CORRESPONDENCES BETWEEN VALUED DIVISION ALGEBRAS AND GRADED DIVISION ALGEBRAS

Y.-S. Hwang[1] and A. R. Wadsworth[2]

Abstract. If D is a tame central division algebra over a Henselian valued field F, then the valuation on D yields an associated graded ring GD which is a graded division ring and is also central and graded simple over GF. After proving some properties of graded central simple algebras over a graded field (including a cohomological characterization of its graded Brauer group), it is proved that the map $[D] \mapsto [GD]_g$ yields an index-preserving isomorphism from the tame part of the Brauer group of F to the graded Brauer group of GF. This isomorphism is shown to be functorial with respect to field extensions and corestrictions, and using this it is shown that there is a correspondence between F-subalgebras of D (with center tame over F) and graded GF-subalgebras of GD.

Introduction

If D is a division ring finite-dimensional over its center F, and the field F has a Henselian valuation v, then v is known to extend uniquely to a valuation on D. The features associated with the valuation on D, especially the residue division algebra $\overline{D}$ and the value group Γ_D carry much information about the structure of D, and often can be used to settle questions such as decomposability, and which fields can be subfields of D. However, $\overline{D}$ and Γ_D do not determine D, and there are many subtleties in the way they interact.

Associated to the valuation on D there is a filtration of D by the principal fractional ideals of the valuation ring, which allows one to build an associated graded ring $GD = \bigoplus_{\gamma \in \Gamma_D} GD_\gamma$, where $GD_0 = \overline{D}$ and the grade group of GD is precisely the value group Γ_D of D. Furthermore, GD is a graded division ring, i.e., its homogeneous elements are all units. In addition, as shown in [B_2], the total ordering on Γ_D allows one to define a valuation on GD which extends to the ring of quotients QGD of GD, which is a division algebra. The valued division algebra QGD is usually not isomorphic to D, not even after Henselization, but we will see that their structures are closely related. The very presence of a valuation on QGD suggests that not so much is lost in the passage from D to its graded ring GD, even though GD appears to have a much simpler structure than D. We will show, in fact, that if D is tame then it is completely determined by GD, and its subalgebra structure is faithfully mirrored in that of GD.

Specifically, let $TBr(F)$ denote the tame part of the Brauer group of the Henselian field F, and let $GBr(GF)$ denote the graded Brauer group of the graded field GF determined by the valuation

[1]Supported in part by the Non-directed Research Fund, Korea.
[2]Supported in part by the NSF.

on F. We will show in Th. 5.3 that the map $[D] \mapsto [GD]_g$ gives a Schur-index-preserving group isomorphism $TBr(F) \to GBr(GF)$, which (see Cor. 5.7 and Th. 6.1) is functorial with respect to scalar extensions and corestrictions. The index-preserving and functorial properties allow us to deduce (see Th. 5.9) that if K is a tame valued field extension of F, and D and A are tame division algebras with center F, then K (resp. A) embeds in D iff GK (resp. GA) embeds in GD.

These results show that much of what is known about tame valued division algebras can be carried over readily to graded division algebras finite-dimensional over their centers, when the grade group is torsion-free. Beyond that, it lays the foundation for proving theorems about valued division algebras by first proving corresponding results in the relatively easier setting of graded division algebras. This approach has previously been applied successfully for wildly ramified valued division algebras by Tignol in [T].

This paper is organized as follows: Before considering connections between valued and graded division algebras, we develop the graded theory in the first three sections. In §1 we recall basic properties of graded division algebras and graded central simple algebras (GCSA's) over a graded field with torsion-free grade group, and point out the analogues of Wedderburn's theorem and the double centralizer theorem. We also prove a version of the Skolem-Noether theorem for GCSA's, which is somewhat delicate. In §2 we prove properties for graded division algebras which are analogous to known properties of tame valued division algebras. This is used in §3 to prove a cohomological characterization of the graded Brauer group $GBr(R)$ of a graded field R. In §4, we show how to get back and forth between tame valued division algebras and graded division algebras. If we start with a graded field R with totally ordered grade group Γ_R, then $GHR \cong_g R$ (graded isomorphism) canonically, where GHR is the graded field obtained from the valuation on the Henselian field HR obtained from the valuation on the quotient field of R determined by the grading on R. But, if we start with a Henselian valued field F, and take the Henselization HGF of the quotient field of the graded field GF (with respect to the valuation determined by the grading on GF), where GF is built from the valuation on F, then usually $HGF \not\cong F$. (These fields need not even have the same characteristic.) Nonetheless, we prove in Th. 4.4 that $TBr(HGF) \cong TBr(F)$. In §5 we prove the isomorphism $TBr(F) \cong GBr(GF)$ mentioned above, and the correspondences between tame subalgebras and graded subalgebras. Finally, the compatibility with the corestriction is given in §6.

§1 Graded division algebras and graded central simple algebras

We begin by setting up notation and recalling some results about graded division algebras and graded central simple algebras. Except for the graded Skolem-Noether theorem, Prop. 1.6, most of what we say in this section can be found in the literature somewhere (see especially [B_2], [CvO], [NvO]), though not always in the generality we need.

