with the above considerations one can find a unit w such that $k(w)|k$ is of odd prime degree and such that

$$\begin{cases} w(w-u_3)\sim\alpha\\ -u_1(w-u_2)\sim\alpha.\end{cases}$$

Then for the minimal polynomial $p(x)$ of $\theta=\pi^{k_3}w$ we have $(\pi,(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2))_{p(x)}\not\sim 1$, so that $(\pi,(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2))$ is also non-trivial.

Thus $\langle[(\pi,x)],\ [(\pi,x-\pi^{k_1}u_1)],\ [(\pi,x-\pi^{k_2}u_2)]\rangle$ is a group of order 8. Since the extension of constants by $\sqrt{\pi}$ leaves us in case $II1,3$, we conclude that $[(\alpha,x)]$ does not belong to this group and the proof of case $II1$ is finished.

In case $II2$ the algebras (π,x), $(\pi,x-\pi^{k_i}u_i)$, $(\pi,x(x+e))$ and $(\pi,(x+e)(x-\pi^{k_i}u_i))$ are non-trivial by the lemma 3.

Let us consider case $II2,1$. There exists $i\in\{1,2\}$ such that k_i is odd. Then we have

$$(\alpha,x)_{x-\pi^{k_i}u_i}\not\sim 1,\ \ (\alpha,x-\pi^{k_3}u_3)_{x-\pi^{k_i}u_i}\not\sim 1,$$

$$(\alpha,x(x-\pi^{k_3}u_3))_x\sim(\alpha,(x+e)(x-\pi^{k_1}u_1)(x-\pi^{k_3}u_3))_x\sim(\pi,\pi^{k_1+k_2})\not\sim 1,$$

so that $\langle[(\alpha,x)],\ \ [(\alpha,x-\pi^{k_3}u_3)]\rangle$ is of order 4. From the above considerations it follows that $\langle[(\pi,x)],\ [(\pi,x+e)]\rangle$ is also of order 4. After the extension of constants to $k(\sqrt{\alpha})$ we have case $II1,1$, so that

$$(\pi,x)\otimes k(\sqrt{\alpha})(x,\sqrt{f(x)})\not\sim 1 \text{ and}$$

$$(\pi,x(x+e))\otimes k(\sqrt{\alpha})(x,\sqrt{f(x)})\sim(\pi,x)\otimes k(\sqrt{\alpha})(x,\sqrt{f(x)})\not\sim 1$$

and therefore $[(\pi,x)],\ [(\pi,x(x+e))]\notin\langle[(\alpha,x)],\ [(\alpha,x-\pi^{k_3}u_3)]\rangle$. To complete the inspection of case $II2,1$ we need only to prove that $[(\pi,x+e)]\notin\langle[(\alpha,x)],\ [(\alpha,x-\pi^{k_3}u_3)]\rangle$.

Let k_j is even, $j\in\{1,2\}$. Then

$$(\pi,x+e)_{x-\pi^{k_j}u_j}\sim(\pi,e)\not\sim 1,\ (\alpha,x)_{x-\pi^{k_j}u_j}\sim(\alpha,\pi^{k_j}u_j)\sim 1,$$

$$(\alpha,x-\pi^{k_3}u_3)_{x-\pi^{k_j}u_j}\sim 1,\ (\alpha,x(x-\pi^{k_3}u_3))_{x-\pi^{k_j}u_j}\sim 1,$$

and case $II2,1$ is considered.

In case $II2,1$ in order to see that the group $\langle[(\pi,x-\pi^{k_1}u_1)],\ [(\pi,x+e)],\ [(\pi,x)]\rangle$ has order 8 it is enough to prove that $(\pi,x(x-\pi^{k_1}u_1)(x+e))\not\sim 1$ and $(\pi,x(x-\pi^{k_1}u_1))\not\sim 1$. This follows from the above considerations. But the extension of constants by $\sqrt{\alpha}$ leads us to case $II1,2$ and we saw that in this case $(\pi,x+e)\sim 1$, $(\pi,x(x-\pi^{k_1}u_1))\not\sim 1$ and therefore $(\pi,x(x-\pi^{k_1}u_1)(x+e))\not\sim 1$ over $k(C)$.

Let us prove now that $(\alpha,x)\not\sim 1$. If $k_2\not\equiv k_3(2)$ and k_i $(i\in\{2,3\})$ is odd, then $(\alpha,x)_{x-\pi^{k_i}u_i}\sim(\alpha,\pi^{k_i})\not\sim 1$ therefore without loss of generality we can assume k_1,k_2,k_3 to be even $(k_2<k_3)$. Then

$$f(\pi^{k_2+1})\sim\pi^{k_2+1}(-\pi^{k_1}u_1)(-\pi^{k_2}u_2)\pi^{k_2+1}e\sim u_1u_2e\sim 1 \text{ and}$$

$$(\alpha,x)_{x-\pi^{k_2+1}}\sim\left(\frac{\alpha\ ,\ \pi^{k_2+1}}{k\langle x-\pi^{k_2+1}\rangle(\sqrt{f(\pi^{k_2+1})})}\right)\sim\left(\frac{\alpha\ ,\ \pi}{k\langle x-\pi^{k_2+1}\rangle}\right)\not\sim 1.$$

To complete the consideration of case $II2,2$ we need to prove that $[(\alpha,x)]\notin\langle[(\pi,x-\pi^{k_1}u_1)],\ [(\pi,x+e)],\ [(\pi,x)]\rangle$.

First of all $(\alpha,x)\not\sim(\pi,x)$ because the extension of constants by $\sqrt{\alpha}$ leads us to case $II1,2$, where $(\pi,x)\not\sim 1$. Furthermore, the same extension of constants shows that $(\alpha,x)\not\sim(\pi,x(x+e))$ since in case $II1,2$ we have $(\pi,x+e)\sim 1$. By the same way we have

$$(\alpha,x)\not\sim(\pi,x-\pi^{k_1}u_1),\ (\alpha,x)\not\sim(\pi,(x-\pi^{k_1}u_1)(x+e)),$$

$$(\alpha,x)\not\sim(\pi,x(x-\pi^{k_1}u_1)) \text{ and } (\alpha,x)\not\sim(\pi,x(x+e)(x-\pi^{k_1}u_1)).$$

Consider the algebra $(\pi,x+e)$. If among of k_1,k_2,k_3 there exists an even k_i, then $(\alpha,x)_{x-\pi^{k_i}u_i}\sim 1$, but $(\pi,x+e)_{x-\pi^{k_i}u_i}\sim(\pi,e)\not\sim 1$. Let now all k_i be odd. Then in view of $k_2+1<k_3$ we have $f(\pi^{k_2+1})\sim u_1u_2e\sim 1$ and $(\alpha,x)_{x-\pi^{k_2+1}}\sim(\alpha,\pi^{k_2+1})\sim 1$, $(\pi,x+e)_{x-\pi^{k_2+1}}\sim(\pi,e)\not\sim 1$ and again $(\alpha,x)\not\sim(\pi,x+e)$. This completes the consideration of case $II2,2$.

Consider case $II2,3$. Let $l_0=x$, $l_j=x-\pi^{k_j}u_j$ $(j=1,2,3)$. Let us observe that all algebras listed in $II2,3$ are non-trivial because of lemma 3, so to prove that the group generated by them is of order 16 it is enough to check that all of them are pairwise non-isomorphic.

First of all it is true for $(\pi, l_i(x+e))$ because of lemma 3. For the algebra $(\pi, l_0 l_3)$ if $k_2 \neq k_3$ we have $(\pi, l_0 l_3)_x \sim (\pi, u_1 u_2 e) \not\sim 1$. If $k_2 = k_3,\ k_1 < k_2$ one can find a unit w such that $k(w)|k$ is of odd prime degree and

$$\begin{cases} w(w-u_3) \sim \alpha \\ -u_1 e(w-u_2) \sim \alpha. \end{cases}$$

Then for the minimal polynomial $p(x)$ of $\pi^{k_3} w$ over k one has $(\pi, l_0 l_3)_{p(x)} \not\sim 1$. In case $k_1 = k_2 = k_3$ one can use a unit w such that

$$\begin{cases} w(w-u_3) \sim \alpha \\ (w-u_1)(w-u_2) \sim \alpha. \end{cases}$$

Then again $(\pi, l_0 l_3)_{p(x)} \not\sim 1$. Similar arguments prove that $(\pi, l_0 l_1)$ and $(\pi, l_0 l_2)$ are non-trivial in case $k_2 < k_3$. Indeed, in first case one can use a unit w such that

$$\begin{cases} w \sim -u_1 \alpha \\ w - u_3 \sim -u_2 \end{cases}$$

and in the second one such a unit that

$$\begin{cases} w \sim -u_2 \alpha \\ w - u_3 \sim -u_1. \end{cases}$$

Then in both cases for the minimal polynomial $p(x)$ of $\pi^{k_3} w$ over k one has $(\pi, l_0 l_1)_{p(x)} \not\sim 1$ and $(\pi, l_0 l_2)_{p(x)} \not\sim 1$. Let now $k_2 = k_3$. Then in the case of the algebra $(\pi, l_0 l_1)$ we need to use a unit w,

$$\begin{cases} w \sim -\alpha u_1 \\ (w-u_2)(w-u_3) \sim 1, \end{cases}$$

if $k_1 < k_2$ and

$$\begin{cases} w(w-u_1) \sim \alpha \\ (w-u_2)(w-u_3) \sim 1, \end{cases}$$

if $k_1 = k_2 = k_3$. In the case of algebra $(\pi, l_0 l_2)$ one can work similarly.

Consider algebras $(\pi, l_1 l_3),\ (\pi, l_2 l_3),\ (\pi, l_1 l_2)$. After replacing $x' = x - \pi^{k_3} u_3$ we are again in case $II2,3$ and with new notations algebras $(\pi, l_1 l_3),\ (\pi, l_2 l_3)$ look like algebras $(\pi, l_0' l_1'),\ (\pi, l_0' l_2')$, where $l_0' = x'$ and $l_i' = l_i + \pi^{k_3} u_3$. But the last algebras are non-trivial in view of the above arguments. In case $k_2 = k_3$ for the algebra $(\pi, l_1 l_2)$ one can renumber l_1, l_2, l_3 as follows: $l_2' = l_3,\ l_3' = l_2,\ l_1' = l_1$, so that $(\pi, l_1 l_2) = (\pi, l_1' l_3')$ and the last algebra is non-trivial. Finally, let $k_1 < k_3$. Then

$$(\pi, l_1 l_2)_{x - \pi^{k_1} u_1} \sim (\pi, l_0 l_3 (x+e))_{x - \pi^{k_1} u_1} \sim (\pi, e) \not\sim 1.$$

The theorem is proved.

If $f(x)$ is of the form II' we have the following statement.

Theorem 5 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of form II'. Then the class of (π, α) and the classes of the following algebras generate ${}_2Br\, C$.*

$II'1$. $(\alpha, x),\ (\alpha, x - \pi^{k_1} u_1),\ (\pi, x - \pi^{k_1} u_1),\ (\pi, x)$.

$II'2,1$. $(\alpha, x),\ (\pi, x - \pi^{k_1} u_1),\ (\pi, x),\ (\pi, x+e)$.

$II'2,2$. $(\pi, x - \pi^{k_1} u_1),\ (\pi, x - \pi^{k_1} u_1'),\ (\pi, x+e),\ (\pi, x),\ u_1' = u_1 + \pi^s v$.

Proof. As in the proof of theorem 4 one can easy prove that $(\alpha, x+e) \not\sim 1$ for any $e \in O_k^*$ and $(\pi, x+e) \not\sim 1$ if and only if $e \in O_k^* \setminus (O_k^*)^2$. Similarly to this proof one can check that the algebras $(\pi, x),\ (\pi, x - \pi^{k_1} u_1),\ (\pi, x - \pi^{k_1} u_1'),\ (\pi, x - \pi^{k_3} u_3)$ are non-trivial.

If $e \sim 1$, then one has as in the proof of theorem 4 that the algebras $(\alpha, x - \pi^{k_1} u_1),\ (\alpha, x)$ are non-trivial. Hence all algebras (except (α, x) in case $II'2,1$) listed in case II' are non-trivial.

Consider case $II'1$. First of all observe that $(\alpha, l_0 l_1) \not\sim 1$. Indeed, if $k_1 \not\equiv k_3 (2)$, then

$$(\alpha, l_0 l_1)_x \sim (\alpha, (x - \pi^{k_1} u_1') l_3 (x+e))_x \sim (\alpha, \pi^{k_1+k_3}) \not\sim 1.$$

If $k_1 \equiv k_3(2)$, then since $f(\pi^{k_1+1}) \sim e \sim 1$ we have

$$(\alpha, l_0 l_1)_{x-\pi^{k_1+1}} \sim \left(\frac{\alpha\ ,\ \pi}{k\langle x-\pi^{k_1+1}\rangle}\right) \not\sim 1.$$

As to the algebra $(\pi, l_0 l_1)$ one can find a unit w generating an odd degree extension over k such that

$$\begin{cases} w \sim (-)^s \alpha u_1 \\ w - v \sim (-1)^s \alpha u_1. \end{cases}$$

Let $\theta = \pi^{k_1}(u_1 + \pi^s w)$, $p = Irr_{k(w)|k}(\theta)$. Then

$$f(\theta) \sim \pi^{k_1} u_1 (\pi^{k_1} u_1 + \pi^{k_1+s} w - \pi^{k_1} u_1)(\pi^{k_1} u_1 + \pi^{k_1+s} w - \pi^{k_1} u_1 - \pi^{k_1+s} v) \sim w(w-v) \sim 1$$

and

$$(\pi, l_0 l_1)_p \sim (\pi, \pi^s w u_1) \sim (\pi, (-1)^s u_1 w) \sim (\pi, \alpha) \not\sim 1.$$

Thus groups $\langle[(\alpha, l_0)],\ [(\alpha, l_1)]\rangle$ and $\langle[(\pi, l_0)],\ [(\pi, l_1)]\rangle$ are of order 4. And since the extension of constants by $\sqrt{\alpha}$ leaves us in case $II'1$ we have that these two groups have trivial intersection, so case $II'1$ is considered.

Now we are in case $II'2$. The extension of k by $\sqrt{\alpha}$ leads us to case $II'1$ and splits the algebra $(\pi, x+e)$, hence the group $\langle[(\pi, l_0)],\ [(\pi, l_1)],\ [(\pi, x+e)]\rangle$ is of order 8.

Now if $i \in \{1, 3\}$ and k_i is odd, then $(\alpha, x)_{x-\pi^{k_i} u_i} \sim (\alpha, \pi^{k_i}) \not\sim 1$ and therefore $(\alpha, x) \not\sim 1$. To complete the consideration of case $II'2, 1$ we need only to prove that $[(\alpha, x)] \notin \langle[(\pi, l_0)],\ [(\pi, l_1)],\ [(\pi, x+e)]\rangle$. The extension of constants by $\sqrt{\alpha}$ shows us that $[(\alpha, x)] \notin \langle[(\pi, l_0)],\ [(\pi, l_1)]\rangle$ and $(\alpha, x) \not\sim (\pi, l_0(x+e))$, $(\alpha, x) \not\sim (\pi, l_1(x+e))$ and $(\alpha, x) \not\sim (\pi, l_0 l_1(x+e))$. Furthermore, for even $k_i \in \{k_2, k_3\}$ $(\alpha, x)_{l_i} \sim 1$ and $(\pi, x+e)_{l_i} \not\sim 1$, so $(\alpha, x) \not\sim (\pi, x+e)$. Case $II'2, 1$ is done.

Consider case $II'2.2$. In this case it is enough to prove that all listed algebras with the algebra $(\pi, x - \pi^{k_3} u_3)$ are pairwise non-isomorphic. Furthermore, $(\pi, l_i(x+e)) \not\sim 1$ and $(\pi, l_1'(x+e)) \not\sim 1$ as was noted before. We also have

$$(\pi, l_0 l_3)_x \sim (\pi, l_1 l_1'(x+e))_x \sim (\pi, e) \not\sim 1,$$

so that $(\pi, l_0 l_3) \not\sim 1$. One can also prove similarly to the previous case that $(\pi, l_0 l_1) \not\sim 1$ and $(\pi, l_0 l_1') \not\sim 1$. To prove that $(\pi, l_1 l_3) \not\sim 1$ (similarly $(\pi, l_1' l_3) \not\sim 1$) it is enough to observe that after replacement $x' = x - \pi^{k_3} u_3$ our algebra looks like $(\pi, \tilde{l}_0 \tilde{l}_1)$, where $\tilde{l}_0 = x'$ and $\tilde{l}_1 = l_1 + \pi^{k_3} u_3$. Finally,

$$(\pi, l_1 l_1')_{l_1} \sim (\pi, l_0 l_3(x+e))_{l_1} \sim (\pi, e) \not\sim 1.$$

The theorem is proved.

In case where $f(x)$ is of the form III we have

Theorem 6 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of form III. Then the class of (π, α) and the classes of the following algebras generate ${}_2Br\, C$.*

$III1$. (α, x), $(\alpha, x - \pi^{k_2} u_2)$, $(\pi, x+e)$, $(\pi, x+e-\pi^l v)$.

$III2$. (α, x), $(\alpha, x - \pi^{k_2} u_2)$, $(\pi, x+e)$, $(\alpha, x+e)$.

$III3$. (α, x), (π, x), $(\pi, x+e)$, $(\pi, x+e-\pi^l v)$.

$III4$. (α, x), (π, x), $(\pi, x+e)$, $(\alpha, x+e)$.

$III5$. (π, x), $(\pi, x - \pi^{k_2} u_2)$, $(\pi, x+e)$, $(\pi, x+e-\pi^l v)$.

$III6$. (π, x), $(\pi, x - \pi^{k_2} u_2)$, $(\pi, x+e)$, $(\alpha, x+e)$.

Proof. Let $G_1 = \langle [(\pi, x+e)],\ [(\pi, x+e')]\rangle$, where $e' = e - \pi^l v$. Let us prove that in cases $III1, III3, III5$ G_1 is of order 4. Indeed, in these cases $(\pi, x+e) \not\sim 1$ and $(\pi, x+e') \not\sim 1$ according to lemma 3. Furthermore,

$$(\pi, (x+e)(x+e'))_{x+e} \sim (\pi, x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2))_{x+e} \sim (\pi, -e) \not\sim 1.$$

Now let $G_2 = \langle [(\pi, x+e)],\ [(\alpha, x+e)]\rangle$, then in cases $III2, III4, III6$ G_2 is also of order 4. Indeed, $(\alpha, x+e) \not\sim 1$ by lemma 4 and $(\pi, x+e) \not\sim 1$ by lemma 3. And $(\alpha, x+e) \not\sim (\pi, x+e)$ since $k(\sqrt{\alpha})(x, \sqrt{f(x)})$ does not split $(\pi, x+e)$.

In cases $III1, III2$ let $G_3 = \langle [(\alpha, x)],\ [(\alpha, x-\pi^{k_2}u_2)]\rangle$. Then G_3 is of order 4. Indeed,

$$(\alpha, x)_{x-\pi^{k_1}u_1} \sim (\alpha, \pi^{k_1}) \not\sim 1,\ (\alpha, x-\pi^{k_2}u_2)_{x-\pi^{k_1}u_1} \sim (\alpha, \pi) \not\sim 1,$$

$$(\alpha, x(x-\pi^{k_2}u_2))_x \sim (\alpha, (x-\pi^{k_1}u_1)(x+e)(x+e'))_x \sim (\alpha, -\pi^{k_1}u_1 ee') \not\sim 1.$$

In cases $III3, III4$ let $G_4 = \langle [(\alpha, x)],\ [(\pi, x)]\rangle$ and let us show that G_4 is of order 4. For the algebra (α, x) we have: if k_2 is odd, then

$$(\alpha, x)_{x-\pi^{k_2}u_2} \sim (\alpha, \pi^{k_2}u_2) \not\sim 1.$$

If k_2 is even, then

$$f(\pi^{k_1+1}) \sim \pi^{k_1+1}(-\pi^{k_1}u_1)\pi^{k_1+1} \sim -u_1 \sim 1$$

and

$$(\alpha, x)_{x-\pi^{k_1+1}} \sim \left(\frac{\alpha\ ,\ \pi}{k\langle x-\pi^{k_1+1}\rangle}\right) \not\sim 1.$$

As for the algebra (π, x), one can find a unit w such that $k(w)|k$ is an unramified odd prime degree extension and

$$\begin{cases} w \sim \alpha \\ w - u_1 \sim 1. \end{cases}$$

In view of $f(\pi^{k_1}w) \sim w - u_1$ we have

$$(\pi, x)_{p(x)} \sim \left(\frac{\pi\ ,\ \pi^{k_1}w}{k(w)\langle p\rangle}\right) \not\sim 1,$$

where $p(x)$ is a minimal polynomial of $\pi^{k_1}w$ over k. Now since $k(\sqrt{\alpha})(x, \sqrt{f(x)})$ does not split (π, x), we have $(\pi, x) \not\sim (\alpha, x)$.

Finally, let $G_5 = \langle [(\pi, x)],\ [(\pi, x-\pi^{k_2}u_2)]\rangle$ in cases $III5$ and $III6$. We prove that G_5 is of order 4. To prove that (π, x) is non-trivial it is enough to repeat previous arguments with the last w. Similarly, with the same w in case $k_1 < k_2$ we have

$$(\pi, x-\pi^{k_2}u_2)_{x-\pi^{k_1}w} \sim (\pi, x)_{x-\pi^{k_1}w} \not\sim 1.$$

If $k_1 = k_2$ let w satisfies the following property

$$\begin{cases} w - u_2 \sim \alpha \\ w(w-u_1) \sim \alpha. \end{cases}$$

Then $(\pi, x-\pi^{k_2}u_2)_{x-\pi^{k_1}w} \not\sim 1$. For the algebra $(\pi, x(x-\pi^{k_2}u_2))$ in case $k_1 \neq k_2$ we have

$$(\pi, x(x-\pi^{k_2}u_2))_x \sim (\pi, (x-\pi^{k_1}u_1)(x+e)(x+e))_x \sim (\pi, -u_1) \not\sim 1.$$

If $k_1 = k_2$ one can find a unit w with the property

$$\begin{cases} w(w-u_2) \sim \alpha \\ w - u_1 \sim \alpha. \end{cases}$$

Then $(\pi, x(x-\pi^{k_2}u_2))_{x-\pi^{k_1}w} \not\sim 1$. Thus G_5 is of order 4.

To complete the proof of the theorem it remains to show that all the following groups are trivial

1. $G_1 \cap G_3$ in case $III1$,
2. $G_2 \cap G_3$ in case $III2$,
3. $G_1 \cap G_4$ in case $III3$,
4. $G_2 \cap G_4$ in case $III4$,
5. $G_1 \cap G_5$ in case $III5$,
6. $G_2 \cap G_5$ in case $III6$.

In case $III6$ the extension of constants by $\sqrt{\pi}$ leaves us in this case, so that $[(\alpha, x)] \notin G_5$ and $[(\alpha, x) \otimes (\pi, x+e)] \notin G_5$. Let $g(x) \in \{x, x-\pi^{k_2}u_2, x(x-\pi^{k_2}u_2)$. Then $(\pi, g(x)(x+e)) \not\sim 1$ in view of $\bar{g}(x)(x+\bar{e})$, $\bar{f}(x)/(\bar{g}(x)(x+\bar{e})) \notin \bar{k}[x]^2$ and by lemma 3. This implies $G_2 \cap G_5$ is trivial.

In case $III4$ since the extension of constants by $\sqrt{\alpha}$ leaves us in case $III4$, then in order to prove that $G_2 \cap G_4$ is trivial it is enough to check that $(\alpha, x(x+e)) \not\sim 1$ and $(\pi, x(x+e)) \not\sim 1$. But $(\alpha, x(x+e)) \not\sim 1$, since $(\bar{\alpha}, x(x+\bar{e})) \not\sim 1$ and $(\pi, x(x+e)) \not\sim 1$, since $x(x+\bar{e})$, $\bar{f}/(x(x+\bar{e})) \notin \bar{k}[x]^2$.

In case $III2$ since the extension of constants by $\sqrt{\alpha}$ leaves us in this case, we have $[(\pi, x+e)]$, $[(\pi\alpha, x+e)] \notin G_3$. And it remains to show that $[(\alpha, x+e)] \notin G_3$. But for any $g \in \{x, x-\pi^{k_2}u_2, x(x-\pi^{k_2}u_2)\}$ $(\alpha, g(x+e)) \not\sim 1$ since $(\bar{\alpha}, \bar{g}(x+\bar{e})) \not\sim 1$.

In case $III1$ the extension of constants by $\sqrt{\alpha}$ leads us to case $III2$, since $[(\pi, x+e)] \notin G_3$. The same is true for $[(\pi, x+e')]$. Now it remains to prove that $[(\pi, (x+e)(x+e'))] \notin G_3$. We have

$$(\alpha, x)_{x-\pi^{k_1}u_1} \sim (\alpha, x-\pi^{k_2}u_2)_{x-\pi^{k_1}u_1} \not\sim 1 \text{ but } (\pi, (x+e)(x+e'))_{x-\pi^{k_1}u_1} \sim (\pi, ee') \sim 1.$$

Furthermore,

$$(\alpha, x(x-\pi^{k_2}u_2))_x \sim (\alpha, (x-\pi^{k_1}u_1)(x+e)(x+e'))_x \sim (\alpha, -\pi^{k_1}u_1ee') \not\sim 1$$

and $(\pi, (x+e)(x+e'))_x \sim (\pi, ee') \sim 1$. Thus $[(\pi, (x+e)(x+e'))] \notin G_3$.

In case $III3$ the extension of constants by $\sqrt{\alpha}$ leads us to case $III4$. So this extension does not split the algebras $(\pi, x+e)$, $(\pi, x+e')$. Let us show that $(\alpha, x) \not\sim (\pi, (x+e)(x+e'))$. In case k_2 is odd we have

$$(\pi, (x+e)(x+e'))_{x-\pi^{k_2}u_2} \sim 1 \text{ and } (\alpha, x)_{x-\pi^{k_2}u_2} \not\sim 1.$$

In case k_2 is even

$$(\pi, (x+e)(x+e'))_{x-\pi^{k_1+1}} \sim 1 \text{ and } (\alpha, x)_{x-\pi^{k_1+1}} \not\sim 1.$$

Thus $[(\alpha, x)] \notin G_1$. We have

$$(\pi, x) \not\sim (\pi, x+e) \text{ and } (\pi, x) \not\sim (\pi, x+e'),$$

since $x(x+\bar{e}), \bar{f}(x)/(x(x+\bar{e})) \notin \bar{k}[x]^2$. Let w be a unit such that $(\pi, x)_{x-\pi^{k_1}w} \not\sim 1$, then $(\pi, (x+e)(x+e'))_{x-\pi^{k_1}w} \sim 1$. This implies $[(\pi, x)] \notin G_1$. The remaining possibility is $[(\pi\alpha, x)] \in G_1$. Observe firstly that $(\pi\alpha, x)_{x-\pi^{k_1}w} \not\sim 1$, hence $(\pi\alpha, x) \not\sim (\pi, (x+e)(x+e'))$. Furthermore, if $(\pi\alpha, x) \sim (\pi, x+e)$, then $(\alpha, x) \sim (\pi, x(x+e))$. But $k(\sqrt{\alpha})(x, \sqrt{f(x)})$ splits (α, x) and does not split $(\pi, x(x+e))$. Similarly $(\pi\alpha, x) \not\sim (\pi, x+e')$, so $G_1 \cap G_4$ is trivial.

Now pass to remaining case $III5$. If $g \in \{x, x-\pi^{k_2}u_2, x(x-\pi^{k_2}u_2)\}$, then $(\pi, g(x+e) \not\sim 1$ by lemma 3. Thus $[(\pi, x+e)] \notin G_5$. Similarly $[(\pi, x+e')] \notin G_5$. Let $(\pi, g) \sim (\pi, (x+e)(x+e'))$. On the other hand, if w is a unit such that $(\pi, x)_{x-\pi^{k_1}w} \not\sim 1$, then since $(\pi, (x+e)(x+e'))_{x-\pi^{k_1}w} \sim 1$ we have $g \neq x$. In the same way one can prove that $g \neq x-\pi^{k_2}u_2$. Finally, if $g = x(x-\pi^{k_2}u_2)$, then in case $k_1 = k_2$ for a unit w such that $(\pi, x(x-\pi^{k_2}u_2))_{x-\pi^{k_1}w} \not\sim 1$ we have $(\pi, (x+e)(x+e'))_{x-\pi^{k_1}w} \sim 1$ and in case $k_1 \neq k_2$ $(\pi, (x+e)(x+e'))_x \sim 1$, $(\pi, x(x-\pi^{k_2}u_2))_x \not\sim 1$. Thus $[(\pi, (x+e)(x+e'))] \notin G_4$. The theorem is proved.

In case where $f(x)$ is of the form IV we have the following statement.

Theorem 7 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of form IV. Then the class of (π, α) and the classes of the following algebras generate ${}_2Br\, C$.*

$IV1.$ (π, x), $(\pi, x-\pi^{k_2}u_2)$, $(\pi, x+e_1)$, $(\pi, x+e_2)$.

$IV2.$ (π, x), $(\pi, x+e_1)$, $(\pi, x+e_2)$, (α, x).

Proof. Observe that $G = \langle [(\pi, x)],\ [(\pi, x+e_1)],\ [(\pi, x+e_2)],\ \rangle$ is of order 8. First of all $x, x+\bar{e}_1, x+\bar{e}_2, x(x+\bar{e}_1), x(x+\bar{e}_2), (x+\bar{e}_1)(x+\bar{e}_2), \bar{f}(x)/x, \bar{f}(x)/(x+\bar{e}_1), \bar{f}(x)/(x+\bar{e}_2), \bar{f}(x)/(x(x+\bar{e}_1)), \bar{f}(x)/(x(x+\bar{e}_2)), \bar{f}(x)/((x+\bar{e}_1)(x+\bar{e}_2)) \notin \bar{k}[x]^2$ and by lemma 3 $(\pi, x) \not\sim 1$, $(\pi, x+e_1) \not\sim 1$, $(\pi, x+e_2) \not\sim 1$, $(\pi, x(x+e_1)) \not\sim 1$, $(\pi, x(x+e_2)) \not\sim 1$, $(\pi, (x+e_1)(x+e_2)) \not\sim 1$, so G is of order 8.

If one of the k_i is odd, we have $(\alpha, x)_{x-\pi^{k_i}u_i} \not\sim 1$. If $k_1 \equiv k_2 \equiv 0(2)$, $k_1 < k_2$ and $-u_1e_1e_2 \in (k^*)^2$, then $k_1 < k_1+1 < k_2$ and $(\alpha, x)_{x-\pi^{k_1+1}} \not\sim 1$, since $f(\pi^{k_1+1}) \sim -u_1e_1e_2$. Thus in case $IV2$ $(\alpha, x) \not\sim 1$. Since the extension of constants by $\sqrt{\alpha}$ leaves us in case IV, we conclude that $[(\alpha, x)] \notin G$.

Consider case $IV1$. If $k_1 \neq k_2$, we have

$$(\pi, x(x-\pi^{k_2}u_2))_x \sim (\pi, (x-\pi^{k_1}u_1)(x+e_1)(x+e_2))_x \sim (\pi, -u_1e_1e_2) \not\sim 1.$$

If $k_1 = k_2$ let w be a unit such that $k(w)|k$ is an unramified prime degree extension and

$$\begin{cases} w(w-u_2) \sim \alpha \\ w-u_1 \sim \alpha e_1 e_2. \end{cases}$$

Then $(\pi, x(x-\pi^{k_2}u_2))_{x-\pi^{k_2}w} \not\sim 1$. This implies $(\pi, x) \not\sim (\pi, x-\pi^{k_2}u_2)$. By lemma 3 the algebra $(\pi, x-\pi^{k_2}u_2)$ is also non-isomorphic to $(\pi, x+e_1), (\pi, x+e_2), (\pi, x(x+e_2)), (\pi, x(x+e_2)), (\pi, x(x+e_1)(x+e_2))$. Finally, let $k_1 \neq k_2$. Then

$$(\pi, (x-\pi^{k_2}u_2)(x+e_1)(x+e_2))_{x-\pi^{k_1}w} \sim (\pi, x(x-\pi^{k_1}u_1))_{x-\pi^{k_1}w} \not\sim 1$$

if w is a unit with the above properties and

$$\begin{cases} w \sim \alpha e_1 e_2 \\ w-u_1 \sim e_1 e_2. \end{cases}$$

In case $k_1 = k_2$ we only need to change the latter conditions by

$$\begin{cases} w(w-u_1) \sim \alpha \\ w-u_2 \sim \alpha e_1 e_2. \end{cases}$$

In any case $(\pi, x-\pi^{k_2}u_2) \not\sim (\pi, (x+e_1)(x+e_2))$. This implies $[(\pi, x-\pi^{k_2}u_2)] \notin G$ and completes the proof of the theorem.

If $f(x)$ is of the form V we have

Theorem 8 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of form V. Then the class of (π, α) and the classes of the following algebras generate ${}_2Br\,C$.*

$V1.$ (π, x), $(\pi, x-\pi^{k_1}v_1)$, $(\pi, x+u)$, $(\pi, x+u-\pi^{k_2}v_2)$.

$V2.$ (π, x), $(\pi, x-\pi^{k_1}v_1)$, $(\pi, x+u)$, $(\alpha, x+u)$.

$V3.$ (π, x), (α, x), $(\pi, x+u)$, $(\pi, x+u-\pi^{k_2}v_2)$.

$V4.$ (π, x), (α, x), $(\pi, x+u)$, $(\alpha, x+u)$.

Proof. Let l_1, l_2, l_3, l_4, l_5 be the different linear monic divisors of $f(x)$. If $l_i \neq x+e$ we have by lemma 3 in view of $\bar{l}_i, \bar{f}(x)/\bar{l}_i \notin \bar{k}[x]^2$ that $(\pi, l_i) \not\sim 1$. By the same reason $(\pi, l_il_j) \not\sim 1$ if $\bar{l}_i \neq \bar{l}_j$. Let now $e \not\sim 1$, then $(\pi, x(x-\pi^{k_1}v_1))_{x+\alpha v_1\pi^{k_1+2}} \not\sim 1$, since $f(-\alpha v_1\pi^{k_1+2}) \sim \alpha e \sim 1$. Thus if $e \not\sim 1$, then $(\pi, x(x-\pi^{k_1}v_1)) \not\sim 1$. The change of variables $x' = x+u$ leads us to the equation

$$y^2 = x'(x'-\pi^{k_2}v_2)(x'-u)(x'-u-\pi^{k_1}v_1)(x'+(e-u))$$

and we have $(\pi, (x+u)(x+u-\pi^{k_2}v_2)) = (\pi, x'(x'-\pi^{k_2}v_2))$, so by the previous arguments $(\pi, (x+u)(x+u-\pi^{k_2}v_2)) \not\sim 1$. Observe also that if $e \sim 1$, then

$$\left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}(x, \sqrt{\bar{f}(x)})}\right)_x \sim \left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}\langle x\rangle}\right) \not\sim 1.$$

It follows by lemma 4 that in case $e \sim 1$ $(\alpha, x) \not\sim 1$. Similarly if $e-u \sim 1$, then $(\alpha, x+u) \not\sim 1$.

Now, to complete the consideration of case VI we need only to prove that $(\pi, l_i l_j l_r) \not\sim 1$, where l_i, l_j, l_r are not equal to each other and to $x+e$ and $(\pi, x(x-\pi^{k_1}v_1)(x+u)(x+u-\pi^{k_2}v_2)) \not\sim 1$ if $e \not\sim 1$. But the last statement is equivalent to $(\pi, x+e) \not\sim 1$ which is valid in view of $(\pi, x+e)_x \not\sim 1$. As to the previous one, we have $\bar{l}_i\bar{l}_j\bar{l}_r,\ \bar{f}(x)/(\bar{l}_i\bar{l}_j\bar{l}_r) \notin \bar{k}[x]^2$, so lemma 3 works.

In case $V2$ it is enough to observe that by above arguments the group $\langle[(\pi, x)],\ [(\pi, x-\pi^{k_1}v_1)],\ [(\pi, x+u)]\rangle$ is of order 8 and the field $k(\sqrt{\pi})(x, \sqrt{f(x)})$ does not split the algebra $(\alpha, x+u)$.

Case $V3$ can be considered by analogy with case $V2$.

We saw that the group $\langle[(\pi, x)],\ [(\pi, x+u)]\rangle$ is of order 4. To prove that $\langle[(\alpha, x)],\ [(\alpha, x+u)]\rangle$ is also of order 4 we need to check that $(\alpha, x(x+u)) \not\sim 1$. But this is true in view of lemma 4 since

$$\left(\frac{\bar{\alpha}\ ,\ x(x+\bar{u})}{\bar{k}(x, \sqrt{\bar{f}(x)})}\right)_x \sim \left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}\langle x\rangle}\right) \not\sim 1.$$

Finally, since the extension of constants by $\sqrt{\alpha}$ leaves us in case $V4$ the groups $\langle[(\pi, x)],\ [(\pi, x+u)]\rangle$ and $\langle[(\alpha, x)],\ [(\alpha, x+u)]\rangle$ have the trivial intersection. The theorem is proved.

In case where $f(x)$ is of the form VI we have

Theorem 9 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of form VI. Then the class of (π, α) and the classes of the following algebras generate ${}_2Br\, C$.*

$VI1$. $\{(\pi, l_i)\}_i$, l_i is a monic linear divisor of f.

$VI2$. $\{(\pi, l_i)\}_i$, l_i is a monic linear divisor of f, $l_i \neq x - \pi^k u$ and (α, x).

Proof. In case $VI1$ it is enough to check that for any monic linear divisor of $f(x)$ $(\pi, l_i) \not\sim 1$ and $(\pi, l_i l_j) \not\sim 1$ if $i \neq j$. We have that $(\pi, l_i) \not\sim 1$ by lemma 3. By the same reason $(\pi, l_i l_j) \not\sim 1$ if $l_i l_j \neq x(x-\pi^k u)$. If $l_i l_j = x(x-\pi^k u)$, then $(\pi, x(x-\pi^k u))_x \sim (\pi, e_1e_2e_3) \not\sim 1$.

In case $V2$ arguments similar to the previous ones shows that the group H generated by $[(\pi, l_i)]$, where l_i runs through the set of all monic linear divisors not equal to $x-\pi^k u$ is of order 8. To complete the consideration of the case let us show that $(\alpha, x) \not\sim 1$. If k is odd we have $(\alpha, x)_{x-\pi^k u} \not\sim 1$, if not, then $(\alpha, x)_{x-\pi} \not\sim 1$, since $f(\pi) \sim \pi(\pi - \pi^k u)e_1e_2e_3 \sim e_1e_2e_3 \sim 1$. Finally, the extension of constants by $\sqrt{\alpha}$ leaves us in case $VI2$, so all non-trivial algebras from H are non-trivial after this extension but it is not the case for (α, x). The theorem is proved.

3 Splitting type (1,1,1,2).

This section is devoted to case $deg\ f_1 = deg\ f_2 = deg\ f_3 = 1, deg\ f_4 = 2$ and the reduction is bad.

Preliminary results. The evident list of all cases under consideration according to the reduction type of $f(x)$ is as follows.

I. $f(x) = (x^2 - \pi^k u)(x-e_1)(x-e_2)(x-e_3)$,
$\pi^k u \not\sim 1,\ u, e_i \in O_k^*,\ k > 0,\ \bar{e}_i \neq \bar{e}_j$ if $i \neq j$.

II. $f(x) = (x^2 - \pi^k u)(x-e)(x-e')(x-e_1)$,
$\pi^k u \not\sim 1,\ u, e, e', e_1 \in O_k^*,\ k > 0,\ \bar{e} = \bar{e}' \neq \bar{e}_1$.

III. $f(x) = (x^2 - \pi^k u)(x-e_1)(x-e_2)(x-e_3)$,
$u \in O_k^*,\ k > 0,\ \pi^k u \not\sim 1,\ \bar{e}_1 = \bar{e}_2 = \bar{e}_3 \neq 0$.

$IV.\ f(x)=(x^2+ax+b)x(x-\pi^k u)(x+e),$

$u,e\in O_k^*,\ k>0,\ x^2+\bar{a}x+\bar{b}$ is irreducible over $\bar{k}$.

$V.\ f(x)=(x^2+ax+b)x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2),$

$u_i\in O_k^*,\ 0<k_1\le k_2$, and if $k_1=k_2$, then $\bar{u}_1\neq\bar{u}_2$,

$x^2+\bar{a}x+\bar{b}$ is irreducible over $\bar{k}$.

$VI.\ f(x)=(x^2-\pi^k u)(x-\pi^m v)(x+e_1)(x+e_2),$

$\pi^k u\not\sim 1,\ k,m>0,\ u,v,e_i\in O_k^*,\ \bar{e}_1\neq\bar{e}_2.$

$VII.\ f(x)=(x^2-\pi^k u)(x-\pi^m v)(x+e)(x+e'),$

$\pi^k u\not\sim 1,\ k,m,l>0,\ u,v,e\in O_k^*,\ \bar{e}=\bar{e}'.$

$VIII.\ f(x)=(x^2-\pi^k u)(x-\pi^{m_1}v_1)(x-\pi^{m_2}v_2)(x+e),$

$\pi^k u\not\sim 1,\ 0<\ m_1\le m_2,\ k>0,\ u,v_i,e\in O_k^*.$

$IX.\ f(x)=(x^2-\pi^k u)(x-\pi^{m_1}v_1)(x-\pi^{m_2}v_2)(x-\pi^{m_3}v_3),$

$\pi^k u\not\sim 1,\ 0<\ m_1\le m_2\le m_3,\ k>0,\ u,v_i\in O_k^*.$

Let us prove firstly some preliminary results.