Let $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ be a graded ring. This means for us that A is an associative ring with 1, Γ is an abelian group, each A_γ is a subgroup of the additive group of A, and $A_\gamma \cdot A_\delta \subseteq A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. Because we are interested in the graded rings associated to valuation rings, *we will assume throughout that* Γ *is torsion-free.* We set

$$\Gamma_A = \{\gamma \in \Gamma \mid A_\gamma \neq (0)\}, \quad \text{the grade set of } A,$$

and

$$A^h = \bigcup_{\gamma\in\Gamma_A} A_\gamma, \quad \text{the set of homogeneous elements of } A.$$

For $a \in A^h$, $a \neq 0$, we write $deg(a) = \gamma$ if $a \in A_\gamma$. Each $c \in A$ is uniquely expressible as $c = \sum_{\gamma\in\Gamma_A} c_\gamma$ with each $c_\gamma \in A_\gamma$. The c_γ are called the homogeneous components of c. Let A^* denote the group of units of A. A subring S of A is a graded subring if $S = \bigoplus_{\gamma\in\Gamma_A} (S \cap A_\gamma)$ (iff for each $s \in S$, all the homogeneous components of s lie in S). Note that if S is a graded subring of A, then its centralizer $C_A(S)$ is also a graded subring of A. In particular, the center of A, $Z(A) = C_A(A)$ is a graded subring of A. A (left, right, or two-sided) ideal I of A is said to be *homogeneous* if $I = \bigoplus_{\gamma\in\Gamma_A} (I \cap A_\gamma)$ (iff I is generated as a left or $\ldots$ ideal of A by homogeneous elements). Suppose $B = \bigoplus_{\gamma\in\Gamma'} B_\gamma$ is another graded ring, and suppose there is a torsion-free group Δ containing Γ and Γ' as subgroups. A graded ring homomorphism $f\colon A \to B$ is a ring homomorphism such that $f(A_\delta) \subseteq B_\delta$ for all $\delta \in \Delta$. (It is understood that $A = \bigoplus_{\delta\in\Delta} A_\delta$, where $A_\delta = (0)$ for $\delta \in \Delta - \Gamma_A$; likewise for B.) If, further, f is bijective, then f is a graded isomorphism, and we write $A \cong_g B$. We frequently abbreviate "graded" by *gr*. The graded ring A is said to be *graded simple* if $|A| > 1$ (i.e., $1_A \neq 0_A$) and the only homogeneous two-sided ideals of A are A and (0).

A graded left A-module M is a left A-module with a direct sum decomposition as abelian groups $M = \bigoplus_{\gamma\in\Gamma'} M_\gamma$, where Γ' is some torsion-free abelian group containing Γ, such that $A_\gamma \cdot M_\delta \subseteq M_{\gamma+\delta}$, for all $\gamma \in \Gamma_A$, $\delta \in \Gamma'$. Then Γ_M, M^h, and graded submodules are defined just as above for rings. We can make M into a graded A-module in other ways by shifting the grading: For any $\gamma \in \Gamma'$, the γ-shift of M, denoted $s_\gamma(M)$ is defined by

$$s_\gamma(M) = M \text{ as an } A\text{-module, and } s_\gamma(M)_\delta = M_{\gamma+\delta}, \qquad \text{for all } \delta \in \Gamma'.$$

So, $\Gamma_{s_\gamma(M)} = -\gamma + \Gamma_M$. Now, let $N = \bigoplus_{\gamma\in\Gamma''} N_\gamma$ be another graded left A-module, such that there is a torsion-free abelian group Δ containing Γ' and Γ'' as subgroups. A graded A-module homomorphism $f\colon M \to N$ is an A-module homomorphism such that $f(M_\delta) \subseteq N_\delta$ for all $\delta \in \Delta$. There is the corresponding notion of graded isomorphism, and when there is one between M and N we write $M \cong_g N$. Let $GHom_A(M, N)$ denote the group of graded A-module homomorphisms from M to N, so $GHom_A(M, N)$ is a subgroup of the group $Hom_A(M, N)$ of all A-module homomorphisms from M to N. For each $\delta \in \Delta$, we have a subgroup of $Hom_A(M,N)$ of δ-shifted homomorphisms

$$Hom_A(M,N)_\delta = \{f \in Hom_A(M,N) \mid f(M_\gamma) \subseteq N_{\gamma+\delta} \quad \text{for all } \gamma, \delta \in \Delta\}.$$

Of course, $Hom_A(M,N)_\delta = GHom_A(M, s_\delta(N)) = GHom_A(s_{-\delta}(M), N)$. Clearly, $\bigoplus_{\delta\in\Delta} Hom_A(M,N)_\delta$ is subgroup of $Hom_A(M,N)$; if M is a finitely-generated A-module, then

$$Hom_A(M,N) = \bigoplus_{\delta\in\Delta} Hom_A(M,N)_\delta$$

(cf. [NvO, Lemma I.6.1, p. 26]). Indeed, for $f \in Hom_A(M,N)$, $\delta \in \Delta$, define $f_\delta \in Hom_A(M,N)_\delta$ by, for $m = \sum_{\gamma\in\Delta} m_\gamma$ with $m_\gamma \in M_\gamma$, setting $f_\delta(m) = \sum_{\varepsilon\in\Delta} (f(m_{\varepsilon-\delta}))_\varepsilon$. When M is finitely-generated, all but finitely many $f_\delta = 0$, and $f = \sum_\delta f_\delta$. In particular, for any finitely-generated graded left A-module M, $End_A(M) = Hom_A(M,M)$ is a graded ring. When A acts on M on the left, we view $End_A(M)$ as acting on M on the right; so M is a graded A-$End_A(M)$-bimodule.

Now, let $M = \bigoplus_{\gamma\in\Gamma'} M_\gamma$ be a graded right A-module and $N = \bigoplus_{\gamma\in\Gamma''} N_\gamma$ a graded left A-module, with $\Gamma', \Gamma'' \subseteq \Delta$ for some torsion-free-abelian group Δ. Then, $M \otimes_A N$ has a natural grading as $Z(A)$-module given by

$$(M \otimes_A N)_\delta = \Big\{ \sum_i m_i \otimes n_i \mid m_i \in M^h,\, n_i \in N^h,\, deg(m_i) + deg(n_i) = \delta \Big\}, \quad \delta \in \Delta.$$

One can see that this gives a grading on $M \otimes_A N$ by observing that the corresponding grading on $M \otimes_{A_0} N$ is clearly well-defined, and the grading on $M \otimes_{A_0} N$ is inherited by $M \otimes_A N \cong (M \otimes_{A_0} N)/J$, since the subgroup J of $M \otimes_{A_0} N$ is generated by the homogeneous elements $\{ma \otimes n - m \otimes an \mid m \in M^h, n \in N^h, a \in A^h\}$.

For example, suppose F is a *graded free* right A-module of finite rank, i.e., F is graded right A-module which is free as an A-module with a finite base $\{b_1, \ldots, b_n\} \subseteq F^h$. Let $\delta_i = deg(b_i) \in \Gamma_F$. Of course, $End_A(F) \cong M_n(A)$ ($n \times n$ matrices over A) if we ignore the grading, and by convention $End_A(F)$ acts on F on the left. In this isomorphism the ij-matrix unit $E_{ij} \in M_n(A)$ corresponds to the map $e_{ij} \in End_A(F)$, defined by $e_{ij}(b_j) = b_i$ and $e_{ij}(b_k) = 0$, for $k \neq j$. Clearly, $e_{ij} \in End_A(F)_{\delta_i - \delta_j}$. So, when we take the grading into account, we find that

$$End_A(F) \cong_g M_n(A)(d), \quad \text{for } d = (\delta_1, \ldots, \delta_n), \tag{1.1}$$

where $M_n(A)(d)$ means $n \times n$ matrices over A but with the degree of the ij-entry shifted by $\delta_i - \delta_j$, i.e.,

$$M_n(A)(d) = \begin{pmatrix} s_{\delta_1-\delta_1}(A) & \ldots & s_{\delta_n-\delta_1}(A) \\ \cdot & \cdot & \cdot \\ s_{\delta_1-\delta_n}(A) & \ldots & s_{\delta_n-\delta_n}(A) \end{pmatrix}. \tag{1.2}$$

So, the ij-entry of $M_n(A)(d)$ is $s_{\delta_j-\delta_i}(A)$ (as $s_{\delta_j-\delta_i}(A)_{\delta_i-\delta_j} = A_0$). Thus, the ε-component of $M_n(A)(d)$ consists of matrices with ij-entry in $A_{\varepsilon+\delta_j-\delta_i}$.

For future reference, we point out a few elementary properties of these shifted graded matrix rings. Let A be any graded ring. Then,

(i) If $\pi \in S_n$ is any permutation, then

$$M_n(A)(\delta_1, \ldots, \delta_n) \cong_g M_n(A)(\delta_{\pi(1)}, \ldots, \delta_{\pi(n)}). \tag{1.3}$$

(ii) If $\gamma_1, \ldots, \gamma_n \in \Gamma_A$, with $\gamma_i = deg(a_i)$ for some unit $a_i \in A^h$, then

$$M_n(A)(\delta_1, \ldots, \delta_n) \cong_g M_n(A)(\delta_1 + \gamma_1, \ldots, \delta_n + \gamma_n). \tag{1.4}$$

(iii) If A is commutative, and $d = (\delta_1, \ldots, \delta_n)$, $e = (\varepsilon_1, \ldots, \varepsilon_m)$, then

$$M_n(A)(d) \otimes_A M_m(A)(e) \cong_g M_{mn}(A)(f), \tag{1.5}$$

where $f = \{\delta_i + \varepsilon_j \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$. (The order of the terms is immaterial, in view of (1.3).)