Lemma 10

$$|\,{}_2Br\,C|=16.$$

Lemma 11 *let $u\in O_k^*$ is a unit such that $u\not\sim 1$, then*
i) for any $w\in O_k^*$ $w\pm\sqrt{u}\sim 1$ in $k(\sqrt{u})$ iff $w^2-u\sim 1$ in k,
ii) for any local field k there exists $w\in O_k^*$ such that $w^2-u\not\sim 1$,
iii) for any $v\in O_k^*$ there exists $w\in O_k^*$ such that $v^2-w\in O_k^*$ and $v^2-w\not\sim 1$.

Proof. i) follows from the surjectivity on units of the norm homomorphism $N_{k(\sqrt{u})|k}$ and for ii) and iii) see [13], lemma 1.

Lemma 12 *Let $g(x)\in k[x]$ be a monic polynomial, and let either n be odd or $\deg g(x)$ be even. Then for any $b\in k(C)^*$ the quaternion algebra*

$$\left(\frac{b\ ,\ g(x)}{k(C)}\right)$$

is not isomorphic to the scalar algebra

$$\left(\frac{\pi\ ,\ \alpha}{k(C)}\right).$$

To check this fact it is enough to observe that

$$\left(\frac{b\ ,\ g(x)}{k(C)}\right)_\infty\sim 1 \text{ and } \left(\frac{\pi\ ,\ \alpha}{k(C)}\right)_\infty\not\sim 1.$$

Lemma 13 *Let K be an algebraically closed field and $f(x),g(x),h(x)\in K[x]$. Then the system*

$$\begin{cases} y^2=f(x),\\ z^2=g(x),\\ t^2=h(x)\end{cases}$$

defines a variety V in $\boldsymbol{P}^4(K)$ if and only if all polynomials $f(x)$, $g(x)$, $h(x)$, $f(x)g(x)$, $f(x)h(x)$, $g(x)h(x)$ and $f(x)g(x)h(x)$ are not in $K[x]^2$.

Proof. Consider the homomorphism

$$\phi: \ K[x,y,z,t] \longrightarrow K(x, \sqrt{f(x)}, \sqrt{g(x)}, \sqrt{h(x)}),$$

defined by the rule

$$x \mapsto x, \ y \mapsto \sqrt{f(x)}, \ z \mapsto \sqrt{g(x)}, \ t \mapsto \sqrt{h(x)}.$$

Then the ideal $I = (y^2 - f(x), z^2 - g(x), t^2 - h(x)) \subset Ker\,\phi$. Let $F = F_0(x) + F_1(x)y + F_2(x)z + F_3(x)t + F_4(x)yz + F_5(x)yt + F_6(x)zt + G(x,y,z,t) \in Ker\,\phi$, where $G(x,y,z,t) \in I$. So $\phi(F) = F_0(x) + F_1(x)\sqrt{f} + F_2(x)\sqrt{g} + F_3(x)\sqrt{h} + F_4(x)\sqrt{fg} + F_5(x)\sqrt{fh} + F_6(x)\sqrt{gh} = 0$. Let us check that the extension $K(x)(\sqrt{f(x)}, \sqrt{g(x)}, \sqrt{h(x)})|K(x)$ is of degree 8. Indeed, $K(x)(\sqrt{f(x)})|K(x)$ is of degree 2. If $\sqrt{g(x)} \in K(x)(\sqrt{f(x)})$, then $g(x) = a(x)^2 + b(x)^2 f(x) + 2a(x)b(x)\sqrt{f(x)}$, where $a(x), b(x) \in K(x)$. So $a(x)b(x) = 0$ and either $g(x) = a(x)^2$ or $g(x) = b(x)^2 f(x)$. And this is impossible in view of $g(x) \notin K[x]^2$ and $f(x)g(x) \notin K[x]^2$. Hence $K(x)(\sqrt{f(x)}, \sqrt{g(x)})|K(x)$ is of degree 4. Let now $\sqrt{h(x)} \in K(x)(\sqrt{f(x)}, \sqrt{g(x)})$, i.e. $\sqrt{h(x)} = a_0(x) + a_1(x)\sqrt{f(x)} + a_2(x)\sqrt{g(x)} + a_3(x)\sqrt{f(x)g(x)}$. We have $h = (a_0^2 + a_1^2 f + a_2^2 g + a_3^2 fg) + 2(a_0a_1 + a_2a_3g)\sqrt{f} + 2(a_0a_2 + a_1a_3f)\sqrt{g} + 2(a_0a_3 + a_1a_2)\sqrt{fg}$. Therefore

$$\begin{cases} a_0a_1 + a_2a_3g = 0, \\ a_0a_2 + a_2a_3f = 0, \\ a_0a_3 + a_1a_2 = 0. \end{cases}$$

We have $h(x) \notin K(x)(\sqrt{f(x)g(x)})$, so that one of a_1, a_2, say, a_1 is not equal to 0. Then

$$\begin{cases} a_2 = -a_0a_3/a_1, \\ a_0(a_3^2 g - a_1^2) = 0, \\ a_3(a_1^2 f - a_0^2) = 0, \end{cases}$$

Furthermore, $a_3^2 g - a_1^2 \neq 0$, since $a_1 \neq 0$ and $g(x) \notin K[x]^2$. Then $a_0 = a_2 = a_3 = 0$ and $\sqrt{h(x)} \in K(x)(\sqrt{f(x)})$. This contradiction proves that the extension

$$K(x)(\sqrt{f(x)}, \sqrt{g(x)}, \sqrt{h(x)})|K(x)$$

is of degree 8. The evident basis of this extension is 1, $\sqrt{f(x)}$, $\sqrt{g(x)}$, $\sqrt{h(x)}$, $\sqrt{f(x)g(x)}$, $\sqrt{f(x)h(x)}$, $\sqrt{g(x)h(x)}$, $\sqrt{f(x)g(x)h(x)}$, so $F_i(x) = 0$ for any i and $Ker\,\phi = I$. Then I is prime and V is a variety.

Lemma 14 *Let $f(x) = (x^2 - \pi^k u)g(x)$, $g(x) \in O_k[x]$ without multiple roots, $\bar{g}(x) \notin \bar{k}[x]^2$, $u \in O_k^*$, $k > 0$, $\pi^k u \not\sim 1$ and $E = g(0) \in O_k^*$. Then the following algebra is nontrivial, unramified and not isomorphic to the scalar algebra (π, α).*

1. $(-\pi E u, x)$ *if* $k \equiv 1\,(mod\,2)$.
2. $(\alpha, x - \pi^{k/2} w)$ *if* $k \equiv 0\,(mod\,2)$ *and* $E \sim 1$, *where* $w \in O_k^*$, $w^2 - u \not\sim 1$.
3. $(\pi, x^2 - \pi^k u)$ *if* $k \equiv 0\,(mod\,2)$ *and* $E \not\sim 1$.

Proof.

$$(-\pi E u, x)_x \sim \left(\frac{-\pi E u \ , \ x}{k\langle x\rangle(\sqrt{-\pi E u})} \right) \sim 1,$$

so this algebra is unramified. Let K be an unramified odd degree extension of k such that there exists a unit $\tau \in O_K^*$ with the property $\tau \not\sim 1$, $g(\tau) \sim 1$ and $p(x) = Irr_{K|k}(\tau)$. Such K exists in view of lemma 6 from the previous section. Then

$$(-\pi E u, x)_p \sim \left(\frac{-\pi E u \ , \ \tau}{K\langle p\rangle(\sqrt{g(\tau)})} \right) \sim \left(\frac{\pi \ , \ \tau}{K\langle p\rangle} \right) \not\sim 1.$$

The unit w in 2) exists in view of lemma 11. We have

$$(\alpha, x - \pi^{k/2} w)_{x - \pi^{k/2} w} \sim \left(\frac{\alpha \ , \ x - \pi^{k/2} w}{k\langle x - \pi^{k/2} w\rangle(\sqrt{(\pi^k w^2 - \pi^k u)E})} \right) \sim$$

$$\sim \left(\frac{\alpha\ ,\ x-\pi^{k/2}w}{k\langle x-\pi^{k/2}w\rangle(\sqrt{\alpha})}\right)\sim 1,$$

so the algebra $(\alpha, x-\pi^{k/2}w)$ is unramified. Furthermore,

$$(\alpha, x-\pi^{k/2}w)_{x^3-\pi}\sim\left(\frac{\alpha\ ,\ \sqrt[3]{\pi}}{k(\sqrt[3]{\pi})\langle x^3-\pi\rangle(\sqrt{(\pi^{2/3}-\pi^k u)E})}\right)\sim\left(\frac{\alpha\ ,\ \sqrt[3]{\pi}}{k(\sqrt[3]{\pi})\langle x^3-\pi\rangle}\right)\not\sim 1.$$

Lemma 2 provides the algebra $(\pi, x^2-\pi^k u)$ to be unramified. Moreover,

$$(\pi, x^2-\pi^k u)_{x-\pi^{k/2}w}\sim\left(\frac{\pi\ ,\ \alpha}{k\langle x-\pi^{k/2}w\rangle(\sqrt{\alpha E})}\right)\sim\left(\frac{\pi\ ,\ \alpha}{k\langle x-\pi^{k/2}w\rangle}\right)\not\sim 1.$$

To complete the proof it is enough to check that all algebras are not isomorphic to the scalar algebra, and this is true in view of lemma 12.

Presentation by algebras.

Theorem 10 *Let f be as in cases I or II. Then the group ${}_2Br\, C$ is generated by the class of the algebra (π,α) and classes of the following algebras.*

- *Case I.*
  1. $(e_1e_1e_3u\pi, x),\ (\pi, x-e_1),\ (\pi, x-e_2)$, *if* $k\equiv 1\,(mod\, 2)$.
  2. $(\alpha, x-\pi^{k/2}w),\ (\pi, x-e_1),\ (\pi, x-e_2)$, *if* $k\equiv 0\,(mod\, 2),\ -e_1e_2e_3\sim 1$.
  3. $(\pi, x^2-\pi^k u),\ (\pi, x-e_1),\ (\pi, x-e_2)$, *if* $k\equiv 0\,(mod\, 2),\ -e_1e_2e_3\not\sim 1$.
- *Case II.*
  1. $(e_1u\pi, x),\ (\pi, x-e),\ (\pi, x-e')$, *if* $k\equiv 1\,(mod\, 2),\ e-e_1\not\sim 1$.
  2. $(e_1u\pi, x),\ (\pi, x-e),\ (\alpha, x-e)$, *if* $k\equiv 1\,(mod\, 2),\ e-e_1\sim 1$.
  3. $(\alpha, x-\pi^{k/2}w),\ (\pi, x-e),\ (\pi, x-e')$, *if* $k\equiv 0\,(mod\, 2),\ -e_1\sim 1,\ e-e_1\not\sim 1$.
  4. $(\alpha, x-\pi^{k/2}w),\ (\pi, x-e),\ (\alpha, x-e)$, *if* $k\equiv 0\,(mod\, 2),\ -e_1\sim 1,\ e-e_1\sim 1$.
  5. $(\pi, x^2-\pi^k u),\ (\pi, x-e),\ (\pi, x-e')$, *if* $k\equiv 0\,(mod\, 2),\ -e_1\not\sim 1,\ e-e_1\not\sim 1$.
  6. $(\pi, x^2-\pi^k u),\ (\pi, x-e),\ (\alpha, x-e)$, *if* $k\equiv 0\,(mod\, 2),\ -e_1\not\sim 1,\ e-e_1\sim 1$,

where $w\in O_k^*,\ w^2-u\not\sim 1$.

Proof. In case I all algebras $(\pi, x-e_i)$ and $(\pi,(x-e_i)(x-e_j)), i\neq j$ are unramified and not isomorphic to (π,α) in view of lemma 2. They are nontrivial by lemma 3.

In case II let us prove that if $e-e_1\not\sim 1$, then the algebras $(\pi, x-e),\ (\pi, x-e')$ and $(\pi,(x-e)(x-e'))$ are unramified, nontrivial and not isomorphic to (π,α). If $e-e_1\sim 1$, then the same is true for the algebras $(\pi, x-e),\ (\alpha, x-e)$ and $(\pi\alpha, x-e)$. Indeed, if $e-e_1\not\sim 1$, then $(\pi,(x-e)(x-e'))_{x-e}\sim(\pi,(x^2-\pi^k u)(x-e_1))_{x-e}\sim(\pi, e-e_1)\not\sim 1$. Let now $e-e_1\sim 1$, then

$$(\bar\alpha, x-\bar e)_{x-\bar e}\sim\left(\frac{\bar\alpha\ ,\ x-\bar e}{\bar k\langle x-\bar e\rangle(\sqrt{\bar e-\bar e_1})}\right)\not\sim 1,$$

and in view of lemma 4 $(\alpha, x-e)\not\sim 1$. Finally, after the replacement $\nu=\pi\alpha$ of prime element we have $(\pi\alpha, x-e)=(\nu, x-e)\not\sim 1$.

Because of the polynomial f in cases I and II satisfies the conditions of lemma 12 it is enough to prove that algebra from this lemma is not isomorphic to the algebras already considered in the current proof.

Consider firstly case I. If $k \equiv 1\,(mod\,2)$, then one can find such w_1, w_2 and w_3 from O_K^* (K as above) that $w_i \sim 1$, $w_i - e_i \not\sim 1$, $(w_i - e_1)(w_i - e_2)(w_i - e_3) \sim 1$. Indeed, for example, in case $i = 1$ it is enough to solve the following system

$$\begin{cases} w_1 = x^2, \\ y^2 = \alpha(x^2 - e_1) \\ z^2 = \alpha(x^2 - e_2)(x^2 - e_3), \end{cases}$$

and lemma 6 works. Let $p_i = Irr_{K|k}(w_i)$. Then for any i we have $(-\pi E u, x)_{p_i} \sim 1$, $(\pi, x - e_i)_{p_i} \not\sim 1$ and $(\pi, (x - e_1)(x - e_2))_{p_3} \not\sim 1$.

If $E = -e_1 e_2 e_3 \sim 1$, then the algebra $(\alpha, x - \pi^{k/2} w)$ splits by the extension of constants by $\sqrt{\alpha}$ but this is not true for the algebras $(\pi, x - e_i)$ and $(\pi, (x - e_1)(x - e_2))$.

Let $k \equiv 0\,(mod\,2)$, $-e_1 e_2 e_3 \not\sim 1$, then $(\pi, x^2 - \pi^k u) \not\sim (\pi, x - e_i)$, $(\pi, (x - e_1)(x - e_2))$ in view of lemma 3.

Consider now case II. The extension by $\sqrt{\alpha}$ does not split $(-\pi E u, x)$, therefore $(-\pi E u, x) \not\sim (\alpha, x - e)$. Let $w \in O_K^*$ such that $w \sim 1$, $w - e \not\sim 1$ and $(w - e)(w - e')(w - e_1) \sim w - e_1 \sim 1$, $p = Irr_{K|k}(w)$. Then $(-\pi E u, x)_p \sim 1$ and $(\pi, x - e)_p \not\sim 1$. Changing the prime element $\nu = \pi\alpha$ one can check that $(-\pi E u, x) \not\sim (\pi\alpha, x - e)$. In the same way we have $(-\pi E u, x) \not\sim (\pi, x - e')$. Let us show finally that $(-\pi E u, x) \not\sim (\pi, (x - e)(x - e'))$. If $q = Irr_{K|k}(w)$, where $w \not\sim 1$, $\bar{w} \neq \bar{e}$ and $w - e_1 \sim 1$, then $(-\pi E u, x)_q \not\sim 1$, $(\pi, (x - e)(x - e'))_q \sim (\pi, (w - e)^2) \sim 1$.

Let now $k \equiv 0\,(mod\,2)$, $E \sim -e_1 \sim 1$. We have $(\alpha, x - \pi^{k/2} w) \not\sim (\pi, x - e)$, $(\pi, x - e')$, $(\pi, (x - e)(x - e'))$, since the extension of constants by $\sqrt{\pi}$ does not split the first algebra but splits the others. After this extension $(\pi\alpha, x - e) \sim (\alpha, x - e)$. It only remains to check that $(\alpha, x - \pi^{k/2} w) \not\sim (\alpha, x - e)$ if $e - e_1 \sim 1$. It is true since $(\alpha, x - \pi^{k/2} w)_{x^3 - \pi} \not\sim 1$, but $(\alpha, (x - e)(x - e'))_{x^3 - \pi} \sim 1$.

Finally, let $k \equiv 0\,(mod\,2)$, $-e_1 \not\sim 1$. We have $(\pi, (x^2 - \pi^k u)(x - e))$, $(\pi, (x^2 - \pi^k u)(x - e')) \not\sim 1$. Moreover, $(\pi, (x^2 - \pi^k u)(x - e)(x - e'))_{x - e} \sim (\pi, e - e_1) \not\sim 1$ if $e - e_1 \not\sim 1$. The extension of constants by $\sqrt{\pi}$ does not split $(\alpha, x - e)$, so that $(\pi, x^2 - \pi^k u) \not\sim (\alpha, x - e)$. After the same extension $(\pi\alpha, x - e) \sim (\alpha, x - e) \not\sim 1$, so $(\pi\alpha, x - e) \not\sim (\pi, x^2 - \pi^k u)$. The theorem is proved.

Without loss of generality one can assume that in case III $f(x) = (x^2 - \pi^k u)(x - e_1)(x - e_2)(x - e_3)$, where $e_1 = e$, $e_2 = e + \pi^{k_1} u_1$, $e_3 = e + \pi^{k_2} u_2$, $e, u_i \in O_k^*$, $0 < k_1 \leq k_2$ and if $k_1 = k_2$, then $\bar{u}_1 \neq \bar{u}_2$.

Theorem 11 *Let f be as in case III. Then the group ${}_2Br\,C$ is generated by the class of the algebra (π, α) and the classes of the following algebras.*

1. $(eu\pi, x)$, $(\alpha, x - e_1)$, $(\alpha, x - e_2)$,
   if $k \equiv 1\,(mod\,2)$, $k_1 \equiv 1\,(mod\,2)$.
2. $(eu\pi, x)$, $(\alpha, x - e_1)$, $(\pi, x - e_1)$,
   if $k \equiv 1\,(mod\,2)$, $k_1 \equiv 0\,(mod\,2)$ *and either* $k_2 \equiv 1\,(mod\,2)$ *or* $k_1 < k_2$, $-u_1 \sim 1$.
3. $(eu\pi, x)$, $(\pi, x - e_1)$, $(\pi, x - e_2)$,
   if $k \equiv 1\,(mod\,2)$, $k_1 \equiv k_2 \equiv 0\,(mod\,2)$ *and either* $k_1 = k_2$ *or* $-u_1 \not\sim 1$.
4. $(\alpha, x - \pi^{k/2} w)$, $(\alpha, x - e_1)$, $(\alpha, x - e_2)$,
   if $k \equiv 0\,(mod\,2)$, $-e \sim 1$, $k_1 \equiv 1\,(mod\,2)$.
5. $(\alpha, x - \pi^{k/2} w)$, $(\alpha, x - e_1)$, $(\pi, x - e_1)$,
   if $k \equiv 0\,(mod\,2)$, $-e \sim 1$, $k_1 \equiv 0\,(mod\,2)$ *and either* $k_2 \equiv 1\,(mod\,2)$ *or* $k_1 < k_2$, $-u_1 \sim 1$.
6. $(\alpha, x - \pi^{k/2} w)$, $(\pi, x - e_1)$, $(\pi, x - e_2)$,
   if $k \equiv 0\,(mod\,2)$, $-e \sim 1$, $k_1 \equiv k_2 \equiv 0\,(mod\,2)$ *and either* $k_1 = k_2$ *or* $-u_1 \not\sim 1$.
7. $(\pi, x^2 - \pi^k u)$, $(\alpha, x - e_1)$, $(\alpha, x - e_2)$,
   if $k \equiv 0\,(mod\,2)$, $-e \not\sim 1$, $k_1 \equiv 1\,(mod\,2)$.

8. $(\pi, x^2-\pi^k u),\ (\alpha, x-e_1),\ (\pi, x-e_1)$,

if $k\equiv 0\,(mod\,2),\ -e\not\sim 1,\ k_1\equiv 0\,(mod\,2)$ *and either* $k_2\equiv 1\,(mod\,2)$ *or* $k_1<k_2,\ -u_1\sim 1$.

9. $(\pi, x^2-\pi^k u),\ (\pi, x-e_1),\ (\pi, x-e_2)$,

if $k\equiv 0\,(mod\,2),\ -e\not\sim 1,\ k_1\equiv k_2\equiv 0\,(mod\,2)$ *and either* $k_1=k_2$ *or* $-u_1\not\sim 1$,

where $w\in O_k^*,\ w^2-u\not\sim 1$.

Let us prove firstly

Lemma 15 *In case III the following algebras are unramified, nontrivial and not isomorphic to* (π,α).

1. $(\alpha, x-e_1),\ (\alpha, x-e_2),\ (\alpha,(x-e_1)(x-e_2))$, *if* $k_1\equiv 1\,(mod\,2)$.
2. $(\alpha, x-e_1),\ (\pi, x-e_1),\ (\alpha\pi, x-e_1)$, *if* $k_1\equiv 0\,(mod\,2)$ *and either* $k_2\equiv 1\,(mod\,2)$ *or* ($k_2\equiv 0\,(mod\,2),\ k_1<k_2$ *and* $-u_1\sim 1$).
3. $(\pi, x-e_1),\ (\pi, x-e_2),\ (\pi,(x-e_1)(x-e_2))$, *if* $k_1\equiv k_2\equiv 0\,(mod\,2)$ *and either* $k_1<k_2$ *and* $-u_1\sim 1$ *or* $k_1=k_2$.

Proof of the lemma. In 1 we have: $(\alpha,x-e_1)_{x-e_2}\not\sim 1$, $(\alpha,x-e_2)_{x-e_1}\not\sim 1$, $(\alpha,(x-e_1)(x-e_2))_{x-e_2}\sim(\alpha,x-e_3)_{x-e_2}\not\sim 1$.
2. Let $\theta=e+\pi^{k_1}w$. Then $f(\theta)\sim w-u_1$. If w satisfies the condition

$$\begin{cases} w_1\not\sim 1,\\ w-u_1\sim 1,\end{cases}$$

then $(\pi,x-e_1)_p\not\sim 1$ and $(\pi\alpha,x-e_1)_p\not\sim 1$, where $p=Irr_{K|k}(\theta)$. Furthermore, in case $k_2\equiv 1\,(mod\,2)$ we have $(\alpha,x-e_1)_{x-e_3}\not\sim 1$ and $(\alpha,x-e_1)_{x-e-\pi^{k_1+1}}\sim(\alpha,\pi^{k_1+1})\not\sim 1$ since $f(e+\pi^{k_1+1})\sim -u_1\sim 1$, otherwise.

Finally, consider subcase 3. Let $\theta_i=e+\pi^{k_1}w_i,\ i=1,2,3$. Then $f(\theta_i)\sim w_i(w_i-u_1)(w_i-u_2)$, provided $k_1=k_2$. If w_i satisfies the conditions

$$\begin{cases} w_1\not\sim 1,\\ w_2-u_1\not\sim 1,\\ w_3-u_2\not\sim 1,\\ (w_1-u_1)(w_1-u_2)\not\sim 1,\\ w_2(w_2-u_2)\not\sim 1,\\ w_3(w_3-u_3)\not\sim 1,\end{cases}$$

then $f(\theta_i)\sim 1$ and $(\pi,x-e_i)_{p_i}\not\sim 1$, where $p_i=Irr_{K|k}(\theta_i)$. Moreover, $(\pi,(x-e_i)(x-e_j))_{p_l}\sim(\pi,(x^2-\pi^k u)(x-e_l))_{p_l}\sim(\pi,x-e_l)_{p_l}\not\sim 1$, where $\{i,j,l\}=\{1,2,3\}$.

Let now $k_1<k_2$ are even and $-u_1\not\sim 1$. Let θ be as in 2. Then $(\pi,x-e_1)_p\not\sim 1$ and $(\pi,(x-e_1)(x-e_2))_p\not\sim 1$, where p is as above. We have also $(\pi,x-e_2)_{x-e_1}\sim(\pi,-u_1)\not\sim 1$. The lemma is proved.

Now to prove the theorem it is enough to check that the algebra from lemma 12 is not isomorphic to all algebras from lemma 14 in the corresponding subcases.

Let firstly $k\equiv 1\,(mod\,2)$. The extension of constants by $\sqrt{\alpha}$ leaves us in case 1 of lemma 12, and in cases 1 and 2 of lemma 13, therefore the algebra $(eu\pi,x)$ does not split by this extension. On the other hand, all algebras from case 1 of lemma 14 split by it. And in case 2 of this lemma one only need to prove that $(eu\pi,x)\not\sim(\pi,x-e_1)$ and $(eu\pi,x)\not\sim(\pi\alpha,x-e_1)$. But if w,θ and p are as in the proof of this case, then after the extension indicated above $(eu\pi,x)_p\sim(\pi,e)\sim 1$. On the other hand, $(\pi\alpha,x-e_1)_p\sim(\pi,x-e_1)_p\not\sim 1$.

Let us take a unit $\eta\in O_K^*$ with the usual properties satisfying the condition $\bar\eta\neq\bar e,\ \eta\not\sim 1,\ \eta-e\sim 1$. Then $f(\eta)\sim\eta-e\sim 1$ and $(eu\pi,x)_q\not\sim 1$, where $q=Irr_{K|k}(\eta)$. We have also $(\pi,x-e_i)_q\sim 1$ and $(\pi,(x-e_1)(x-e_2))_q\sim 1$. Thus case $k\equiv 1\,(mod\,2)$ is considered.

If k is even and $-e\sim 1$, then for any $A\in\{\pi,\alpha,\pi\alpha\}$ $(A,x-e_i)_{x^3-\pi}\sim(A,-e)\sim 1$, $(A,(x-e_1)(x-e_1))_{x^3-\pi}\sim 1$, but $(\alpha,x-\pi^{k/2}w)_{x^3-\pi}\not\sim 1$.

Finally, let k be even, $-e \not\sim 1$, then $(\pi, x^2 - \pi^k u)_{x-\pi^{k/2}w} \not\sim 1$ and $(\alpha, x - e_i)_{x-\pi^{k/2}w} \sim 1$, $(\alpha, (x - e_1)(x - e_2))_{x-\pi^{k/2}w} \sim 1$. If p_i are the points from the proof of lemma 14 in which the corresponding algebras are nontrivial, then $(\pi, x^2 - \pi^k u)_{p_i} \sim 1$ and we are done.

Let us pass to the next two cases.

Theorem 12 *Let f be as in cases IV or V. Then the group ${}_2Br\, C$ is generated by the classes of the algebras (π, α), $(\pi, x^2 + ax + b)$, and classes of the following algebras.*

- *Case IV.*
  1. (π, x), $(\pi, x - \pi^k u)$, *if* $be \not\sim 1$.
  2. (π, x), (α, x), *if* $be \sim 1$.
- *Case V.*
  1. (α, x), $(\alpha, x - \pi^{k_1} u_1)$, *if* $k_1 \equiv 1\,(mod\,2)$.
  2. (α, x), (π, x), *if* $k_1 \equiv 0\,(mod\,2)$ *and either* $k_2 \equiv 1\,(mod\,2)$ *or* $k_1 < k_2$ *and* $-bu_1 \sim 1$.
  3. (π, x), $(\pi, x - \pi^{k_1} u_1)$, *if* $k_1 \equiv k_2 \equiv 0\,(mod\,2)$ *and either* $k_1 = k_2$ *or* $-bu_1 \not\sim 1$.

Proof. In case IV $(\pi, x) \not\sim 1$, $(\pi, x - \pi^k u) \not\sim 1$ by lemma 3. For the same reason $(\pi, x) \not\sim 1$ in $V.2$, $V.3$ and $(\pi, x - \pi^{k_1} u_1) \not\sim 1$ in $V.3$. In IV if $be \not\sim 1$, then for $w \in O_K^*$ we have $f(\pi^k w) \sim \alpha w(w - e)$ and $(\pi, x(x - \pi^k u))_{Irr_{K|k}(\pi^k w)} \sim (\pi, w(w - e)) \not\sim 1$, provided $w(w - e) \not\sim 1$. If $be \sim 1$, then

$$\left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}(\tilde{C})}\right)_x \sim \left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}\langle x\rangle}\right) \not\sim 1,$$

so $(\alpha, x) \not\sim 1$. Replacing $\nu = \pi\alpha$ one can check that $(\pi\alpha, x) \not\sim 1$.

In case V if $k_1 \equiv 1\,(mod\,2)$, then $(\alpha, x)_{x-\pi^{k_1}u_1} \not\sim 1$ and $(\alpha, x - \pi^{k_1} u_1)_x \not\sim 1$. Moreover, $(\alpha, x(x - \pi^{k_1} u_1))_{x-\pi^{k_1}u_1} \sim (\alpha, (x^2 + ax + b)(x - \pi^{k_2} u_2))_{x-\pi^{k_1}u_1} \sim (\alpha, b(u_1 - \pi^{k_2-k_1} u_2)\pi^{k_1}) \not\sim 1$.

Now we are in subcase $V.2$. If $k_2 \equiv 1\,(mod\,2)$, then $(\alpha, x)_{x-\pi^{k_2}u_2} \not\sim 1$. In case $k_2 \equiv 0\,(mod\,2)$ we have $f(\pi^{k_1+1}) \sim -bu_1 \sim 1$ and $(\alpha, x)_{x-\pi^{k_1+1}} \not\sim 1$. Replacing the prime element one can check that $(\alpha\pi, x) \not\sim 1$.

Consider the last subcase in V. We have $f(\pi^{k_1} w) \sim bw(w - u_1)(w - \pi^{k_2-k_1} u_2)$ and $(\pi, x(x - \pi^{k_1} u_1))_{Irr_{K|k}(\pi^{k_1} w)} \sim (\pi, w(w - u_1)) \not\sim 1$, provided that

$$\left\{\begin{array}{l} w(w - u_1) \not\sim 1, \\ b(w - \pi^{k_2-k_1} u_2) \not\sim 1. \end{array}\right.$$

It remains to prove that the algebra $(\pi, x^2 + ax + b)$ is nontrivial and not isomorphic to the algebras listed above. In case IV we have $(\pi, x^2+ax+b) \not\sim 1$, $(\pi, x(x^2+ax+b)) \not\sim 1$, $(\pi, (x-\pi^k u)(x^2+ax+b)) \not\sim 1$ by lemma 3. In $IV.1$ $(\pi, x(x - \pi^k u)(x^2 + ax + b)) \not\sim 1$ by the same reason. The extension of constants by $\sqrt{\pi}$ leaves us in case $IV.2$, so that $(\pi, x^2 + ax + b) \not\sim (\alpha, x)$. After this extension we also have that $(\pi\alpha, x) \sim (\alpha, x) \not\sim (\pi, x^2 + ax + b)$.

Finally, let us compare the algebra $(\pi, x^2 + ax + b)$ with the others in case V. First of all, $(\pi, x^2 + ax + b) \not\sim 1$ and $(\pi, (x^2 + ax + b)x(x - \pi^{k_1} u_1)) \not\sim 1$ by analogy with IV. After the extension of constants by $\sqrt{\alpha}$ we are not in the same case because the polynomial $x^2 + ax + b$ splits after this extension. But the algebra $(\pi, x^2 + ax + b)$ is still nontrivial in view of lemma 3. So this algebra is not isomorphic to (α, x), $(\alpha, x - \pi^{k_1} u_1)$ and $(\alpha, x(x - \pi^{k_1} u_1))$. To complete the proof one only need to check that $(\pi, x^2 + ax + b)$ is not isomorphic to (π, x), $(\pi\alpha, x)$ in $V.2$ and (π, x), $(\pi, x - \pi^{k_1} u_1))$ in $V.3$. In $V.2$ we have $f(\pi^{k_1} w) \sim b(w - u_1)$, where w is as above. Let $p = Irr_{K|k}(\pi^{k_1} w)$. Then $(\pi, x)_p \sim (\pi\alpha, x)_p \sim (\pi, w)$, $(\pi, x^2 + ax + b)_p \sim (\pi, b)$. If we require

$$\left\{\begin{array}{l} b(w - u_1) \sim 1, \\ bw \not\sim 1, \end{array}\right.$$

then $(\pi, x)_p \sim (\pi\alpha, x)_p \not\sim (\pi, x^2 + ax + b)_p$.

Consider subcase $V.3$. Let $k_1 = k_2$. To prove that $(\pi, x^2+ax+b) \not\sim (\pi, x), (\pi, x-\pi^{k_1}u_1)$ we need to solve the systems

$$\begin{cases} bw_1 \not\sim 1, \\ (w_1-u_1)(w_1-u_2) \not\sim 1 \end{cases} \text{ and } \begin{cases} b(w_2-u_1) \not\sim 1, \\ w_2(w_2-u_2) \not\sim 1. \end{cases}$$

By the same way one can prove $(\pi, x^2+ax+b) \not\sim (\pi, x)$, provided $k_1 < k_2$. Finally, in this case $(\pi, (x^2+ax+b)(x-\pi^{k_1}u_1))_x \sim (\pi, -bu_1) \not\sim 1$.

Theorem 13 *Let f be as in cases VI or VII. Then the group ${}_2Br\, C$ is generated by the classes of the algebra (π, α), and classes of the following algebras.*

- *Case VI. $(\pi, x+e_1)$, $(\pi, x+e_2)$.*
- *Case VII. $(\pi, x+e)$ and $(\alpha, x+e)$ if $-e \sim 1$, $(\pi, x+e')$ if $-e \not\sim 1$.*

and

1. *$(\alpha, x^2-\pi^k u)$ if $m \geq k/2$, $k \equiv 1\,(mod\,2)$.*
2. *$(\pi, x^2-\pi^k u)$ if $m \geq k/2$, $k \equiv 0\,(mod\,4)$.*
3. *$(\pi E, x-\pi^{k/2}w)$ if $m \geq k/2$, $k \equiv 2\,(mod\,4)$.*
4. *$(\pi uve_1e_2, x)$ if $m < k/2$, $k \equiv 1\,(mod\,2)$, $m \equiv 0\,(mod\,2)$.*
5. *$(\pi E, x-\pi^{k/2}w)$ if $m < k/2$, $k \equiv 0\,(mod\,2)$, $m \equiv 1\,(mod\,2)$.*
6. *$(\alpha, x-\pi^m v)$ if $m < k/2$, $k \equiv m \equiv 1\,(mod\,2)$, $uve_1e_2 \sim 1$.*
7. *(α, x) if $m < k/2$, $k \equiv m \equiv 1\,(mod\,2)$, $uve_1e_2 \not\sim 1$.*
8. *$(\pi, x^2-\pi^k u)$ if $m < k/2$, $k \equiv m \equiv 0\,(mod\,2)$, $-ve_1e_2 \not\sim 1$.*
9. *$(\alpha, x-\pi^{k/2}w)$ if $m < k/2$, $k \equiv m \equiv 0\,(mod\,2)$, $-ve_1e_2 \sim 1$,*

where $w \in O_k^*$, $w^2-u \not\sim 1$, E is chosen such that $f(\pi^{k/2}w) \sim \pi E$, and in case VII the units e_1 and e_2 must be omitted in all algebras and conditions listed above.

Proof. The algebras $(\pi, x+e_1)$, $(\pi, x+e_2)$, $(\pi, (x+e_1)(x+e_2))$ are nontrivial in case VI. The same is true for the algebras $(\pi, x+e)$ and $(\pi, x+e')$ in case VII. In the last case we also have

$$\left(\frac{\bar\alpha\,,\; x+\bar e}{\bar k(\tilde E)}\right)_{x+\bar e} \sim \left(\frac{\bar\alpha\,,\; x+\bar e}{\bar k(\sqrt{x})}\right)_{x+\bar e} \sim \left(\frac{\bar\alpha\,,\; x+\bar e}{\bar k\langle x+\bar e\rangle(\sqrt{-\bar e})}\right) \not\sim 1,$$

provided $-e \sim 1$. Moreover, then the extension by $\sqrt{\alpha}$ does not split $(\pi, x+e)$, so $(\pi\alpha, x+e) \not\sim 1$. If $-e \not\sim 1$, then $(\pi, (x+e)(x+e'))_{x+e} \sim (\pi, (x^2-\pi^k u)(x-\pi^m v))_{x+e} \sim (\pi, -e) \not\sim 1$.

Let us consider now the remaining algebra simultaneously in both cases.

1. $(\alpha, x^2-\pi^k u)_{x-\pi^m v} \not\sim 1$. Furthermore, $(\alpha, x^2-\pi^k u) \not\sim (\pi, x+e_i), (\pi, (x+e_1)(x+e_1))$ in case VI since the last two algebras do not split after the extension by $\sqrt{\alpha}$. For the same reason $(\alpha, x^2-\pi^k u) \not\sim (\pi, x+e)$, $(\pi, x+e')$, and $(\pi\alpha, x+e)$ in case VII. We have also $(\alpha, x+e)_{x-\pi^m v} \sim 1$, so that $(\alpha, x+e) \not\sim (\alpha, x^2-\pi^k u)$ and $(\pi, (x+e)(x+e'))_{x-\pi^m v} \sim (\pi, ee') \sim 1$, so that $(\alpha, x^2-\pi^k u) \not\sim (\pi, (x+e)(x+e'))$.
2. Let $w \in O_K^*$ such that

$$\begin{cases} w^2-u \not\sim 1, \\ a(w-v\pi^{m-k/2}) \not\sim 1, \end{cases} \quad \text{where } a = \begin{cases} e_1e_2 \text{ in } VI, \\ 1, \text{ in } VII. \end{cases}$$

Then $f(\pi^{k/2}w) \sim 1$ and $(\pi, x^2-\pi^k u)_p \not\sim 1$, where $p = Irr_{K|k}(\pi^{k/2}w)$. Let now $w_0 \in O_K^*$ such that

$$\begin{cases} w_0^2-u \sim 1, \\ a(w_0-v\pi^{m-k/2}) \sim 1, \end{cases}$$

Then $f(\pi^{k/2}w) \sim 1$ and $(\pi, x^2 - \pi^k u)_{p_0} \sim 1$, where $p_0 = Irr_{K|k}(\pi^{k/2}w_0)$. In case $e_i \sim 1$ we have $(\pi, x^2 - \pi^k u)_p \not\sim (\pi, x+e_i)_p$ and if $e_i \not\sim 1$, then $(\pi, x^2 - \pi^k u)_{p_0} \not\sim (\pi, x+e_i)_{p_0}$. In any case $(\pi, x^2 - \pi^k u) \not\sim (\pi, x+e_i)$. By the same way one can check that $(\pi, x^2 - \pi^k u) \not\sim (\pi, (x+e_1)(x+e_2))$, $(\pi, x+e)$, $(\pi, x+e')$ and $(\pi, (x+e)(x+e'))$. Finally, $(\pi, x^2 - \pi^k u) \not\sim (\pi\alpha, x+e)$, $(\alpha, x+e)$ since two last algebras do not split by $\sqrt{\pi}$.
3.
$$(\pi E, x - \pi^{k/2}w)_{x^2 - \pi^k u} \sim \left(\frac{\pi \,,\, w - \sqrt{u}}{k(\sqrt{u})\langle D(x^2 - \pi^k u)\rangle} \right) \not\sim 1.$$
To finish the consideration of this subcase it is enough to observe that the completion in $x^2 - \pi^k u$ splits any other algebra from the list.
4. $f(0) \sim \pi uve_1e_2$, so the algebra $(\pi uve_1e_2, x)$ is unramified. Let w be from O_K^* and $q = Irr_{K|k}(\pi^m w)$. Then $f(\pi^m w) \sim e_1e_2(w - v)$ in VI or $w - v$ in VII. Furthermore, $(\pi uve_1e_2, x)_q \sim (\pi uve_1e_2, w)$. If we require $w \not\sim 1$ and $e_1e_2(w-v) \sim 1$, then $(\pi uve_1e_2, x)_q \not\sim 1$. Similarly, if $we_i \not\sim 1$ and $e_1e_2(w-v) \sim 1$, then $(\pi uve_1e_2, x)_q \not\sim (\pi, x+e_i)_q$. By analogy we can prove that $(\pi uve_1e_2, x)$ is not isomorphic to the remaining algebras.
5. This case can be considered similarly to case 3.
6. $f(0) \sim uve_1e_2$ (or uv in VII) ~ 1. Therefore $(\alpha, x - \pi^m v)_x \sim (\alpha, \pi^m) \not\sim 1$. The extension by $\sqrt{\alpha}$ does not split the algebras $(\pi, x+e_i)$, $(\pi, x+e)$, $(\pi, x+e')$, $(\pi, (x+e_1)(x+e_2))$, $(\pi\alpha, x+e)$. So it is sufficient to check that $(\alpha, x - \pi^m v) \not\sim (\alpha, x+e), (\pi, (x+e)(x+e'))$. But this is true in view of $(\alpha, x+e)_x \sim 1$ and $(\pi, (x+e)(x+e'))_x \sim 1$.
7. The algebra (α, x) is unramified since $f(0) \sim \alpha$. $(\alpha, x)_{x - \pi^m v} \sim (\alpha, \pi^m v) \not\sim 1$. In case VI the extension by $\sqrt{\alpha}$ does not split the algebras $(\pi, x+e_i)$, $(\pi, (x+e_1)(x+e_2))$ since this extension leaves us in case VI. The same holds for the algebras $(\pi, x+e)$, $(\pi\alpha, x+e)$ and $(\pi, x+e')$. Indeed, after this extension $(\pi\alpha, x+e) \sim (\pi, x+e) \not\sim 1$ since we are in VII again. And $(\pi, x+e')$ is also nontrivial after the extension because of the units e and e' are symmetrical. Finally, $(\alpha, x+e)_{x - \pi^m v} \sim (\alpha, e) \sim 1$ and $(\pi, (x+e)(x+e'))_{x - \pi^m v} \sim 1$.
8. $f(\pi^{k/2}w) \sim -ve_1e_2(w^2 - u) \sim 1$, provided $w^2 - u \not\sim 1$. Thus $(\pi, x^2 - \pi^k u)_{x - \pi^{k/2}w} \not\sim 1$. The algebras $(\pi, (x^2 - \pi^k u)(x+e_i))$, $(\pi, (x^2 - \pi^k u)(x+e))$, $(\pi, (x^2 - \pi^k u)(x+e'))$, $(\pi, (x^2 - \pi^k u)(x+e_1)(x+e_2))$ are nontrivial by lemma 3. We only need to prove $(\pi, x^2 - \pi^k u) \not\sim (\pi, (x+e)(x+e')), (\alpha, x+e), (\pi\alpha, x+e)$. But two last algebras do not split after the extension by $\sqrt{\pi}$ because this extension leaves us in case $VII, -e \sim 1$. Finally, $(\pi, (x+e)(x+e'))_{x - \pi^{k/2}w} \sim 1$.
9. $f(\pi^{k/2}w) \sim \alpha$, so the algebra $(\alpha, x - \pi^{k/2}w)$ is unramified. Let $q = Irr_{k(\sqrt[3]{\pi})|k}(\pi^m \sqrt[3]{\pi})$. Then $f(\pi^m \sqrt[3]{\pi}) \sim -ve_1e_2 \sim 1$. Hence
$$(\alpha, x - \pi^{k/2}w)_q \sim \left(\frac{\alpha \,,\, \sqrt[3]{\pi}}{k(\sqrt[3]{\pi})\langle q\rangle} \right) \not\sim 1.$$
Since the extension by $\sqrt{\pi}$ leaves us in the current subcase, in order to complete the proof one only need to check that $(\alpha, x - \pi^{k/2}w) \not\sim (\alpha, x+e)$. But this holds since $(\alpha, x+e)_q \sim (\alpha, e) \sim 1$.

Theorem 14 *Let f be as in case $VIII$. Then the group ${}_2Br\,C$ is generated by the classes of the algebra (π, α), and the classes of the following algebras ($l_i = x - \pi^{m_i}v_i$).*

- (π, l_1), $(\pi, x+e)$ *if* $e \not\sim 1$.
- (π, l_1), (α, l_1) *if* $e \sim 1$.

and

1. $(\alpha, x^2 - \pi^k u)$ *if* $k/2 < m_1$, $e \sim 1$, $k \equiv 1\,(mod\,2)$.
2. $(\pi, x^2 - \pi^k u)$ *if* $k/2 < m_1$, $e \sim 1$, $k \equiv 0\,(mod\,2)$ *and* $-1 \sim 1$.
3. $(\alpha, x - \pi^{k/2}w)$ *if* $k/2 < m_1$, $e \sim 1$, $k \equiv 0\,(mod\,2)$ *and* $-1 \not\sim 1$.
4. $(\pi, x^2 - \pi^k u)$ *if* $k/2 < m_1$, $e \not\sim 1$, $k \equiv 0\,(mod\,2)$ *and* $-1 \not\sim 1$.

5. (α, l_1) *if* $k/2 < m_1$, $e \not\sim 1$, $k \equiv 0\,(mod\,2)$ *and* $-1 \sim 1$.

6. (α, l_1) *if* $k/2 < m_1$, $e \not\sim 1$, $k \equiv 1\,(mod\,2)$.

7. $(\pi, x^2 - \pi^k u)$ *if* $k/2 = m_1$ *and either* $m_1 < m_2$ *or* $\bar{v}_1 \neq \bar{v}_2$.

8. $(\pi, x^2 - \pi^k u)$ *if* $k/2 = m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ *and* $e(v_1^2 - u) \not\sim 1$.

9. (α, l_1) *if* $k/2 = m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ *and* $e \not\sim 1$, $v_1^2 - u \not\sim 1$.

10. $(\alpha, x - \pi^{k/2} w)$ *if* $k/2 = m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ *and* $e \sim 1$, $v_1^2 - u \sim 1$.

11. $(\alpha, x^2 - \pi^k u)$ *if* $m_1 < k/2 \leq m_2$, $k \equiv 1\,(mod\,2)$.

12. $(\pi, x^2 - \pi^k u)$ *if* $m_1 < k/2 \leq m_2$, $k \equiv 0\,(mod\,2)$ *and* $m_1 + k/2 \equiv 0\,(mod\,2)$.

13. $(\pi E, x - \pi^{k/2} w)$ *if* $m_1 < k/2 \leq m_2$, $k \equiv 0\,(mod\,2)$ *and* $m_1 + k/2 \equiv 1\,(mod\,2)$.

14. $(\pi E, x - \pi^{k/2} w)$ *if* $m_2 < k/2$, $m_1 + m_2 \equiv 1\,(mod\,2)$, $k \equiv 0\,(mod\,2)$.

15. $(\alpha, x^2 - \pi^k u)$ *if* $m_2 < k/2$, $m_1 + m_2 \equiv 1\,(mod\,2)$, $k \equiv 1\,(mod\,2)$ *and* $-ev_1v_2u \sim 1$.

16. (α, x) *if* $m_2 < k/2$, $m_1 + m_2 \equiv 1\,(mod\,2)$, $k \equiv 1\,(mod\,2)$ *and* $-ev_1v_2u \not\sim 1$.

17. $(\pi F, x)$ *if* $m_2 < k/2$, $m_1 + m_2 \equiv 0\,(mod\,2)$, $k \equiv 1\,(mod\,2)$, *where* $F \in O_k^*$, $\pi F \sim f(0)$.

18. $(\pi, x^2 - \pi^k u)$ *if* $m_2 < k/2$, $m_1 + m_2 \equiv 0\,(mod\,2)$, $k \equiv 0\,(mod\,2)$, $ev_1v_2 \not\sim 1$.

19. $(\alpha, x - \pi^{k/2} w)$ *if* $m_2 < k/2$, $m_1 + m_2 \equiv 0\,(mod\,2)$, $k \equiv 0\,(mod\,2)$, $ev_1v_2 \sim 1$,

where $w \in O_k^*$, $w^2 - u \not\sim 1$ *and* E *is chosen such that* $f(\pi^{k/2} w) \sim \pi E$.

Proof. In view of lemma 3 the algebras (π, l_1) and $(\pi, l_1(x+e))$ are nontrivial. Furthermore, $(\pi, x+e)_{l_1} \not\sim 1$, provided $e \not\sim 1$. After the extension by $\sqrt{\alpha}$ we have $(\pi\alpha, l_1) \sim (\pi, l_1) \not\sim 1$. Finally, $f(\sqrt[3]{\pi}) \sim e$, so in case $e \sim 1$

$$(\alpha, l_1)_{x^3-\pi} \sim \left(\frac{\alpha\ ,\ \sqrt[3]{\pi}}{k(\sqrt[3]{\pi})\langle x^3 - \pi\rangle}\right) \not\sim 1.$$

Consider now the third generator.
1. The algebra $(\alpha, x^2 - \pi^k u)$ is nontrivial since $(\alpha, x^2 - \pi^k u)_{l_1} \not\sim 1$. $(\alpha, x^2 - \pi^k u)_{x^3-\pi} \sim 1$, on the other hand $(\alpha, l_1)_{x^3-\pi} \not\sim 1$, therefore these algebras are not isomorphic. The algebra $(\alpha, x^2 - \pi^k u)$ is not isomorphic to (π, l_1), $(\pi\alpha, l_1)$ because the extension by $\sqrt{\alpha}$ does not split two last algebras.
2. $(\pi, x^2 - \pi^k u)_{l_1} \sim (\pi, -u) \not\sim 1$. By lemma 3 $(\pi, x^2 - \pi^k u) \not\sim (\pi, l_1)$. Since after the extension by $\sqrt{\pi}$ one has $(\pi\alpha, l_1) \sim (\alpha, l_1) \not\sim 1$, it follows that $(\pi, x^2 - \pi^k u)$ is not isomorphic to these algebras.
3. Because of $f(\pi^{k/2} w) \sim \alpha$ the algebra $(\alpha, x - \pi^{k/2} w))$ is unramified. Moreover, $(\alpha, x - \pi^{k/2} w))_{x^3-\pi} \not\sim 1$. Let $q = Irr_{k(\sqrt[3]{\pi})|k}(\pi^s)$, where $s = k/2 + 1/3$. Then $f(\pi^s) \sim 1$, therefore

$$(\alpha, x - \pi^{k/2} w)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{k/2}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \text{ and } (\alpha, l_1)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{s}}{k(\sqrt[3]{\pi})\langle q\rangle}\right),$$

so $(\alpha, x - \pi^{k/2} w)_q \not\sim (\alpha, l_1)_q$. To complete the consideration of this subcase it is enough to observe that after the extension by $\sqrt{\pi}$ we have $(\pi, l_1) \sim 1 \not\sim (\alpha, x - \pi^{k/2} w)$ and $(\pi\alpha, l_1) \sim (\alpha, l_1) \not\sim (\alpha, x - \pi^{k/2} w)$.
4. $f(\pi^{k/2} w)) \sim e(w^2 - u) \sim 1$ and $(\pi, x^2 - \pi^k u)_{x - \pi^{k/2} w} \not\sim 1$. By lemma 3 $(\pi, x^2 - \pi^k u) \not\sim (\pi, l_1), (\pi, l_1(x + e))$. Finally, $(\pi, x^2 - \pi^k u)_{l_1} \sim 1 \not\sim (\pi, x + e)_{l_1}$.
5. Let

$$\varepsilon = \begin{cases} 1, & \text{if } k \equiv 0\,(mod\,4), \\ 2, & \text{if } k \equiv 2\,(mod\,4), \end{cases}$$

$q = Irr_{k(\sqrt[3]{\pi})|k}(\pi^{k/2+\varepsilon/3})$. Then $f(\pi^{k/2+\varepsilon/3}) \sim 1$ and

$$(\alpha, l_1)_q \sim \left(\frac{\alpha\ ,\ \pi^{k/2+\varepsilon/3}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \sim \left(\frac{\alpha\ ,\ \sqrt[3]{\pi}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \not\sim 1.$$

The algebra (α, l_1) is not isomorphic to the remaining algebras since the extension by $\sqrt{\pi}$ leaves us in the current subcase and therefore it does not split this algebra.
6. Since the extension by $\sqrt{\alpha}$ leaves us in case $VIII$, the algebras (π, l_1) and $(\pi, l_1(x+e))$ do not split by this extension, so they are not isomorphic to (α, l_1). Furthermore, if $m_1 < m_2$ or $\bar{v}_1 \neq \bar{v}_2$, then $(\alpha, l_1)_{l_2} \sim (\alpha, \pi^{m_1})$ and $(\alpha, l_1)_{l_1} \sim (\alpha, (x^2 - \pi^k u) l_2(x+e))_{l_1} \sim (\alpha, \pi^{m_1+1})$. One of these two algebras is nontrivial. Thus $(\alpha, l_1) \not\sim 1$. Let now $m_1 = m_2$ and $v_2 = v_1 + \pi^s \tau,\ s > 0, \tau \in O_k^*$. Then $(\alpha, l_1)_{l_2} \sim (\alpha, \pi^{m_1+s})$ and $(\alpha, l_1)_{l_1} \sim (\alpha, \pi^{m_1+s+1})$, so that the algebra (α, l_1) is nontrivial again. It remains to check that $(\alpha, l_1) \not\sim (\pi, x+e)$. But $(\pi, x+e)_{l_1} \not\sim 1$ and $(\pi, x+e)_{l_2} \not\sim 1$. On the other hand, among the algebras $(\alpha, l_1)_{l_1}$ and $(\alpha, l_1)_{l_2}$ a trivial one exists.
7. $f(\pi^{k/2} w) \sim (w^2 - u)(w - v_1)(w - v_2 \pi^{m_2 - m_1})$. If we require

$$\begin{cases} w^2 - u \not\sim 1, \\ (w - v_1)(w - v_2 \pi^{m_2 - m_1}) \not\sim 1, \end{cases}$$

$w \in O_K^*,\ \bar{w} \neq \bar{v}_1, \bar{v}_2,\ p = Irr_{K|k}(\pi^{k/2} w)$, then $f(\pi^{k/2} w) \sim 1$ and $(\pi, x^2 - \pi^k u)_p \not\sim 1$. As we already had seen above, $(\pi, x^2 - \pi^k u) \not\sim (\pi, l_1), (\pi, l_1(x+e))$ by lemma 3. If

$$\begin{cases} w_0^2 - u \sim 1, \\ (w_0 - v_1)(w_0 - v_2 \pi^{m_2 - m_1}) \sim 1, \end{cases}$$

$w_0 \in O_K^*,\ \bar{w}_0 \neq \bar{v}_1, \bar{v}_2,\ p_0 = Irr_{K|k}(\pi^{k/2} w_0)$, then $f(\pi^{k/2} w_0) \sim 1$ and $(\pi, x^2 - \pi^k u)_{p_0} \sim 1$. But $(\pi, x+e)_{p_0} \sim (\pi, e) \not\sim 1$, provided $e \not\sim 1$. Thus in case $e \not\sim 1$ we are done. Case $e \sim 1$ can be considered by analogy with subcase 2.
8. Let $e \not\sim 1,\ v_1^2 - u \sim 1$. There exists $w \in O_k^*$ such that $w^2 - u \not\sim 1$. So $(\pi, x^2 - \pi^k u)_{x - \pi^{k/2} w} \sim (\pi, w^2 - u) \not\sim 1$ since $f(\pi^{k/2} w) \sim (w^2 - u)e \sim 1$. If $e \sim 1,\ v_1^2 - u \not\sim 1$, then $(\pi, x^2 - \pi^k u)_{x - \pi^{m_1} v_1} \sim (\pi, v_1^2 - u) \not\sim 1$. Let us check that the algebra $(\pi, x^2 - \pi^k u)$ is not isomorphic to the other ones. If $e \not\sim 1$, then $(\pi, x^2 - \pi^k u) \not\sim (\pi, l_1), (\pi, l_1(x+e))$ as above. We also have $(\pi, x^2 - \pi^k u)_{x - \pi^{m_1} v_1} \sim (\pi, v_1^2 - u) \sim 1$ and $(\pi, x+e)_{x - \pi^{m_1} v_1} \sim (\pi, e) \not\sim 1$. Let now $e \sim 1$. Then one can argue as in subcase 2.
9. Let

$$\varepsilon = \begin{cases} 1, & \text{if } k \equiv 0\,(mod\,4), \\ 2, & \text{if } k \equiv 2\,(mod\,4), \end{cases}$$

$\theta = \pi^{k/2}(v_1 + (\sqrt[3]{\pi})^{\varepsilon}),\ q = Irr_{k(\sqrt[3]{\pi})|k}(\theta)$. Then $f(\theta) \sim (v_1^2 - u)e \sim 1$ and

$$(\alpha, l_1)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m_1+\varepsilon}}{k(\sqrt[3]{\pi})\langle q \rangle} \right) \not\sim 1.$$

$(\alpha, l_1) \not\sim (\pi, l_1), (\pi, x+e), (\pi, l_1(x+e))$ since the extension by $\sqrt{\pi}$ leaves us in the current subcase and therefore it does not split the algebra (α, l_1).
10. The algebra $(\alpha, x - \pi^{k/2} w)$ is unramified. $f(\sqrt[3]{\pi}) \sim e \sim 1$ so $(\alpha, x - \pi^{k/2} w)_{x^3 - \pi} \not\sim 1$. After the extension by $\sqrt{\pi}$ we are in the current subcase again, therefore it is enough only to prove that $(\alpha, x - \pi^{k/2} w) \not\sim (\alpha, l_1)$. To check this fact assume $\theta_0 = \pi^{k/2}(v_1 + \sqrt[3]{\pi}),\ q_0 = Irr_{k(\sqrt[3]{\pi})|k}(\theta_0)$. Then $f(\theta_0) \sim 1$ and

$$(\alpha, l_1)_{q_0} \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{k/2+1}}{k(\sqrt[3]{\pi})\langle q_0 \rangle} \right) \text{ and } (\alpha, x - \pi^{k/2} w)_{q_0} \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{k/2}}{k(\sqrt[3]{\pi})\langle q_0 \rangle} \right).$$

11. $(\alpha, x^2 - \pi^k u)_{l_2} \sim (\alpha, \pi^k) \not\sim 1$. If $e \sim 1$, then $\sqrt{\alpha}$ does not split (π, l_1) and $(\pi\alpha, l_1)$ so that these algebras are not isomorphic to $(\alpha, x^2 - \pi^k u)$. Moreover, $(\alpha, x^2 - \pi^k u)_{x^3 - \pi} \not\sim (\alpha, l_1)_{x^3 - \pi}$. Let $e \not\sim 1$. Then we have $(\alpha, x^2 - \pi^k u) \not\sim (\pi, l_1), (\pi, l_1(x+e))$. Finally, $(\alpha, x^2 - \pi^k u)_{l_1} \sim 1$, but $(\pi, x+e)_{l_1} \not\sim 1$.
12. Let $w \in O_K^*$ such that

$$\begin{cases} w^2 - u \not\sim 1, \\ -v_1 e(w - v_2 \pi^{m_2 - k/2}) \not\sim 1, \end{cases}$$

then $f(\pi^{k/2} w) \sim 1$ and $(\pi, x^2 - \pi^k u)_{Irr_{K|k}}(\pi^{k/2} w) \not\sim 1$. If $e \not\sim 1$, then $(\pi, x^2 - \pi^k u) \not\sim (\pi, l_1), (\pi, l_1(x+e))$. We also have $(\pi, x^2 - \pi^k u)_{l_1} \sim 1$ and $(\pi, x+e)_{l_1} \not\sim 1$. If $e \sim 1$, then the extension by $\sqrt{\pi}$ leaves us in subcase 12, so we can argue as in subcase 2.

13. The algebra $(\pi E, x-\pi^{k/2}w)$ is unramified by the choice of E.

$$(\pi E, x-\pi^{k/2}w)_{x^2-\pi^k u} \sim \left(\frac{\pi E\ ,\ \pi^{k/2}(\sqrt{u}-w)}{k(\sqrt{u})\langle x^2-\pi^k u\rangle(\sqrt{\pi(x^2-\pi^k u)})}\right) \sim$$

$$\sim \left(\frac{\pi\ ,\ w-\sqrt{u}}{k(\sqrt{u})\langle\sqrt{\pi(x^2-\pi^k u)}\rangle}\right) \not\sim 1.$$

On the other hand,

$$(\pi, l_1)_{x^2-\pi^k u} \sim \left(\frac{\pi\ ,\ -\pi^{m_1}v_1}{k(\sqrt{u})\langle\sqrt{\pi(x^2-\pi^k u)}\rangle}\right) \sim 1.$$

The same is true for the algebras $(\pi, x+e)$, $(\pi, l_1(x+e))$, (α, l_1) and $(\pi\alpha, l_1)$.

14. This case is totally similar to the previous one.

15. $f(0)\sim 1$, so $(\alpha, x^2-\pi^k u)_x \sim (\alpha, \pi^k) \not\sim 1$. The extension by $\sqrt{\alpha}$ does not split (π, l_1), $(\pi\alpha, l_1)$, $(\pi, l_1(x+e))$ in the corresponding cases, therefore $(\alpha, x^2-\pi^k u)$ is not isomorphic to them. If $e\sim 1$, then $(\alpha, l_1)_{x^3-\pi} \not\sim (\alpha, x^2-\pi^k u)_{x^3-\pi}$. Finally, if $e\not\sim 1$, then $(\pi, x+e)_{l_1} \not\sim (\alpha, x^2-\pi^k u)_{l_1}$.

16. $f(0)\sim -ev_1v_2u\sim\alpha$. Thus the algebra (α, x) is unramified. We have also

$$(\alpha, x)_{x^2-\pi^k u} \sim \left(\frac{\alpha\ ,\ \sqrt{\pi^k u}}{k(\sqrt{\pi u})\langle x^2-\pi^k u\rangle(\sqrt{-\alpha(x^2-\pi^k u)})}\right) \sim$$

$$\sim \left(\frac{\alpha\ ,\ \sqrt{\pi u}}{k(\sqrt{\pi u})\langle\sqrt{-\alpha(x^2-\pi^k u)}\rangle}\right) \not\sim 1.$$

As it was already noticed, the other algebras are trivial after completion at $x^2-\pi^k u$.

17. The algebra $(F\pi, x)$ is unramified by the choice of F. Let $\eta\in O_K^*$, K be as above, $\eta\not\sim 1$, $\eta+e\sim 1$, and $r = Irr_{K|k}(\eta)$. Then

$$(F\pi, x)_r \sim \left(\frac{F\pi\ ,\ \eta}{K\langle r\rangle}\right) \not\sim 1,$$

since $f(\eta)\sim\eta+e\sim 1$. On the other hand, $(\alpha, l_1)_q\sim(\alpha,\eta)\sim 1$ and $(\pi, x+e)_q\sim(\pi,\eta+e)\sim 1$. Let firstly $m_1<m_2$ or $\bar v_1\neq\bar v_2$. Then for $w\in O_K^*$, $\bar w\neq\bar v_1, \bar v_2$ we have $f(\pi^{m_2}w)\sim -e(w-v_2)(v_1-w\pi^{m_2-m_1})$. Let $p = Irr_{K|k}(\pi^{m_2}w)$. Then $(\pi, l_1)_p \sim (\pi, (-1)^{m_1+1}(v_1-w\pi^{m_2-m_1}))$ and $(F\pi, x)_p\sim(\pi, (-F)^{m_2}w)$. We require

$$\begin{cases} -e(w-v_2)(v_1-w\pi^{m_2-m_1})\sim 1,\\ -F^{m_2}w(v_1-w\pi^{m_2-m_1})\not\sim 1.\end{cases}$$

Then $(\pi, l_1)_p\not\sim(F\pi, x)_p$. Similarly one can check $(\pi\alpha, l_1)\not\sim(F\pi, x)$ and $(\pi, l_1(x+e))\not\sim(F\pi, x)$. Consider now the case $m_1=m_2$, $v_2=v_1+\pi^r\mu, r>0, \mu\in O_k^*$. Suppose $w\in O_K^*$, $\bar w\neq\bar\mu$, $s = Irr_{K|k}(\pi^{m_2}(v_1+\pi^r w))$. Then $f(\pi^{m_2}(v_1+\pi^r w))\sim ew(w-\mu)$, $(\pi, l_1)_s\sim(\pi, (-1)^{m_1+r}w)$ and $(F\pi, x)_s\sim(\pi, (-F)^{m_1}v_1)$. If we require

$$\begin{cases} ew(w-\mu)\sim 1,\\ (-1)^r F^{m_1}v_1 w\not\sim 1,\end{cases}$$

then $(\pi, l_1)_s\not\sim(F\pi, x)_s$. For the remaining algebras the arguments are similar.

18. If $w\in O_k^*$, $w^2-u\not\sim 1$, then $(\pi, x^2-\pi^k u)_{x-\pi^{k/2}w}\not\sim 1$ in view of $f(\pi^{k/2}w)\sim ev_1v_2(w^2-u)\sim 1$. Let η and r be from the previous case, then $(\pi, l_1)_r\not\sim 1$, $(\pi, l_1(x+e))_r\not\sim 1$ and $(\pi\alpha, l_1)_r\not\sim 1$, but $(\pi, x^2-\pi^k u)_r\sim 1$. The extension by $\sqrt{\pi}$ leaves us in the current case, so it does not split (α, l_1), provided $e\sim 1$, but it splits $(\pi, x^2-\pi^k u)$. Finally, $(\pi, x+e)_{l_1}\not\sim(\pi, x^2-\pi^k u)_{l_1}$ if $e\not\sim 1$.

19. $f(\pi^{k/2}w)\sim ev_1v_2(w^2-u)\sim\alpha$, hence the algebra $(\alpha, x-\pi^{k/2}w)$ is unramified. If

$$\varepsilon = \begin{cases} 1, \text{ if } m_2\equiv 0\,(mod\,2),\\ 2, \text{ if } m_2\equiv 1\,(mod\,2),\end{cases}$$

then $f(\pi^{m_2+\varepsilon/3}) \sim ev_1v_2 \sim 1$. Assume $p = Irr_{k(\sqrt[3]{\pi})|k}(\pi^{m_2+\varepsilon/3})$. Then

$$(\alpha, x-\pi^{k/2}w)_p \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m_2+\varepsilon}}{k(\sqrt[3]{\pi})\langle p\rangle}\right) \not\sim 1.$$

Since the extension by $\sqrt{\pi}$ leaves us in case 19, it does not split the algebra $(\alpha, x-\pi^{k/2}w)$. Hence $(\alpha, x-\pi^{k/2}w) \not\sim (\pi, l_1)$, $(\pi, x+e)$, $(\pi, l_1(x+e))$. Since after this extension $(\pi\alpha, l_1) \sim (\alpha, l_1)$, it is enough to show that $(\alpha, x-\pi^{k/2}w) \not\sim (\alpha, l_1)$, provided $e \sim 1$. Let $p_0 = Irr_{k(\sqrt[3]{\pi})|k}(\pi^{m_2+1/3})$. Then $f(\pi^{m_2+1/3}) \sim 1$ as above and

$$(\alpha, x-\pi^{k/2}w)_{p_0} \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m_2+1}}{k(\sqrt[3]{\pi})\langle p_0\rangle}\right), \qquad (\alpha, l_1)_{p_0} \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m_1}}{k(\sqrt[3]{\pi})\langle p_0\rangle}\right).$$

Because of $m_1 \equiv m_2\,(mod\,2)$ we have $(\alpha, x-\pi^{k/2}w)_{p_0} \not\sim (\alpha, l_1)_{p_0}$ and the theorem is proved.

Without loss of generality one can assume that in case IX either $0 < k < 4$ and $k/2 \le m_1$ (case $IX.1$) or $m_1 = 1$, $k > 2$ (case $IX.2$). Let $l_i = x - \pi^{k_i}v_i$, $i = 1, 2, 3$.

Theorem 15 *Let f be as in case $IX.1$ and as soon as $m_1 = m_2 = m_3$, then $\bar{v}_3 \ne \bar{v}_i$, $i = 1, 2$. Then the group ${}_2Br\,C$ is generated by the class of the algebra (π, α) and the classes of the following algebras.*

1. $(\alpha, x^2-\pi^k u)$, (α, l_1), (α, l_2), *if*
   $k \equiv 1\,(mod\,2)$, $m_1 \equiv 0\,(mod\,2)$.
2. $(\pi E, p_w)$, (α, l_1), (α, l_2), *if*
   $k \equiv 0\,(mod\,2)$, $m_1 \equiv 1\,(mod\,2)$.
3. $(\pi E, p_w)$, (α, l_2), (π, l_2), *if*
   $k \equiv m_1 \equiv 0\,(mod\,2)$, $m_1 < m_2$, $uv_1 \sim 1$.
4. $(\alpha, x^2-\pi^k u)$, (α, l_2), (π, l_2), *if*
   $k \equiv m_1 \equiv 1\,(mod\,2)$, $m_1 < m_2$, $uv_1 \sim 1$.
5. $k \equiv m_1 \equiv 1\,(mod\,2)$, *and if* $m_1 < m_2$, *then* $uv_1 \not\sim 1$.
   a) $(\alpha, x^2-\pi^k u)$, (π, l_1), (α, l_1),
   if $m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ *and* $\xi = -u(v_1 - v_3\pi^{m_3-m_1}) \sim 1$.
   b) $(\alpha, x^2-\pi^k u)$, (π, l_1), (π, l_2), *otherwise.*
6. $k \equiv m_1 \equiv 0\,(mod\,2)$, *and if* $m_1 < m_2$, *then* $uv_1 \not\sim 1$.
   a) $(\pi E, p_w)$, (π, l_1), (α, l_1),
   if $m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ *and* $\xi \sim 1$.
   b) $(\pi E, p_w)$, (π, l_1), (π, l_2), *otherwise,*

where $w \in O_K^*$ satisfies the following conditions. If $m_1 > 1$, then $w \in O_k^*$, $w^2 - u \not\sim 1$, $E = \alpha w$, and $p_w = x - \pi w$. If $m_1 = 1$, then

- *If* $1 < m_2$, *then*
$$\begin{cases} w^2 - u \not\sim 1, \\ w(w-v_1) \sim 1. \end{cases}$$
- *If* $1 = m_2 < m_3$, *then*
$$\begin{cases} (w^2-u)(w-v_1)w \not\sim 1, \\ (w^2-u)(w-v_2)w \not\sim 1, \\ (w^2-u)(w-v_1)(w-v_2) \not\sim 1. \end{cases}$$

- *If $1 = m_2 = m_3$, then*

$$\begin{cases} (w^2-u)(w-v_1)(w-v_2) \not\sim 1, \\ (w^2-u)(w-v_1)(w-v_3) \not\sim 1, \\ (w^2-u)(w-v_2)(w-v_3) \not\sim 1 \end{cases}$$

for a suitable unramified odd degree extension $K|k$, $p_w = Irr_{K|k}(\pi w)$, $E = N_{K|k}(E_0)$, and $E_0 \in O_K^*$ is such that $f(\pi w) \sim \pi E_0$ in K.

It is convenient to divide the proof into several parts. We assume in the following lemmas that all conditions of the above theorem are satisfied.

Lemma 16 *If $k \equiv 1\,(mod\,2)$, then the algebra $(\alpha, x^2-\pi^k u)$ is unramified and nontrivial. Otherwise, the same is true for the algebra $(\pi E, p_w)$, where w and p_w are from the above theorem.*

We have $(\alpha, x^2-\pi^k u)_{l_i} \sim (\alpha, \pi^k) \not\sim 1$, since $k < 2m_i$, $i = 1,2,3$. If $k = 2$, $m_1 > 1$, then $f(\pi w) \sim \pi w(w^2-u) \sim \pi E$ and the algebra $(\pi E, x - \pi w)$ is unramified. $(\pi E, x-\pi w)_{l_i} \sim (\pi\alpha w, -\pi w) \not\sim 1$. Let now $k = 2$, $m_1 = 1$. Then

$$f(\pi w) \sim \pi(w^2-u)(w-v_1)(w-\pi^{m_2-1}v_2)(w-\pi^{m_3-1}v_3).$$

Let $\bar{w} \neq \bar{v}_1$ and if $m_i = 1$, then $\bar{w} \neq \bar{v}_i$. Then $f(\pi w) \sim \pi E_0$, $E_0 \in O_K^*$, where

$$E_0 = \begin{cases} (w^2-u)(w-v_1), \text{ if } 1 < m_2, \\ (w^2-u)(w-v_1)(w-v_2)w, \text{ if } 1 = m_2 < m_3, \\ (w^2-u)(w-v_1)(w-v_2)(w-v_3), \text{ if } 1 = m_2 = m_3. \end{cases}$$

We have

$$(\pi E, p_w)_{p_w} \sim \left(\frac{\pi E_0\ ,\ p_w}{K(\sqrt{\pi E_0})\langle p_w\rangle}\right) \sim 1,$$

since $E = N_{K|k}(E_0) \sim E_0$ in K. So the algebra $(\pi E, p_w)$ is unramified. Let us require $(\pi E, p_w)_{l_i} \not\sim 1$, $i = 1,2,3$. Then

$$(\pi E, p_w)_{l_i} \sim (\pi, N_{K|k}(E_0(w-v_i))),$$

$$(\pi E, p_w)_{l_2} \sim (\pi, N_{K|k}(E_0(w-v_1\pi^{m_i-1}))),\ i = 2,3,$$

and these algebras are nontrivial iff $E_0(w-v_1) \not\sim 1$ and $E_0(w-v_i) \not\sim 1$, provided $m_i = 1$ or $E_0 w \not\sim 1$, provided $m_i > 1$, $i = 2,3$. It just gives the conditions on w from the theorem. These conditions can always be satisfied by lemma 6. The lemma is proved.

Lemma 17 *If $k \not\equiv m_1\,(mod\,2)$, then the classes of the algebras (π,α), (α, l_1), (α, l_2) together with the class of the algebra from lemma 16 give all generators of ${}_2Br\,C$.*

Proof. Let firstly $k/2 < m_1$. Then $(\alpha, l_1)_{l_1} \sim (\alpha, (x^2-\pi^k u)l_2 l_3)_{l_1} \sim (\alpha, \pi^{k+m_1} l_2(\pi^{m_1}v_1)) \sim (\alpha, \pi l_2(\pi^{m_1}v_1))$ and $(\alpha, l_1)_{l_2} \sim (\alpha, l_2(\pi^{m_1}v_1))$. Thus exactly one of the algebras $(\alpha, l_1)_{l_1}$ and $(\alpha, l_1)_{l_2}$ is nontrivial. Since the algebra from lemma 16 is nontrivial at l_1, l_2, the algebra (α, l_1) is not isomorphic to it. Let now $k = 2$, $m_1 = 1$. Then $(\alpha, l_1)_{l_1} \sim (\alpha, \pi l_2(\pi^{m_1}v_1)) \not\sim (\alpha, l_1)_{l_2}$ again.

Consider the algebras (α, l_2) and $(\alpha, l_1 l_2)$. We have

$$(\alpha, l_2)_{l_1} \sim (\alpha, l_2(\pi^{m_1}v_1)),\ (\alpha, l_2)_{l_2} \sim (\alpha, \pi^{k+m_2} l_2(\pi^{m_1}v_1)),\ (\alpha, l_2)_{l_3} \sim (\alpha, \pi^{m_2}),$$

$$(\alpha, l_1 l_2)_{l_2} \sim (\alpha, (x^2-\pi^k u)l_3)_{l_2} \sim (\alpha, \pi^{k+m_2}),\ (\alpha, l_1 l_2)_{l_3} \sim (\alpha, \pi^{m_1+m_2}).$$

Hence $(\alpha, l_1 l_2)_{l_2} \not\sim (\alpha, l_1 l_2)_{l_3}$. This means that $(\alpha, l_1 l_2) \not\sim 1$ and $(\alpha, l_1 l_2)$ is not isomorphic to the algebra from lemma 16. Moreover, if $m_1 \equiv m_2\,(mod\,2)$, then $(\alpha, l_2)_{l_2} \sim (\alpha, \pi l_2(\pi^{m_1}v_1)) \not\sim (\alpha, l_2)_{l_1}$ and $(\alpha, l_2)_{l_2} \sim (\alpha, l_2(\pi^{m_1}v_1)) \sim (\alpha, \pi^{m_1}) \not\sim (\alpha, l_2)_{l_3}$ otherwise, since in this case $m_1 < m_2$. The lemma is proved.

Lemma 18 *If $k \equiv m_1\,(mod\,2)$, $m_1 < m_2$, $uv_1 \sim 1$, then the classes of the algebras (π,α), (α, l_2), (π, l_2) together with the class of the algebra from lemma 16 give all generators of ${}_2Br\,C$.*

Proof. Let $\varepsilon \in \{1,2\}$, $\tau \in O_k^*$, $\theta = \pi^{m_1}(\sqrt[3]{\pi})^{\varepsilon}\tau$, and $p = Irr_{k(\sqrt[3]{\pi})|k}(\theta)$. Then $f(\theta) \sim \pi^{k+m_1}uv_1 \sim 1$. Assume $A \in \{\pi, \alpha, \pi\alpha\}$. Then

$$(A, l_2)_p \sim \left(\frac{A\ ,\ \theta}{k(\sqrt[3]{\pi})\langle p\rangle}\right) \sim \left(\frac{A\ ,\ (\sqrt[3]{\pi})^{m_1+\varepsilon}\tau}{k(\sqrt[3]{\pi})\langle p\rangle}\right),$$

and for any A we are always able to find ε and τ such that $(A, l_2)_p \not\sim 1$. Let now compare the algebras (A, l_2) with the algebra from lemma 16. Assume that ε and τ are such that $(A, l_2)_p \sim 1$. In view of $k/2 < m_1 + \varepsilon/3$ and $k/2 < m_2$ we have that the algebra from lemma 16 is nontrivial at p because it is nontrivial at l_2. This completes the proof of the lemma.

Lemma 19 *Let $k \equiv m_1 \,(mod\, 2)$ and if $m_1 < m_2$, then $uv_1 \not\sim 1$. Then the classes of the algebras (π, α), (π, l_1), the class of the algebra from lemma 16 and the class of the following algebra give all generators of ${}_2Br\, C$.*

- (π, l_2), *if* $m_1 < m_2$ *or* $\bar{v}_1 \neq \bar{v}_2$.
- (π, l_2), *if* $m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ *and* $\xi \not\sim 1$.
- (α, l_1), *if* $m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ *and* $\xi \sim 1$.

Proof. In this case $k/2 < m_1$. If $m_1 < m_2$, then $(\pi, l_1)_{l_1} \sim (\pi, -\pi^k u)$ and $(\pi, l_1)_{l_2} \sim (\pi, -\pi^{m_1}v_1) \sim (\pi, -\pi^k v_1)$. Since $uv_1 \not\sim 1$ we have that among the algebras $(\pi, l_1)_{l_1}$ and $(\pi, l_1)_{l_2}$ there is just one trivial algebra. So $(\pi, l_1) \not\sim 1$ and (π, l_1) is not isomorphic to the algebra from lemma 16. Let now $m_1 = m_2$ and $\mu_i \in O_K^*$. Then $f(\pi^{m_1}\mu_i) \sim -u(\mu_i - v_1)(\mu_i - v_2)(\mu_i - v_3\pi^{m_3-m_1})$. If $p_i = Irr_{K|k}(\pi^{m_1}\mu_i)$, then $(\pi, l_1)_{p_i} \sim (\pi, (-1)^{m_1}(\mu_i - v_1))$. One can find K, μ_1, and μ_2 such that $f(\pi^{m_1}\mu_1) \sim f(\pi^{m_1}\mu_2) \sim 1$ and $(\pi, l_1)_{p_1} \not\sim 1$, $(\pi, l_1)_{p_2} \sim 1$. Since the algebra from lemma 16 is not trivial at p_2 we have again that $(\pi, l_1) \not\sim 1$ and (π, l_1) is not isomorphic to this algebra.

Let us describe the last generator. Consider firstly the case $m_1 < m_2$. Then $f(\pi^{m_1}\mu_i) \sim -u(\mu_i - v_1)$. If μ_i and p_i are such that $-u(\mu_i - v_1) \sim 1$, $i = 1,2,3,4$, $(-1)^{m_1}\mu_1 \sim 1$, $(-1)^{m_1}\mu_2 \not\sim 1$, $\mu_3 \not\sim -u$, and $\mu_4 \sim -u$, then $(\pi, l_2)_{p_1} \sim (\pi, (-1)^{m_1}\mu) \sim 1$, $(\pi, l_2)_{p_2} \not\sim 1$, $(\pi, l_1 l_2)_{p_3} \sim (\pi, \mu_3(\mu_3 - v_1)) \sim (\pi, -u\mu_3) \not\sim 1$ and $(\pi, l_1 l_2)_{p_4} \sim 1$. Thus $(\pi, l_2) \not\sim 1$, $(\pi, l_2) \not\sim (\pi, l_1)$, (π, l_2) is not isomorphic to the algebra from lemma 16 and $(\pi, l_1 l_2)$ is not.

Let $m_1 = m_2$. Then $(\pi, l_2) \not\sim 1$ and (π, l_2) is not isomorphic to the algebra from lemma 16 in view of the symmetry for the algebras (π, l_1) and (π, l_2). Let $\bar{v}_1 \neq \bar{v}_2$. Then one can find μ_1, μ_2 such that $f(\pi^{m_1}\mu_1) \sim 1$, $i = 1,2$ and $(\mu_1 - v_1)(\mu_1 - v_2) \not\sim 1$, $(\mu_2 - v_1)(\mu_2 - v_2) \sim 1$. This means that $(\pi, l_1 l_2)_{p_1} \not\sim 1$ and $(\pi, l_1 l_2)_{p_2} \sim 1$.

Let now $v_2 = v_1 + \pi^s\tau$, $q = Irr_{K|k}(\pi^{m_1}(v_1 + \pi^s\eta))$, $\eta \in O_K^*$. Then $f(\pi^{m_1}(v_1 + \pi^s\eta)) \sim \xi\eta(\eta - \tau)$. If $\xi \not\sim 1$, then we require $\eta(\eta - \tau) \not\sim 1$. In this case $f(\pi^{m_1}(v_1 + \pi^s\eta)) \sim 1$, $(\pi, l_1 l_2)_q \sim (\pi, \eta(\eta - \tau)) \not\sim 1$. Moreover, $(\pi, l_1 l_2)_{l_3} \sim (\pi, (\pi^{m_3-m_1}v_3 - v_1)(\pi^{m_3-m_1}v_3 - v_2)) \sim 1$.