For, $M_n(A)(\delta_1, \ldots, \delta_n) \cong_g End_A(F)$, where F is a graded free graded right A-module with homogeneous base $b_1, \ldots, b_n$ with $deg(b_i) = \delta_i$. Since $b_{\pi(1)}, \ldots, b_{\pi(n)}$ is also a homogeneous base of F, we also have $M_n(A)(\delta_{\pi(1)}, \ldots, \delta_{\pi(n)}) \cong_g End_A(F)$, yielding (1.3). Likewise, if a_i is a homogeneous unit of A with $deg(a_i) = \gamma_i$, then $b_1a_1, \ldots, b_na_n$ is another homogeneous base of F, with $deg(b_ia_i) = \delta_i + \gamma_i$. So, $M_n(A)(\delta_1 + \gamma_1, \ldots, \delta_n + \gamma_n) \cong_g End_A(F)$, proving (1.4). Now, assuming A is commutative, let F' be another graded free A-module, with base $c_1, \ldots, c_m$ with $deg(c_j) = \varepsilon_j$. Then, $F \otimes_A F'$ is a free graded A-module with base $\{b_i \otimes c_j\}$, where $deg(b_i \otimes c_j) = \delta_i + \varepsilon_j$. So

$$M_n(A)(d) \otimes_A M_m(A)(e) \cong_g End_A(F) \otimes_A End_A(F') \cong_g End_A(F \otimes_A F') \cong_g M_{mn}(A)(f),$$

showing (1.5).

A graded ring $E = \bigoplus_{\gamma \in \Gamma_E} E_\gamma$ is called a *graded division ring* if every nonzero homogeneous element of E is a unit, and $1_E \neq 0_E$. Note that the grade set Γ_E is actually a group. Further, since Γ_E is torsion-free, it follows that E has no zero divisors and $E^* = E^h - \{0\}$. (This is easy to see by recalling that the torsion-free abelian group Γ_E can be given a total ordering compatible with the group operation. Thus, if $a \neq 0$, $a = a_\gamma +$ terms of higher degree and $b \neq 0$, $b = b_\delta +$ terms of higher degree, then $ab = a_\gamma b_\delta +$ terms of higher degree, so $ab \neq 0$.) Also, E_0 must be a division ring, and for each $\gamma \in \Gamma_E$, the group E_γ is a one-dimensional left and right vector space over E_0. Note further that every graded left (resp. right) E-module M is a graded free E-module (cf. [B_1, Th. 3, p. 29]). For, it is easy to check that a maximal homogeneous E-linearly independent subset of M is actually a base. We call M a *graded vector space* over E, and write $dim_E(M)$ for the rank of M as a graded free E-module. (This is well-defined, since one can apply the usual exchange argument to see that any two homogeneous bases of M have the same cardinality.) Note that if N is a graded submodule of M, then

$$dim_E(N) + dim_E(M/N) = dim_E(M). \tag{1.6}$$

Consequently, if $dim_E(M) < \infty$ and N is a proper submodule of M, then $dim_E(N) < dim_E(M)$. Let S be a graded subring of E such that S is also a graded division ring, and let $[E : S] = dim_S(E)$ (left dimension) and likewise $[E_0 : S_0] = dim_{S_0} E_0$ (left dimension). Note the easy but fundamental formula (cf. [B$_2$, p. 4278])

$$[E : S] = [E_0 : S_0] \cdot |\Gamma_E : \Gamma_S|. \tag{1.7}$$

This holds since if $\{a_i\}$ is a base of E_0 as left S_0-vector space and if $\{b_j\} \subseteq E^h - \{0\}$ is chosen so that $\{deg(b_j)\}$ is a set of coset representatives for Γ_S in Γ_E, then $\{a_i b_j\}$ is a homogeneous base of E as a left S-vector space.

A commutative graded division ring is called a *graded field.* For example, if L is any field and Γ is any torsion-free abelian group, then the group ring $R = L[\Gamma]$ is a graded field with $R_0 = L$ and $\Gamma_R = \Gamma$. In fact, Γ is a free abelian group, then every graded field S with $\Gamma_S = \Gamma$ is a group ring (cf. [HW, Prop. 1.1]). However, there do exist graded fields which are not group rings (cf. [HW, Ex. 1.2]).

Let R be a graded field. A *graded R-algebra* A is graded ring which is an R-algebra such that the associated ring homomorphism $\varphi\colon R \to Z(A)$ is a gr-homomorphism. This φ is necessarily injective (assuming $1 \neq 0$ in A), as R is a graded field. We have A_0 is an R_0-algebra. Also, while Γ_A need not be a group, it is a union of cosets of the group Γ_R in some ambient torsion-free abelian group Γ'. We write

$$|\Gamma_A : \Gamma_R| = \text{the number of cosets of } \Gamma_R \text{ in } \Gamma_A$$

and

$$[A : R] = dim_R(A).$$

It is easy to check that

$$[A : R] \geq [A_0 : R_0] \cdot |\Gamma_A : \Gamma_R|, \tag{1.8}$$

but equality often does not hold (see Prop. 1.4 below).