Finally, let $\xi \sim 1$. After the replacement of prime element $\nu = \pi\alpha$ one can check that the algebra $(\pi\alpha, l_1)$ is nontrivial and not isomorphic to the algebra from lemma 16. To complete the proof it is sufficient to show that the same is true for the algebra (α, l_1). If $\theta_i = \pi^{m_1}(v_1 + (\sqrt[3]{\pi})^i)$, $i = 1,2$, $t_i = Irr_{k(\sqrt[3]{\pi})|k}(\theta_i)$, then $f(\theta_i) \sim \xi \sim 1$, and

$$(\alpha, l_1)_{t_i} \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m_1+i}}{k(\sqrt[3]{\pi})\langle t_i\rangle}\right).$$

So among the algebras $(\alpha, l_1)_{t_i}$, $i = 1,2$ there is exactly one nontrivial algebra. Since the algebra from lemma 16 is nontrivial at t_1 and t_2, the lemma is proved.

Now the proof of the theorem follows immediately from lemmas 16–19.

Let us consider the remaining subcase of case $IX.1$. In this subcase one can assume that $f(x) = (x^2 - \pi^k u)(x - \pi^m v)(x - \pi^m v_1)(x - \pi^m v_2)$, $v_i = v + \pi^{s_i}\tau_i$, $\tau_i \in O_k^*$, $i = 1,2$, $0 < s_1 \leq s_2$ and if $s_1 = s_2$, then $\bar{\tau}_1 \neq \bar{\tau}_2$. Let $l = x - \pi^m v$, $l_i = x - \pi^m v_i$, and

$$\xi = \begin{cases} u\tau_1, & \text{if } k/2 < m_1, \\ (u - v^2)\tau_1, & \text{otherwise.} \end{cases}$$

Theorem 16 *Let f be as in case IX.1, $m_1 = m_2 = m_3$, $\bar{v}_1 = \bar{v}_2 = \bar{v}_3$. Then the group ${}_2Br\, C$ is generated by the class of the algebra (π, α) and the classes of the following algebras.*

- $A = (\alpha, x^2 - \pi^k u)$, *if* $k \equiv 1\,(mod\,2)$.
- $A = (\pi E, x - \pi w)$, *if* $k = 2$,

where $w \in O_k^*$, $w^2 - u \not\sim 1$,

$$E = \begin{cases} \alpha w, \text{ if } m > 1, \\ \alpha(w - v), \text{ if } m = 1, \end{cases}$$

and

1. (α, l), (α, l_1), *if*
   $k + m + s_1 \equiv 1\,(mod\,2)$.
2. (π, l), (π, l_1), *if*
   $k + m + s_1 \equiv 0\,(mod\,2)$, $s_1 = s_2$.
3. (π, l), (π, l_2), *if*
   $k + m + s_1 \equiv 0\,(mod\,2)$, $s_1 < s_2$, $\xi \not\sim 1$.
4. (π, l), (α, l), *if*
   $k + m + s_1 \equiv 0\,(mod\,2)$, $s_1 < s_2$, $\xi \sim 1$,

Proof. If k is odd, then $(\alpha, x^2 - \pi^k u)_l \not\sim 1$ and $(\alpha, x^2 - \pi^k u)_{l_i} \not\sim 1$, $i = 1, 2$. Let $k = 2$. Then $f(\pi w) \sim \pi E$ and the algebra A is unramified. We also have $(\pi E, x - \pi w)_l \sim (\pi, w^2 - u) \not\sim 1$. By the same argument $(\pi E, x - \pi w)_{l_i} \not\sim 1$. So A is nontrivial at l, l_1, l_2 in any case.

Observe that $(\alpha, l)_l \sim (\alpha, \pi^{k+s_1+s_2})$, $(\alpha, l)_{l_1} \sim (\alpha, \pi^{m+s_1})$, $(\alpha, l)_{l_2} \sim (\alpha, \pi^{m+s_2})$, $(\alpha, l_1)_l \sim (\alpha, \pi^{m+s_1})$, $(\alpha, l_1)_{l_1} \sim (\alpha, \pi^k)$, $(\alpha, l_1)_{l_2} \sim (\alpha, \pi^{m+s_1})$. Let $k + m + s_1 \equiv 1\,(mod\,2)$. Then among the algebras $(\alpha, l)_l$ and $(\alpha, l)_{l_2}$ there exists exactly one nontrivial algebra, so that $(\alpha, l) \not\sim 1$ and $(\alpha, l) \not\sim A$. The same is true for the algebras $(\alpha, l_1)_{l_1}$ and $(\alpha, l_1)_{l_2}$. Finally, $(\alpha, ll_1)_l \sim (\alpha, \pi^{s_1+s_2+1})$ and $(\alpha, ll_1)_{l_2} \sim (\alpha, \pi^{s_1+s_2})$.

Let now $k + m + s_1 \equiv 0\,(mod\,2)$, $\theta_i = \pi^m(v + \pi^{s_1}\eta_i)$, $\eta_i \in O_K^*$, $p_i = Irr_{K|k}(\theta_i)$,

$$\delta = \begin{cases} -u, \text{ if } k/2 < m, \\ v^2 - u, \text{ otherwise.} \end{cases}$$

Then

$$f(\theta_i) \sim \begin{cases} \delta(\eta_i - \tau_1), \text{ if } s_1 < s_2, \\ \delta\eta_i(\eta_i - \tau_1)(\eta_i - \tau_2) \text{ otherwise.} \end{cases}$$

Let us require $f(\theta_i) \sim 1$, then $(\pi, l)_{p_i} \sim (\pi, (-1)^k \eta_i)$. One can check that $A_{p_i} \not\sim 1$. There exist η_1, η_2 such that $(-1)^k \eta_1 \not\sim 1$, $(-1)^k \eta_2 \sim 1$. Then $(\pi, l)_{p_1} \not\sim 1$ and $(\pi, l)_{p_2} \not\sim A_{p_2}$. Consider the case $s_1 = s_2$. We have $(\pi, l_1)_{p_i} \sim (\pi, (-1)^k(\eta_i - \tau_1))$. One can always find an element η_i such that

$$\begin{cases} (-1)^k(\eta_1 - \tau_1) \sim 1, \\ \delta(-1)^k \eta_1(\eta_1 - \tau_2) \sim 1 \end{cases} \Longrightarrow (\pi, l_1)_{p_1} \not\sim A_{p_1}.$$

$$\begin{cases} (-1)^k(\eta_2 - \tau_1) \not\sim 1, \\ \delta(-1)^k \eta_2(\eta_2 - \tau_2) \not\sim 1 \end{cases} \Longrightarrow (\pi, l_1)_{p_2} \not\sim 1.$$

$$\begin{cases} \eta_3(\eta_3 - \tau_1) \not\sim 1, \\ \delta(\eta_3 - \tau_2) \not\sim 1 \end{cases} \Longrightarrow (\pi, l_1)_{p_3} \not\sim (\pi, l_1)_{p_3}.$$

$$\begin{cases} \eta_4(\eta_4 - \tau_1) \sim 1, \\ \delta(\eta_4 - \tau_2) \sim 1 \end{cases} \Longrightarrow (\pi, l_1)_{p_4} \not\sim A_{p_4} \otimes (\pi, l_1)_{p_4}.$$

Let now $k + m + s_1 \equiv 0\,(mod\,2)$, $s_1 < s_2$, $\xi \not\sim 1$. Consider the algebra (π, l_2). If $v' = v + \pi^{s_2}\tau_2$, then $f(x) = (x^2 - \pi^k u)(x - \pi^m v')(x - \pi^m(v' + \pi^{s_1}\tau_1'))(x - \pi^m(v' + \pi^{s_2}\tau_2'))$. So $(\pi, l_2) = (\pi, l') \not\sim 1, A$ since we are in the case $k+m+s_1 \equiv 0\,(mod\,2)$ again. Furthermore, $(\pi, ll_2)_{l_1} \sim (\pi, (\pi^{m+s_1}\tau_1)^2) \sim 1$, therefore $(\pi, ll_2) \not\sim A$. We have also $(\pi, ll_2)_l \sim (\pi, \xi) \not\sim 1$.

Finally, let $k+m+s_1 \equiv 0\,(mod\,2)$, $s_1 < s_2$, $\xi \sim 1$. If $\theta_i = \pi^m(v + \pi^{s_1}(\sqrt[3]{\pi})^i)$, $q_i = Irr_{k(\sqrt[3]{\pi})|k}(\theta_i)$, $i = 1, 2$, then $f(\theta) \sim \xi \sim 1$ and

$$(\alpha, l)_{q_i} \sim \left(\frac{\alpha\,,\,(\sqrt[3]{\pi})^{m+s_1+i}}{k(\sqrt[3]{\pi})\langle q_i\rangle}\right), \quad A_{q_i} \not\sim 1.$$

Thus $(\alpha, l) \not\sim 1, A$. To finish the proof one only need to check that $(\pi\alpha, l) \not\sim 1, A$. But it is evident after the replacing the prime element by $\nu = \alpha\pi$.

Theorem 17 *Let f be as in case IX.2 Then the group ${}_2Br\,C$ is generated by the class of the algebra (π, α) and the classes of the following algebras.*

1. *$k/2 \le m_2$. Two generators are (α, l_1), (α, l_2). The third one is*

   a) (π, l_2) *if* $k \equiv 1\,(mod\,2)$,

   b) $(\pi E, x - \pi^{k/2}w)$ *otherwise.*

2. *$m_2 < k/2 \le m_3$. Two generators are (α, l_1), (α, l_2). The third one is*

   a) $(\alpha, x^2 - \pi^k u)$ *if* $k \equiv 1\,(mod\,2)$,

   b) (π, l_3) *if* $k \equiv 0\,(mod\,2)$, $m_2 + k/2 \equiv 1\,(mod\,2)$,

   c) $(\pi E, x - \pi^{k/2}w)$ *if* $k \equiv 0\,(mod\,2)$, $m_2 + k/2 \equiv 0\,(mod\,2)$,

3. *$m_3 < k/2$ and if $m_2 = m_3 = 1$, then $\bar{v}_3 \ne \bar{v}_1$, $\bar{v}_3 \ne \bar{v}_2$. Two generators are (α, l_1), (α, l_2). The third one is*

   a) $(\pi E, x - \pi^{k/2}w)$ *if* $m_2 + m_3 \equiv k \equiv 0\,(mod\,2)$,

   b) $(\alpha, x - \pi^{k/2}w)$ *if* $m_2 + m_3 \equiv 1\,(mod\,2)$, $k \equiv 0\,(mod\,2)$ *and* $-v_1v_2v_3 \sim 1$,

   c) $(\pi, x^2 - \pi^k u)$ *if* $m_2 + m_3 \equiv 1\,(mod\,2)$, $k \equiv 0\,(mod\,2)$ *and* $-v_1v_2v_3 \not\sim 1$,

   d) $(\alpha, x^2 - \pi^k u)$ *if* $m_2 + m_3 \equiv 0\,(mod\,2)$, $k \equiv 1\,(mod\,2)$ *and* $uv_1v_2v_3 \sim 1$,

   e) $(f(0), x)$, *otherwise.*

4. *$m_2 = m_3 = 1 < k/2$, $v_i = v_1 + \pi^{s_i}\tau_i$, $i = 2, 3$, $s_2 \le s_3$ and if $s_1 = s_2$, then $\bar{\tau}_2 \ne \bar{\tau}_3$. Two generators are*

   a) (α, l_1), (α, l_2) *if* $s_2 \equiv 0\,(mod\,2)$,

   b) (π, l_1), (π, l_2) *if* $s_2 = s_3 \equiv 1\,(mod\,2)$,

   c) (π, l_1), (α, l_1) *if* $s_2 \equiv 1\,(mod\,2)$, $s_2 < s_3$ *and* $-\tau_2 \sim 1$,

   d) (π, l_1), (π, l_3) *if* $s_2 \equiv 1\,(mod\,2)$, $s_2 < s_3$ *and* $-\tau_2 \not\sim 1$.

   The third one is

   α) $(\alpha, x^2 - \pi^k u)$, *if* $k \equiv 1\,(mod\,2)$, $uv_1 \sim 1$,

   β) (α, x), *if* $k \equiv 1\,(mod\,2)$, $uv_1 \not\sim 1$,

   γ) $(\pi E, x - \pi^{k/2}w)$, *if* $k \equiv 0\,(mod\,2)$,

where $w \in O_k^*$, $w^2 - u \not\sim 1$, and $E \in O_k^*$ can be found from the condition $f(\pi^{k/2}w) \sim \pi E$.

Proof.
1. We have $(\alpha, l_1)_{l_2} \sim (\alpha, -\pi v_1) \not\sim 1$, $(\alpha, l_2)_{l_1} \sim (\alpha, \pi v_1) \not\sim 1$ and $(\alpha, l_1 l_2)_{l_1} \sim (\alpha, (x^2 - \pi^k u) l_3)_{l_1} \sim (\alpha, (\pi v_1)^2 \pi v_1) \not\sim 1$.

Assume firstly $k \equiv 1\,(mod\,2)$. Suppose $\varepsilon \in \{1,2\}$, $\varepsilon \equiv m_2\,(mod\,2)$, $\theta = \alpha(\sqrt[3]{\pi})^{3m_2-\varepsilon}$ and $p = Irr_{k(\sqrt[3]{\pi})|k}(\theta)$. Then $3k/2 < v_{k(\sqrt[3]{\pi})}(\theta) < 3m_2$. Let $uv_1 \sim 1$. Then we have $f(\theta) \sim uv_1 \sim 1$. Thus

$$(\pi, l_2)_p \sim \left(\frac{\alpha\ ,\ \theta_1}{k(\sqrt[3]{\pi})\langle p\rangle}\right) \sim \left(\frac{\sqrt[3]{\pi}\ ,\ \alpha d^2}{k(\sqrt[3]{\pi})\langle p\rangle}\right) \not\sim 1.$$

So $(\pi, l_2) \not\sim 1$. If now uv_1 is arbitrary, then after the extension by $\sqrt{\alpha}$ we are still in subcase 1.a and $uv_1 \sim 1$, therefore the algebra (π, l_2) is nontrivial again and it does not split by $\sqrt{\alpha}$. Hence $(\pi, l_2) \not\sim (\alpha, l_1), (\alpha, l_2), (\alpha, l_1 l_2)$.

Let now $k \equiv 0\,(mod\,2)$. We have

$$(\pi E, x - \pi^{k/2} w)_{x^2 - \pi^k u} \sim \left(\frac{\pi\ ,\ w - \sqrt{u}}{k(\sqrt{u})}\right) \not\sim 1.$$

Since $(\alpha, l_i)_{x^2-\pi^k u} \sim 1$ and $(\alpha, l_1 l_2)_{x^2-\pi^k u} \sim 1$, we are done.
2. $(\alpha, l_1)_{l_3} \not\sim 1$, $(\alpha, l_1 l_2)_{l_1} \not\sim 1$. If $m_2 > 1$, then $(\alpha, l_2)_{l_1} \not\sim 1$. If $m_2 = 1$, then $(\alpha, l_2)_{l_3} \not\sim 1$. Let us find the last algebra. Consider the case $k \equiv 1\,(mod\,2)$. $(\alpha, x^2 - \pi^k u)_{l_3} \not\sim 1$. We have also $(\alpha, x^2 - \pi^k u)_{l_1} \sim 1$, $(\alpha, l_1 l_1)_{l_1} \not\sim 1$. If either $m_2 > 1$ or $\bar{v}_1 \neq \bar{v}_2$, then $(\alpha, l_1)_{l_2} \not\sim (\alpha, x^2 - \pi^k u)_{l_2}$ and $(\alpha, l_2)_{l_1} \not\sim (\alpha, x^2 - \pi^k u)_{l_1}$. Let $m_2 = 1$ and $\bar{v}_1 = \bar{v}_2$. Suppose $\theta = \pi\sqrt[3]{\pi}$. Then $f(\theta) \sim v_1 v_2 \sim 1$. So

$$(\alpha, l_1)_{x^3-\pi^4} \sim \left(\frac{\alpha\ ,\ -\pi v_1}{k(\sqrt[3]{\pi})\langle x^3 - \pi^4\rangle}\right) \not\sim 1,\quad (\alpha, l_2)_{x^3-\pi^4} \not\sim 1$$

but

$$(\alpha, x^2 - \pi^k u)_{x^3-\pi^4} \sim (\alpha, \theta^2) \sim 1.$$

Let now $k \equiv 0\,(mod\,2)$ and $m_2 + k/2 \equiv 0\,(mod\,2)$. Then

$$(\pi E, x - \pi^{k/2} w)_{x^2-\pi^k u} \sim \left(\frac{\pi\ ,\ w - \sqrt{u}}{k(\sqrt{u})\langle x^2 - \pi^k u\rangle}\right) \not\sim 1.$$

To complete the consideration of subcase 2.c it is enough to observe that all other algebras split at $x^2 - \pi^k u$.

Finally, if $m_2 + k/2 \equiv 1\,(mod\,2)$, then $f(\pi^{k/2} w_i) \sim v_1 v_2 (w_i^2 - u)(w - \pi^{m_3 - k/2} v_3)$, where $w_i \in O_K^*$ and $\bar{w}_i \neq \bar{v}_3$, provided $m_3 = k/2$. For $p_i = Irr_{K|k}(\pi^{k/2} w_i)$ we have $(\pi, l_3)_{p_i} \sim (\pi, (-1)^{k/2}(w - \pi^{m_3-k/2} v_3))$. Let us require

$$\begin{cases} (-1)^{k/2}(w_1 - \pi^{m_3-k/2} v_3) \not\sim 1, \\ (-1)^{k/2} v_1 v_2 (w_1^2 - u) \not\sim 1, \end{cases} \quad \begin{cases} (-1)^{k/2}(w_2 - \pi^{m_3-k/2} v_3) \sim 1, \\ (-1)^{k/2} v_1 v_2 (w_2^2 - u) \sim 1. \end{cases}$$

Then $(\pi, l_3)_{p_1} \not\sim 1$, $(\pi, l_3)_{p_2} \sim 1$. Furthermore, the algebras $(\pi, l_1)_{p_1}$ and $(\pi, l_1)_{p_2}$ are either both trivial or both nontrivial. The same is true for the algebras $(\pi, l_2)_{p_1}$, $(\pi, l_2)_{p_2}$ and for the algebras $(\pi, l_1 l_2)_{p_1}$, $(\pi, l_1 l_2)_{p_2}$. So (π, l_3) is not isomorphic to (π, l_i) and $(\pi, l_1 l_2)$.
3. We have $(\alpha, l_1)_{l_3} \not\sim 1$, $(\alpha, l_2)_{l_3} \not\sim 1$ and $(\alpha, l_1 l_2)_{l_1} \not\sim 1$. If $m_2 + m_3$ and k are even, then one can argue as in subcase 2.c.

Let $m_2 + m_3 \equiv 1\,(mod\,2)$, $k \equiv 0\,(mod\,2)$ and $-v_1 v_2 v_3 \sim 1$. Then $f(\pi^{k/2} w) \sim -v_1 v_2 v_3 (w^2 - u) \sim \alpha$, so the algebra $(\alpha, x - \pi^{k/2} w)$ is unramified. Then $f(\theta_i) \sim \theta_i^2 \sim 1$ and

$$(\alpha, x - \pi^{k/2} w)_{p_i} \sim \left(\frac{\alpha\ ,\ \theta_i}{k(\sqrt[3]{\pi})\langle p_i\rangle}\right),$$

therefore there is just one nontrivial algebra among $(\alpha, x - \pi^{k/2} w)_{p_i}$, $i = 1, 2$. In since the algebras $(\alpha, l_1)_{p_1}$, $(\alpha, l_1)_{p_2}$ are either simultaneously trivial or simultaneously nontrivial, we have $(\alpha, x - \pi^{k/2} w) \not\sim (\alpha, l_1)$. By analogy $(\alpha, x - \pi^{k/2} w) \not\sim (\alpha, l_2), (\alpha, l_1 l_2)$.

Let now $-v_1 v_2 v_3 \not\sim 1$. We have $f(\pi^{k/2} w) \sim -v_1 v_2 v_3 (w^2 - u) \sim 1$ and $(\pi, x^2 - \pi^k u)_{x - \pi^{k/2} w} \sim (\pi, w^2 - u) \not\sim 1$. For any i $(\pi, x^2 - \pi^k u)_{l_i} \sim 1$. On the other hand, $(\alpha, l_1)_{l_3} \not\sim 1$, $(\alpha, l_1 l_2)_{l_3} \not\sim 1$ and either $(\alpha, l_2)_{l_1} \not\sim 1$ or $(\alpha, l_2)_{l_3} \not\sim 1$.

The case of even k is done. Let k be odd. Then $f(0) \sim uv_1v_2v_3\pi^{m_2+m_3}$. If we are in subcase 3.d, then $f(0) \sim 1$ and $(\alpha, x^2-\pi^k u)_x \not\sim 1$. Moreover, $(\alpha, x^2-\pi^k u) \not\sim (\alpha, l_i), (\alpha, l_1l_2)$ as in the previous subcase.

Finally, in 3.e the algebra $(f(0), x)$ is unramified. Assume firstly that $m_2+m_3 \equiv 0\,(mod\,2)$, so $f(0) \sim \alpha$,

$$(\alpha, x)_{x^2-\pi^k u} \sim \left(\frac{\alpha\ ,\ \sqrt{\pi u}}{k(\sqrt{\pi u})\langle -\alpha(x^2-\pi^k u)\rangle}\right) \not\sim 1.$$

On the other hand,

$$(\alpha, l_1)_{x^2-\pi^k u} \sim \left(\frac{\alpha\ ,\ -\pi v_1}{k(\sqrt{\pi u})\langle -\alpha(x^2-\pi^k u)\rangle}\right) \sim 1 \text{ and } (\alpha, l_2)_{x^2-\pi^k u} \sim (\alpha, l_1l_2)_{x^2-\pi^k u} \sim 1.$$

Let now $m_2+m_3 \equiv 1\,(mod\,2)$. To finish the consideration of case 3 it is sufficient to prove that the algebra $(\pi uv_1v_2v_3, x)$ does not split after the extension by $\sqrt{\alpha}$. This extension leaves us in subcase 3.e, so we can assume that $-1 \sim u \sim v_i \sim 1$. If $\theta = \pi^{m_3}\sqrt[3]{\pi}w$ and $p = Irr_{k(\sqrt[3]{\pi})|k}(\theta)$, then

$$(\pi, x)_p \sim \left(\frac{\sqrt[3]{\pi}\ ,\ w}{k(\sqrt[3]{\pi})\langle p\rangle}\right) \not\sim 1,$$

since $f(\theta) \sim -v_1v_2v_3 \sim 1$.

4. Consider case $s_2 \equiv 0\,(mod\,2)$. We have $(\alpha, l_1)_{l_2} \sim (\alpha, \pi^{s_2+1}) \not\sim 1$, $(\alpha, l_2)_{l_1} \sim (\alpha, \pi^{s_2+1}) \not\sim 1$, and $(\alpha, l_1l_2)_{l_2} \sim (\alpha, l_3(x^2-\pi^k u))_{l_2} \sim (\alpha, \pi^{s_2+1}) \not\sim 1$.

If $s_2 = s_3 \equiv 1\,(mod\,2)$, let $\theta_i = \pi(v_1+\pi^{s_2}\mu_i)$, $\mu_i \in O_K^*$, $p_i = Irr_{K|k}(\theta_i)$, then $f(\theta_i) \sim \mu_i(\mu_i-\tau_2)(\mu_i-\tau_3)$,

$$(\pi, l_1)_{p_i} \sim \left(\frac{\pi\ ,\ \mu_i}{K(\sqrt{f(\theta_i)})\langle p_i\rangle}\right),\quad (\pi, l_2)_{p_i} \sim \left(\frac{\pi\ ,\ \mu_i-\tau_2}{K(\sqrt{f(\theta_i)})\langle p_i\rangle}\right),$$

$$(\pi, l_1l_2)_{p_i} \sim \left(\frac{\pi\ ,\ \mu_i(\mu_i-\tau_2)}{K(\sqrt{f(\theta_i)})\langle p_i\rangle}\right).$$

If we require

$$\begin{cases} \mu_1 \not\sim 1, \\ (\mu_1-\tau_2)(\mu_1-\tau_3) \not\sim 1, \end{cases} \quad \begin{cases} \mu_2-\tau_2 \not\sim 1, \\ \mu_2(\mu_2-\tau_3) \not\sim 1, \end{cases} \quad \begin{cases} \mu_3(\mu_3-\tau_2) \not\sim 1, \\ \mu_3-\tau_3 \not\sim 1, \end{cases}$$

then $(\pi, l_1)_{p_1} \not\sim 1$, $(\pi, l_2)_{p_2} \not\sim 1$, and $(\pi, l_1l_2)_{p_3} \not\sim 1$.

Let s_1 be odd, $s_1 < s_2$ and $\theta = \pi(v_1+\pi^{s_2}\mu)$, $\mu \in O_K^*$, $p = Irr_{K|k}(\theta)$, then $f(\theta) \sim \mu-\tau_2$, $(\pi, l_1)_p \sim (\pi, \mu)$. If

$$\begin{cases} \mu \not\sim 1, \\ \mu-\tau_2 \not\sim 1, \end{cases}$$

then $(\pi, l_1)_p \not\sim 1$. Suppose $\eta = \pi(v_1+\pi^{s_2}\sqrt[3]{\pi})$, $q = Irr_{k(\sqrt[3]{(\pi)})|k)}(\eta)$. In this case $f(\eta) \sim -\tau_2$ and

$$(\alpha, l_1)_q \sim \left(\frac{\alpha\ ,\ \pi^{s_2+1}\sqrt[3]{\pi}}{k(\sqrt[3]{\pi})|k}\right) \not\sim 1,$$

provided $-\tau_2 \sim 1$. After replacing the prime element $\nu = \pi\alpha$ we have $-\tau_2' = -\tau_2\alpha^{-s_2-1} \sim 1$, therefore we are in the same subcase, hence $(\pi\alpha, l_1) = (\nu, l_1) \not\sim 1$. Let now $-\tau_2 \not\sim 1$. The replacement $v_1' = v_1+\pi^{s_3}\tau_3$ leaves us in the current subcase, so $(\pi, l_3) \sim (\pi, l_1') \not\sim 1$. $(\pi, l_1l_3)_{l_1} \sim (\pi, l_2)_{l_1} \sim (\pi, -\tau_2) \not\sim 1$.

So, two generators are found. To find the third one we can argue just by the same way as in subcase 3. To complete the proof it remains only to show that the third generator does not belong to the group generated by the first two ones. This is true for the algebra from α), because it is trivial everywhere in this proof, where the algebras from the group, generated by algebras a)–d) are nontrivial. Finally, the algebras from β and γ) are nontrivial at $x^2-\pi^k u$, but the algebras from a)–d) are trivial there. The theorem is proved.

4 Splitting type (1,2,2).

This section is devoted to case $deg\ f_1 = deg\ f_2 = 2,\ deg\ f_3 = 1$ and the reduction is bad.

The evident list of all cases under consideration according to the reduction type of $f(x)$ is following ($f(x) = (x^2 - \pi^k u)((x-a)^2 - \delta)(x-e),\ k \geq 0,\ u \in O_k^*,\ a, \delta, e \in O_k,\ \pi^k u, \delta \not\sim 1$).

I. $k = 0,\ a \in M_k,\ \delta \in O_k^*,\ \bar{u} = \bar{\delta}$.

II. $k > 0,\ \delta \in O_k^*,\ e \in O_k^*$.

III. $k > 0,\ \delta \in O_k^*,\ e \in M_k$.

IV. $k > 0,\ \delta \in M_k,\ a \in O_k^*,\ e \in O_k^*,\ \bar{a} \neq \bar{e},\ k \leq v(\delta)$.

V. $k > 0,\ \delta \in M_k,\ a \in O_k^*,\ e \in M_k$.

VI. $k > 0,\ \delta \in M_k,\ a \in M_k,\ e \in O_k^*,\ k \leq v(\delta)$.

VII. $k > 0,\ \delta \in M_k,\ a \in M_k,\ e \in M_k,\ k \leq v(\delta)$.

Lemma 20

$$|\,{}_2Br\, C| = 8.$$

Theorem 18 *Let f be as in case I. Then the group ${}_2Br\, C$ is generated by the classes of the algebras (π, α), $(\pi, x^2 - u)$ and the class of the following algebra.*

1. *$(\pi, x - e)$, if either $e \in O_k^*,\ e^2 - u \not\sim 1$ or $e \in M_k,\ -1 \sim 1$.*
2. *$(\alpha(x^2 - u), x - w)$, otherwise,*

where $w \in O_k^*,\ w^2 - u \not\sim 1$.

Remark 1 *The algebra $(\alpha(x^2 - u), x - w)$ does not split by any quadratic extension of scalars.*

Proof of the theorem. The algebra $(\pi, x^2 - u)$ is nontrivial by lemma 3. We have also

$$(\pi, x - e)_{x^2 - u} \sim \left(\frac{\pi\ ,\ \sqrt{u} - e}{k(\sqrt{u})\langle x^2 - u \rangle (\sqrt{D(x^2 - u)})} \right),\ D \in k^*.$$

The last algebra is nontrivial iff $\sqrt{u} - e \not\sim 1$ in $k(\sqrt{u})$. And this is true if either $e \in O_k^*,\ e^2 - u \not\sim 1$ or $e \in M_k,\ -1 \sim 1$. The algebra $(\pi, (x^2 - u)(x - e))$ is nontrivial by lemma 3 again.

Consider now the algebra $A = (\alpha(x^2 - u), x - w)$. It is unramified because $A_{x-w} \sim (\alpha(w^2 - u), x - w) \sim 1$. Let $\theta = (1 + \sqrt[3]{\pi})\sqrt{u}$ and $p = Irr_{k(\sqrt[3]{\pi}, \sqrt{u})|k}(\theta)$. Then $f(\theta) \sim \sqrt{u} - e \sim 1$, provided we are in subcase 2. So

$$A_p \sim \left(\frac{(1 + \sqrt[3]{\pi})^2 u - u\ ,\ \sqrt{u} - w}{k(\sqrt[3]{\pi}, \sqrt{u})\langle p \rangle} \right) \sim \left(\frac{\sqrt[3]{\pi}\ ,\ \sqrt{u} - w}{k(\sqrt[3]{\pi}, \sqrt{u})\langle p \rangle} \right) \not\sim 1.$$

On the other hand,

$$(\pi, x^2 - u)_p \sim \left(\frac{\sqrt[3]{\pi}\ ,\ \sqrt[3]{\pi}}{k(\sqrt[3]{\pi}, \sqrt{u})\langle p \rangle} \right) \sim 1.$$

The theorem is proved.

Theorem 19 *Let f be as in cases II or III. Then the group ${}_2Br\, C$ is generated by the class of the algebra (π, α) and the classes of the following algebras.*

- *Case I. The first algebra is $(\pi, x - e)$ and the second one is*
  1. *$(e\xi u\pi, x)$, if $k \equiv 1\,(mod\,2)$.*
  2. *$(\pi, x^2 - \pi^k u)$, if $k \equiv 0\,(mod\,2)$ and $e\xi \not\sim 1$.*
  3. *$(\alpha, x - \pi^{k/2} w)$, if $k \equiv 0\,(mod\,2)$ and $e\xi \sim 1$.*

- *Case II* ($e = \pi^m v$).

  1. $(\pi, x - \pi^m v)$, $(\pi, x^2 - \pi^k u)$, *if* $m \leq k/2$, $m, k \equiv 0\,(mod\,2)$ *and either* $\xi v \not\sim 1$ *or* $m = k/2$.
  2. $(\pi, x - \pi^m v)$, $(\alpha, x - \pi^{k/2} w)$, *if* $m \leq k/2$, $m, k \equiv 0\,(mod\,2)$, $\xi v \sim 1$ *and* $m < k/2$.
  3. $(\pi, (x-a)^2 - \delta)$, $(\alpha\xi\eta\pi, x - \pi^{k/2} w)$, *if* $m \leq k/2$, $k \equiv 0\,(mod\,2)$, $m \equiv 1\,(mod\,2)$,
  $$\eta = \begin{cases} v, & \text{if } m < k/2, \\ v - w, & \text{if } m = k/2. \end{cases}$$
  4. $(\pi, x - \pi^m v)$, $(\alpha, x^2 - \pi^k u)$, *if* $m \leq k/2$, $m, k \equiv 1\,(mod\,2)$, $-\xi uv \sim 1$.
  5. $(\pi, x - \pi^m v)$, (α, x), *if* $m \leq k/2$, $m, k \equiv 1\,(mod\,2)$, $-\xi uv \not\sim 1$.
  6. $(\pi, x - \pi^m v)$, $(-\xi uv\pi, x)$, *if* $m \leq k/2$, $k \equiv 1\,(mod\,2)$, $m \equiv 0\,(mod\,2)$.
  7. $(\pi, x - \pi^m v)$, $(\alpha, x^2 - \pi^k u)$, *if* $m > k/2$, $k \equiv 1\,(mod\,2)$.
  8. $(\pi, x - \pi^m v)$, $(\pi, x^2 - \pi^k u)$, *if* $m > k/2$, $k \equiv 0\,(mod\,4)$.
  9. $(\pi, (x-a)^2 - \delta)$, $(-\xi w\alpha\pi, x - \pi^{k/2} w)$, *if* $m > k/2$, $k \equiv 2\,(mod\,4)$.

where $w \in O_k^*$, $w^2 - u \not\sim 1$ *and*
$$\xi = \begin{cases} \delta, & \text{if } a \in M_k, \\ \delta - a^2, & \text{otherwise.} \end{cases}$$

Proof. In II and III $(\pi, x - e) \not\sim 1$ by lemma 3.
In $II.1$ we have $f(0) \sim -e\xi u\pi$, so the algebra $(-e\xi u\pi, x)$ is unramified. Let $\tau \in O_K^*$ such that $K|k$ is an unramified odd degree extension and the following conditions hold:
$$\begin{cases} \tau \not\sim 1, \\ \tau - e \sim 1, \\ (\tau - a)^2 - \delta \sim 1. \end{cases}$$
Such K and τ exist in view of lemmas 5 and 6. Let $p = Irr_{K|k}(\tau)$. Then $f(\tau) \sim 1$. Thus
$$(\pi, x - e)_p \sim \left(\frac{\pi\ ,\ \tau - e}{K\langle p\rangle}\right) \sim 1 \text{ and } (-e\xi u\pi, x)_p \sim \left(\frac{\pi\ ,\ \tau}{K\langle p\rangle}\right) \not\sim 1.$$
Consider subcase $II.2$. We have $f(w\pi^{k/2}) \sim e\xi(w^2 - u) \sim 1$, so $(\pi, x^2 - \pi^k u)_{x - \pi^{k/2} w} \not\sim 1$. Finally, $(\pi, x - e) \not\sim (\pi, x^2 - \pi^k u)$ by lemma 3.
If we are in $II.3$, then $f(w\pi^{k/2}) \sim \alpha$, so that the algebra $(\alpha, x - w\pi^{k/2})$ is unramified. Let $\varepsilon \in \{1, 2\}$, $\varepsilon \equiv k/2\,(mod\,2)$, $\theta = \pi^{k/2-1}(\sqrt[3]{\pi})^\varepsilon$ and $p = Irr_{k(\sqrt[3]{\pi})|k}(\theta)$. Then $f(\theta) \sim e\xi \sim 1$ and
$$(\alpha, x - w\pi^{k/2})_p \sim \left(\frac{\alpha\ ,\ \pi^{k/2-1}(\sqrt[3]{\pi})^\varepsilon}{k(\sqrt[3]{\pi})\langle p\rangle}\right) \not\sim 1.$$
To prove $(\pi, x - e) \not\sim (\alpha, x - w\pi^{k/2})$ it is enough to observe that the extension by $\sqrt{\pi}$ leaves us in subcase $II.3$, hence this extension does not split the algebra $(\alpha, x - w\pi^{k/2})$.
In III the algebras $(\pi, x - \pi^m v)$ and $(\pi, (x-a)^2 - \delta)$ are nontrivial.
$III.1$. Since $f(\pi^{k/2} w) \sim \xi v \sim 1$, provided $m < k/2$, then $(\pi, x^2 - \pi^k u)_{x - \pi^{k/2} w} \not\sim 1$. Let now $k/2 = m$. Then we require
$$\begin{cases} \tau^2 - u \sim \alpha, \\ v - \tau \sim \alpha\xi, \end{cases}$$
$\tau \in O_K^*$, $K|k$ is unramified of odd degree, then $f(\pi^{k/2}\tau) \sim 1$ and $(\pi, x^2 - \pi^k u)_{Irr(\pi^{k/2}\tau)} \not\sim 1$. Furthermore, $(\pi, x^2 - \pi^k u) \otimes (\pi, x - \pi^m v) \sim (\pi, (x-a)^2 - \delta) \not\sim 1$.
$III.2$. $f(\pi^{k/2} w) \sim \alpha$, so that $(\alpha, x - \pi^{k/2} w)$ is unramified. Now we can argue as in $II.3$.
$III.3$. In view of $f(\pi^{k/2} w) \sim \xi\eta(w^2 - u)\pi$ the algebra $A = (\xi\eta(w^2 - u)\pi, x - \pi^{k/2} w)$ is unramified.
$$A_{x^2 - \pi^k u} \sim \left(\frac{\pi\ ,\ w - \sqrt{u}}{k(\sqrt{u})\langle x^2 - \pi^k u\rangle(\sqrt{D(x^2 - \pi^k u)})}\right) \not\sim 1,\ (D \in k^*).$$

On the other hand,

$$(\pi,(x-a)^2-\delta)_{x^2-\pi^k u}\sim\left(\frac{\pi\ ,\ -\xi}{k(\sqrt{u})\langle x^2-\pi^k u\rangle(\sqrt{D(x^2-\pi^k u)})}\right)\sim 1.$$

$III.4$. We have $f(0)\sim -\xi uv$. Thus $(\alpha,x^2-\pi^k u)_x\not\sim 1$. Moreover, although the extension by $\sqrt{\alpha}$ leads us to case $(1,1,1,2)$, it does not split the algebra $(\pi,x-\pi^m v)$. Therefore $(\alpha,x^2-\pi^k u)\not\sim(\pi,x-\pi^m v)$.
$III.5$. $f(0)\sim\alpha$, so the algebra (α,x) is unramified. $(\alpha,x)_{x-\pi^m v}\not\sim 1$. The further arguments are similar to the previous subcase.
$III.6$. The algebra $A=(-\xi uv\pi,x)$ is unramified. Let $\theta=\pi^m\tau$, $\tau\in O_K^*$, $p=Irr_{K|k}(\theta)$. Then $f(\theta)\sim\xi(v-\tau)$ and $A_p\sim(\pi,\tau)$. If we require

$$\begin{cases}\tau\not\sim 1,\\ \xi(v-\tau)\sim 1,\end{cases}$$

then $A_p\not\sim 1$. And if we require

$$\begin{cases}\tau\not\sim -\xi,\\ \xi(v-\tau)\sim 1,\end{cases}$$

then $A_p\not\sim(\pi,x-\pi^m v)_p$ since in this case $(\pi,x-\pi^m v)_p\sim(\pi,-\xi)$.
$III.7$. $(\alpha,x^2-\pi^k u)_{x-\pi^m v}\not\sim 1$. Moreover, the extension by $\sqrt{\alpha}$ does not split the algebra $(\pi,x-\pi^m v)$.
$III.8$. We have $f(\pi^{k/2}\tau)\sim -\xi\tau(\tau^2-u)$. If $\tau^2-u\not\sim 1$ and $-\xi\tau\not\sim 1$, then $(\pi,x^2-\pi^k u)_{Irr(\pi^{k/2}\tau)}\not\sim 1$.
$III.9$. $f(\pi^{k/2}w)\sim -\xi w\alpha\pi$, so the algebra $A=(-\xi w\alpha\pi,x-\pi^{k/2}w)$ is unramified. Finally,

$$A_{x^2-\pi^k u}\sim\left(\frac{\pi\ ,\ w-\sqrt{u}}{k(\sqrt{u})\langle x^2-\pi^k u\rangle(D(x^2-\pi^k u))}\right)\not\sim 1,$$

but

$$(\pi,(x-a)^2-\delta)_{x^2-\pi^k u}\sim\left(\frac{\pi\ ,\ -\xi}{k(\sqrt{u})\langle x^2-\pi^k u\rangle(D(x^2-\pi^k u))}\right)\sim 1.$$

The theorem is proved.

Theorem 20 *Let f be as in case IV. $f(x)=(x-e)(x^2-\pi^k u)((x-v)^2-\pi^m\gamma)$, $e,u,v,\gamma\in O_k^*$, $\bar{v}\neq\bar{\gamma}$, $0<k\le m$. Then the group ${}_2Br\,C$ is generated by the classes of the algebra (π,α) and the classes of the following algebras. The first algebra is*

1. $(\pi,x-e)$, *if* $k\equiv 0\,(mod\,2)$, $-e\not\sim 1$.

2. $(\alpha,x-\pi^{k/2}w)$, *if* $k\equiv 0\,(mod\,2)$, $-e\sim 1$.

3. $(\pi ue,x)$, *if* $k\equiv 1\,(mod\,2)$.

The second one is

a. $(\pi,(x-v)^2-\pi^m\gamma)$, *if* $m\equiv 0\,(mod\,2)$, $v-e\not\sim 1$.

b. $(\alpha,x-v-\pi^{m/2}\tau)$, *if* $m\equiv 0\,(mod\,2)$, $v-e\sim 1$.

c. $(\pi(e-v)\gamma,x-v)$, *if* $m\equiv 1\,(mod\,2)$,

where $w,\tau\in O_k^*$, $w^2-u\not\sim 1$, $\tau^2-\gamma\not\sim 1$.

Proof. If $k\equiv 0\,(mod\,2)$, then $f(\pi^{k/2}w)\sim -e\alpha$. In case 1 we have $f(\pi^{k/2}w)\sim 1$ and $(\pi,x-e)_{x-\pi^{k/2}w}\sim(\pi,-e)\not\sim 1$. In addition let us consider the algebra $(\pi,x^2-\pi^k u)$. It is also nontrivial at this place. Indeed, $(\pi,x^2-\pi^k u)_{x-\pi^{k/2}w}\sim(\pi,w^2-u)\not\sim 1$.

In case 2 $f(\pi^{k/2}w)\sim\alpha$ and the algebra $(\alpha,x-\pi^{k/2}w)$ is unramified. Now one can prove $(\alpha,x-\pi^{k/2}w)\not\sim 1$ by analogy with the theorem 19, subcase $II.3$.

Let now $k \equiv 1\,(mod\,2)$, $\mu \in O_K^*$, K be as above and

$$\left\{\begin{array}{l} \mu \sim \alpha u e, \\ u - \mu \sim \alpha u. \end{array}\right.$$

Assume $\theta = \sqrt{\pi^k \mu} \in K(\sqrt{\pi\mu})$ and $q = Irr_{K(\sqrt{\pi\mu})|k}(\theta)$. Then $f(\theta) \sim e(u-\mu)\pi \sim e\mu(u-\mu) \sim 1$ and

$$(\pi u e, x)_q \sim \left(\frac{\pi u e\,,\ \theta}{K(\sqrt{\pi\mu})\langle q\rangle}\right) \sim \left(\frac{u e \tau\,,\ \sqrt{\pi\mu}}{K(\sqrt{\pi\mu})\langle q\rangle}\right) \not\sim 1.$$

After replacing $x' = x - v$ we have

$$f(x') = (x' - (e-v))((x')^2 - \pi^m\gamma)((x' - (-v))^2 - \pi^k u).$$

The above arguments gives that the algebras from items a, b, c are unramified and nontrivial. (Because nowhere in this arguments we used that $k/2 < m$.) To finish the proof it is enough only to check, that these algebras are not isomorphic to the algebras from items $1, 2, 3$ in the corresponding subcases.
1.a. We have $(\pi, x-e) \otimes (\pi, (x-v)^2 - \pi^m\gamma) \sim (\pi, x^2 - \pi^k u) \not\sim 1$.
2.b. If A is the algebra from 2 and A' is the one from b, then $A_p \not\sim 1$ and $A'_p \sim (\alpha, -v) \sim 1$, where p is constructed by analogy with the proof of theorem 19, subcase II.3..
1.b, 2.a. In view of symmetry one need to consider only one case, say, 2.a. In this case for p as above $A_p \not\sim 1$, but $(\pi, (x-v)^2 - \pi^m\gamma)_p \sim (\pi, v^2) \sim 1$.
3.a. $(\pi u e, x)_q \not\sim 1$, but $(\pi, (x-v)^2 - \pi^m\gamma)_q \sim 1$.
3.b. $A'_q \sim (\alpha, -v) \sim 1$.
3.c. Finally, we have

$$(\pi(e-v)\gamma, x-v)_q \sim \left(\frac{\pi(e-v)\gamma\,,\ -v}{K(\sqrt{\pi\mu})\langle q\rangle}\right) \sim \left(\frac{\tau(e-v)\gamma\,,\ -v}{K(\sqrt{\pi\mu})\langle q\rangle}\right) \sim 1,$$

and this completes the proof.

Theorem 21 *Let f be as in case V. $f(x) = ((x-v)^2 - \pi^m\gamma)(x - \pi^r\delta)(x^2 - \pi^k u)$ $v, \gamma.\delta, u \in O_k^*$, $m, r, k > 0$, $\pi^m\gamma, \pi^k u \not\sim 1$. Then the group ${}_2Br\, C$ is generated by the classes of the algebra (π, α) and the classes of the following algebras.*
The first algebra is

i) $(\pi, (x-v)^2 - \pi^m\gamma)$, *if* $m \equiv 0\,(mod\,2)$, $v \not\sim 1$.

ii) $(\alpha, x - v - \pi^{m/2}w)$, *if* $m \equiv 0\,(mod\,2)$, $v \sim 1$.

iii) $(-v\gamma\pi, x-v)$, *if* $m \equiv 1\,(mod\,2)$.

The second one is

a. $(\alpha, x - \pi^r\delta)$, *if* $k/2 < r$, $k \equiv 1\,(mod\,2)$.

b. $(\pi, x - \pi^r\delta)$, *if* $k/2 < r$, $k \equiv 0\,(mod\,4)$.

c. $(\tau(\tau^2 - u)\pi, x - \pi^{k/2}\tau)$, *if* $k/2 < r$, $k \equiv 2\,(mod\,4)$.

d. $(\pi\delta u, x)$, *if* $k/2 > r$, $k \equiv 1\,(mod\,2)$, $r \equiv 0\,(mod\,2)$.

e. $(\pi, x - \pi^r\delta)$, *if* $k \equiv 0\,(mod\,2)$, $r \equiv 0\,(mod\,2)$ *and either* $k/2 = r$ *or* $k/2 > r$, $-\delta \not\sim 1$.

f. $(\alpha, x - \pi^{k/2}\tau)$, *if* $k \equiv 0\,(mod\,2)$, $r \equiv 0\,(mod\,2)$, $k/2 > r$ *and* $-\delta \sim 1$.

g. $(\xi\alpha\pi, x - \pi^{k/2}\tau)$, *if* $k \equiv 0\,(mod\,2)$, $r \equiv 1\,(mod\,2)$, $k/2 \geq r$,

$$\xi = \left\{\begin{array}{l} -\delta,\ \textit{if } k/2 > r, \\ \tau - \delta - a^2,\ \textit{if } k/2 = r. \end{array}\right.$$

h. $(\alpha, x-\pi^r\delta)$, *if* $k \equiv 1\,(mod\,2)$, $r \equiv 1\,(mod\,2)$, $k/2 > r$ *and* $\delta u \sim 1$.

k. (α, x), *if* $k \equiv 1\,(mod\,2)$, $r \equiv 1\,(mod\,2)$, $k/2 > r$ *and* $\delta u \not\sim 1$,

where $w, \tau \in O_k^*$, $w^2 - \gamma \not\sim 1$, $\tau^2 - u \not\sim 1$.

Proof.
i. $f(\pi^{m/2}w + v) \sim v(w^2-u) \sim 1$, so $(\pi, (x-v)^2 - \pi^m\gamma)_{x-v-\pi^{m/2w}} \not\sim 1$.
ii. $f(\pi^{m/2}w+v) \sim v(w^2-u) \sim \alpha$, so the algebra $(\alpha, x-v-\pi^{m/2}w)$ is unramified. We have also $f(v+\sqrt[3]{\pi}) \sim 1$, therefore

$$(\alpha, x-v-\pi^{m/2}w)_{(x-v)^3-\pi} \sim \left(\frac{\alpha\ ,\ \sqrt[3]{\pi}}{k(\sqrt[3]{\pi})\langle(x-v)^3-\pi\rangle}\right) \not\sim 1.$$

iii. $f(v) \sim -v\gamma\pi$, $(-v\gamma\pi, x-v)$ is unramified. As usual, let $\chi \in O_K^*$ satisfy the conditions

$$\begin{cases} \chi \sim -\alpha v\gamma, \\ \chi - \gamma \sim -\alpha\gamma. \end{cases}$$

Let $\theta = v + \sqrt{\pi^m}\chi$, $q = Irr_{K(\sqrt{\pi\chi})|k}(\theta)$. Then $f(\theta) \sim \chi v(\chi-\gamma) \sim 1$, hence

$$(-v\gamma\pi, x-v)_q \sim \left(\frac{-v\gamma\chi\ ,\ \sqrt{\pi\chi}}{K(\sqrt{\pi\chi})\langle q\rangle}\right) \not\sim 1.$$

a. $(\alpha, x-\pi^r\delta)_{x-\pi^r\delta} \sim (\alpha, ((x-v)^2-\pi^m\gamma)(x^2-\pi^k u)_{x-\pi^r\delta}) \sim (\alpha, -\pi^k u) \not\sim 1$.
b. We have $f(\mu\pi^{k/2}) \sim -\mu(\mu^2-u)$ for some $\mu \in O_K^*$. Thus $(\pi, x-\pi^r\delta)_{Irr(\mu\pi^{k/2})} \sim (\pi,\mu) \not\sim 1$, provided

$$\begin{cases} \mu \not\sim 1, \\ u - \mu^2 \not\sim 1. \end{cases}$$

c. $f(\pi^{k/2}\tau) \sim \tau\alpha\pi$. Furthermore,

$$(\tau\alpha\pi, x-\pi^{k/2}\tau)_{x^2-\pi^k u} \sim \left(\frac{\pi\ ,\ \tau-\sqrt{u}}{k(\sqrt{u})\langle x^2-\pi^k u\rangle}\right) \not\sim 1.$$

d. Let $\nu \in O_K^*$ satisfy the conditions

$$\begin{cases} \nu \sim \alpha\delta u, \\ \nu - u \sim -\alpha u, \end{cases}$$

and $t = Irr_{K(\sqrt{\pi\nu})|k}(\sqrt{\pi^k\nu})$. Then $f(\sqrt{\pi^k\nu}) \sim -\delta\nu(\nu-u) \sim 1$, so that

$$(\pi\delta u, x)_t \sim \left(\frac{\nu\delta u\ ,\ \sqrt{\pi\nu}}{K(\sqrt{\pi\nu})\langle t\rangle}\right) \not\sim 1.$$

e. If $k/2 = r$, then $f(\pi^{k/2}\lambda) \sim (\lambda-\delta)(\lambda^2-u)$ for some $\lambda \in O_K^*$. Moreover, if $s = Irr_{K|k}(\pi^{k/2}\lambda)$, then

$$(\pi, x-\pi^r\delta)_s \sim \left(\frac{\pi\ ,\ \lambda-\delta}{K(\sqrt{(\lambda-\delta)(\lambda^2-u)})\langle s\rangle}\right).$$

If we require

$$\begin{cases} \lambda-\delta \not\sim 1, \\ \lambda^2 - u \not\sim 1, \end{cases}$$

then $(\pi, x-\pi^r\delta)_s \not\sim 1$.

Let now $k/2 > r$. We have $f(\pi^{k/2}\tau) \sim -\delta(\tau^2-u) \sim 1$. So, $(\pi, x-\pi^r\delta)_{x-\pi^{k/2}\tau} \sim (\pi, -\delta) \not\sim 1$.
f. $f(\pi^{k/2}\tau) \sim -\delta(\tau^2-u) \sim \alpha$ and the algebra $(\alpha, x-\pi^{k/2}\tau)$ is unramified. To prove $(\alpha, x-\pi^{k/2}\tau) \not\sim 1$ one can follow the proof of theorem 19, subcase $II.3$.
g. By analogy with subcase c we have $(\xi\alpha\pi, x-\pi^{k/2}\tau)_{x^2-\pi^k u} \not\sim 1$.

h. $f(0) \sim \pi^{k+r}\delta u \sim 1$, therefore $(\alpha, x-\pi^r\delta)_x \not\sim 1$.
k. $f(0) \sim \pi^{k+r}\delta u \sim \alpha$, hence the algebra (α, x) is unramified. Finally, $(\alpha, x)_{x-\pi^r\delta} \not\sim 1$.

Now, to complete the proof one only need to check that the algebra from *i)–iii)* is not isomorphic to the algebra from *a)–k)*. Note, that in all subcases *a)–k)* the nontriviality of the corresponding algebra is showed by the completion of this algebra at the place with prime element $l \in k[x]$ such that $l = Irr(\theta)$, $\theta \in M_{k_{alg}}$. Thus, in *i)* and *ii)* we have $(\pi, (x-v)^2 - \pi^m\gamma)_l \sim (\pi, v^2) \sim 1$ and $(\alpha, x-v-\pi^{m/2}w)_l \sim (\alpha, -v) \sim 1$. The nontriviality of the algebra $(-v\gamma\pi, x)$ from *iii* is proved by the completion at $q = Irr_{K(\sqrt{\pi\chi})|k}(\theta)$, $\theta = v + \sqrt{\pi^m\chi}$. Any algebra from *a)–k)* completed in q can be obtained from some algebra over k by extending of scalars. So it is trivial at q, since the extension $k(\sqrt{\pi\chi})$ splits any algebra over k. The theorem is proved.

Consider the next case.

Theorem 22 *Let f be as in case VI. $f(x) = (x-e)(x^2 - \pi^k u)((x-\pi^m v)^2 - \pi^r\delta)$, $u, v, \delta, e \in O_k^*$, $0 < m, 0 < k \le r$, $\pi^k u, \pi^r\delta \not\sim 1$. Then the group ${}_2Br\,C$ is generated by the class of the algebra (π, α) and the classes of the following algebras.*

Let if $k, r \equiv 1\,(mod\,2)$, then $2m \le k$, and if $k = r \equiv 0\,(mod\,2)$, $2m > k$, then $\bar{u} \ne \bar{\delta}$. Then the first algebra is

i. $(\pi E, x)$, *if* $k \equiv 1\,(mod\,2)$.

ii. $(\pi, x^2 - \pi^k u)$, *if* $k \equiv 0\,(mod\,2)$ *and* $-e \not\sim 1$.

iii. $(\alpha, x - \pi^{k/2}w)$, *if* $k \equiv 0\,(mod\,2)$, *either* $k < r$ *or* $k = r > 2m$, *and* $-e \sim 1$.

iv. $(\alpha, Irr_{K|k}(\pi^{k/2}\beta))$, *if* $k \equiv 0\,(mod\,2)$, $k = r \le 2m$, $-e \sim 1$, *where* $\beta \in O_K^*$ *and*

$$(\beta^2 - u) \times \begin{cases} \beta^2 - \delta, & \text{if } k < 2m, \\ (\beta - v)^2 - \delta, & \text{if } k = 2m. \end{cases}$$

The second one is

a. $(\pi F, x - \pi^m v)$, *if* $r \equiv 1\,(mod\,2)$.

b. $(\pi, (x-\pi^m v)^2 - \pi^r\delta)$ *if* $r \equiv 0\,(mod\,2)$ *and*

- $k = r$ *and either* $k \le 2m$ *or* $-e \not\sim 1$,
- $2m < k < r$ *and* $-e \not\sim 1$,
- $2m = k < r$ *and* $-e(v^2 - u) \not\sim 1$,
- $k < r, k < 2m$, $k \equiv 0\,(mod\,2)$, *and* $ue \not\sim 1$.

c. $(\alpha, x - \pi^m v - \pi^{r/2}\tau)$ *if* $r \equiv 0\,(mod\,2)$ *and*

- $k = r > 2m$, $-e \sim 1$,
- $2m < k < r$ *and* $-e \sim 1$,
- $2m = k < r$ *and* $-e(v^2 - u) \sim 1$,
- $k < r, k < 2m$, $k \equiv 0\,(mod\,2)$, *and* $ue \sim 1$.

d. $(ueN_{K|k}(\gamma^2 - \delta)\pi, Irr_{K|k}(\pi^m v + \pi^{r/2}\gamma))$, *if* $r \equiv 0\,(mod\,2)$, $k < r, k < 2m$ *and* $k \equiv 1\,(mod\,2)$, *where* $\gamma \in O_K^*$, $K|k$ *is of odd degree and unramified,*

$$\gamma^2 - \delta \not\sim \begin{cases} 1 & , \text{if } 2m < r, \\ v^2 - \delta & , \text{if } 2m = r, \\ -\delta & , \text{if } 2m > r. \end{cases}$$

Otherwise we have

1. $(\pi, x^2-\pi^k u)$, $(\alpha(x^2-\pi^k u), \pi^{k/2}(x-\pi^{k/2}w))$, *if* $k=r\equiv 0\,(mod\,2)$, $2m>k$, $\bar u=\bar\delta$.

2. *The first algebra:* $(eu\pi, x)$, *the second one: either* $(\alpha, x^2-\pi^k u)$, *provided* $-ue\delta\sim 1$, *or* $(\alpha, x-\pi^m v)$ *otherwise, if* $k,r\equiv 1\,(mod\,2)$, $k<2m<r$.

3. *The first algebra: either* $(\alpha, x^2-\pi^k u)$, *provided* $-ue\delta\sim 1$, *or* (α, x) *otherwise, the second one:* $(\pi ue, x-\pi^s)$, $k/2<s<r/2$, *if* $k,r\equiv 1\,(mod\,2)$, $k<r<2m$.

4. *The first algebra from 3, and* $(x, -\pi u(x^2-\pi^k u))$, *if* $k=r\equiv 1\,(mod\,2)$, $k<2m$.

where $E,F,w,tau\in O_k^*$ *can be found from the conditions* $f(0)\sim\pi E$, $f(\pi^m v)\sim\pi F$, $w^2-u\not\sim 1$, $\tau^2-\delta\not\sim 1$.

Remark 2 *The algebra* $(\alpha(x^2-\pi^k u), \pi^{k/2}(x-\pi^{k/2}w))$ *from 1. does not split by any quadratic extension of scalars.*

Proof of the theorem.
i. If $\mu\in O_K^*$, then

$$f(\sqrt{\pi^k\mu})\sim\begin{cases} -e(\mu-u)(\mu-\delta) & \text{, if } k=r<2m,\\ -e\mu(\mu-u) & \text{, otherwise.}\end{cases}$$

Let $t=Irr_{K(\sqrt{\pi\mu})|k}(\sqrt{\pi^k\mu})$. If $E\mu\not\sim 1$ in K and $f(\sqrt{\pi^k\mu})\sim 1$ in K, then

$$(\pi E, x)_t\sim\left(\frac{E\mu\,,\,\sqrt{\pi\mu}}{K(\sqrt{\pi\mu})\langle t\rangle}\right)\not\sim 1.$$

The conditions on K and μ always can be satisfied, provided that in case $k=r<2m$ we have $\bar u\neq\bar\delta$.
ii. We have

$$f(\pi^{k/2}\xi)\sim\begin{cases} -e(\xi^2-u), & \text{if } k<r \text{ or } k=r>2m,\\ -e(\xi^2-u)(\xi^2-\delta), & \text{if } k=r<2m,\\ -e(\xi^2-u)((\xi-v)^2-\delta), & \text{if } k=r=2m.\end{cases}$$

Thus if $k<r$ or $k=r>2m$, then $(\pi, x^2-\pi^k u)_{x-\pi^{k/2}w}\not\sim 1$. Let now $k=r\le 2m$. Then $(\pi, x^2-\pi^k u)_{Irr_{K|k}(\pi^{k/2}\xi)}\not\sim 1$, provided $\xi^2-u\not\sim 1$ and $\xi^2-\delta\sim 1$ if $k<2m$, $(\xi-v)^2-\delta\sim 1$ if $k=2m$.
iii. To prove $(\alpha, x-\pi^{k/2}\tau)\not\sim 1$ one can follow the proof of theorem 19, subcase $II.3$.
iv. From the conditions on β we have the algebra to be unramified. The nontriviality can be proved as in *iii*.
a. Similarly as in subcase *i* for $t=Irr_{K(\sqrt{\pi\mu})|k}(\theta)$, $\theta=\pi^m v+\sqrt{\pi^r\mu}$ we can find K and μ such that $(\pi F, x-\pi^m v)_t\not\sim 1$. Indeed,

$$f(\theta)\sim\begin{cases} -e\mu(\mu-\delta), & \text{if } 2m<k,\\ ue\mu(\mu-\delta), & \text{if } 2m,r>k,\\ -e\mu(\mu-\delta)(v^2-u), & \text{if } k=2m<r,\end{cases}$$

and the conditions $f(\theta)\sim 1$ and $F\mu\not\sim 1$ can be satisfied in some unramified extension $K|k$ of odd degree.
b,c. By replacing $x'=x-\pi^m v$ case $k=r$ is reduced to *ii* and *iii*. Furthermore,

$$f(\pi^m v+\pi^{r/2}\tau)\sim -e\alpha\times\begin{cases} 1, & \text{if } 2m<k,\\ v^2-u, & \text{if } 2m=k,\\ -u\pi^k, & \text{if } 2m<k.\end{cases}$$

So, in *b* the algebra $(\pi, (x-\pi^m v)^2-\pi^r\delta)_{x-\pi^m v+\pi^{r/2}\tau}$ is nontrivial, and the algebra from *c* is unramified. The nontriviality of the last algebra can be checked by analogy with the proof of theorem 19, subcase $II.3$. Indeed, this algebra is nontrivial at $Irr_{k(\sqrt[3]{\pi})|k}(\theta)$, where $\theta=\pi^m v+\pi^{r/2-1}(\sqrt[3]{\pi})^\varepsilon$, $\varepsilon\in\{1,2\}$, $\varepsilon\equiv r/2\,(mod\,2)$.
d. We have $f(\pi^m v+\pi^{r/2}\gamma)\sim ue\pi(\gamma^2-\delta)$. Since $K|k$ is of odd degree $\gamma^2-\delta\sim N_{K|k}(\gamma^2-\delta)$ in K, so that the algebra $A=(ueN_{K|k}(\gamma^2-\delta)\pi, Irr_{K|k}(\pi^m v+\pi^{r/2}\gamma))$ is unramified. Let $\mu\in O_{K'}^*$, $K'|k$ is of odd degree and unramified such that

$$\begin{cases} \mu\not\sim ue(\gamma^2-\delta),\\ \mu-u\not\sim -u(\gamma^2-\delta).\end{cases}$$

Let $t = Irr_{K'(\sqrt{\pi\mu})|k}(\sqrt{\pi^k\mu})$. Then $f(\sqrt{\pi^k\mu}) \sim -e\mu(\mu - u) \sim 1$ and

$$A_t \otimes K \sim \left(\frac{ue(\gamma^2-\delta)\mu\ ,\ \prod_{\sigma\in GK|k}(\sqrt{\pi^k\mu}-\pi^m v-\pi^{r/2}\gamma)}{KK'(\sqrt{\pi\mu})\langle t\rangle}\right) \sim \left(\frac{\alpha\ ,\ \sqrt{\pi\mu}}{KK'(\sqrt{\pi\mu})\langle t\rangle}\right) \not\sim 1.$$

Let us check now that the algebra A from $i - iv$ is not isomorphic to the algebra B from $a)$–$d)$.
Let firstly k and r be odd. Then $2m < k$, $A = (\pi E, x)$, and $A_t \sim 1$. On the other hand,

$$B_t \sim (\pi F, x - \pi^m v)_t \sim \left(\frac{\pi F\ ,\ -\pi^m v}{K(\sqrt{\pi\mu})\langle t\rangle}\right) \sim 1.$$

Let now k be odd, r be even. We have

$$B_t \sim (\pi, (x-\pi^m v)^2 - \pi^r\delta)_t \sim \begin{cases} \left(\dfrac{\pi\ ,\ \pi^{2m}v^2}{K(\sqrt{\pi\mu})\langle t\rangle}\right), & \text{if } 2m < k, \\ \left(\dfrac{\pi\ ,\ \pi^k\mu}{K(\sqrt{\pi\mu})\langle t\rangle}\right), & \text{otherwise} \end{cases} \sim 1.$$

In c we have $B = (\alpha, x - \pi^m v - \pi^{r/2}\delta)$ and $2m < k$. So

$$B_t \sim \left(\frac{\alpha\ ,\ -\pi^m v}{K(\sqrt{\pi\mu})\langle t\rangle}\right) \sim 1.$$

Finally, if case d takes place, then $2m > k$ and

$$B_t \otimes K' \sim \left(\frac{ue(\gamma^2-\delta)\pi\ ,\ \sqrt{\pi^k\mu}}{KK'(\sqrt{\pi\mu})\langle t\rangle}\right) \sim \left(\frac{\alpha ue\mu\ ,\ \sqrt{\pi\mu}}{KK'(\sqrt{\pi\mu})\langle t\rangle}\right) \sim 1,$$

since from the proof of $i)$ we have $\alpha ue\mu \sim 1$.

Conversely, let k be even and r be odd. From the proof of $a)$ $B_t \not\sim 1$, $t = Irr_{K\sqrt{\pi\mu}|k}(\theta)$, $\theta = \pi^m v + \sqrt{\pi^r\mu}$. On the other hand, if $A = (\pi, x^2 - \pi^k u)$, then

$$A_t \sim \left(\frac{\pi\ ,\ \pi^{2m}v^2 - \pi^k u}{K(\sqrt{\pi\mu})\langle t\rangle}\right) \sim 1$$

in view of $\min\{2m, r\} < \min\{r, m + r/2\}$. For $A = (\alpha, x - \pi^{k/2}w)$

$$A_t \sim \left(\frac{\alpha\ ,\ \pi^m v - \pi^{k/2}w}{K(\sqrt{\pi\mu})\langle t\rangle}\right) \sim 1,$$

provided that if $k = 2m$, then $\bar{w} \neq \bar{v}$.

Finally, let k and r be even. The extension by $\sqrt{\pi}$ leaves us in case VI. Moreover, if the algebra $(\alpha, x - \pi^{k/2}w)$ is nontrivial (subcase $iii)$), then it does not split by this extension, since we are still in $iii)$. The same is true for the algebras from $iv)$ and $c)$. So it is enough to prove that in $ii), b)$ $(\pi, x - e) \not\sim 1$, and in $iii), c)$ $(\alpha, x - \pi^{k/2}w) \not\sim (\alpha, x - \pi^m v - \pi^{r/2}\tau)$. Case $iv), c)$ is impossible.

In $ii), b)$ we have $(\pi, x - e)_{Irr(\pi^{k/2}\xi)} \sim (\pi, -e) \not\sim 1$. Consider now $iii), c)$. Then $-e \sim 1$. If $k = r$, then $k > 2m$. For θ from the proof of $iii)$ satisfying the conditions $\theta = \pi^{k/2-1}(\sqrt[3]{\pi})^\varepsilon$, $\varepsilon \in \{1, 2\}$, $\varepsilon \equiv m + k/2\,(mod\,2)$ we have $(q = Irr_{k(\sqrt[3]{\pi})|k}(\theta))$

$$(\alpha, x - \pi^{k/2}w)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m-1}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \text{ and } (\alpha, x - \pi^m v - \pi^{r/2}\tau)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m}}{k(\sqrt[3]{\pi})\langle q\rangle}\right),$$

so $A_q \not\sim B_q$.

Let now $k < r$. Case $2m < k$ is similar to the previous one. In case $2m = k$ consider $\theta = \pi^m v + \pi^{r/2-1}(\sqrt[3]{\pi})^\varepsilon$, $\varepsilon \in \{1, 2\}$, $\varepsilon \equiv k/2 + r/2\,(mod\,2)$. Then

$$(\alpha, x - \pi^{k/2}w)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{k/2}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \text{ and } (\alpha, x - \pi^m v - \pi^{r/2}\tau)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{k/2-1}}{k(\sqrt[3]{\pi})\langle q\rangle}\right).$$

Finally, in case $2m > k$ let $\theta = \pi^{k/2}(\sqrt[3]{\pi})$. We have

$$(\alpha, x - \pi^{k/2}w)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{k/2}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \text{ and } (\alpha, x - \pi^m v - \pi^{r/2}\tau)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{k/2+1}}{k(\sqrt[3]{\pi})\langle q\rangle}\right).$$

Consider now cases 1–4 step by step.

1. There exists $a \in O_k^*$ such that $a^2 - 4u \not\sim 1$. If $p = Irr_{k(\sqrt{u})|k}(\pi^{k/2}(a + \sqrt{u}))$, then

$$(\pi, x^2 - \pi^k u)_p \sim \left(\frac{\pi\ ,\ a + 2\sqrt{u}}{k(\sqrt{u})\langle p\rangle}\right),$$

since $f(\pi^{k/2}(a + \sqrt{u})) \sim -e(a^2 + 2a\sqrt{u})^2 \sim 1$ in $k(\sqrt{u})$. So, $(\pi, x^2 - \pi^k u)_p \not\sim 1$.

Furthermore, $(\alpha(x^2 - \pi^k u), \pi^{k/2}(x - \pi^{k/2}w))_{x-\pi^{k/2}w} \sim (\alpha(w^2 - u), \pi^{k/2}(x - \pi^{k/2}w)) \sim 1$, so that this algebra is unramified. Let us prove the nontriviality. Suppose $\theta = (1 + \sqrt[3]{\pi})\pi^{k/2}\sqrt{u} \in k(\sqrt{u}, \sqrt[3]{\pi})$ and $q = Irr(\theta)$. Then $\theta^2 - \pi^k u \sim (\theta - \pi^m v)^2 - \pi^k\delta \sim \sqrt[3]{\pi}$ in $k(\sqrt{u})$. Therefore $f(\theta) \sim 1$ and

$$(\alpha(x^2 - \pi^k u), \pi^{k/2}(x - \pi^{k/2}w))_q \sim \left(\frac{\alpha\sqrt[3]{\pi}\ ,\ \pi^{k/2}((1 + \sqrt[3]{\pi})\pi^{k/2}\sqrt{u} - \pi^{k/2}w)}{k(\sqrt{u}, \sqrt[3]{\pi})\langle q\rangle}\right) \sim$$

$$\sim \left(\frac{\sqrt[3]{\pi}\ ,\ \sqrt{u} - w}{k(\sqrt{u}, \sqrt[3]{\pi})\langle q\rangle}\right) \not\sim 1.$$

Thus $\sqrt{\alpha}$ does not split the algebra. To check that $\sqrt{\pi}$ (and $\sqrt{\pi\alpha}$) does not split it, one only need to observe that this extension leaves us in subcase $VI.1$.

2. The first algebra is unramified and nontrivial by analogy with subcase i). We have $f(\pi^m v) \sim -ue\delta$. If $-ue\delta \sim 1$, then $(\alpha, x^2 - \pi^k u)_{x-\pi^m v} \not\sim 1$. Otherwise, the algebra $(\alpha, x - \pi^m v)$ is unramified. Moreover, $(\alpha, x)_{x^2-\pi^k u} \sim (\alpha, \sqrt{\pi u}) \not\sim 1$. Finally, the extension by $\sqrt{\alpha}$ leaves us in subcase $VI.2$, therefore it does not split the first algebra.

3. If $f(0) \sim -ue\delta \sim 1$, then $(\alpha, x^2 - \pi^k u)_x \not\sim 1$. If $-ue\delta \not\sim 1$, then (α, x) is unramified and $(\alpha, x)_{x^2-\pi^k u} \sim (\alpha, \sqrt{\pi u}) \not\sim 1$. Let $\theta = (\beta - \sqrt{\alpha})\sqrt[3]{\pi}$, where $\beta^2 - \alpha \not\sim 1$, $q = Irr_{k(\sqrt{\alpha}, \sqrt[3]{\pi})|k}(\theta)$. Then $f(\theta) \sim 1$ and

$$(\pi ue, x - \pi^s)_q \sim \left(\frac{\sqrt[3]{\pi}\ ,\ (\beta - \sqrt{\alpha})\sqrt[3]{\pi}}{k(\sqrt{\alpha}, \sqrt[3]{\pi})\langle q\rangle}\right) \not\sim 1.$$

4. $(x, -\pi u(x^2 - \pi^k u))_x \sim (x, \pi^{k+1}u^2) \sim 1$, so the second algebra B is unramified. To prove that B is nontrivial let us do it for the algebra $B' = B \otimes k(C)(\sqrt{\alpha})$. If $\theta = \pi^{k/2}\mu \in K(\sqrt{\pi\mu})$, $\mu \in O_K^*$, K is of odd degree and unramified over $k(\sqrt{\alpha})$, then $f(\theta) \sim (\mu - u)(\mu - \delta)$ and

$$B'_{Irr(\theta)} \sim \left(\frac{\sqrt{\pi\mu}\ ,\ \mu - u}{K(\sqrt{\pi\mu})(\sqrt{f(\theta)})\langle Irr(\theta)\rangle}\right).$$

Anyway, we can find such K and μ that $\mu - u \sim \mu - \delta \not\sim 1$, $\bar{u} \neq \bar{\mu}$, $\bar{\delta} \neq \bar{\mu}$. Then $B'_{Irr(\theta)} \not\sim 1$. Case VI is considered.

It is convenient to divide the last case into several parts.

Lemma 21 *Let case VII take place, i.e.*

$$f(x) = (x - \pi^s e)(x^2 - \pi^k u)((x - \pi^m v)^2 - \pi^r\delta),$$

where $e, u, v, \delta \in O_k^*$, $s, k, m, r > 0$, $\pi^k u, \pi^r\delta \not\sim 1, k < r$. Then the following subcases do not intersect and cover case VII.

VII_1**.** *$s = 1$, $k > 1$ and if $m = 1$, then $\bar{v} \neq \bar{e}$.*

VII_1'**.** *$s = 1$, $k > 1$, $m = 1$, and $\bar{v} = \bar{e}$.*

VII_2**.** *$s \geq 2$, $k \geq 4$, $m = 1$.*

VII_3. $1 \le k \le 3$ *and* $k < 2s$.

Proof. If $s > 1$, $m > 1$, and $k > 3$, then one can use replacement $x = x'\pi^2$, $y = y'\pi^5$.

Theorem 23 *Let* f *be as in case* VII_1. *Then the group* ${}_2Br\,C$ *is generated by the class of the algebra* (π, α) *and the classes of the following algebras.*

Let if $k \equiv 1\,(mod\,2)$, *then* $2m < k$, *and if* $k = r \equiv 0\,(mod\,2)$, $2m > k$, *then* $\bar{u} \ne \bar{\delta}$. *Then the first algebra is*

i. $(-\pi\alpha N_{K|k}(e - \pi^{k/2-1}\tau), p)$, *if* $k \equiv 0\,(mod\,2)$,

where $\tau \in O_K^*$ *such that* $K|k$ *is of odd degree and unramified,* $p = Irr_{K|k}(\pi^{k/2}\tau)$ *and* τ *satisfies the conditions:* $(\tau^2 - u) \not\sim 1$, $\phi(\tau) \sim 1$,

$$\phi(x) = \begin{cases} 1, & \text{if } k < r \text{ or } 2m < r, \\ x^2 - \delta, & \text{if } k = r \text{ and } 2m > r, \\ (x - v)^2 - \delta, & \text{if } k = r = 2m. \end{cases}$$

ii. $(\alpha, x^2 - \pi^k u)$, *if* $k \equiv 1\,(mod\,2)$ *and* $ue\psi \sim 1$, *where*

$$\psi = \begin{cases} 1, & \text{if } 2m < r, \\ v^2 - \delta, & \text{if } 2m = r, \\ -\delta, & \text{if } 2m > r,\ r \text{ is even.} \end{cases}$$

iii. (α, x), *if* $k \equiv 1\,(mod\,2)$ *and* $ue\psi \not\sim 1$.

The second one is

a. $(-\pi(\mu^2 - \delta)\chi(\mu)N_{L|k}(e - v\pi^{m-1} - \mu\pi^{r/2-1}), q)$, *if* $r \equiv 0\,(mod\,2)$,

where $\mu \in O_L^*$ *such that* $L|k$ *is of odd degree and unramified,* $q = Irr_{L|k}(\pi^m v + \pi^{r/2}\mu)$ *and* μ *satisfies the condition:* $(\mu^2 - \delta) \not\sim 1$ *if* k *is even, and* $(\mu^2 - \delta)\chi(\mu) \not\sim 1$, *otherwise,*

$$\chi(x) = \begin{cases} 1, & \text{if } 2m < k, \\ v^2 - u, & \text{if } 2m = k < r, \\ -u, & \text{if } 2m > k, k < r, \\ x^2 - u, & \text{if } 2m > k = r, \\ (x + v)^2 - u, & \text{if } k = r = 2m. \end{cases}$$

b. $(\alpha, (x - \pi^m v)^2 - \pi^r\delta)$, *if* $r \equiv 1\,(mod\,2)$ *and* $\delta(e - \pi^{m-1}v)\chi(\mu) \sim 1$.

c. $(\alpha, x - \pi^m v)$, *if* $r \equiv 1\,(mod\,2)$ *and* $\delta(e - \pi^{m-1}v)\chi(\mu) \not\sim 1$.

Otherwise we have

1. *The first algebra is from ii) or iii), and the second one is* $(-\pi ue\delta, x - \pi^m v)$, *if* $k, r \equiv 1\,(mod\,2)$, $k < 2m < r$.

2. *The first algebra is* $(-\pi ue\delta, x)$, *the second one is either* $(\alpha, x^2 - \pi^k u)$, *provided* $ue \sim 1$ *or* $(\alpha, x - \pi^{(k+1)/2})$, *provided* $ue \not\sim 1$, *if* $k, r \equiv 1\,(mod\,2)$, $k < r < 2m$.

3. *If* $k \equiv 1\,(mod\,2)$, $r \equiv 0\,(mod\,2)$, *and* $2m > k$, *then the first algebra is from ii) or iii), and the second one is*

- $(\pi, (x - \pi^m v)^2 - \pi^r\delta)$ *in the following cases:*
  1. $2m < r$ *and* $ue \not\sim 1$.
  2. $2m = r$, $ue \not\sim 1$, *and* $v^2 - \delta \sim 1$.
  3. $2m > r$, $ue \not\sim 1$, *and* $-1 \not\sim 1$.

- (π, t) *in cases*
  1. $2m = r$, $ue \sim 1$, *and* $v^2 - \delta \not\sim 1$.
  2. $2m > r$, $ue \sim 1$, *and* $-1 \sim 1$,

  where $t = Irr_{K(\sqrt{\pi w})|k}(\sqrt{\pi^k w})$, K *is as above,* $w \in O_K^* \setminus (O_K^*)^2$, *and* $-e(w-u) \not\sim 1$ *in* K.

- $(\alpha, Irr_{K|k}(\pi^m v + \pi^{r/2}\lambda))$ *in cases*
  1. $2m < r$, $ue \sim 1$. $v^2 - \delta \not\sim 1$.
  2. $2m > r$, $ue \not\sim -1$,

  where $\lambda \in O_K^*$, *and* $\lambda^2 - \delta \not\sim 1$ *in case* $2m < r$, $\lambda^2 - \delta \sim -1$ *in case* $2m > r$. *Note that if* $2m < r$, *then one can suppose* $K = k$, *and if* $2m > r$, *then this is true as soon as* $\bar{k} \neq \mathbf{F}_3$.

- $(\alpha, Irr_K(\pi^m \eta))$ *if* $2m = r$ *and* $ue \sim v^2 - \delta$, *where* $\eta \in O_K^*$, $ue((\eta - v)^2 - \delta) \not\sim 1$.

4. $(-\pi(e - \pi^{k-2}w), x - \pi^{k/2}w)$, $(\alpha(x^2 - \pi^k u), \pi^{k/2}(x - \pi^{k/2}w))$, $w^2 - u \not\sim 1$, *if* $k = r \equiv 0\,(mod\,2)$, $2m > k$, $\bar{u} = \bar{\delta}$ *and if* $k = 2$, *then* $e^2 - u \sim 1$.

5. $(\pi, x - \pi e)$, $(\pi, x^2 - \pi^k u)$, *if* $k = r = 2$, $m > 1$, $\bar{u} = \bar{\delta}$ *and* $e^2 - u \not\sim 1$.

6. *If* $k = r \equiv 1\,(mod\,2)$, $2m > k$, *then the first algebra is* $(x, -\pi u(x^2 - \pi^k u))$ *and the second one is*

- $(-\pi u e \delta, x)$, *if* $\bar{u} \neq \bar{\delta}$.
- $(\alpha, x^2 - \pi^k u)$, *if* $\bar{u} = \bar{\delta}$ *and either* $-ue \sim 1$ *or* $2m = k+1$.
- $(\alpha, Irr_{k(\sqrt{\pi u})|k}(\theta))$, $\theta = \sqrt{\pi^k u}(1 + \sqrt{\pi u})$, *if* $\bar{u} = \bar{\delta}$, $-ue \not\sim 1$, *and* $2m > k+1$.

Remark 3 *The algebra* $(\alpha(x^2 - \pi^k u), \pi^{k/2}(x - \pi^{k/2}w))$ *from 4 does not split by any quadratic extension of scalars.*

Proof of the theorem.
$i)$. In view of $f(\pi^{k/2}\tau) \sim -\pi\alpha(e - \pi^{k/2}\tau)$ in K the algebra is unramified. We also have $(-\pi\alpha N_{K|k}(e - \pi^{k/2-1}\tau), p)_{x-\pi e} \sim (-\pi\alpha N(e - \pi^{k/2-1}\tau), \prod_\sigma(\pi e - \pi^{k/2}\tau^\sigma))(-\pi\alpha N(e - \pi^{k/2-1}\tau), \pi N(\pi e - \pi^{k/2}\tau)) \not\sim 1$.
$ii), iii)$. $f(0) \sim ue\psi$, so in $iii)$ the algebra (α, x) is unramified. Moreover, in $ii)$ we have $(\alpha, x^2 - \pi^k u)_x \not\sim 1$ and in $iii)$ $(\alpha, x)_{x-\pi e} \not\sim 1$.
$a)$. $f(\pi^m v + \pi^{r/2}\mu) \sim -\pi(e - \pi^{m-1}v - \pi^{r/2-1}\mu)(\mu^2 - \delta)\chi(\mu)$, so the algebra is unramified. If k is even, then

$$(-\pi\alpha\chi(\mu)N_{L|k}(e - v\pi^{m-1} - \pi^{r/2-1}\mu), q)_{x^2 - \pi^k u} \sim \left(\frac{\pi\ ,\ \prod_\lambda(\pi^{k/2}\sqrt{u} - \pi^m v - \pi^{r/2}\mu^\lambda)}{k(\sqrt{\alpha})\langle x^2 - \pi^k u\rangle(\sqrt{C(x^2 - \pi^k u)})}\right).$$

The latter algebra is nontrivial if and only if $\chi(\mu)$ is not a square in L. We also have $(-\pi(\mu^2 - \delta)\chi(\mu)N_{L|k}(e - v\pi^{m-1} - \pi^{r/2-1}\mu), q)_{x-\pi e} \sim (-\pi(\mu^2 - \delta)\chi(\mu)N(e - v\pi^{m-1} - \pi^{r/2-1}\mu), \prod_\lambda(\pi e - \pi^m v - \pi^{r/2}\mu^\lambda)) \sim (\pi, (\mu^2 - \delta)\chi(\mu))$, and this algebra is nontrivial iff $(\mu^2 - \delta)\chi(\mu)$ is not a square in L. Anyway, the algebra from $a)$ is nontrivial.
$b), c)$. We have $f(\pi^m v) \sim \delta(e - \pi^{k-2}v)\chi(\mu)$ and $(\alpha, x - \pi^m v)_{x-\pi e} \not\sim 1$.