For our graded field R, let Δ_R be the divisible hull of the torsion-free abelian group Γ_R, so

$$\Delta_R \cong \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_R,$$

and fix some $\mathbb{Q}$-vector space Δ' containing Δ_R with $dim_{\mathbb{Q}}(\Delta'/\Delta_R) = \infty$. Then (1.8) shows that if A is any finite dimensional graded R-algebra, then Γ_A is Γ_R-isomorphic to a subset of Δ'. Indeed, if Γ_A is a group (which occurs, e.g., whenever A is a graded division algebra) then, as Γ_A is torsion-free and Γ_A/Γ_R is torsion by (1.8), there is a unique group homomorphism $\Gamma_A \to \Delta_R$ which restricts to the identity on Γ_R. So we will assume henceforth that all graded R-algebras A satisfy $\Gamma_A \subseteq \Delta'$.

Note that if A and B are graded R-algebras, then $A \otimes_R B$ is also a graded R-algebra. If A' is a graded R-subalgebra of A and B' a graded R-subalgebra of B, then it is easy to check that

$$C_{A \otimes_R B}(A' \otimes_R B') = C_A(A') \otimes_R C_B(B'). \tag{1.9}$$

A graded algebra A over a graded field R is said to be a *graded central simple algebra* (GCSA) over R if A is a simple graded ring, $[A : R] < \infty$, and $Z(A) = R$. There is a theory of GCSA's over a graded field analogous to the theory of central simple algebras (CSA's) over a field, and we recall some basic properties here.

Proposition 1.1. *Let A be a GCSA over a graded field R, and let B be any graded R-algebra. If I is a homogeneous ideal of $A \otimes_R B$, then $I = A \otimes_R J$, where $J = I \cap B$, and J is a homogeneous ideal of B. Hence, if B is graded simple, then $A \otimes_R B$ is a GCSA over $Z(B)$.*

Of course, in Prop 1.1 we are identifying B with its gr-isomorphic copy $R \otimes_R B$ in $A \otimes_R B$. This proposition can be proved analogously to the ungraded result. One can first show the special case: if $I \cap B = (0)$, then $I = (0)$. The general result follows by applying the special case to $B' = B/J$ (after noting that J is homogeneous, so B'/J is graded); since $A \otimes_R B' \cong_g (A \otimes_R B)/(A \otimes_R J)$ and $(I/(A \otimes_R J)) \cap B' \cong (I \cap B)/J = (0)$, we obtain $I/(A \otimes_R J) = (0)$, as desired. Formula (1.9) shows $Z(A \otimes_R B) = Z(A) \otimes_R Z(B) \cong_g Z(B)$.

Corollary 1.2 (cf. [B_2, Prop. 5.1])**.** *Let A be an algebra over a graded field R. Then, A is a GCSA over R iff A is both an Azumaya algebra over R and also a graded R-algebra.*

Proof. Suppose A is a GCSA over R. Then, A is a free R-module of finite rank, and the graded R-algebra homomorphism $A \otimes_R A^{\mathrm{op}} \to End_R(A)$ is injective, since the domain is graded simple by Prop. 1.1, and surjective by dimension count (using (1.6)). Hence, by [DI, Th. 3.4, p. 52], A is an Azumaya algebra over R. Conversely, suppose A is an Azumaya algebra over R such that A is also a graded R-algebra. We identify R with its gr-isomorphic copy in A. Since A is Azumaya over R, by [DI, Prop. 21, p. 47; Cor. 3.7, p. 54], A is a finitely-generated R-module, so $[A : R] < \infty$, and $Z(A) = R$, and every ideal I of A has the form $I = A(I \cap R)$. If I is a homogeneous ideal of A, then $I \cap R$ is a homogeneous ideal of R. Hence, A is graded simple since R is graded simple. $\square$

A *graded central division algebra* (GCDA) over a graded field R is a GCSA E over R such that E is also a graded division ring. Observe that the usual matrix calculations show that for any GCDA E over R, any n, and any $d = (\delta_1, \ldots, \delta_n)$, $\delta_i \in \Delta'$, we have $M_n(E)(\delta)$ is a GCSA over R. Our next proposition is the graded Wedderburn theorem, which says that all GCSA's over R have this form.

Proposition 1.3. *Let A be a GCSA over a graded field R. Then,*

(a) There is a GCDA E over R such that $A \cong_g M_n(E)(d)$ for some $d = (\delta_1, \dots, \delta_n)$. Moreover, if $A \cong_g M_{n'}(E')(d')$ for some GCDA E' over R, then $n' = n$ and $E' \cong_g E$.

(b) Every graded left (or right) A-module is a direct sum of graded simple A-modules.