Let us check now that the algebra from $i), ii), iii)$ is not isomorphic to the algebra from $a), b), c)$ in the corresponding subcases.
1). Let firstly $k, r \equiv 1\,(mod\,2)$. We have $(\alpha, x^2 - \pi^k u)_{x - \pi^m v} \sim 1 \not\sim (\alpha, (x - \pi^m v)^2 - \pi^r \delta)_{x - \pi^m v}$. Furthermore,

$$(\alpha, x)_{x^2 - \pi^k u} \sim \left(\frac{\alpha\ ,\ \sqrt{\pi u}}{k(\sqrt{\pi u})\langle x^2 - \pi^k u\rangle(\sqrt{D(x^2 - \pi^k u)})}\right) \not\sim 1,$$

but

$$(\alpha, x - \pi^m v)_{x^2 - \pi^k u} \sim \left(\frac{\alpha\ ,\ -\pi^m u}{k(\sqrt{\pi u})\langle x^2 - \pi^k u\rangle(\sqrt{D(x^2 - \pi^k u)})}\right) \sim 1.$$

Similarly one can check that $(\alpha,(x-\pi^m v)^2-\pi^r\delta)_{x^2-\pi^k u}\sim 1$. Finally, we obtain

$$(\alpha, x-\pi^m v)_{(x-\pi^m v)^2-\pi^r\delta}\sim(\alpha,\sqrt{\pi\delta})\not\sim 1, \text{ and } (\alpha, x^2-\pi^k u)_{(x-\pi^m v)^2-\pi^r\delta}\sim 1.$$

2). $k\equiv 0\,(mod\,2)$, $r\equiv 1\,(mod\,2)$. Then the algebra from i is nontrivial at $x-\pi e$, but $(\alpha,(x-\pi^m v)^2-\pi^r\delta)_{x-\pi e}\sim 1$. Moreover,

$$(-\pi\alpha N_{K|k}(e-\pi^{k/2-1}\tau),p)_{(x-\pi^m v)^2-\pi^r\delta}\sim$$

$$\left(\frac{-\delta\alpha N(e-\pi^{k/2-1}\tau)\ ,\ \prod_\sigma(\pi^m v+\sqrt{\pi^r\delta}-\pi^{k/2}\tau^\sigma)}{k(\sqrt{\pi\delta})\langle(x-\pi^m v)^2-\pi^r\delta\rangle(\sqrt{C((x-\pi^m v)^2-\pi^r\delta)})}\right).$$

One can choose τ such that if $2m=k$, then $\bar\tau\neq\bar v$. For this τ the latter algebra is trivial in view of $v_K(\pi^m v-\pi^{k/2}\tau^\sigma)<r/2$. On the other hand, $(\alpha,x-\pi^m v)_{(x-\pi^m v)^2-\pi^r\delta}\not\sim 1$.
3). $k\equiv 1\,(mod\,2)$, $r\equiv 0\,(mod\,2)$. The algebra from $a)$ is nontrivial at $x-\pi e$, but $(\alpha,x^2-\pi^k u)$ is trivial at this place. We also have $(\alpha,x)_{x^2-\pi^k u}\not\sim 1$ and

$$(-\pi\alpha N_{L|k}(e-v\pi^{m-1}-\pi^{r/2-1}\mu),q)_{x^2-\pi^k u}\sim$$

$$\left(\frac{-\alpha u(e-v\pi^{m-1})\ ,\ \prod_\lambda(\sqrt{\pi^k u}-\pi^m v-\pi^{r/2}\mu^\lambda)}{k(\sqrt{\pi u})\langle x^2-\pi^k u\rangle(\sqrt{D(x^2-\pi^k u)})}\right)\sim$$

$$\left(\frac{-\alpha u(e-v\pi^{m-1})\ ,\ -\pi^m v}{k(\sqrt{\pi u})\langle x^2-\pi^k u\rangle(\sqrt{D(x^2-\pi^k u)})}\right)\sim 1.$$

4). $k,r\equiv 0\,(mod\,2)$. We have

$$(-\pi\alpha N_{K|k}(e-\pi^{k/2-1}),p)_{x^2-\pi^k u}\sim\left(\frac{\pi\ ,\ \prod_\sigma(\tau^\sigma-\sqrt{u})}{k(\sqrt{\alpha})\langle x^2-\pi^k u\rangle(\sqrt{C(x^2-\pi^k u)})}\right).$$

This algebra is nontrivial since $\tau^2-u\not\sim 1$ in K. So the algebra from $i)$ is nontrivial either at $x-\pi e$ or at $x^2-\pi^k u$. But the algebra from $a)$ is nontrivial at one of these places, and trivial at the other. Therefore, these algebras are not isomorphic.

Consider the remaining cases.
1. Let w be a unit from a sufficiently large unramified extension K of odd degree over k, such that

$$\begin{cases} w\not\sim -ue\delta,\\ w-u\sim -e,\end{cases}$$

and $t=Irr_{K(\sqrt{\pi w})|k}(\sqrt{\pi^k w})$. Then $f(\sqrt{\pi^k w})\sim -e(w-u)\sim 1$ and

$$(-\pi ue\delta, x-\pi^m v)_t\sim\left(\frac{-wue\delta\ ,\ \sqrt{\pi w}}{K(\sqrt{\pi w})\langle t\rangle}\right)\not\sim 1.$$

To finish the proof of this subcase it is enough to check that the extension of scalars by $\sqrt{\alpha}$ does not split algebra $(-\pi ue\delta, x-\pi^m v)$. Indeed, this extension leaves us in subcase $VII.7$.
2. All algebras are unramified. Since $f(\pi^{(k+1)/2})\sim ue$ we obtain $(x^2-\pi^k u)_{x-\pi^{(k+1)/2}}\not\sim 1$. Furthermore,

$$(\alpha,x-\pi^{(k+1)/2})_{x^2-\pi^k u}\sim\left(\frac{\alpha\ ,\ \sqrt{\pi u}}{k(\sqrt{\pi u})\langle x^2-\pi^k u\rangle(\sqrt{D(x^2-\pi^k u)})}\right)\not\sim 1.$$

The nontriviality of the algebra $(-\pi ue\delta,x)$ can be checked similarly as in the previous subcase. Finally, the extension by $\sqrt{\alpha}$ leaves us in the same subcase, so it does not split the latter algebra.
3. The nontriviality of algebras from $ii)$ and $iii)$ was already shown. Let firstly $2m<r$. Then $f(\pi^m v+\pi^{r/2}\lambda)\sim ue(\lambda^2-\delta)$, so if $ue\sim 1$, $\lambda^2-\delta\not\sim 1$, then $(\alpha, Irr(\pi^m v+\pi^{r/2}\delta))$ is unramified. In this case the first algebra is $(\alpha,x^-\pi^k u)$, it is trivial at $x-\pi e$, but $(\alpha, Irr(\pi^m v+\pi^{r/2}\delta))_{x-\pi e}\not\sim 1$.

If $ue \not\sim 1$, then $f(\pi^m v+\pi^{r/2}\lambda) \sim 1$, provided $\lambda^2-\delta \not\sim 1$. Therefore $(\pi,(x-\pi^v)^2-\pi^r\delta)_{Irr(\pi^m v+\pi^{r/2}\lambda)} \not\sim 1$. The first algebra is (α, x), $(\alpha,x)_{x-\pi e} \not\sim 1$. On the other hand, $(\pi,(x-\pi^v)^2-\pi^r\delta)_{x-\pi e} \sim 1$.

Let now $2m = r$. We have $f(\pi^m\eta) \sim ue((\eta-v)^2-\delta)$, so algebra $(\alpha, Irr(\pi^m\eta))$ is unramified. It is nontrivial at $x-\pi e$. Since $ue(v^2-\delta) \sim 1$, the first algebra is trivial at $x-\pi e$.

If $ue \not\sim 1$, $v^2-\delta \sim 1$, then $(\pi,(x-\pi^m v)^2-\pi^r\delta)_{Irr(\pi^m\eta)} \not\sim 1$. In this case the first algebra is nontrivial at $x-\pi e$, and this is not true for the second one.

Suppose $ue \sim 1$ and $v^2-\delta \not\sim 1$. We have $f(\sqrt{\pi^k w}) \sim -e(w-u) \sim \alpha$ in $K(\sqrt{\pi w})$. So

$$(\pi,t)_t \sim \left(\frac{w\ ,\ t}{K(\sqrt{\pi w},\sqrt{\alpha})}\right)\langle t\rangle \sim 1,$$

and algebra (π,t) is unramified. It is not isomorphic to (α,x) at $x-\pi e$. It is easy to check that $f(\pi^{m-1/3}) \sim ue \sim 1$. Hence $(\pi,t)_{Irr_{k(\sqrt[3]{\pi})|k}(\pi^{m-1/3})} \sim (\sqrt[3]{\pi}, w) \not\sim 1$.

Finally, let $2m > r$. One can consider this case by analogy to the previous ones.

4. We have $f(\pi^{k/2}w) \sim -\pi(e-\pi^{k-2}w)$ and $\alpha((\pi^{k/2}w)^2-\pi^k u) \sim 1$, so the both algebras are unramified. $(-\pi(e-\pi^{k-2}), x-\pi^{k/2}w)_{x^2-\pi^k u} \sim (\pi, w-\sqrt{u}) \not\sim 1$. Let us check that the second algebra is nontrivial, not isomorphic to the first one, and does not split by any quadratic extension of the scalars. To prove this it is sufficient to show that

$$B = (\alpha(x^2-\pi^k u), \pi^{k/2}(x-\pi^{k/2}w)) \otimes_{k(C)} k(\sqrt{\pi},\sqrt{\alpha})(C) \not\sim 1.$$

Assume $\nu = \sqrt{\pi}$, $\theta = \nu^k\sqrt{u}(1+\nu)$. Then $f(\theta) \sim 1$ in view of $\delta = u+\nu^{2l}\mu, l > 0$ and if $k=2$, then $e-\sqrt{u} \not\sim 1$. Thus

$$B_{x-\theta} \sim \left(\frac{\nu\ ,\ w-\sqrt{u}}{k(\sqrt{\pi},\sqrt{\alpha})\langle x-\theta\rangle}\right) \not\sim 1.$$

5. In this case $(\pi, x-\pi e)_{x^2-\pi^2 u} \sim (\pi, e-\sqrt{u}) \not\sim 1$ and $(\pi, x^2-\pi^2 u)_{x-\pi e} \sim (\pi, e^2-u) \not\sim 1$. Moreover, $((\pi,x-\pi e)_{\otimes}(\pi,x^2-\pi^2 u))_{x-\pi e} \sim (\pi,(x-\pi^m v)^2-\pi^r\delta)_{x-\pi e} \sim (\pi, e^2-\delta) \not\sim 1$.

6. We have $(x, -\pi u(x^2-\pi^k u))_x \sim (x, \pi^{k+1}u^2) \sim 1$, therefore the algebra $(x,-\pi u(x^2-\pi^k u))$ is unramified. Let K, w, and t be as in subcase 1, but

$$\begin{cases} uew(w-\delta) \not\sim 1, \\ -u(w-u) \not\sim 1. \end{cases}$$

Then $f(\sqrt{\pi^k w}) \sim -ew(w-u)(w-\delta) \sim 1$ and

$$(x,-\pi u(x^2-\pi^k u))_t \sim \left(\frac{\sqrt{\pi w}\ ,\ -u(w-u)}{K(\sqrt{\pi w}\langle t\rangle)}\right) \not\sim 1.$$

Consider the second algebra. For $\bar{u} \neq \bar{\delta}$ one always can find w and K such that

$$\begin{cases} -ue\delta w \not\sim 1, \\ -ew(w-u)(w-\delta) \sim 1. \end{cases}$$

Then $f(\sqrt{\pi^k w}) \sim 1$ and $(-\pi ue\delta, x)_t \sim (-ue\delta w, \sqrt{\pi w}) \not\sim 1$. So the second algebra is nontrivial. We also can find such K, w, and t that

$$\begin{cases} w \sim -ue\delta, \\ w-u \not\sim -u, \\ w-\delta \not\sim -\delta. \end{cases}$$

For such w the first algebra is nontrivial at t and the second one is trivial at t.

Let now $\bar{u} = \bar{\delta}$ and $\theta_w = \sqrt{\pi^k u}(1+\sqrt{\pi u w})$. Then $f(\theta_w) \sim -ew(uw-\pi^{m-(k+1)/2})$ (if $2m = k+1$ let $\bar{u}\bar{w} \neq \bar{v}$). In case $2m > k+1$ we have $f(\theta_w) \sim -ue \sim 1$ for any w, and otherwise we can find w such that $f(\theta_w) \sim 1$. Thus for $q = Irr_{K(\sqrt{\pi w})|k}(\theta_w)$

$$(\alpha, x^2-\pi^k u)_q \sim \left(\frac{\alpha\ ,\ 2w\sqrt{\pi u}}{K(\sqrt{\pi w})\langle q\rangle}\right) \not\sim 1.$$

Let $\bar{u} = \bar{\delta}$, $2m > k+1$, and $-ue \not\sim 1$. Then for $\theta = \theta_w, K = k, w = 1$ we find $f(\theta) \sim -ue \sim \alpha$ and the algebra (α, q) is unramified.

$$(\alpha, q)_{x^2 - \pi^k u} \sim \left(\frac{\alpha\ ,\ (\sqrt{\pi^k u} - \sqrt{\pi^k u}(1 + \sqrt{\pi u}))(\sqrt{\pi^k u} + \sqrt{\pi^k u}(1 - \sqrt{\pi u}))}{k(\sqrt{\pi u})\langle\sqrt{D(x^2 - \pi^k u)}\rangle} \right) \not\sim 1.$$

Both algebras $(\alpha, x^2 - \pi^k u)$ and (α, q) are not isomorphic to the first algebra from the list. Indeed, the extension by $\sqrt{\alpha}$ leaves us in the current case and subcase. So it does not split the first algebra.

The theorem is proved.

Theorem 24 *Let f be as in case VII_1', i.e.*

$$f(x) = (x - \pi v(1 + \pi^l \tau))(x^2 - \pi^k u)((x - \pi v)^2 - \pi^r \delta), \quad 1 < k \le r,\ \pi^k u, \pi^r \delta \not\sim 1.$$

Then the group ${}_2Br\,C$ is generated by the class of the algebra (π, α) and the classes of the following algebras.
The first algebra A is

i . $(\pi E(w), x - \pi^{k/2} w)$, *if* $k \equiv 0\,(mod\,2)$,

ii . $(\alpha, x^2 - \pi^k u)$, *if* $k \equiv 1\,(mod\,2)$ *and* $uv \sim 1$,

iii .(α, x), *if* $k \equiv 1\,(mod\,2)$ *and* $uv \not\sim 1$,

where $w \in O_k^*$, $w^2 - u \not\sim 1$, *and*

$$E(w) = \alpha \left\{ \begin{array}{ll} -v, & if\ k > 2, \\ w - v, & if\ k = 2. \end{array} \right\} \left\{ \begin{array}{ll} 1, & if\ r > 2, \\ (w - v)^2 - \delta, & if\ r = 2. \end{array} \right\}.$$

The second algebra B is

a . $(\alpha, x - \pi v)$,
if $r \equiv 1\,(mod\,2)$, $l \equiv 0\,(mod\,2)$, $\tau v \delta \xi \not\sim 1$, *where*

$$\xi = \left\{ \begin{array}{l} 1,\ if\ k > 2, \\ v^2 - u\ if\ k = 2. \end{array} \right.$$

b . $(\alpha, (x - \pi v)^2 - \pi^r \delta)$,
if $r \equiv 1\,(mod\,2)$, $l \equiv 0\,(mod\,2)$, $\tau v \delta \xi \sim 1$.

c . $(\pi v \delta \tau \xi, x - \pi v)$,
if $r, l \equiv 1\,(mod\,2)$, $2l + 2 < r$.

d . $(\pi \alpha N_{K|k}(\eta(\gamma)\zeta(\gamma)), p)$,
if either $r, l \equiv 0\,(mod\,2)$, $2l + 2 \le r$, *or* $r \equiv 2\,(mod\,4)$, $2l + 2 > r$, *where*

$$\eta(x) = \left\{ \begin{array}{l} 1,\ if\ k > 2, \\ v^2 - u\ if\ k = 2 < r, \\ (x + v)^2 - \delta,\ if\ k = r = 2, \end{array} \right. \qquad \zeta(x) = \left\{ \begin{array}{l} -v\tau\ if\ 2l + 2 < r, \\ x - v\tau,\ if\ 2l + 2 = r, \\ x,\ if\ 2l + 2 > r, \end{array} \right.$$

$\gamma \in O_K^*$, *$K|k$ is of odd degree and unramified,* $p = Irr_{K|k}(\pi v + \pi^{r/2}\gamma)$, $\gamma^2 - \delta \not\sim 1$, *and if* $r = 2$, *then* $(\gamma + v)^2 - u \sim 1$.

e . $(\pi, (x - \pi v)^2 - \pi^r \delta)$,
if $r \equiv 0\,(mod\,2)$, $l \equiv 1\,(mod\,2)$, *and either* $2l + 2 = r$ *or* $2l + 2 < r$ *and* $-\tau v \xi \not\sim 1$.

f . $(\alpha, x - \pi v - \pi^{r/2}\mu)$,
if $r \equiv 0\,(mod\,2)$, $l \equiv 1\,(mod\,2)$, $2l + 2 < r$, *and* $-\tau v \xi \sim 1$, *where* $\mu \in O_k^*$ *and* $\mu^2 - \delta \not\sim 1$.

g . $(\pi, (x-\pi v)^2 - \pi^r\delta)$,

if $r \equiv 0\,(mod\,4)$, $2l+2 > r$.

h . $(\alpha, (x-\pi v)^2 - \pi^r\delta)$,

if $r, l \equiv 1\,(mod\,2)$, $2l+2 > r$.

Note that in d) if $r > 2$, *then one can assume* $K = k$, $p = x - \pi v - \pi^{r/2}\gamma$.

Proof.
i). We have $f(\pi^{k/2}w) \sim \pi E(w)$, so A is unramified.

$$A_{x^2-\pi^k u} \sim \left(\frac{\pi\ ,\ w-\sqrt{u}}{k(\sqrt{u})\langle x^2-\pi^k u\rangle(\sqrt{D(x^2-\pi^k u)})}\right) \not\sim 1.$$

ii),iii). Then $f(0) \sim uv$, so $(\alpha, x^2-\pi^k u)_x \not\sim 1$ in case $uv \sim 1$, and (α, x) is unramified otherwise. Finally, $(\alpha, x)_{x^2-\pi^k u} \not\sim 1$.

It easy to check that $f(\pi v) \sim \pi^{r+l+1}\tau v\delta\xi$ and for even r and arbitrary $\mu \in O_K^*$ $f(\pi^v + \pi^{r/2}\mu) \sim (\mu^2-\delta)\eta(\mu)\zeta(\mu)\pi^{\min\{l+1, r/2\}}$.
a. We have $B_{(x-\pi v)^2-\pi^r\delta} \not\sim 1$ and

$$(\alpha, x)_{(x-\pi v)^2-\pi^r\delta} \sim \left(\frac{\alpha\ ,\ \pi v}{k(\sqrt{\pi\delta})\langle\ldots\rangle}\right) \sim 1,$$

$$(\alpha, x^2-\pi^k u)_{(x-\pi v)^2-\pi^r\delta} \sim \left(\frac{\alpha\ ,\ \xi}{k(\sqrt{\pi\delta})\langle\ldots\rangle}\right) \sim 1,$$

$$(\pi E(w), x-\pi^{k/2}w)_{(x-\pi v)^2-\pi^r\delta} \sim \left(\frac{\delta E(w)\ ,\ \pi v-\pi^{k/2}w}{k(\sqrt{\pi\delta})\langle\ldots\rangle}\right) \sim 1,$$

so $A \not\sim B$.
b. $f(\pi v) \sim 1$, therefore $B_{x-\pi v} \not\sim 1$. On the other hand, $(\alpha, x^2-\pi^k u)_{x-\pi v} \sim 1$. For the remaining A we have $A_{x^2-\pi^k u} \not\sim 1$, but $B_{x^2-\pi^k u} \sim (\alpha, (\pi\mu)^2) \sim 1$.
c. Let $\theta = \pi v + \sqrt{\pi^r\lambda}$. Then $f(\theta) \sim -v\tau\xi\lambda(\lambda-\delta)$ in $k(\sqrt{\pi\lambda})$. $B_{Irr(\theta)} \sim (v\delta\tau\xi\lambda, \sqrt{\pi\lambda})$. The conditions $-v\tau\xi\lambda(\lambda-\delta) \sim 1$ and $v\delta\tau\xi\lambda \not\sim 1$ can be satisfied in a sufficiently large unramified extension of odd degree of k. Thus $B \not\sim 1$. We have also $(\alpha, x)_{Irr(\theta)} \sim (\alpha, \pi v) \sim 1$, $(\alpha, x^2-\pi^k u)_{Irr(\theta)} \sim (\alpha, (\pi v)^2) \sim 1$, and $(\pi E(w))_{Irr(\theta)} \sim (\lambda E(w), \pi v - \pi^{k/2}w) \sim 1$. So $A \not\sim B$.
d. B is unramified since $f(\pi v + \pi^{r/2}\gamma) \sim \alpha\eta(\gamma)\zeta(\gamma)\pi$.

$$B_{(x-\pi v)^2-\pi^r\delta} \sim \left(\frac{\pi\ ,\ \prod_\sigma(\gamma^\sigma-\sqrt{\delta})}{k(\sqrt{\delta})\langle\ldots\rangle}\right) \not\sim 1.$$

The algebras (α, x), $(\alpha, x^2-\pi^k u)$ split by $\sqrt{\delta}$, so they are not isomorphic to B. Finally, $(\pi E(w), x-\pi^{k/2}w)_{(x-\pi v)^2-\pi^r\delta} \sim (\pi, \pi v - \pi^{k/2}w + \pi^{r/2}\sqrt{\delta})$. If $r > 2$, then the latter algebra is trivial, and $A \not\sim B$ again. Let $k = r = 2$. Then $A_{x^2-\pi^2 u} \not\sim 1$ and $B_{x^2-\pi^2 u} \sim (\pi, \prod_\sigma(\gamma^\sigma + v - \sqrt{u})) \sim 1$ in view of $(\gamma+v)^2 - u \sim 1$ in K.
e. In e) and f) we have

$$f(\pi v + \pi^{r/2}\mu) \sim (\mu^2-\delta)\eta\mu\zeta\mu \sim (\mu^2-\delta)\xi\left\{\begin{array}{ll} -v\tau, & \text{if } 2l+2 < r, \\ \mu - v\tau, & \text{if } 2l+2 = r. \end{array}\right\}.$$

If either $2l+2 = r$ or $-\tau v\xi \not\sim 1$, then there exists $\mu \in O_K^*$ such that $\mu^2-\delta \not\sim 1$ and $f(\pi v + \pi^{r/2}\mu) \sim 1$. For such μ we have $B_{Irr(\pi v+\pi^{r/2}\mu)} \sim (\pi, \mu^2-\delta) \not\sim 1$. Since $r \geq 2l+2 \geq 4$, then $B_x \sim 1$ and $B_{x^2-\pi^k u} \sim 1$. So $B \not\sim A$.

f. In this case $f(\pi v+\pi^{r/2}\mu)\sim\alpha$, so B is unramified. Let $\theta=\pi v(1+\pi^{l+1/3})\in k(\Pi)$, $\Pi=\sqrt[3]{\pi}$. Then $f(\theta)\sim-\tau v\xi(\pi^{2l+8/3}v^2-\pi^r\delta)\sim 1$.

$$B_{Irr(\theta)}\sim\left(\frac{\alpha\ ,\ \pi^{l+4/3}v-\pi^{r/2}\mu}{k(\Pi)\langle Irr(\theta)\rangle}\right)\sim\left(\frac{\alpha\ ,\ \Pi^{3l+4}}{k(\Pi)\langle Irr(\theta)\rangle}\right)\not\sim 1.$$

Moreover, $A\not\sim B$. Indeed, if k is even, then $B_{x^2-\pi^k u}\sim 1$ since B splits by $\sqrt{u}$, and if k is odd, then $(\alpha,x^2-\pi^k u)_{Irr(\theta)}\sim 1$. Finally, for odd k $(\alpha,x)_{x^2-\pi^k u}\not\sim 1$ and $B_{x^2-\pi^k u}\sim(\alpha,-\pi v)\sim 1$.
g. Let $\mu\in O_K^*$ such that $\mu^2-\delta\not\sim 1$, $\mu\eta(\mu)\not\sim 1$. Then $f(\pi v+\pi^{r/2}\mu)\sim 1$ and $B_{Irr(\pi v+\pi^{r/2}\mu)}\not\sim 1$. But $B_x\sim 1$, so $B\not\sim(\alpha,x^2-\pi^k u)$. The remaining algebra A is not trivial at $x^2-\pi^k u$, but $B_{x^2-\pi^k u}\sim(\pi,(\sqrt{\pi^k u}-\pi v)^2-\pi^r\delta)\sim 1$ since $r>2$.
h. $B_{x-\pi v(1+\pi^l\tau)}\sim(\alpha,\pi^{2l+2}v^2\tau^2)-\pi^r\delta\not\sim 1$. We have also $B_x\sim(\alpha,\pi^2v^2)\sim 1$. If k is odd, then $B_{x^2-\pi^k u}\sim(\alpha,(\sqrt{\pi^k u}-\pi v)^2-\pi^r\delta)\sim 1$ since $r>2$. Finally, if k is even, then $B_{x^2-\pi^k u}\sim 1$ in view of B splits by $\sqrt{u}$.

Consider the next case.

Theorem 25 *Let f be as in case VII_2, i.e.*

$$f(x)=(x-\pi^s e)(x^2-\pi^k u)((x-\pi v)^2-\pi^r\delta),\quad s\geq 2,\ 4\leq k\leq r,\ \ \pi^k u,\pi^r\delta\not\sim 1.$$

Then the group ${}_2Br\,C$ is generated by the class of the algebra (π,α) and the classes of the following algebras.

If $2s<k$, then the first algebra A is

1 . (α,x), *if* $k,s\equiv 1\,(mod\,2)$ *and* $ue\not\sim 1$.

2 . $(\alpha,x^2-\pi^k u)$, *if* $k,s\equiv 1\,(mod\,2)$ *and* $ue\sim 1$.

3 . $(-\pi\alpha e,x-\pi^{k/2}w)$, *if* $k\equiv 0\,(mod\,2)$, $s\equiv 1\,(mod\,2)$, *where* $w\in O_k^*$, $w^2-u\not\sim 1$.

4 . $(\pi,x^2-\pi^k u)$, *if* $k,s\equiv 0\,(mod\,2)$ *and* $-e\not\sim 1$.

5 . $(\alpha,x-\pi^{k/2}w)$, *if* $k,s\equiv 0\,(mod\,2)$ *and* $-e\sim 1$, $w^2-u\not\sim 1$.

6 . $(ue\pi,x)$, *if* $k\equiv 1\,(mod\,2)$, $s\equiv 0\,(mod\,2)$.

If $2s\geq k$, then the first algebra A is

7 . $(\alpha,x^2-\pi^k u)$, *if* $k\equiv 1\,(mod\,2)$.

8 . $(\alpha\xi(w)\pi,x-\pi^{k/2}w)$, *if* $k\equiv 2\,(mod\,4)$, $w^2-u\not\sim 1$,

$$\xi(x)=\begin{cases} w, & \text{if } 2s>k,\\ w-e, & \text{if } 2s=k, \bar{w}\neq\bar{e}.\end{cases}$$

9 . $(\pi,x^2-\pi^k u)$, *if* $k\equiv 0\,(mod\,4)$.

And the second algebra B is

a . $(\alpha,x-\pi v)$, *if* $r\equiv 1\,(mod\,2)$, $-v\delta\not\sim 1$.

b . $(\alpha,(x-\pi v)^2-\pi^r\delta)$, *if* $r\equiv 1\,(mod\,2)$, $-v\delta\sim 1$.

c . $(\pi\alpha v,x-\pi v-\pi^{r/2}\mu)$, *if* $r\equiv 0\,(mod\,2)$, *where* $\mu\in O_k^*$, $\mu^2-\delta\not\sim 1$.

Proof. Let $2s<k$. We have $f(0)\sim ue\pi^{k+s}$, $f(\pi v)\sim -v\delta\pi^{r+1}$. If k is even and $w^2-u\not\sim 1$, then $f(\pi^{k/2}w)\sim -e\alpha\pi^s$. If r is even and $\mu^2-\delta\not\sim 1$, then $f(\pi v+\pi^{r/2}\mu)\sim v\alpha\pi$.
1. $(\alpha,x)_{x-\pi^s e}\not\sim 1$.
2. $(\alpha,x^2-\pi^k u)_x\not\sim 1$.
3. $(-\pi\alpha e,x-\pi^{k/2}w)_{x-\pi^s e}\not\sim 1$.

4. $(\pi, x^2-\pi^k u)_{x-\pi^{k/2}w} \not\sim 1$.
5. $f(\pi^{s+1}) \sim -\pi^s e \sim 1$, so $(\alpha, x-\pi^{k/2}w)_{x-\pi^{s+1}} \sim (\alpha, \pi^{s+1}) \not\sim 1$.
6. $f(\sqrt{\pi^k\gamma}) \sim -e\gamma(\gamma-u)$ in $k(\sqrt{\pi\gamma})$. So $(ue\pi, x)_{Irr(\sqrt{\pi^k\gamma})} \sim (ue\gamma, \sqrt{\pi\gamma}) \not\sim 1$, provided $\gamma \in O_K^*$ satisfies the condition $\gamma \not\sim ue$, $\gamma - u \not\sim -u$.

Let now $k \le 2s$. If k is even and $w^2-u \not\sim 1$, then $f(\pi^{k/2}w) \sim \pi^{k/2}\alpha\xi(w)$.
7. $(\alpha, x^2-\pi^k u)_{x-\pi^s e} \not\sim 1$.
8. $(\alpha\xi(w)\pi, x-\pi^{k/2}w)_{x^2-\pi^k u} \not\sim 1$.
9. Let $\gamma \in O_K^*$, $\gamma^2-u \not\sim 1$, and $\xi(\gamma) \not\sim 1$. Then $f(\pi^{k/2}\gamma) \sim 1$ and $(\pi, x^2-\pi^k u)_{Irr(\pi^{k/2}\gamma)} \not\sim 1$.
a. $(\alpha, x-\pi v)_{(x-\pi v)^2-\pi^r\delta} \not\sim 1$. On the other hand, $A_{(x-\pi v)^2-\pi^r\delta} \sim 1$.
b. $(\alpha, (x-\pi v)^2-\pi^\delta)_{x-\pi v} \not\sim 1$. $A \not\sim B$ since B is trivial at all the places from 1–9, at which A is nontrivial.
c. $(\pi\alpha v, x-\pi v-\pi^{r/2}\mu)_{(x-\pi v)^2-\pi^r\delta} \sim (\pi, \mu-\sqrt{\delta}) \not\sim 1$. On the other hand, $A_{(x-\pi v)^2-\pi^r\delta} \sim 1$.

The theorem is proved.

Let us pass to the last case.

Theorem 26 *Let f be as in case VII_3, i.e.*

$$f(x) = (x^2-\pi^k u)(x-\pi^s e)((x-\pi^m v)^2-\pi^r\delta),$$

where $k \in \{1,2,3\}$, $k < 2s$, $k \le r$, $\pi^k u, \pi^r\delta \not\sim 1$. Then the group ${}_2Br\,C$ is generated by the class of the algebra (π,α) and the classes of the following algebras.
The first algebra A is

i. $(\alpha, x^2-\pi^k u)$, *if $k \equiv 1\,(mod\,2)$.*

ii. $(\pi E(w), x-\pi w)$, *if $k=2$, where $w^2-u \not\sim 1$,*

$$E(x) = \begin{cases} \alpha x \ \textit{if}\ r > 2, \\ \alpha x(x^2-\delta) \ \textit{if}\ r=2, m>1, \bar{u} \ne \bar{\delta} \\ x, \ \textit{if}\ r=2, m>1, \bar{u}=\bar{\delta} \\ \alpha x((x-v)^2-\delta), \ \textit{if}\ r=2, m=1. \end{cases}$$

1. *If $r < 2s$, $r \le 2m$, then the second algebra B is*

- **a.** $(\alpha, (x-\pi^m v)^2-\pi^r\delta)$, *if $r \equiv 1\,(mod\,2)$.*
- **b.** $(\pi, (x-\pi^m v)^2-\pi^r\delta)$, *if $r \equiv 0\,(mod\,2)$, $k+r/2 \equiv 0\,(mod\,2)$.*
- **c.** $(\pi N_{K|k}(S(\mu)), Irr_{K|k}(\pi^m v+\pi^{r/2}\mu))$, *if $r \equiv 0\,(mod\,2)$, $k+r/2 \equiv 1\,(mod\,2)$, and if $k=r=2, m>1$, then $\bar{u} \ne \bar{\delta}$. Here $\mu \in O_K^*$, $K|k$ is of odd degree and unramified, $\mu^2-\delta \not\sim 1$. If $k=r=2$, then the additional condition on μ is*

$$\begin{cases} \mu^2-u \sim 1, \ \textit{if}\ m>1, \\ (\mu+v)^2-u \sim 1, \ \textit{if}\ m=1. \end{cases}$$

  Note that if $k=r=2$ does not hold, then one can assume $K=k$.

$$S(x) = \alpha\left\{\begin{array}{l} x, \ \textit{if}\ 2m>r, \\ x+v, \ \textit{if}\ 2m=r, \end{array}\right\}\left\{\begin{array}{l} -u, \ \textit{if}\ k<r, \\ x^2-u, \ \textit{if}\ k=r<2m, \\ (x+v)^2-u, \ \textit{if}\ k=r=2m, \end{array}\right\}$$

2. *If $k \le 2m < r$, $m < s$, then the second algebra B is*

- **a.** $(\alpha v\xi\pi, x-\pi^m v-\pi^{r/2}\mu)$, *if $m+k \equiv 1\,(mod\,2)$, $r \equiv 0\,(mod\,2)$, where $\mu^2-\delta \not\sim 1$ and*

$$\xi = \begin{cases} -u, \ \textit{if}\ 2m>k, \\ v^2-u, \ \textit{if}\ 2m=k. \end{cases}$$

b. $(-\delta v\xi\pi, x-\pi^m v)$, *if* $m+k \equiv 0\,(mod\,2)$, $r \equiv 1\,(mod\,2)$.

c. $(\alpha, (x-\pi^m v)^2 - \pi^r\delta)$, *if* $m+k, r \equiv 1\,(mod\,2)$ *and* $-\delta v\xi \sim 1$.

d. $(\alpha, x-\pi^m v)$, *if* $m+k, r \equiv 1\,(mod\,2)$ *and* $-\delta v\xi \not\sim 1$.

e. $(\pi, (x-\pi^m v)^2 - \pi^r\delta)$, *if* $m+k, r \equiv 0\,(mod\,2)$, $v\xi = -uv \not\sim 1$.

f. $(\alpha, x-\pi^m v - \pi^{r/2}\mu)$, *if* $m+k, r \equiv 0\,(mod\,2)$, $v\xi = -uv \sim 1$, $\mu^2 - \delta \not\sim 1$.

3. *Let* $2s \le r$, $s \le m$, *and if* $s = m$, *then* $\bar{v} \ne \bar{e}$. *The second algebra* B *is*

a. $(u\delta\psi\pi, x-\pi^m v)$, *if* $k+s \equiv 0\,(mod\,2)$, $r \equiv 1\,(mod\,2)$.

b. $(\alpha, x-\pi^m v)$, *if* $k+s, r \equiv 1\,(mod\,2)$, $\psi u\delta \not\sim 1$.

c. $(\alpha, (x-\pi^m v)^2 - \pi^r\delta)$, *if* $k+s, r \equiv 1\,(mod\,2)$, $\psi u\delta \sim 1$.

d. $(-\pi u\alpha\psi(\mu), x-\pi^m v - \pi^{r/2}\mu)$, *if* $k+s \equiv 1\,(mod\,2)$, $r \equiv 0\,(mod\,2)$, *where* $\mu^2 - \delta \not\sim 1$ *and*

$$\psi(x) = \begin{cases} -e, & \text{if } s < m,\ 2s < r, \\ v-e, & \text{if } s = m,\ 2s < r, \\ x-e, & \text{if } s < m,\ 2s = r, \\ x+v-e, & \text{if } s = m,\ 2s = r, \end{cases}$$

if $2s < r$, *then* $\psi(x) = \psi$ *does not depend on* x.

e. $(\pi, (x-\pi^m v)^2 - \pi^r\delta)$, *if* $k+s, r \equiv 0\,(mod\,2)$, $2s < r$, $-u\psi \not\sim 1$.

f. $(\alpha, x-\pi^m v - \pi^{r/2}\mu)$, *if* $k+s, r \equiv 0\,(mod\,2)$, $2s < r$, $-u\psi \sim 1$, $\mu^2 - \delta \not\sim 1$.

g. $(\pi, x-\pi^s e)$, *if* $k+s, r \equiv 0\,(mod\,2)$, $2s = r$.

4. *If* $2m < k$ *(i.e.* $m = 1$, $k = 3$, $s \ge 2$*), then the second algebra* B *is*

a . $(\pi\alpha v, x-\pi v - \pi^{r/2}\mu)$, $r \equiv 0\,(mod\,2)$, $\mu^2 - \delta \not\sim 1$.

b. $(\alpha, (x-\pi v)^2 - \pi^r\delta)$, $r \equiv 1\,(mod\,2)$, $-v\delta \sim 1$.

c. $(\alpha, x-\pi v)$, $r \equiv 1\,(mod\,2)$, $-v\delta \not\sim 1$.

5. *If* $k < 2m = 2s < r$, $e = v(1+\pi^l\tau)$, $l > 0$, *then the second algebra* B *is*

a. $(\alpha, x-\pi^m v)$, *if* $k+m+l, r \equiv 1\,(mod\,2)$, $-uv\tau\delta \not\sim 1$.

b. $(\alpha, (x-\pi^m v)^2 - \pi^r\delta)$, *if* $k+m+l, r \equiv 1\,(mod\,2)$, $-uv\tau\delta \sim 1$.

c. $(\pi, (x-\pi^m v)^2 - \pi^r\delta)$, *if* $m+l \le r/2$, $k+m+l, r \equiv 0\,(mod\,2)$, *and either* $m+l < r/2$, $uv\tau \not\sim 1$, *or* $m+l = r/2$.

d. $(\alpha, x-\pi^m v - \pi^{r/2}\mu)$, *if* $m+l < r/2$, $k+m+l, r \equiv 0\,(mod\,2)$, *and* $uv\tau \sim 1$. $\mu^2 - \delta \not\sim 1$.

e. $(-\pi uv\tau\delta, x-\pi^m v)$, *if* $m+l < r/2$, $k+m+l \equiv 0\,(mod\,2)$, $r \equiv 1\,(mod\,2)$.

f. $(\pi T(\mu), x-\pi^m v - \pi^{r/2}\mu)$, *if* $m+l \le r/2$, $k+m+l \equiv 1\,(mod\,2)$, $r \equiv 0\,(mod\,2)$, *where* $\mu^2 - \delta \not\sim 1$, *and*

$$T(x) = \begin{cases} \alpha v\tau u, & \text{if } m+l < r/2, \\ -\alpha u(x - v\tau), & \text{if } m+l = r/2. \end{cases}$$

g. $(-\pi u\mu\alpha, x-\pi^m v - \pi^{r/2}\mu)$, *if* $m+l > r/2$, $k+r/2 \equiv 1\,(mod\,2)$, $r \equiv 0\,(mod\,2)$.

h. $(\pi, (x-\pi^m v)^2 - \pi^r\delta)$, *if* $m+l > r/2$, $k+r/2, r \equiv 0\,(mod\,2)$.

j. $(\alpha, (x-\pi^m v)^2 - \pi^r\delta)$, *if* $m+l > r/2$, $r \equiv 1\,(mod\,2)$, $k+l+m \equiv 0\,(mod\,2)$.

6. *If* $k = r = 2$, $m, s \ge 2$, $\bar{u} = \bar{\delta}$, *then the second algebra* B *is*

a. $(\pi, x^2 - \pi^k u)$, *if* $-1 \sim 1$.

b. $(\alpha(x^2 - \pi^k u), x-\pi w)$, *if* $-1 \not\sim 1$, *where* $w^2 - u \not\sim 1$.