(c) If L is a minimal nonzero homogeneous left ideal of A and N is a graded simple A-module, then $N \cong_g s_\delta(L)$ i.e., N is the δ-shift of L for some δ. Hence,

$$dim_R(N) = n[E : R] = [A : R]/n. \tag{1.10}$$

This can be proved analogously to the usual Wedderburn theorem. Here is a sketch. Take a minimal nonzero homogeneous left ideal L of A (which exists as $[A : R] < \infty$), and let $E = End_A(L)$. Since L is a graded simple A-module (i.e., it has no nonzero proper graded A-submodule), the graded Schur's Lemma shows that E is a graded division ring, and $[E : R] \le [End_R(L) : R] < \infty$. Let $b_1, \dots, b_n$ be a homogeneous base of L as a graded free right E-module, so $L = b_1E \oplus \ldots \oplus b_nE$. Then, $End_E(L) \cong_g M_n(E)(d)$, where $d = (deg(b_1), \dots, deg(b_n))$, as noted in (1.1) above. Rieffel's proof of Wedderburn's Theorem ([Ri], or see [L, Th. 5, p. 449]) can be applied here to see that the graded R-algebra homomorphism $A \to End_E(L)$ ($a \mapsto$ left multiplication by a) is an isomorphism, so $A \cong_g End_E(L) \cong_g M_n(E)(d)$. If N is any graded simple left A-module, there is a $b \in N^h - \{0\}$ with $L \cdot b \ne (0)$. Then the A-module homomorphism $\lambda\colon L \to N$, $\ell \mapsto \ell \cdot b$ is an isomorphism since $im(\lambda)$ and $ker(\lambda)$ are graded submodules. Since λ shifts degrees by $deg(b)$, we have $N \cong_g s_{-deg(b)}(L)$. This yields (c) and also the uniqueness part of (a). For, if $A \cong_g M_{n'}(E')(d')$ and L' is the set of first columns of elements of $M_{n'}(E')(d')$, then L' is a graded simple left $M_{n'}(E')(d')$-module with endomorphism ring E'. Since L' is graded simple when viewed as an A-module, $L' \cong_g s_\delta(L)$, for some δ, so $E = End_A(s_\delta(L)) \cong_g End_A(L') \cong_g E'$; then $n' = n$ by dimension count. Finally, for (b), since A is a sum of simple graded left A-modules (corresponding to the columns of $M_n(E)(d)$), every graded left A-module M is a sum $M = \sum N_i$ where the N_i are graded simple submodules of M. The usual argument shows that M is a direct sum of some subset of $\{N_i\}$.

For any GCSA A over R, we define the Schur index of A, $ind(A)$, analogously to the ungraded case: We have $A \cong M_n(E)(d)$ for a GCDA E over R; set

$$ind(A) = \sqrt{[E : R]}, \quad \text{a positive integer.}$$

The graded Wedderburn theorem yields a description of Γ_A and of A_0 for a GCSA. Let $A = M_n(E)(d)$ where E is a GCDA over R and $d = (\delta_1, \ldots, \delta_n)$, $\delta_i \in \Delta'$. Let $\varepsilon_1 + \Gamma_E, \ldots, \varepsilon_k + \Gamma_E$ be the distinct cosets of Γ_E of the form $\delta_i + \Gamma_E$, $1 \le i \le n$, and for each ε_ℓ let r_ℓ be the number of i with $\delta_i \equiv \varepsilon_\ell \,(mod\, \Gamma_E)$.

Proposition 1.4. *Let E be a graded division algebra, and let $A = M_n(E)(d)$, for $d = (\delta_1, \dots, \delta_n)$. Then,*

(a) $\Gamma_A = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} (\delta_i - \delta_j) + \Gamma_E$.

(b) $A_0 \cong M_{r_1}(E_0) \times \ldots \times M_{r_k}(E_0)$, *with the r_i as defined above. In particular, A_0 is simple iff $k = 1$ iff $\Gamma_A = \Gamma_E$.*

Proof. (a) is immediate from the description of the grading on $M_n(E)(d)$ (see (1.2)). For (b), observe that by (1.3) and (1.4) above, $A \cong_g M_n(E)(e)$, where $e = (\varepsilon_1, \dots, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_2, \dots, \varepsilon_k, \dots, \varepsilon_k)$ with each ε_ℓ occurring r_ℓ times. In $M_n(E)(e)_0$ there is a contribution of E_0 in the ij-entry when the same ε_ℓ appears in the i-th and the j-th position in e. This accounts for all of $M_n(E)(e)_0$ since $0 \notin \varepsilon_\ell - \varepsilon_m + \Gamma_E$ when $\ell \neq m$. Thus, $A_0 \cong M_n(E)(e)_0 \cong M_{r_1}(E_0) \times \ldots \times M_{r_k}(E_0)$. $\square$

The double centralizer theorem is also available in the graded context:

Proposition 1.5. *Let A be a GCSA over a graded field R and let B be a graded simple graded R-subalgebra of A. Set $C = C_A(B)$. Then,*

(a) *C is a graded simple R-subalgebra of A with $Z(C) = Z(B)$, and $C_A(C) = B$.*
(b) $[C : R] \cdot [B : R] = [A : R]$.
(c) *$B \otimes_{Z(B)} C \cong_g C_A(Z(B))$. In particular, if $Z(B) = R$, then $B \otimes_R C \cong_g A$.*

This is proved analogously to the ungraded version. We give a sketch. Let L be a minimal homogeneous left ideal of A, and let $E = End_A(L)$, so E is a GCDA over R and $A \cong_g End_E(L)$ as we saw in the discussion of Prop. 1.3. Let $T = B^{\mathrm{op}} \otimes_R E$, which is a GCSA over $Z(B)$ by Prop. 1.1, and view L as a graded right T-module. Now, $End_T(L) \subseteq End_E(L) \cong_g A$, and A acts faithfully on L. Hence $End_T(L)$ is gr-isomorphic to the set of elements of A acting on L compatibly with the B-action, i.e., $End_T(L) \cong_g C_A(B) = C$. Let N be a minimal homogeneous right ideal of T, and let $U = End_T(N)$, which is a GCDA over $Z(T)$. Then $L \cong_g s_{\delta_1}(N) \oplus \ldots \oplus s_{\delta_k}(N)$ as graded T-modules by Prop. 1.3(b) and (c). Hence, $C \cong_g End_T(L) \cong_g End_T(s_{\delta_1}(N) \oplus \ldots \oplus s_{\delta_k}(N)) \cong_g M_k(U)(d)$, where $d = (\delta_1, \dots, \delta_k)$, so C is graded simple with $Z(C) \cong_g Z(U) \cong_g Z(T) \cong_g Z(B)$. The formula in (b) follows from $[C : R] = k^2[U : R]$, $dim_R(L) = k\, dim_R(N)$, $[T : R] = [B : R] \cdot [E : R]$, together with (by (1.10)) $[A : R] \cdot [E : R] = dim_R(L)^2$ and $[T : Z(B)] \cdot [U : Z(B)] = dim_{Z(B)}(N)^2$. Then $C_A(C) = B$, since $C_A(C) \supseteq B$ and (b) shows $[C_A(C) : R] = [B : R]$. The graded homomorphism $B \otimes_{Z(B)} C \to C_A(Z(B))$ given by $b \otimes c \mapsto bc$ is injective as its domain is graded simple by Prop. 1.1, and surjective by dimension count.

There is a partial graded analogue to the Skolem-Noether theorem theorem. One would prefer to be able to conjugate a GCSA by a homogeneous unit, since then the grading is preserved. We will see that this is possible in some significant cases, but not always.

Proposition 1.6. *Let A be a GCSA over a graded field R, let B and B' be graded simple R-subalgebras of A, and let $C = C_A(B)$, $Z = Z(B)$, and $C' = C_A(B')$, $Z' = Z(B')$. Suppose there is a graded R-algebra isomorphism $\alpha\colon B \to B'$. Then,*

(a) There is $a \in A^*$ such that $\alpha(b) = aba^{-1}$ for all $b \in B$.

(b) The a of part (a) can be chosen to be homogeneous iff there is a gr-isomorphism $\gamma\colon C \to C'$ such that $\gamma|_Z = \alpha|_Z$.

(c) If C_0 is a division ring, then the a of part (a) can be chosen to be homogeneous.

In particular, every graded R-algebra automorphism of A is given by conjugation by a homogeneous unit of A.

Proof. The proof of part (a) is analogous to the ungraded theorem (cf. [R, pp. 103–104]): Let L, E, T, N be as in the proof above of the double centralizer theorem, so $T = B \otimes_R E^{\mathrm{op}}$. We make L into a graded left T-module in two ways, first by $(b \otimes e^{\mathrm{op}}) \cdot \ell = b\,\ell\,e$, and second by $(b\otimes e^{\mathrm{op}})\cdot \ell = \alpha(b)\,\ell\,e$. Write L' for L with the second T-module action, while L unadorned denotes L with the first T-action. By Prop. 1.3, $L \cong_g s_{\delta_1}(N)\oplus\ldots\oplus s_{\delta_n}(N)$ and $L' \cong_g s_{\varepsilon_1}(N)\oplus\ldots\oplus s_{\varepsilon_m}(N)$ as graded T-modules, and $m = n$ by dimension count. Since each $s_\delta(N) = N$ as T-modules when we ignore the grading, we have $L' \cong L$ as ungraded T-modules. The ungraded argument as in [R] then shows there is $a \in A^*$ with

$$ab = \alpha(b)a \tag{1.11}$$

for all $b \in B$. Thus, $\alpha(b) = aba^{-1}$, for all $b \in B$, proving part (a). We proceed to the proof of (c). Let $c = a^{-1}$, and let $a = \sum a_\gamma$ and $c = \sum c_\delta$ be the homogeneous decompositions of a and c. Because α is a graded homomorphism, for each $b \in B^h$, formula (1.11) yields by comparing homogeneous components

$$a_\gamma b = \alpha(b)a_\gamma, \tag{1.12}$$

for each a_γ. Hence, (1.12) holds for all $b \in B$. Likewise, since $bc = c\alpha(b)$ for all $b \in B$, we find that $bc_\delta = c_\delta\alpha(b)$ for all c_δ. These equations show $c_\delta a_\gamma \in C$ for all a_γ and c_δ. So, the equation $1 = ca = \sum\limits_\delta \sum\limits_\gamma c_\delta a_\gamma$ has all its summands in C^h. Therefore, there must be a nonzero summand $c_\delta a_\gamma \in C_0$. When C_0 is a division ring, $c_\delta a_\gamma \in C_0^*$, so $a_\gamma \in A^*$. Then (1.12) shows $\alpha(b) = a_\gamma b a_\gamma^{-1}$ for all $b \in B$, proving (c).