Proof.
i),ii). $(\alpha, x^2-\pi^k u)_{x-\pi^s e} \not\sim 1$, provided k is odd. Otherwise we have $f(\pi w) \sim \pi E(w)$, so A is unramified. Moreover,

$$(\pi E(w), x-\pi w)_{x^2-\pi^k u} \sim \left(\frac{\pi \ , \ w-\sqrt{u}}{k(\sqrt{u})\langle x^2-\pi^k u\rangle (D(x^2-\pi^k u))}\right) \not\sim 1.$$

1.a. $B_{x-\pi^s e} \sim (\alpha, \pi^r) \not\sim 1$. If k is odd, then $(A \otimes B)_{x^2-\pi^k u} \sim (x-\pi^s e)_{x^2-\pi^k u} \sim (\alpha, \sqrt{\pi u}) \not\sim 1$, so $A \not\sim B$. Let k is even. Then $B_{x^2-\pi^k u} \sim 1$, and $A \not\sim B$ again.
1.b. If r is even, then $f(\pi^m v + \pi^{r/2}\mu) \sim \pi^{k+r/2} S(\mu)$. In case 1.b) one can find K, μ such that $S(\mu) \sim 1$ and $\mu^2 - \delta \not\sim 1$. Therefore $B_{Irr(\pi^m v+\pi^{r/2}\mu)} \sim (\pi, \mu^2-\delta) \not\sim 1$. For odd k $A_{x^2-\pi^k u} \sim (\alpha, \sqrt{\pi u}) \not\sim 1$, but $B_{x^2-\pi^k u} \sim 1$. Let k be even. Then $r/2$ is even, so $r > k$ and $B_{x^2-\pi^k u} \sim 1$. Anyway, $A \not\sim B$.
1.c. We have $B_{(x-\pi^m v)^2-\pi^r \delta} \not\sim 1$. On the other hand, for odd k $A_{(x-\pi^m v)^2-\pi^r\delta} \sim 1$. Let k is even. If $r > 2$, then $A_{(x-\pi^m v)^2-\pi^r\delta} \sim (\pi, -\pi w) \sim 1$. Let $k = r = 2$.

$$B_{x^2-\pi^k u} \sim \left(\frac{\pi \ , \ \prod_\sigma ((\mu^\sigma + \pi^{m-1} v) - \sqrt{u})}{k(\sqrt{u})\langle \ldots \rangle}\right) \sim 1.$$

Thus, $A \not\sim B$.

It is easy to check that in 2) $f(\pi^m v) \sim -\pi m + k + r\delta\xi v$ and for even r $f(\pi^m v + \pi^{r/2}\mu) \sim (\mu^2 - \delta)\xi v \pi^{m+k}$. Moreover, $E(w) \sim \alpha w$, since $k < r$. So in 2) $A_{x-\pi^s e} \not\sim 1$ and $A_{x^2-\pi^k u}$ in case i) as well as in case ii).
2.a. Let firstly m be even and k be odd. Then $B_{x^2-\pi^k u} \sim (\sqrt{\pi u}, -\alpha v)$ and $B_{x-\pi^s e} \sim (\pi, -v)$. There is exactly one nontrivial algebra among two last algebras. So $B \not\sim 1$ and $A \not\sim B$.

Let now m be odd and k be even. Then $B_{x-\pi^s e} \sim (\alpha v \xi \pi, -\pi v) \sim (\pi, \alpha\xi)$. Furthermore, the algebra $B_{x^2-\pi^k u}$ is nontrivial iff the algebra (π, ξ) is nontrivial over k. Thus, $B \not\sim 1$, $A \not\sim B$ again.
2.b. Assume $\theta = \pi^m v + \sqrt{\pi^r \lambda}$, $\lambda \in O_K^*$, $p = Irr_{K(\sqrt{\pi\lambda})|k}(\theta)$. We have $f(\theta) \sim v\xi\lambda(\lambda-\delta)$ in $K(\sqrt{\pi\lambda})$.

$$B_p \sim \left(\frac{\sqrt{\pi\lambda} \ , \ -\delta v \xi \lambda}{K(\sqrt{\pi\lambda})\langle p \rangle}\right).$$

One always can find K, λ such that $B_p \not\sim 1$. If k is odd, then $A_p \sim (\alpha, \pi) \sim 1$ over $K(\sqrt{\pi\lambda})$ and $A \not\sim B$. Let k be even. $A_p \sim (\alpha\lambda w, \pi^m v + \sqrt{\pi^r\lambda} - \pi w) \sim 1$.
2.c. $B_{x-\pi^m v} \not\sim 1$. In addition, $B_{x-\pi^s e} \sim 1$, so $A \not\sim B$.
2.d. If k is odd and m is even, then $B_{x^2-\pi^k u} \sim (\alpha, \sqrt{\pi u}) \not\sim 1$ and $B_{x-\pi^s e} \sim (\alpha, \pi^m) \sim 1$. Otherwise, $B_{x-\pi^s e} \not\sim 1$ and $B_{x^2-\pi^k u} \sim 1$.
2.e. Let $\mu^2 - \delta \not\sim 1$, then $f(\pi^m v + \pi^{r/2}\mu) \sim 1$ and $B_{Irr(\pi^m v+\pi^{r/2}\mu)} \not\sim 1$. $B_{x-\pi^s e} \not\sim 1$.
2.f. Assume $\Pi = \sqrt[3]{\pi}$, $\theta = \pi^m v + \pi^m \Pi$, and $p = Irr_{k(\Pi)|k}(\theta)$. We have $f(\theta) \sim 1$, $B_p \sim (\alpha, \Pi\pi^m)$, $B_{x-\pi^s e} \sim (\alpha, \pi^m)$. There is exactly one nontrivial algebra among the two last algebras. On the other hand, $A_{x-\pi^s e} \not\sim 1$ and $A_p \not\sim 1$. Indeed, if k is odd, then $A_p \sim (\alpha, -\pi^k u)$ and $A_p \sim (\alpha w \pi, -\pi w)$. So $B \not\sim 1$ and $B \not\sim A$.

In 3) $f(\pi^m v) \sim \pi^{k+r+s} u\delta\psi$ and $f(\pi^m v + \pi^{r/2}\mu) \sim -\pi^{k+s} u\psi(\mu)(\mu^2-\delta)$, provided r is even. If k is odd, then $A_{x^2-\pi^k u} \sim (\alpha, \sqrt{\pi u}) \not\sim 1$. If k is even, then $A_{x-\pi^s e} \sim (\pi\alpha w, -\pi w) \not\sim 1$. Anyway, A is nontrivial at two these places. On the other hand, $A_{(x-\pi^m v)^2-\pi^r\delta} \sim 1$.
3.a. This case can be considered by analogy with 2.b).
3.b. We have $B_{(x-\pi^m v)^2-\pi^r\delta} \not\sim 1$ and $A_{(x-\pi^m v)^2-\pi^r\delta} \sim 1$.
3.c. $B \not\sim 1$ since $B_{x-\pi^m v} \not\sim 1$, and $B \not\sim A$ in view of $B_{x-\pi^s e} \sim 1$.
3.d. $B_{(x-\pi^m v)^2-\pi^r\delta} \sim (\pi, \mu - \sqrt{\delta}) \not\sim 1$, provided $\mu^2 - \delta \not\sim 1$. So $B \not\sim 1, A$.
3.e. For $\mu^2 - \delta \not\sim 1$ $f(\pi^m v + \pi^{r/2}\mu) \sim 1$ and $B_{x-\pi^m v - \pi^{r/2}\mu} \not\sim 1$. $B_{x-\pi^s e} \sim 1$.
3.f. Let $\theta_i = \pi^m v + \pi^s \Pi^i \in k(\Pi)$, $\Pi = \sqrt[3]{\pi}$, $p_i = Irr_{k(\Pi)|k}(\theta_i)$, $i = 1, 2$. Then $f(\theta_i) \sim -\pi^k u \pi^s \psi(\pi^{2s}\Pi^{2i} - \pi^r\delta) \sim -u\psi \sim 1$ and $B_{p_i} \sim (\alpha, \pi^s \Pi^i)$ over $k(\Pi)$. So there is exactly one nontrivial algebra among B_{p_i}, $i = 1, 2$. On the other hand, $A_{p_i} \not\sim 1$. Therefore, $B \not\sim 1, A$.
3.g. Let $\eta_i = \pi^s \gamma_i$, $\gamma_i \in O_K^*$, $\bar{\gamma}_i \neq \bar{e}$, $q_i = Irr_{K|k}(\eta_i)$, $i = 1, 2$. Then

$$f(\eta_i) \sim -u(\gamma_i - e) \left\{ \begin{array}{l} \gamma_i^2 - \delta, \text{ if } m > s, \\ (\gamma_i - v)^2 - \delta, \text{ if } m = s. \end{array} \right\}$$

We require $\gamma_1 - e \not\sim \gamma_2 - e$ and $f(\eta_i) \sim 1$, $i = 1, 2$. Then $B_{q_i} \sim (\pi, (-1)^s(\gamma_i - e))$ and $A_{q_i} \sim 1$, $i = 1, 2$. Since there is exactly one nontrivial algebra among B_{q_i}, $i = 1, 2$, $B \not\sim 1, A$.

In 4) we have $f(\pi v) \sim -v\delta\pi^{r+1}$ and $f(\pi v + \pi^{r/2}\mu) \sim v(\mu^2 - \delta)\pi$, provided r is even. In this case $A = (\alpha, x^2 - \pi^3 u)$ and $A_{x-\pi^s e} \not\sim 1$, $A_{x^2-\pi^3 u} \not\sim 1$.
4.a. $B_{x-\pi^s e} \sim (\alpha v\pi, -\pi v) \not\sim 1$, $B_{x^2-\pi^k u} \sim 1$, so that $B \not\sim 1, A$.
4.b. $B \not\sim 1$ in view of $B_{x-\pi v} \not\sim 1$, and $B \not\sim A$ since $B_{x-\pi^s e} \sim 1$.
4.c. Finally, in this case $B_{x-\pi^s e} \not\sim 1$ and $B_{x^2-\pi^3 u} \sim 1$.

In 5) $f(\pi^m v) \sim -\pi^{k+m+l+r} uv\tau\delta$ and if r is even, then

$$f(\pi^m v + \pi^{r/2}\mu) \sim (\mu^2 - \delta) \left\{ \begin{array}{l} \pi^{k+l+m} uv\tau, \text{ if } m + l < r/2, \\ -\pi^{k+l+m} u(\mu - v\tau), \text{ if } m + l = r/2, \\ -\pi^{k+r/2} u\mu, \text{ if } m + l > r/2. \end{array} \right\}$$

We have also $A_{x-\pi^s e} \not\sim 1$ and $A_{x^2-\pi^k u} \not\sim 1$.
5.a. $B_{(x-\pi^m v)^2-\pi^r\delta} \not\sim 1$ and $A_{(x-\pi^m v)^2-\pi^r\delta} \sim 1$.
5.b. $B_{x-\pi^m v} \not\sim 1$, $B_{x^2-\pi^k u} \sim 1$.
5.c. If $m + l < r/2$, then $f(\pi^m v + \pi^{r/2}\mu) \sim 1$, provided $\mu^2 - \delta \not\sim 1$. Otherwise, $f(\pi^m v + \pi^{r/2}\mu) \sim (\mu^2 - \delta)(v\tau - \mu)$, and we always can find such K, μ that $f(\pi^m v + \pi^{r/2}\mu) \sim 1$ and $\mu^2 - \delta \not\sim 1$ again. Anyway, $B_{Irr(\pi^m v + \pi^{r/2}\mu)} \not\sim 1$. $B \not\sim A$ in view of $B_{x^2-\pi^k u} \sim 1$.
5.d. Let $\theta_i = \pi^m v + \pi^{(m+l)}\Pi^i$, $p_i = Irr_{k(\Pi)|k}(\theta_i)$, $i = 1, 2$. Then $f(\theta_i) \sim uv\tau \sim 1$ and $B_{p_i} \sim (\alpha, \Pi^{m+l+i})$. So there is exactly one nontrivial algebra among B_{p_i}, $i = 1, 2$. On the other hand, $A_{p_i} \not\sim 1$. Therefore, $B \not\sim 1, A$.
5.e. Let $\theta = \pi^m v + \sqrt{\pi^r\lambda} \in K(\sqrt{\pi\lambda})$ and $p = Irr(\theta)$. Then $f(\theta) \sim uv\tau\lambda(\lambda - \delta)$ and

$$B_p \sim \left(\frac{\sqrt{\pi\lambda}\,,\ -\lambda uv\tau\delta}{K(\sqrt{\pi\lambda})\langle p\rangle} \right).$$

One always can choose K and λ such that $B_p \not\sim 1$. Furthermore, $B \not\sim A$ since $A_p \sim 1$.
5.f. $f(\pi^m v + \pi^{r/2}\mu) \sim \pi T(\mu)$, so B is unramified. $A_{(x-\pi^m v)^2-\pi^r\delta} \sim 1$, but $B_{(x-\pi^m v)^2-\pi^r\delta} \sim (\pi, \mu - \sqrt{\delta}) \not\sim 1$.
5.g. $B_{(x-\pi^m v)^2-\pi^r\delta} \not\sim 1$ and $A_{(x-\pi^m v)^2-\pi^r\delta} \sim 1$.
5.h. In view of $f(\pi^m v + \pi^{r/2}\mu) \sim -u\mu(\mu^2 - \delta)$ we can find μ such that $f(\pi^m v + \pi^{r/2}\mu) \sim 1$ and $(\mu^2 - \delta) \not\sim 1$. Then $B_{Irr(\pi^m v + \pi^{r/2}\mu)} \not\sim 1$. Moreover, $B_{x^2-\pi^k u} \sim 1$.
5.j. $B_{x-\pi^s e} \sim (\alpha, \pi^r) \not\sim 1$. Finally, $B \not\sim A$ since $A_{x^2-\pi^k u} \sim 1$.
6. $A = (\pi w, x - \pi w)$, $w^2 - u \not\sim 1$. So, $A_{x^2-\pi^k u} \not\sim 1$ and $A_{x-\pi^s e} \sim (\pi w, -\pi w) \sim 1$. Let $-1 \sim 1$. Then $B_{x-\pi^s e} \sim (\pi, -u) \not\sim 1$. Thus, $B \not\sim 1, A$. Let now $-1 \not\sim 1$. In this case $B_{x-\pi w} \sim (\alpha(w^2 - u), x - \pi w) \sim 1$, so B is unramified. Finally, $B_{x-\pi^s e} \sim (-u\alpha, -\pi w) \not\sim 1$.

The theorem is proved.

5 Splitting type (1,4).

This section is devoted to case $deg\ f_1 = 4$, $deg\ f_2 = 1$ and the reduction is bad.

In our further considerations we will need lemmas from the previous sections.

Lemma 22 $|\,{}_2Br\,C| = 4$.

So it is enough to find one unramified and nontrivial algebra that is not isomorphic to scalar algebra (π, α). This algebra is not isomorphic to (π, α) by lemma 12.

Lemma 23 *By an appropriate replacing of the variable x the polynomial f can be reduced to the form $f(x) = g(x)(x - \pi^l e)$, $l \geq 0, e \in O_k^*$, g is irreducible, and either*
I. $g(x) = (x^2 - u)^2 + \pi^m vx + \pi^n\delta$, or
II. $g(x) = x^4 + \pi^k ux^2 + \pi^m vx + \pi^r\delta$,
where $k, m, n > 0$, $u, v, \delta \in O_k^*$, and in case I $u \notin (O_k^*)^2$.

Proof. Since $g(x)$ is irreducible, then by Hensel's lemma we have only the following two possibilities for $\bar{g}(x)$.
I. $\bar{g}(x) = (x^2 + px + q)^2$, $p, q \in \bar{k}$, $p^2 - 4q \notin \bar{k}^2$,
II. $\bar{g}(x) = (x - \gamma)^4$, $\gamma \in \bar{k}$.

In the second case one can assume that $\gamma = 0$, i.e. $g(x) = x^4 + ax^3 + bx^2 + cx + d$, $a, b, c, d \in M_k$. Moreover, by replacing $x \mapsto x - a/4$ we can remove the monomial ax^3.

In the first case the same replacement gives $(x^2 + px + q)^2 = x^4 + \bar{b}x^2 + \bar{c}x + \bar{d}$, so $p = 0$, $\bar{b} = 2q$, $\bar{d} = q^2$. Suppose $b = -2u$, $\bar{u} = -q$, then $d = u^2 + \lambda$, $\lambda \in M_k$, and $g(x) = (x^2 - u)^2 + cx + \lambda$. The lemma is proved.

Theorem 27 *Let the curve C correspond to case I from lemma 23. Then the following algebra A is nontrivial, unramified, and not isomorphic to the scalar one. (We define $\Delta = g(\sqrt{u}) = \pi^m v\sqrt{u} + \pi^n \delta \in k(\sqrt{u})$).*

1. $v(\Delta) \equiv 1\,(2)$, $\pi\Delta\xi(\sqrt{u}) \sim 1$ *in* $k(\sqrt{u})$, *where*

$$\xi(x) = \begin{cases} x, & \text{if } l > 0, \\ x - e, & \text{if } l = 0. \end{cases}$$

$A = (\pi, x^2 - u)$.

2. $v(\Delta) \equiv 1\,(2)$, $\pi\Delta\xi(\sqrt{u}) \not\sim 1$.
$A = (\alpha(x^2 - u), \pi(x - \beta))$, $\beta \in O_k^*$, $\beta^2 - u \not\sim 1$.

3. $v(\Delta) \equiv 0\,(2)$, $\xi(\sqrt{u}) \not\sim 1$ *in* $k(\sqrt{u})$.
$A = (\pi, x - \pi^l e)$.

4. $v(\Delta) \equiv 0\,(2)$, $\xi(\sqrt{u}) \sim 1$.
$A = (\alpha p(x), x - \beta)$, *where* $p = Irr_{k(\sqrt{u}|k)}(\theta)$, $\theta = \sqrt{u}(1 + \pi^s w)$, $w \in O^*_{k(\sqrt{u})}$ *such that* $\mu + 4u^2w^2 \in O^*_{k(\sqrt{u})} \setminus (O^*_{k(\sqrt{u})})^2$, $\Delta = \pi^{2s}\mu$.

Proof.
1. We have $f(\sqrt{u}) \sim \Delta\xi(\sqrt{u})$, so A is unramified. Let $K|k$ be unramified extension of odd degree and $\eta \in O_K^*$ such that

$$\begin{cases} \xi(\eta) \sim 1, \\ \eta^2 - u \not\sim 1. \end{cases}$$

Then $f(\eta) \sim \xi(\eta) \sim 1$ and

$$A_{Irr_{K|k}(\eta)} \sim \left(\frac{\pi\,,\,\eta^2 - u}{K\langle\ldots\rangle}\right) \not\sim 1.$$

2. We have $A_{x-\beta} \sim (\alpha^2, \pi(x - \beta)) \sim 1$ and

$$A_{x^2-u} \sim \left(\frac{x^2 - u\,,\,\pi(\beta - \sqrt{u})}{k(\sqrt{u})(\sqrt{\Delta\xi(\sqrt{u})})\langle x^2 - u\rangle}\right) \sim 1,$$

since $\Delta\xi(\sqrt{u}) \sim \pi(\beta - \sqrt{u})$ in $k(\sqrt{u})$.

Thus, A is unramified. If $\eta \in O_K^*$ satisfies the conditions

$$\begin{cases} \xi(\eta) \sim 1, \\ \eta^2 - u \sim 1, \end{cases}$$

then $f(\eta) \sim \xi(\eta) \sim 1$ and

$$A_{Irr_{K|k}(\eta)} \sim \left(\frac{\alpha(\eta^2 - u)\,,\,\pi(\eta - \beta)}{K\langle\ldots\rangle}\right) \sim (\alpha, \pi) \not\sim 1.$$

3. Let μ, s, θ, p be from item 4 of this theorem. Then $g(\theta) \sim \mu + 4u^2w^2 \not\sim 1$, and $f(\theta) \sim g(\theta)\xi(\sqrt{u}) \sim 1$. So

$$A_p \sim \left(\frac{\pi\ ,\ \theta - \pi^l e}{k(\sqrt{u})\langle p\rangle}\right) \sim \left(\frac{\pi\ ,\ \xi(\sqrt{u})}{k(\sqrt{u})\langle p\rangle}\right) \not\sim 1.$$

4. In view of $\alpha p(\beta) \sim \alpha(\beta - \sqrt{u} - \pi^s\sqrt{u}w)(\beta + \sqrt{u} + \pi^s\sqrt{u}w^\sigma) \sim \alpha(\beta^2 - u) \sim 1$ we have $A_{x-\beta} \sim 1$. Furthermore, $f(\theta) \sim \mu + 4u^2w^2 \not\sim 1$, therefore $f(\theta) \sim \beta - \sqrt{u}$ in $k(\sqrt{u})$ and

$$A_p \sim \left(\frac{p\ ,\ \beta - \sqrt{u}}{k(\sqrt{u})(\sqrt{f(\theta)})\langle p\rangle}\right) \sim 1.$$

Thus A is unramified. We prove its nontriviality. Assume that $\zeta = \sqrt{u}(1+\Pi) \in k(\sqrt{u}, \Pi)$, $\Pi = \sqrt[3]{\pi}$. Then $f(\zeta) \sim (u^2(2+\Pi)^2\Pi^2 + \pi^{2s}\mu)\xi(\sqrt{u}) \sim 1$ and

$$A_{Irr_{k(\sqrt{u},\Pi)}}(\zeta) \sim \left(\frac{\alpha p(\zeta)\ ,\ \zeta - \beta}{k(\sqrt{u}, \Pi)\langle\ldots\rangle}\right) \sim \left(\frac{p(\zeta)\ ,\ \beta - \sqrt{u}}{k(\sqrt{u}, \Pi)\langle\ldots\rangle}\right) \not\sim 1,$$

since $p(\zeta) \sim (\zeta - \theta)(\zeta - \theta^\sigma) \sim 2\sqrt{u}(\Pi\sqrt{u} - \pi^s w\sqrt{u}) \sim \Pi$.

The theorem is proved.

Let us pass to case II. It is convenient to introduce the following notations.

$$d = \min\left\{\frac{k}{2}, \frac{m}{3}, \frac{n}{4}\right\} \in \frac{1}{4}\boldsymbol{Z} \bigcup \frac{1}{3}\boldsymbol{Z}, \quad h(t) = t^4 + \pi^{k-2d}ut^2 + \pi^{m-3d}vt + \pi^{n-4d}\delta \in k(\pi^d)[t].$$

Then for a suitable unramified $K|k$ of odd degree and $w \in O_K^*$ we have

$$g(\pi^d w) \sim \pi^{4d}h(w) \sim h(w) \in O^*_{K(\pi^d)}.$$

In addition, $\bar{h}(t) \in \bar{k}[t]$ is a monic polynomial which is of degree 4 and does not coincide with t^4.

Theorem 28 *Assume we are in case II, $d \le l$, and $\bar{h}(t) \notin \bar{k}[t]^2$. Then the following algebra A is unramified, nontrivial, and not isomorphic to the scalar one.*

1. $d \in 2\boldsymbol{Z}$, $A = (\pi, x - \pi^l e)$.

2. $d \in 1 + 2\boldsymbol{Z}$, $A = (\pi, Irr_{K|k}(\pi^d w))$, $w \in O_K^*$ *such that*

$$\left\{\begin{array}{l} \xi(w) \not\sim 1, \\ h(w) \not\sim 1, \end{array}\right. \quad \textit{where } \xi(x) = \left\{\begin{array}{ll} x, & \textit{if } d < l \\ x - e, & \textit{if } d = l. \end{array}\right.$$

3. $d \in 2/3 + \boldsymbol{Z}$, $A = (\pi, x - \pi^l e)$.

4. $d \in 1/3 + \boldsymbol{Z}$, $A = (\pi, Irr_{K(\sqrt[3]{\pi})|k}(\pi^d w))$, K *and* w *are from 2.*

5. $d \in 1/4 + \boldsymbol{Z} \cup 3/4 + \boldsymbol{Z}$, $A = (\alpha, x - \pi^l e)$.

6. $d \in 1/2 + \boldsymbol{Z}$, $A = (-\pi Irr_{K(\sqrt{\pi})|k}(\pi^d w), \alpha x)$, $w \in O_K^*$, $h(w) \not\sim 1$.

Proof. Since $\bar{h}(t) \notin \bar{k}[t]^2$ the equation $y^2 = a\bar{h}(t)$, $a \in \bar{k}^*$ gives a variety in $\boldsymbol{A}^2(\bar{k}_s)$.

Therefore by Leng-Weil theorem for any $N \in \boldsymbol{N}$ there exists a sufficiently large extension $\tilde{K}|\bar{k}$ such that $|V(\tilde{K})| > N$. So there exists $\tilde{w}_i \in \tilde{K}$, $i = 1, 2$ such that $\bar{h}(\tilde{w}_1) \sim 1$, $\bar{h}(\tilde{w}_2) \not\sim 1$ in $\tilde{K}$. After lifting $\tilde{K}$ to K and $\tilde{w}_i$ to $w_i \in O_K^*$ ($\bar{K} = \tilde{K}$, $\bar{w}_i = \tilde{w}_i$) we have $h(w_1) \sim 1$, $h(w_2) \not\sim 1$. If w is required to have some additional property (for example, $\xi(w) \not\sim 1$), then one can consider a variety V in $\boldsymbol{A}^3(\bar{k}_s)$

$$\left\{\begin{array}{l} y^2 = a\bar{h}(t), \\ z^2 = \bar{a}\bar{\xi}(t) \text{ (lemma 6)}. \end{array}\right.$$

1. Let K, w be as in item 2 of this theorem. Then $f(\pi^d w) \sim h(w)\xi(w) \sim 1$, so

$$A_{Irr(\pi^d w)} \sim \left(\frac{\pi\ ,\ \xi(w)}{K\langle Irr(\pi^d w)\rangle}\right) \not\sim 1.$$

2. $f(\pi^d w) \sim h(w)\xi(w)\pi \sim \pi$, so A is unramified. We have also

$$A_{x-\pi^l e} \sim (\pi, \prod_{\sigma \in G(K|k)} (\pi^l e - \pi^d w^\sigma)) \sim (\pi, N_{K|k}(\xi(w))) \not\sim 1.$$

3. $f(\pi^d w) \sim wh(w)$ in $K(\sqrt[3]{\pi})$. Let $w, h(w) \not\sim 1$, then

$$A_{Irr(\pi^d w)} \sim \left(\frac{\sqrt[3]{\pi}\ ,\ w}{K(\sqrt[3]{\pi})\langle\ldots\rangle}\right) \not\sim 1.$$

4. $f(\pi^d w) \sim \pi w h(w) \sim \pi$ and A is unramified.

$$A_{x-\pi^l e} \sim (\pi, \prod_{\sigma \in G(K|k)} (x^3 - \pi^{3d}(w^\sigma)^3))_{x-\pi^l e} \sim (\pi, -\pi N_{K|k}(w)) \not\sim 1.$$

5. In this case $n \equiv 1\,(2)$ and $d = n/4 < \min\{k/2, m/3, l\}$. Let $d < d' < \min\{k/2, m/3, l\}$, $d' \in \boldsymbol{Q}$ such that $d' = p/q$, $p, q \in \boldsymbol{N}$, $(p, q) = 1$, $p, q \equiv 1\,(2)$. Under these assumptions $f(\pi^{d'}\delta) \sim \pi^{(p+nq)/q} \sim 1$ in $k(\pi^{1/q})$. Therefore

$$A_{Irr(\pi^{d'}\delta)} \sim (\alpha, \pi^{d'}\delta) \sim \left(\frac{\alpha\ ,\ \pi^{1/q}}{k(\pi^{1/q})\langle\ldots\rangle}\right) \not\sim 1.$$

6. We have

$$A_x \sim (-\pi \prod_\sigma -\pi^{2d}(w^\sigma)^2, \alpha x) \sim 1.$$

$f(\pi^d w) \sim \pi^d w h(w) \sim \alpha w \pi^d$ in $k(\sqrt{\pi})$, hence

$$A_{Irr(\pi^d w)} \sim \left(\frac{-\pi Irr(\pi^d w)\ ,\ \alpha w \pi^d}{K(\sqrt{\pi})(\sqrt{f(\pi^d w)})\langle\ldots\rangle}\right) \sim 1.$$

Thus, A is unramified. It is nontrivial in view of $f(\pi^{2/7}) \sim \pi^{2/7} - \pi^l e \sim 1$ in $k(\pi^{1/7})$, so

$$A_{x^7-\pi^2} \sim \left(\frac{-\pi \prod_\sigma (\pi^{4/7} - \pi^{2d}(w^\sigma)^2)\ ,\ \alpha\pi^{2/7}}{k(\pi^{1/7})\langle x^7 - \pi^2\rangle}\right) \sim \left(\frac{-\pi^{1/7}\ ,\ \alpha}{k(\pi^{1/7})\langle x^7 - \pi^2\rangle}\right) \not\sim 1.$$

The theorem is proved.

Theorem 29 *Let case II take place and $d > l$, $\bar{h}(t) \notin \bar{k}[t]^2$. Then the following algebra A is unramified, nontrivial, and not isomorphic to the scalar one.*

1. $l \equiv 0\,(2)$, $d \in (1/3)\boldsymbol{Z}$, $-e \sim 1$.
$A = (\alpha, Irr_{K|k}(\pi^d w))$, $h(w) \not\sim 1$.

2. $l \equiv 0\,(2)$, $d \in (1/3)\boldsymbol{Z}$, $-e \not\sim 1$.
$A = (\alpha, x^q - \pi^p)$, *where* $l < p/q < d$, $p, q \in \boldsymbol{N}$, $(p, q) = 1$, $p, q \equiv 1\,(2)$.

3. $l \equiv 0\,(2)$, $n \equiv 1\,(2)$. *(This is always true if $d \in 1/4 + \boldsymbol{Z} \cup 3/4 + \boldsymbol{Z}$).*
$A = (-\pi\delta e, x)$.

4. $l \equiv 0\,(2)$, $n \equiv 0\,(2)$, $d \in 1/2 + \boldsymbol{Z}$, $-\delta e \not\sim 1$.
$A = (\alpha, x)$.

5. $l \equiv 0\,(2)$, $n \equiv 0\,(2)$, $d \in 1/2 + \boldsymbol{Z}$, $-\delta e \sim 1$.
$A = (\alpha, Irr_{K(\sqrt{\pi})|k}(\pi^d w))$, $h(w) \not\sim 1$.

6. $l \equiv 1\,(2)$, $n \equiv 1\,(2)$, $-\delta e \sim 1$.
$A = (\alpha, x - \pi^l e)$.

7. $l \equiv 1\,(2)$, $n \equiv 1\,(2)$, $-\delta e \not\sim 1$.
$A = (\alpha, x)$.

8. $l \equiv 1\,(2)$, $n \equiv 0\,(2)$, $d \in (1/3)\boldsymbol{Z}$.
$A = (-\alpha e\pi, Irr_{K(\pi^d)|k}(\pi^d w))$, $h(w) \not\sim 1$.

9. $l \equiv 1\,(2)$, $n \equiv 0\,(2)$, $d \in 1/2 + \boldsymbol{Z}$.
$A = (-\pi\delta e, x)$.

Proof.
1. $f(\pi^d w) \sim h(w)(-\pi^l e) \sim \alpha$, so A is unramified. $f(\pi^{p/q}) \sim -\pi^l e \sim 1$. Then

$$A_{x^p - \pi^q} \sim \left(\frac{\alpha\ ,\ \prod_\sigma(\pi^{p/q} - \pi^d w^\sigma)}{k(\pi^{1/q})\langle\ldots\rangle}\right) \sim \left(\frac{\alpha\ ,\ \pi^{1/q}}{k(\pi^{1/q})\langle\ldots\rangle}\right) \not\sim 1.$$

2. $f(\pi^{p/q}) \sim -e$, so A is unramified. Moreover, $f(\pi^d w) \sim -eh(w) \sim 1$, so that

$$A_{Irr(\pi^d w)} \sim \left(\frac{\alpha\ ,\ -\pi^p}{k(\pi^d)\langle\ldots\rangle}\right) \not\sim 1.$$

3. $f(0) \sim -\pi\delta e$, i.e. A is unramified. Let $\tau \in O_K^*$, $\tau \not\sim 1$, $\tau - e \sim 1$. Then $f(\pi^l\tau) \sim \tau - e \sim 1$ and $A_{Irr(\pi^l\tau)} \sim (-\pi\delta e, \pi^l\tau) \sim (\pi, \tau) \not\sim 1$.

4. $f(0) \sim -\delta e$, so that A is unramified. If w satisfies $h(w) \not\sim \delta$, then $f(\pi^d w) \sim -eh(w) \sim 1$. We have

$$A_{Irr_{K(\sqrt{\pi})|k}}(\pi^d w) \sim \left(\frac{\alpha\ ,\ \pi^d w}{K(\sqrt{\pi})\langle\ldots\rangle}\right) \sim (\alpha, \sqrt{\pi}) \not\sim 1.$$

5. In this case $f(\pi^d w) \sim \alpha$ and $f(0) \sim 1$. Then A is unramified and $A_x \sim (\alpha, -\pi^{2d} w) \not\sim 1$.

6. $f(0) \sim 1$, $A_x \sim (\alpha, -\pi^l) \not\sim 1$

7. $A_{x - \pi^l e} \sim (\alpha, \pi^l) \not\sim 1$.

8. $f(\pi^d w) \sim -\alpha e\pi$, i.e. A is unramified. $A_{x - \pi^l e} \sim (-\alpha e\pi, \pi e) \not\sim 1$.

9. Let $\theta = \pi^{d-1/2}\sqrt{\pi w} \in K(\sqrt{\pi w})$. Since $d \in 1/2 + \boldsymbol{Z}$, then $d < m/3$, $h(\sqrt{w}) \sim w^2 + \pi^{k-2d} uw + \pi^{n-4d}\delta = h^0(w)$, and $f(\theta) \sim -h^0(w)we$. Note that in view of $\bar{h}(t) = \bar{h}^0(t^2)$ we have $\bar{h}^0(t) \notin \bar{k}[t]^2$. So the condition

$$\begin{cases} w \not\sim -\delta e, \\ h^0(w) \not\sim \delta \end{cases}$$

can be satisfied in a suitable K. Under these conditions we have $f(\theta) \sim 1$ and

$$A_{Irr_{K(\sqrt{\pi w})|k}} \sim \left(\frac{-\pi\delta e\,,\ w^{d-1/2}\sqrt{\pi w}}{K(\sqrt{\pi w})\langle\ldots\rangle}\right) \sim \left(\frac{\sqrt{\pi w}\,,\ -w\delta e}{K(\sqrt{\pi w})\langle\ldots\rangle}\right) \not\sim 1.$$

Thus, case $\bar{h}(t) \notin \bar{k}[t]^2$ is completely considered. Let us make the form of polynomial $g(t)$ more precise, provided $\bar{h}(t) \in \bar{k}[t]^2$. If $x^4 + \overline{\pi^{k-2d}u}x^2 + \overline{\pi^{m-3d}v}x + \overline{\pi^{n-4d}\delta} = (x^2 + \alpha x + \beta)^2$, then $\alpha = 0$, $\overline{\pi^{m-3d}v} = 0$, $\overline{\pi^{k-2d}u} = 2\beta$, and $\overline{\pi^{n-4d}\delta} = \beta^2$, $\beta \in \bar{k}^*$. Then $n = 2k$, $m > 3k/2$, $d = k/2 \in (1/2)\boldsymbol{Z}$, $\delta = \gamma^2 + \pi^s\tau$, $u = -2\gamma$, $\tau, \gamma \in O_k^*$, $s > 0$. So $g(x)$ has the following form.

$$g(x) = (x^2 - \pi^k\gamma)^2 + \pi^m vx + \pi^{2k+s}\tau.$$

Let $\Delta = g(\sqrt{\pi^k\gamma}) = \pi^{2k+s}\tau + \pi^m v\sqrt{\pi^k\gamma} \in k(\sqrt{\pi^k\gamma})$. Note that one always can suppose $v(\Delta) = \min\{2k+s, m+k/2\}$. Indeed, let $v(\Delta) > \min\{2k+s, m+k/2\}$. This means that $k \equiv 0\,(2)$, $\gamma = \beta^2 \in (O_k^*)^2$, $2k + s = m + k/2$ and $\bar{\tau} + \bar{v}\bar{\beta} = 0$. Then $\bar{\tau} - \bar{v}\bar{\beta} \neq 0$, and replacement β by $-\beta$ gives us the required Δ. Moreover, if $k \equiv 0\,(2)$, then equality $v(\Delta) = 2m - k$ (or $v(\Delta)/2 - k = m - 3k/2$) is impossible. Indeed, then $2m - k \leq m + k/2$, $m \leq 3k/2$.

From now on we will consider only such $g(x)$ and Δ.

Theorem 30 *Let $d < l$. Then the following algebra A is unramified, nontrivial, and not isomorphic to the scalar one.*

1. $k \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta \sim \pi^d\beta \sim \alpha$.
$A = (\pi, x - \pi^l e)$.

2. $k \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta \sim \pi^d\beta \sim \pi\rho$, $\rho \in O_k^*$.
$A = (\alpha, x - \pi^l e)$.

3. $k \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta \sim \pi^d\beta \sim 1$.
Let $\theta = \pi^d\beta(1 + \pi^r w)$, $r = \min\{v(\Delta)/2 - k, m - 3k/2\}$, $w \in O_K^*$ such that $\phi(w) \in O_K^* \setminus (O_K^*)^2$, where

$$\phi(x) = \begin{cases} 4\gamma^2x^2 + \mu, & \text{if } v(\Delta)/2 - k < m - 3k/2 \\ 4\gamma^2x^2 + v\beta x, & \text{if } v(\Delta)/2 - k > m - 3k/2, \end{cases}$$

$\Delta = \pi^{v(\Delta)}\mu$, *and* $p = Irr_{K|k}(\theta)$.
Then $A = (\alpha, p)$.

4. $k \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta\pi^d\beta \sim \alpha$, $d \equiv 1\,(2)$.
$A = (\alpha, x - \pi^d\beta)$.

5. $k \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta\pi^d\beta \sim \alpha$, $d \equiv 0\,(2)$, $\beta \sim 1$.
$A = (\alpha, x - \pi^d\beta)$.

6. $k \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta\pi^d\beta \sim \alpha$, $d \equiv 0\,(2)$, $\beta \not\sim 1$.
$A = (\alpha, Irr_{k(\sqrt[3]{\pi})|k}(\theta'))$, *where* $\theta' = \pi^d\beta(1 + \sqrt[3]{\pi})$.

7. $k \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta\pi^d\beta \sim \pi\rho$, $v(\Delta) \equiv 1\,(2)$.
$A = (\pi\beta\mu, x - \pi^d\beta)$, $\Delta = \pi^{v(\Delta)}\mu$.

8. $k \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta\pi^d\beta \sim \pi\rho$, $v(\Delta) \equiv 0\,(2)$.
$A = (\pi\beta\alpha, p)$, *p is from item 3.*

9. $k \equiv 1\,(2)$, $m + k/2 < 2k + s$, $\gamma^m v \sim 1$.
$A = (\alpha, x - \pi^l e)$.

10. $k \equiv 1\,(2),\ m + k/2 < 2k + s,\ \gamma^m v \not\sim 1.$

$A = (\alpha, x^2 - \pi^k\gamma).$

11. $k \equiv 1\,(2),\ m + k/2 > 2k + s.$

Let $\theta = \sqrt{\pi^k\gamma}(1 + (\sqrt{\pi\gamma})^s w) \in K(\sqrt{\pi\gamma})$, $w \in O_K^*$ such that $\gamma^s(4\gamma^{s+2}w^2 + \tau) \in O_K^* \setminus (O_K^*)^2$, $q = Irr_{K(\sqrt{\pi\gamma})|k}(\theta)$.

Then $A = (\alpha x, q(0)q(x))$.

12. $k \equiv 0\,(2),\ \gamma \not\sim 1,\ v(\Delta) \equiv 0\,(2),\ d \equiv 1\,(2),\ -1 \sim 1.$

Let similarly as in item 3 $\theta = \pi^d\sqrt{\gamma}(1+\pi^r w) \in K(\sqrt{\gamma})$, $w \in O_{K(\sqrt{\gamma})}^*$ such that $\phi(w) \not\sim 1$ in $K(\sqrt{\gamma})$, $p = Irr_{K(\sqrt{\gamma})|k}(\theta)$.

Then $A = (\pi, p(x))$.

13. $k \equiv 0\,(2),\ \gamma \not\sim 1,\ v(\Delta) \equiv 0\,(2),\ d \equiv 1\,(2),\ -1 \not\sim 1.$

Let p be as in item 12, $\mu \in O_k^*$, $\mu^2 - \gamma \not\sim 1$.

Then $A = (\alpha p(x), x - \pi^d\mu)$.

14. $k \equiv 0\,(2),\ \gamma \not\sim 1,\ v(\Delta) \equiv 0\,(2),\ d \equiv 0\,(2),\ -1 \not\sim 1.$

$A = (\alpha p(x), x - \pi^d\mu)$, p and μ are as in 13.

15. $k \equiv 0\,(2),\ \gamma \not\sim 1,\ v(\Delta) \equiv 0\,(2),\ d \equiv 0\,(2),\ -1 \sim 1.$

$A = (\pi, x - \pi^l e)$.

16. $k \equiv 0\,(2),\ \gamma \not\sim 1,\ v(\Delta) \equiv 1\,(2),\ d \equiv 0\,(2),\ \mu \sim \sqrt{\gamma}$ *in* $k(\sqrt{\gamma}),\ \Delta = \pi^{v(\Delta)}\mu.$

$A = (\pi, x^2 - \pi^k\gamma)$.

17. $k \equiv 0\,(2),\ \gamma \not\sim 1,\ v(\Delta) \equiv 1\,(2),\ d \equiv 0\,(2),\ \mu \not\sim \sqrt{\gamma}.$

$A = (\alpha(x^2 - \pi^k\gamma), \pi(x - \pi^d\eta))$, $\eta \in O_k^*$, $\eta^2 - \gamma \not\sim 1$.

18. $k \equiv 0\,(2),\ \gamma \not\sim 1,\ v(\Delta) \equiv 1\,(2),\ d \equiv 1\,(2),\ -1 \sim 1.$

$A = (\pi, p(x))$, p is as in item 12.

19. $k \equiv 0\,(2),\ \gamma \not\sim 1,\ v(\Delta) \equiv 1\,(2),\ d \equiv 1\,(2),\ -1 \not\sim 1.$

$A = (\alpha p(x), x - \pi^d\eta)$, p is from item 12, $\eta \in O_k^*$, $\eta^2 - \gamma \not\sim 1$.

Proof.
1. $f(\pi^d\beta) \sim \Delta\pi^d\beta \sim 1$, so $A_{x-\pi^d\beta} \sim (\pi, \pi^d\beta) \not\sim 1$.

2. $A_{x-\pi^d\beta} \sim (\alpha, \pi^d\beta) \not\sim 1$.

3. $f(\theta) \sim g(\theta)\theta \sim g(\theta) \sim 4\pi^{2k+2r}\gamma^2 w^2(1+\pi^r w/2)^2 + \pi^{v(\Delta)}\mu + \pi^{m+d+r}v\beta w \sim 4\gamma^2 w^2 + \pi^{2[(v(\Delta)/2-k)-r]}\mu + \pi^{(m-3k/2)-r}v\beta w \sim \phi(w) \sim \alpha$. Thus A is unramified. Let $\theta' = \pi^d\beta(1+\Pi) \in k(\Pi)$, $\Pi = \sqrt[3]{\pi}$, $p' = Irr(\theta')$. Then $f(\theta') \sim g(\theta') \sim 4\gamma^2 + \pi^{2[(v(\Delta)/2-k)-1/3]}\mu + \pi^{(m-3k/2)-1/3} \sim 1$, and

$$A_{p'} \sim \left(\frac{\alpha\ ,\ \prod_{\sigma\in G(K|k)}(\pi^d\beta(1+\Pi) - \pi^d\beta(1+\pi^r w^\sigma))}{k(\Pi)\langle p'\rangle}\right) \sim (\alpha, \Pi) \not\sim 1.$$

4. $f(\pi^d\beta) \sim \alpha$, so A is unramified. $A_{x-\pi^l e} \sim (\alpha, \pi^d\beta) \not\sim 1$.

5. Let θ', p' be from item 3 of this proof. Then $f(\theta') \sim 1$ and $A_{p'} \not\sim 1$.

6. $f(\theta') \sim g(\theta')(\pi^d\beta - \pi^l e) \sim \pi^d\beta \sim \alpha$, A is unramified. Let θ, p be as in 3. Then $f(\theta) \sim \alpha\phi(w) \sim 1$ and $A_p \not\sim 1$.

7. $f(\pi^d\beta) \sim \pi\mu\beta$, A is unramified. If $\eta \in O_K^*$ satisfies the conditions $\eta \sim 1$, $\bar{\eta} \neq \bar{\beta}$, $\eta - \beta \not\sim 1$, then $f(\pi^d\eta) \sim \pi^d\eta \sim 1$ and $A_{Irr(\pi^d\eta)} \sim (\pi\beta\mu, \eta - \beta) \not\sim 1$.

8. $f(\theta) \sim g(\theta)\pi^d\beta \sim \pi\beta\phi(w)$, so A is unramified.

$$A_{x-\pi^l e} \sim (\pi\alpha\beta, \prod_{\sigma\in G}(\pi^l e - \pi^d\beta(1+\pi^r w^\sigma))) \sim (\pi\alpha\beta, -\pi\beta) \not\sim 1.$$

9. $f(\sqrt{\pi^k\gamma}) \sim \Delta\sqrt{\pi^k\gamma} \sim \gamma^m v \sim 1$ in $k(\sqrt{\pi\gamma})$. Therefore

$$A_{x^2-\pi^k\gamma} \sim \left(\frac{\alpha\ ,\ \sqrt{\pi\gamma}}{k(\sqrt{\pi\gamma})\langle x^2-\pi^k\gamma\rangle}\right) \not\sim 1.$$

10. $f(\sqrt{\pi^k\gamma}) \sim \alpha$, A is unramified. $A_{x-\pi^l e} \sim (\alpha, -\pi^k\gamma) \not\sim 1$.

11. We have $g(\theta) \sim \gamma^s(4\gamma^{s+2}w^2 + \tau) \sim \alpha$, then $f(\theta) \sim \alpha\theta$.
A is unramified. Indeed, $A_x \sim (\alpha x, q(0)^2) \sim 1$ and

$$A_q \sim \left(\frac{\alpha\theta\ ,\ q(0)q}{K(\sqrt{\pi\gamma})(\sqrt{\alpha\theta})\langle q\rangle}\right) \sim 1.$$

$A \not\sim (\pi, \alpha)$ in view of $(\pi,\alpha)_{x-\pi^l e} \not\sim 1$, but $A_{x-\pi^l e} \sim (\alpha\pi^l e, q(0)^2) \sim 1$. Finally,

$$A_\infty \sim \left(\frac{\alpha\ ,\ q(0)}{k\langle\sqrt{1/x}\rangle}\right) \not\sim 1,$$

since $v_k(q(0)) \equiv 1\,(2)$.

12. $f(\theta) \sim \pi\sqrt{\gamma}g(\theta) \sim \pi$, A is unramified.

$$A_{x-\pi^l e} \sim (\pi, \prod_{\sigma\in G(K|k)} \pi^d\sqrt{\gamma}(-\pi^d\sqrt{\gamma})) \sim (\pi, -\gamma) \not\sim 1.$$

13. $p(\pi^d\mu) \sim \prod_\sigma(\mu - \sqrt{\gamma})(\mu + \sqrt{\gamma}) \sim \mu^2 - \gamma \sim \alpha$, so that $A_{x-\pi^d\mu} \sim (\alpha^2, x - \pi^d\mu) \sim 1$.
$f(\theta) \sim \pi g(\theta)\sqrt{\gamma} \sim \pi(\mu - \sqrt{\gamma})$ in $k(\sqrt{\gamma})$. Then

$$A_p \sim \left(\frac{\alpha p\ ,\ \pi(\mu-\sqrt{\gamma})}{K(\sqrt{\gamma})(\sqrt{f(\theta)})\langle p\rangle}\right) \sim 1.$$

Thus, A is unramified.

$$A_{x-\pi^l e} \sim (\alpha\prod_\sigma \pi^d\sqrt{\gamma}(-\pi^d\sqrt{\gamma}), -\pi^d w) \sim (\pi, -\alpha\gamma) \not\sim 1.$$

14. $f(\theta) \sim g(\theta) \sim \mu - \sqrt{\gamma}$, hence A is unramified. Let $\theta' = \pi^d\sqrt{\gamma}(1+\Pi)$, $\Pi = \sqrt[3]{\pi}$. Then $f(\theta') = g(\theta') \sim 1$ and

$$A_{Irr_{k(\Pi,\sqrt{\gamma})}(\theta')} \sim \left(\frac{\prod_{\sigma\in G(K(\sqrt{\gamma})|k)}((\sqrt{\gamma}-(\sqrt{\gamma})^\sigma)+\Pi\gamma-(w\sqrt{\gamma})^\sigma\pi^r)\ ,\ \mu-\sqrt{\gamma}}{k(\Pi,\sqrt{\gamma})\langle\ldots\rangle}\right) \sim$$

$$\sim \left(\frac{\Pi\ ,\ \mu-\sqrt{\gamma}}{k(\Pi,\sqrt{\gamma})\langle\ldots\rangle}\right) \not\sim 1.$$

15. $f(\theta) \sim g(\theta)\sqrt{\gamma} \sim 1$, then

$$A_p \sim \left(\frac{\pi\ ,\ \sqrt{\gamma}}{K(\sqrt{\gamma})\langle p\rangle}\right) \not\sim 1.$$

16. $f(\pi^d\sqrt{\gamma}) \sim \pi\mu\sqrt{\gamma} \sim \pi$, A is unramified. Let $\zeta \in O_K^*$, $\zeta^2 - \gamma \not\sim 1$, $\zeta \sim 1$. Then $f(\pi^d\zeta) \sim g(\pi^d\zeta) \sim 1$ and

$$A_{Irr(\pi^d\zeta)} \sim \left(\frac{\pi\ ,\ \zeta^2-\gamma}{K\langle\ldots\rangle}\right) \not\sim 1.$$

17. $A_{x-\pi^d\eta} \sim (\alpha^2, \pi(x-\pi^d\eta)) \sim 1$.

$$A_{x^2-\pi^k\gamma} \sim \left(\frac{x^2-\pi^k\gamma\ ,\ \pi(\eta-\sqrt{\gamma})}{k(\sqrt{\gamma})(\sqrt{f(\pi^d\sqrt{\gamma})})\langle x^2-\pi^k\gamma\rangle}\right) \sim 1,$$

because $f(\pi^d\sqrt{\gamma}) \sim \pi\mu\sqrt{\gamma} \sim \pi(\eta-\sqrt{\gamma})$, i.e. A is unramified.

$$A_\infty \sim \left(\frac{\alpha\ ,\ \pi}{k\langle\sqrt{1/x}\rangle}\right) \not\sim 1.$$

It remains to prove that $A \not\sim (\pi,\alpha)$. Choose ζ as in 16. Then $A_{Irr(\pi^d\zeta)} \sim (\alpha(\zeta^2-\gamma),\ldots) \sim 1$. On the other hand, $(\pi,\alpha)_{Irr(\pi^d\zeta)} \not\sim 1$.

18. $f(\theta) \sim \pi\sqrt{\gamma}g(\theta) \sim \pi$, A is unramified.

$$A_{x-\pi^l e} \sim (\pi, \prod_{\sigma\in G(K(\sqrt{\gamma})|k)} -\pi^d(\sqrt{\gamma})^\sigma(1+\pi^r w^\sigma)) \sim (\pi,-\gamma) \not\sim 1.$$

19. We have

$$A_{x-\pi^d\eta} \sim (x-\pi^d\eta, \alpha \prod_{\sigma\in G(K(\sqrt{\gamma})|k)} (\eta-(\sqrt{\gamma})^\sigma(1+\pi^r w^\sigma)) \sim (x-\pi^d\eta, \alpha(\eta^2-\gamma)) \sim 1,$$

$f(\theta) \sim \pi\sqrt{\gamma}g(\theta) \sim \pi(\eta-\sqrt{\gamma})$ in $k(\sqrt{\gamma})$. So

$$A_p \sim \left(\frac{p\ ,\ \pi^d(\sqrt{\gamma}-\eta)}{k(\sqrt{\gamma})(\sqrt{\pi(\eta-\sqrt{\gamma})})\langle p\rangle}\right) \sim 1,$$

i.e. A is unramified. To prove the theorem it is enough to check that A is nontrivial.

$$A_{x-\pi^l e} \sim (\alpha \prod_{\sigma\in G(K(\sqrt{\gamma})|k)} -\pi^d(\sqrt{\gamma})^\sigma(1+\pi^r w^\sigma), -\pi^d\eta) \sim (-\alpha\gamma, -\pi\eta) \not\sim 1.$$

Theorem 31 *Let $d \geq l$ and if $d = l$, $\pi^k\gamma = (\pi^d\beta)^2 \in k^2$, then $\bar{b} \neq \bar{e}$. Assume*

$$\xi(x) = \begin{cases} -e, & \text{if } d > l, \\ x-e, & \text{if } d = l. \end{cases}$$

Then the following algebra is unramified, nontrivial, and not isomorphic to the scalar one.

1. $\pi^k\gamma = (\pi^d\beta)^2 \sim 1$, $v(\Delta) \equiv 0\,(2)$, $l \equiv 0\,(2)$, $\xi(\beta) \not\sim 1$.
$A = (\pi, x-\pi^l e)$.

2. $\pi^k\gamma \sim 1$, $v(\Delta) \equiv 0\,(2)$, $l \equiv 0\,(2)$, $\xi(\beta) \sim 1$.
$A = (\alpha, p(x))$, *where p is defined in item 3 of the previous theorem.*

3. $\pi^k\gamma \sim 1$, $v(\Delta) \equiv 0\,(2)$, $l \equiv 1\,(2)$.

$A = (\pi\alpha\xi(\beta), p(x))$.

4. $\pi^k\gamma \sim 1$, $v(\Delta) \equiv 1\,(2)$, $l \equiv 1\,(2)$, $\mu \sim \xi(\beta)$, *where* $\Delta = \pi^{v(\Delta)}\mu$.

Let $\Pi = \sqrt[5]{\pi}$, $t \in \{1,2\}$, $t \equiv d\,(2)$, $\theta' = \pi^d\beta(1 - \Pi^t\alpha\beta) \in k(\Pi)$, $p' = Irr(\theta')$.

Then $A = (\pi\xi(\beta), p'(x))$.

5. $\pi^k\gamma \sim 1$, $v(\Delta) \equiv 1\,(2)$, $l \equiv 1\,(2)$, $\mu \not\sim \xi(\beta)$.

$A = (\alpha, x - \pi^d\beta)$.

6. $\pi^k\gamma \sim 1$, $v(\Delta) \equiv 1\,(2)$, $l \equiv 0\,(2)$.

$A = (\pi\mu\xi(\beta), x - \pi^d\beta)$.

7. $k \equiv 1\,(2)$, $v(\Delta) \equiv 0\,(2)$ *in* $k(\sqrt{\pi\gamma})$, $-e\gamma^l \sim 1$.

Let $\theta = \sqrt{\pi^k\gamma}(1 + (\sqrt{\pi\gamma})^r w) \in K(\sqrt{\pi\gamma})$, $w \in O_K^*$ *such that* $g(\theta) \sim \alpha$ *in* $K(\sqrt{\pi\gamma})$, $p = Irr(\theta)$.

Then $A = (\alpha, p(x))$.

8. $k \equiv 1\,(2)$, $v(\Delta) \equiv 0\,(2)$ *in* $k(\sqrt{\pi\gamma})$, $-e\gamma^l \not\sim 1$.

Let $\theta' = \sqrt{\pi^k\gamma}(1 + \sqrt[6]{\pi\gamma}) \in k(\sqrt[6]{\pi\gamma})$, $q = Irr(\theta')$.

Then $A = (\alpha, q(x))$.

9. $k \equiv 1\,(2)$, $v(\Delta) \equiv 1\,(2)$.

$A = (\rho\pi^{(k-1)/2}x, -\pi\gamma(x^2 - \pi^k\gamma))$, $\rho = -ev^{l+m+(k-1)/2}$.

10. $k \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 0\,(2)$, $l \equiv 1\,(2)$, $l = d$, $\xi(\sqrt{\gamma}) \not\sim 1$ *in* $k(\sqrt{\gamma})$.

Let $\theta = \pi^d\sqrt{\gamma}(1 + \pi^r w)$, $w \in O^*_{K(\sqrt{\gamma})}$ *such that* $g(\theta) \in O^*_{K(\sqrt{\gamma})} \setminus (O^*_{K(\sqrt{\gamma})})^2$, $p = Irr(\theta)$.

Then $A = (\pi, p(x)$.

11. $k \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 0\,(2)$, $l \equiv 0\,(2)$, $l = d$, $\xi(\sqrt{\gamma}) \not\sim 1$ *in* $k(\sqrt{\gamma})$.

$A = (\pi, x - \pi^l e)$.

12. $k \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 0\,(2)$, $\xi(\sqrt{\gamma}) \sim 1$ *in* $k(\sqrt{\gamma})$ *(the last condition always holds if* $l < d$*).*

$A = (\alpha p, \pi^{l+d}(x - \pi^d\beta))$, *where* $\beta \in O_k^*$, $\beta^2 - \gamma \not\sim 1$, p *is from item 10.*

13. $k \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 1\,(2)$, $\mu \not\sim \xi(\sqrt{\gamma})$ *in* $k(\sqrt{\gamma})$.

$A = (\alpha(x^2 - \pi^k\gamma), \pi^{l+d+1}(x - \pi^d\beta))$, *where* $\beta \in O_k^*$, $\beta^2 - \gamma \not\sim 1$.

14. $k \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 1\,(2)$, $\mu \sim \xi(\sqrt{\gamma})$ *in* $k(\sqrt{\gamma})$, $l \equiv 0(2)$.

$A = (\pi, x^2 - \pi^k\gamma)$.

15. $k \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 1\,(2)$, $\mu \sim \xi(\sqrt{\gamma}) \sim 1$, $l \equiv 1(2)$.

Let $\Pi = \sqrt[3]{\pi}$, $\theta' = \pi^d\sqrt{\gamma}(1 + \Pi\eta)$, $\eta \in O^*_{k(\sqrt{\gamma})} \setminus (O^*_{k(\sqrt{\gamma})})^2$, $q = Irr_{k(\Pi,\sqrt{\gamma})|k}(\theta')$.

Then $A = (\pi, q(x))$.

16. $k \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 1\,(2)$, $\mu \sim \xi(\sqrt{\gamma}) \not\sim 1$, $l \equiv 1(2)$.

$A = (\pi, x - \pi^l e)$.

Proof.

1. If θ, p are the same as in item 3 of the previous theorem, then $g(\theta) \sim \alpha$ and $f(\theta) \sim \alpha\pi^l\xi(\beta) \sim 1$, so $A_p \sim (\pi, \xi(\beta)) \not\sim 1$.

2. $f(\theta) \sim \alpha$, A is unramified. Let $\theta' = \pi^d\beta(1+\Pi^t)$, $\Pi = \sqrt[5]{\pi}$, $t \in \{1,2\}$, $t \not\equiv d\,(2)$, $p' = Irr_{k(\Pi)|k}(\theta')$. Then $f(\theta') \sim \xi(\beta) \sim 1$ and

$$A_{p'} \sim \left(\frac{\alpha\ ,\ \prod_\sigma[\pi^d\beta(1+\Pi^t) - \pi^d\beta(1+\pi^r w^\sigma)]}{k(\Pi)\langle p'\rangle}\right) \sim \left(\frac{\alpha\ ,\ \Pi^{d+t}}{k(\Pi)\langle p'\rangle}\right) \not\sim 1.$$

3. A is unramified because of $f(\theta) \sim \pi\alpha\xi(\beta)$. We have also

$$A_{x-\pi^l e} \sim (\pi\alpha\xi(\beta), \prod_\sigma (\pi^l e - \pi^d\beta(1+\pi^r w^\sigma))) \sim (\pi\alpha\xi(\beta), -\pi\xi(\beta)) \not\sim 1.$$

4. $f(\pi^d\beta) \sim \Delta\pi^l\xi(\beta) \sim 1$, $f(\theta') \sim \pi\xi(\beta)$, so that A is unramified. If $\chi^5 = 1$, χ is a primitive root, then

$$A_{x-\pi^d\beta} \sim (\pi\xi(\beta), \prod_{i=0}^{5}(\pi^d\beta - \pi^d\beta(1-\Pi^t\chi^{it}\alpha\beta))) \sim (\Pi\xi(\beta), \Pi^{d+t}\alpha)) \not\sim 1.$$

5. $f(\pi^d\beta) \sim \alpha$, A is unramified. $A_{x-\pi^l e} \sim (\alpha, \pi^l\xi(\beta)) \not\sim 1$.

6. $f(\pi^d\beta) \sim \pi\mu\xi(\beta)$. Find $\eta \in O_K^*$ satisfying the following conditions

$$\begin{cases} \bar\eta \neq \bar\beta, \\ \eta - e \sim 1, \\ \eta \not\sim 1, \text{ if } l < d, \\ \eta - \beta \not\sim 1, \text{ if } l = d. \end{cases}$$

Then $f(\eta\pi^l) \sim (\eta - e)g(\eta\pi^l) \sim 1$ and $A_{Irr(\eta\pi^l)} \sim (\pi\mu\xi(\beta), \eta - \pi^{d-l}\beta) \not\sim 1$.

7,8. In these cases we have $l < d$, $d \in 1/2 + \boldsymbol{Z}$.
$f(\theta) \sim g(\theta)(-\pi^l e) \sim -e\gamma^l\alpha$ in $K(\sqrt{\pi\gamma})$,
$f(\theta') \sim g(\theta')(-\pi^l e) \sim -e\gamma^l$ in $K(\sqrt{\pi\gamma})$.
If $-e\gamma^l \sim 1$, then $A = (\alpha, p)$ is nontrivial and $p(\theta') \sim \prod_{\sigma\in G(K|k)}(\sqrt{\pi^k\gamma}(1+\sqrt[6]{\pi\gamma}) - \sqrt{\pi^k\gamma}(1+(\sqrt{\pi\gamma})^r w^\sigma))(\sqrt{\pi^k\gamma}(1+\sqrt[6]{\pi\gamma}) + \sqrt{\pi^k\gamma}(1+(-1)^r\sqrt{(\pi\gamma)^r w^\sigma)}) \sim 2\sqrt[6]{\pi\gamma}$, i.e.

$$A_q \sim \left(\frac{\alpha\ ,\ \sqrt[6]{\pi\gamma}}{k(\sqrt[6]{\pi\gamma})\langle q\rangle}\right) \not\sim 1.$$

Let now $-e\gamma^l \not\sim 1$. Then $A = (\alpha, q)$ is unramified and $q(\theta) \sim -2\sqrt{\pi\gamma}$. So $A_p \not\sim 1$.

9. $A_x \sim (\rho\pi^{(k-1)/2}x, (-\pi\gamma)^2) \sim 1$.

$$A_{x^2-\pi^k\gamma} \sim \left(\frac{-(x^2-\pi^k\gamma)\ ,\ \rho\sqrt{\pi\gamma}}{k(\sqrt{\pi\gamma})(\sqrt{f(\sqrt{\pi^k\gamma})})\langle x^2-\pi^k\gamma\rangle}\right).$$

Since $v(\Delta) \equiv 1\,(2)$ in $k(\sqrt{\pi\gamma})$, then $2k+s < m+k/2$ and $\Delta \sim \pi^m v\sqrt{\pi^k\gamma} \sim \gamma^{m+(k-1)/2}v\sqrt{\pi\gamma}$ in $k(\sqrt{\pi\gamma})$. So $f(\sqrt{\pi^k\gamma}) \sim \rho\sqrt{\pi\gamma}$ in $k(\sqrt{\pi\gamma})$ and $A_{x^2-\pi^k\gamma} \sim 1$. Thus A is unramified. Furthermore,

$$A_\infty \sim \left(\frac{\pi\ ,\ \rho\gamma^{(k-1)/2}}{k\langle\sqrt{1/x}\rangle}\right).$$

If $l \equiv 0\,(2)$, then choose $\eta \in O_K^*$ satisfying the conditions

$$\begin{cases} \eta - e \sim 1, \\ \eta \not\sim \rho\gamma^{(k-1)/2}. \end{cases}$$

Then $f(\pi^l\eta) \sim (\rho\pi^{(k-1)/2}\eta, -\pi\gamma(\pi^{2l}\eta^2 - \pi^k\gamma)) \sim (\pi, \rho\eta\gamma^{(k-1)/2})$, i.e. among the algebras A_∞, $A_{Irr(\pi^l\eta)}$ there is exactly one nontrivial. Therefore $A \not\sim 1, (\pi, \alpha)$.

Let now $l \equiv 1\,(2)$. One can find $\eta \in O_K^*$ such that

$$\begin{cases} \gamma - \eta \sim 1, \\ \eta \sim -e. \end{cases}$$

Then we have $f(\sqrt{\pi^k\eta}) \sim -\pi^l e \sim -\eta e \sim 1$ in $K(\sqrt{\pi\eta})$ and

$$A_{Irr(\sqrt{\pi^k\eta})} \sim \left(\frac{\rho\pi^{(k-1)/2}\sqrt{\pi^k\eta}\,,\, -\pi\gamma\pi^k(\eta-\gamma)}{K(\sqrt{\pi\eta})\langle\ldots\rangle}\right) \sim \left(\frac{\sqrt{\pi\gamma}\,,\,\gamma}{K(\sqrt{\pi\eta})\langle\ldots\rangle}\right) \not\sim 1.$$

On the other hand,

$$(\pi\alpha)_{Irr(\sqrt{\pi^k\eta})} \sim \left(\frac{\pi\,,\,\alpha}{K(\sqrt{\pi\eta})\langle\ldots\rangle}\right) \sim 1.$$

10,11. Since $f(\theta) \sim \pi^l\xi(\sqrt{\gamma})\delta$, $\delta \in O^*_{K(\sqrt{\gamma})} \setminus (O^*_{K(\sqrt{\gamma})})^2$, then in 10 A is unramified and in 11

$$A_p \sim \left(\frac{\pi\,,\,\pi^l\xi(\sqrt{\gamma})}{K(\sqrt{\gamma})\langle p\rangle}\right) \not\sim 1.$$

Moreover, in the first case $A_{x-\pi^l e} \sim (\pi, e^2-\gamma) \not\sim 1$.

12. $p(\pi^d\beta) \sim \beta^2 - \gamma \sim \alpha$, therefore $A_{x-\pi^d\beta} \sim 1$. $f(\theta) \sim g(\theta)\xi(\sqrt{\gamma})\pi^l \sim \delta\pi^l$, $\delta \not\sim 1$ in $k(\sqrt{\gamma})$. Then

$$A_p \sim \left(\frac{p\,,\,\pi^{l+d}(\pi^d\sqrt{\gamma} - \pi^d\beta)}{K(\sqrt{\gamma})(\sqrt{\pi^l\delta})\langle p\rangle}\right) \sim \left(\frac{p\,,\,\pi^l(\beta-\sqrt{\gamma})}{K(\sqrt{\gamma})(\sqrt{\pi^l\delta})\langle p\rangle}\right) \sim 1.$$

So, A is unramified. We have also

$$A_\infty \sim \left(\frac{\alpha\,,\,\pi^{l+d}}{k\langle\sqrt{1/x}\rangle}\right), \quad A_{x-\pi^l e} \sim (\alpha, \pi^d),$$

the latter equality because of $p(\pi^l e) \sim 1$. If $l \equiv 1\,(2)$, then exactly one among these algebras is nontrivial, so $A \not\sim 1, (\pi, \alpha)$.

Let $l \equiv 0\,(2)$. Suppose $\theta' = \pi^d\sqrt{\gamma}(1+\Pi)$, $\Pi = \sqrt[3]{\pi}$. Then $f(\theta') \sim \xi(\sqrt{\gamma}) \sim 1$ and

$$p(\theta') \sim \prod_\sigma (\pi^d\sqrt{\gamma}(1+\Pi) - (\sqrt{\gamma})^\sigma\pi^d(1+\pi^r w^\sigma)) \sim \Pi.$$

We have

$$A_{Irr_{k(\Pi,\sqrt{\gamma})|k}(\theta')} \sim \left(\frac{\Pi\,,\,\beta-\sqrt{\gamma}}{k(\Pi,\sqrt{\gamma})\langle\ldots\rangle}\right) \not\sim 1.$$

Note, that (π, α) is trivial at this place.

13. $A_{x-\pi^d\beta} \sim (\alpha^2, \ldots) \sim 1$. $f(\pi^d\sqrt{\gamma}) \sim \pi^{l+1}\mu\xi(\sqrt{\gamma})$, then

$$A_{x^2-\pi^k\gamma} \sim \left(\frac{x^2-\pi^k\gamma\,,\,\pi^{l+d+1}(\pi^d\sqrt{\gamma}-\pi^d\beta)}{k(\sqrt{\gamma})(\sqrt{f(\pi^d\sqrt{\gamma})})\langle x^2-\pi^k\gamma\rangle}\right) \sim 1,$$

i.e. A is unramified.

$$A_\infty \sim (\alpha, \pi^{l+d+1}), \quad A_{x-\pi^l e} \sim \left(\alpha \left\{ \begin{array}{ll} 1, & l<d \\ e^2-\gamma, & l=d \end{array} \right\}, \pi^{d+1}\right).$$

If $l \equiv 1\,(2)$, then we have just one nontrivial algebra, therefore $A \not\sim 1, (\pi, \alpha)$. The same is true in case $l \equiv 0\,(2)$, $\xi(\sqrt{\gamma}) \not\sim 1$, since then $l = d$ and $e^2 - \gamma \not\sim 1$. Let $l \equiv 0\,(2)$, $\xi(\sqrt{\gamma}) \sim 1$. For the θ' from the previous item we have

$$A_{Irr_{k(\Pi,\sqrt{\gamma})|k}(\theta')} \sim \left(\frac{(\theta')^2 - \pi^k\gamma\ ,\ \pi^{d+1}(\theta' - \pi^d\beta)}{k(\Pi, \sqrt{\gamma})\langle\ldots\rangle}\right) \sim \left(\frac{2\Pi\ ,\ \pi^{d+1}(\Pi(\beta - \sqrt{\gamma})}{k(\Pi, \sqrt{\gamma})\langle\ldots\rangle}\right) \not\sim 1.$$

14,15,16. $f(\pi^d\sqrt{\gamma}) \sim \pi^{l+1}\mu\xi(\sqrt{\gamma}) \sim \pi^{l+1}$. If $\mu \sim \xi(\sqrt{\gamma})$, $l \equiv 0\,(2)$ (case 14), then $(\pi, x^2 - \pi^k u)$ is unramified. If $\mu \sim \xi(\sqrt{\gamma})$, $l \equiv 1\,(2)$, then $f(\sqrt{\pi^k\gamma}) \sim 1$ and

$$(\pi, x - \pi^l e)_{x^2-\pi^k\gamma} \sim \left(\frac{\pi\ ,\ \pi^l\xi(\sqrt{\gamma})}{k(\sqrt{\gamma})\langle x^2 - \pi^k\gamma\rangle}\right) \sim (\pi, \xi(\sqrt{\gamma})) \not\sim 1,$$

provided $\xi(\sqrt{\gamma}) \not\sim 1$ (case 16). In case 15 we have

$$q(\pi^d\sqrt{\gamma}) \sim \prod_{i=0}^{2}(\pi^d\sqrt{\gamma} - \pi^d\sqrt{\gamma}(1+\chi^i\Pi\eta))(\pi^d\sqrt{\gamma} + \pi^d\sqrt{\gamma}(1+\chi^i\Pi\eta^\sigma)) \sim \prod_{i=0}^{2}\chi^i\Pi\eta \sim \pi\eta,$$

where $\chi^3 = 1$, χ is a primitive root. So

$$A_{x^2-\pi^k\gamma} \sim \left(\frac{\pi\ ,\ \pi\eta}{k(\sqrt{\gamma})\langle x^2 - \pi^k\gamma\rangle}\right) \not\sim 1.$$

To complete the proof one only need to check that $A \not\sim 1$ in case 14 and A is nontrivial in 15. In these cases $f(\theta') \sim \pi^l\xi(\sqrt{\gamma})$. Then $f(\theta') \sim \pi$ in 15 and A is unramified. If case 14 takes place and $l < d$, then $f(\theta') \sim -e \sim 1$ in $k(\sqrt{\gamma})$. Therefore

$$A_q \sim \left(\frac{\pi\ ,\ \pi^k\gamma(1+\Pi\eta)^2 - \pi^k\gamma}{k(\Pi,\gamma)\langle q\rangle}\right) \sim \left(\frac{\Pi\ ,\ \eta}{k(\Pi,\gamma)\langle q\rangle}\right) \not\sim 1.$$

Finally, let $l = d$. Assume $\zeta \in O_K^*$ such that $\bar{\zeta} \neq \bar{e}$, $\zeta - e \sim 1$, and $\zeta^2 - \gamma \not\sim 1$. Then $f(\pi^d\zeta) \sim \zeta - e \sim 1$ and $A_{Irr_{K|k}(\pi^d\zeta)} \sim (\pi, \zeta^2 - \gamma) \not\sim 1$.

Theorem 32 *Let $d = l$, $\pi^k\gamma = (\pi^d\beta)^2 \in k^2$, and $e = \beta + \pi^n\delta$, $n > 0$, $\delta \in O_k^*$.*

Let firstly $v(\Delta) \equiv 0\,(2)$ and $\theta = \pi^d\beta(1+\pi^r w)$, $w \in O_K^*$ such that $\phi(w) \in O_K^* \setminus (O_K^*)^2$, $p = Irr_{K|k}(\theta)$, $r = \min\{v(\Delta)/2 - k, m - 3k/2\}$ (see theorem 30, item 3). Then the following algebra A is unramified, nontrivial, and not isomorphic to the scalar one.

1. *$r \le n$, $r + d \equiv 1\,(2)$. $A = (\alpha N_{K|k}(\chi(w))\pi, p(x))$, where*

$$\chi(x) = \left\{ \begin{array}{ll} \beta x, & \textit{if } r < n, \\ \beta x - \delta, & \textit{if } r = n. \end{array} \right.$$

2. *$r \le n$, $r + d \equiv 0\,(2)$. $A = (\pi, x - \pi^d e)$.*

3. *$r > n$, $n + d \equiv 1\,(2)$. $A = (-\pi\delta\alpha, p(x))$.*

4. *$r > n$, $n + d \equiv 0\,(2)$, $-\delta \not\sim 1$. $A = (\pi, x - \pi^d e)$.*

5. *$r > n$, $n + d \equiv 0\,(2)$, $-\delta \sim 1$. $A = (\alpha, p(x))$.*

Let now $v(\Delta) \equiv 1\,(2)$, $\Delta = \pi^{v(\Delta)}\mu$. Then the following algebra A is unramified, nontrivial, and not isomorphic to the scalar one.

6. $n + d \equiv 1\,(2)$, $-\mu\delta \not\sim 1$. $A = (\alpha, x - \pi^d\beta)$.

7. $n + d \equiv 1\,(2)$, $-\mu\delta \sim 1$. $A = (\alpha, x - \pi^d e)$.

8. $n + d \equiv 0\,(2)$, $n \le r$. $A = (-\mu\delta\pi, x - \pi^d\beta)$.

9. $n + d \equiv 0\,(2)$, $n > r$. $A = (-\pi\mu\alpha, x - \pi^d\beta(1 + \pi^n w))$, *where* $w \in O_k^*$ *such that* $\delta - \beta w \not\sim 1$. *(If there is no such* w *in* k, *then* $A = (-\pi\mu\alpha, Irr_{K|k}(\pi^d\beta(1 + \pi^n w)))$, $w \in O_K^*$*)*

Proof. Let $v(\Delta) \equiv 0\,(2)$. Then

$$f(\theta) \sim \alpha(\pi^d\beta(1 + \pi^r w) - \pi^d(\beta + \pi^n\delta)) \sim \begin{cases} \pi^{d+r}\alpha\chi(w), & \text{if } r \le n, \\ -\pi^{d+n}\alpha\delta, & \text{if } r > n, \end{cases}$$

$\chi(w) \sim N_{K|k}(\chi(w))$ in K.
So, in cases 1,3,5 the algebra A is unramified.
In 4 $f(\theta) \sim -\alpha\delta \sim 1$, then

$$A_p \sim \left(\frac{\pi\,,\ \pi^d\beta(1 + \pi^r w) - \pi^d(\beta + \pi^n\delta)}{K\langle p\rangle}\right) \sim (\pi, -\pi^{n+d}\delta) \not\sim 1.$$

In 2 $f(\theta) \sim \alpha\chi(w)$. System

$$\begin{cases} \phi(w) \not\sim 1, \\ \chi(w) \not\sim 1 \end{cases}$$

is always solvable in a sufficiently large extension $K|k$. For such w and p we have

$$A_p \sim (\pi, \pi^d\beta(1 + \pi^r w) - \pi^d(\beta + \pi^n\delta)) \sim \left(\frac{\pi\,,\ \chi(w)}{K\langle p\rangle}\right) \not\sim 1.$$

In 1,3

$$p(\pi^d e) \sim \pi^d \prod_{\sigma \in G(K|k)} (\pi^n\delta - \pi^r\beta w^\sigma) \sim \begin{cases} -\pi^{d+r}N_{K|k}(\chi(w)), r \le n, \\ \pi^{d+n}\delta, r = n. \end{cases}$$

Then in
1. $A_{x-\pi^d e} \sim (\alpha N_{K|k}(\chi(w))\pi, -\pi N_{K|k}(\chi(w))) \not\sim 1$.
2. $A_{x-\pi^d e} \sim (-\pi\delta\alpha, \pi\delta) \not\sim 1$. Thus, for case $v(\Delta) \equiv 0\,(2)$ it is enough to check that in 5 $A \not\sim 1$. To do this, assume $\theta' = \pi^d\beta(1 + \Pi^t) \in k(\Pi)$, $\Pi = \sqrt[5]{\pi}$, $t \in \{1, 2\}$, $t \not\equiv d\,(2)$, $p' = Irr_{k(\Pi)|k}(\theta')$. Then $f(\theta') \sim g(\theta') \sim 1$ and

$$p(\theta') \sim \prod_{\sigma \in G(K|k)} (\pi^d\beta(1 + \Pi^t) - \pi^d\beta(1 + \pi^r w^\sigma)) \sim \pi^d\beta\Pi^t.$$

So,

$$A_{p'} \sim \left(\frac{\alpha\,,\ \Pi^{d+t}}{k(\Pi)\langle p'\rangle}\right) \not\sim 1.$$

Consider now case $v(\Delta) \equiv 1\,(2)$. We have $f(\pi^d\beta) \sim -\pi^{v(\Delta)}\mu\pi^{d+n}\beta\delta \sim -\mu\delta\pi^{d+n+1}$, therefore in 6 A is unramified and in 7 $f(\pi^d\beta) \sim 1$, $A_{x-\pi^d\beta} \sim (\alpha, \pi^{d+n}) \not\sim 1$. Furthermore, in 6 A is nontrivial. Indeed, then $A_{x-\pi^d e} \sim (\alpha, \pi^{d+n}) \not\sim 1$.
In 8 $f(\pi^d\beta) \sim -\mu\delta\pi^{d+n+1}$ and A is unramified. Let $r = \min\{v(\Delta)/2 - k, m - 3k/2\} \in (1/2)\boldsymbol{Z}$, $\theta = \pi^d\beta(1 + (-\alpha\mu\delta\pi)^r w) \in K((-\alpha\mu\delta\pi)^r)$, $w \in O^*_{K((-\alpha\mu\delta\pi)^r)}$, $p = Irr_{K((-\alpha\mu\delta\pi)^r)|k}(\theta)$. Then

$$g(\theta) \sim 4\gamma^2(-\alpha\mu\delta)^{2r}w^2 + \pi^{v(\Delta)-2k-2r}\mu + \pi^{m-3k/2-2r}v\beta w(-\alpha\mu\delta\pi)^r,$$

$$f(\theta) \sim g(\theta)(-\delta + \beta w(-\alpha\mu\delta\pi)^{r-n}) \sim -g(\theta)\delta \left\{ \begin{array}{l} 1, r > n, \\ 1 + \mu\beta w, r = n. \end{array} \right\}$$

Moreover,

$$A_p \sim \left(\frac{-\mu\delta\pi \ , \ \pi^d \beta(-\alpha\mu\delta\pi)^r}{K((-\alpha\mu\delta\pi)^r)(\sqrt{f(\theta)})\langle p\rangle} \right).$$

If $r \in \boldsymbol{Z}$, then

$$A_p \sim \left(\frac{\pi \ , \ (-\alpha)^r(\mu\delta)^d \beta w}{K(\sqrt{f(\theta)})\langle p\rangle} \right).$$

One always can find K, w such that $f(\theta) \sim 1$ and $w \not\sim (-\alpha)^r(\mu\delta)^d\beta$. Then we have $A_p \not\sim 1$.

Consider $r \in 1/2 + \boldsymbol{Z}$. In this case

$$A_p \sim \left(\frac{\alpha \ , \ \sqrt{-\alpha\mu\delta\pi}}{K(\sqrt{-\alpha\mu\delta\pi})(\sqrt{f(\theta)})\langle p\rangle} \right).$$

In addition, $f(\theta) \sim -\delta(4\gamma^2(-\alpha\mu\delta)^{2r}w^2 + \mu)$. So, $A_p \not\sim 1$, provided $f(\theta) \sim 1$.

9. Since $v(\Delta) = \min\{2k + s, m + k/2\}$, then $v(\Delta) < m + n + k/2$. Suppose $\theta_0 = \pi^d\beta(1 + \pi^n w)$. $g(\theta_0) \sim \gamma^2(2\pi^n w + \pi^{2n}w^2)^2 + \pi^{v(\Delta)-2k}\mu + \pi^{m-3k/2+n}v\beta w$. In view of $v(\Delta) - 2k < m - 3k/2 + n$ and $v(\Delta) - 2k < 2n$ (the latter inequality holds because of $v(\Delta)/2 - k < r < n$) we have $g(\theta_0) \sim \pi\mu$. Therefore $f(\theta_0) \sim \pi\mu(\beta w - \delta)$ and A is unramified. Finally, $A_{x-\pi^d e} \sim (-\pi\mu\alpha, \delta - \beta w) \not\sim 1$.

The theorem is proved.

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