For part (b), observe first that if the a of part (a) is homogeneous, then a^{-1} is also homogeneous, so conjugation by a is a graded automorphism of A. Since this map sends B to B', it also sends C to C'. Hence, we can take γ to be the restriction to C of conjugation by a. Conversely, suppose there is $\gamma\colon C \to C'$ as in (b). Let $Y = C_A(Z)$ and $Y' = C_A(Z')$. Then, as $Y = BC \cong_g B \otimes_Z C$ by Prop. 1.5(c), we obtain a graded R-algebra isomorphism $\beta\colon Y \to Y'$ as the composition

$$Y \cong_g B \otimes_Z C \xrightarrow{\alpha\otimes\gamma} B' \otimes_Z C' \cong_g Y'.$$

Now, Y is graded simple and $C_A(Y) = Z$ by Prop. 1.5, and Z is a graded field. Therefore, we may apply parts (a) and (c) with Y, γ replacing B, α, to see that there is a homogeneous unit $a' \in A$ such that $\beta(y) = a' y a'^{-1}$ for all $y \in Y$. Since $\beta|_B = \alpha$, we can use a' for the a of part (a) for B, α, as desired.

The final assertion of the proposition follows by taking $B = A$ (so $C = R$) and invoking (a) and (c) (or (b)). $\square$

Remark 1.7. Note that for the C of Prop. 1.6, C_0 is a division ring iff C is a division ring, by Prop. 1.4(b).

Example 1.8. Let R be a graded field with $\Gamma_R = \mathbb{Z}$. Let $A = M_4(R)(0, 0, \frac{1}{2}, \frac{1}{2})$, $B_1 = C_1 = M_2(R)(0, \frac{1}{2})$, and $C_2 = M_2(R)(0, 0)$. Then, by (1.5), (1.3), (1.4) above,

$$B_1 \otimes_R C_1 \cong_g A \cong_g B_1 \otimes_R C_2.$$

Let B be the copy of B_1 in A given by the first graded isomorphism, and B' the copy of B_1 in A given by the second. Then $C_A(B) \cong_g C_1$ by (1.9), and $C_A(B') \cong_g C_2$. However, $C_1 \not\cong_g C_2$, e.g. since $\Gamma_{C_1} = \frac{1}{2}\mathbb{Z}$ while $\Gamma_{C_2} = \mathbb{Z}$. Thus, Prop. 1.6(a) and (b) show that although B' is obtainable from B by conjugating by some $a \in A^*$, there is no homogeneous such a. Furthermore, a graded R-isomorphism $B \to B'$ cannot be extended to a graded R-automorphism of A.

§2 Valuation-like properties of graded division algebras

Let $R = \bigoplus_{\gamma \in \Gamma_R} R_\gamma$ be a graded field (with Γ_R torsion-free, as we are always assuming), and let E be a GCDA over R. In this section we will describe some properties of E which are analogous to known properties for tame division algebras over a Henselian valued field. We will use them in §3 in proving a cohomological characterization of $GBr(R)$, see Prop. 3.3 below.

Before considering an arbitrary GCDA over R, we note a couple of extreme cases. First, a GCDA I over R is said to be *unramified* (or inertial) if $\Gamma_I = \Gamma_R$ (iff by (1.7), $[I : R] = [I_0 : R_0]$). In this situation the graded homomorphism $I_0 \otimes_{R_0} R \to I$ is actually an isomorphism, since it is clearly surjective, and a dimension comparison then shows it is also injective. Since $Z(I_0 \otimes_{R_0} R) = Z(I_0) \otimes_{R_0} R$, it follows that $Z(I_0) = R_0$. Thus, there is a one-to-one correspondence ($I \leftrightarrow I_0$) between isomorphism classes of unramified GCDA's over R and isomorphism classes of central division algebras (CDA's) over R_0. Also, if S is a graded R-subalgebra of I, then $\Gamma_S = \Gamma_R$, so $S = S_0 \otimes_R R$. Thus, graded R-subalgebras of I are in canonical one-to-one correspondence (not just up to isomorphism) with R_0-subalgebras of I_0.

At the other extreme, a GCDA T over R is said to be *totally ramified* if $T_0 = R_0$ (iff $|\Gamma_T : \Gamma_R| = [T : R]$, by (1.7)). In this case, there is a pairing $\gamma_T \colon T^* \times T^* \to R_0^*$ given by

$(s,t) \mapsto [s,t]$ $(= sts^{-1}t^{-1})$. (Recall that $T^* = T^h - \{0\}$.) The pairing is clearly skew-symmetric, and since the image of γ_T is central, the commutator identity $[s,tu] = [s,t]t[s,u]t^{-1}$ shows that γ_T is bimultiplicative as well. Because $\gamma_T(s,t) = 1$ if s or t is central, the pairing is actually well-defined on $T^*/R^* \times T^*/R^*$. But, as T is totally ramified $T^*/R^* \cong \Gamma_T/\Gamma_R$ which is finite; so every element of $im(\gamma_T)$ has finite order in R_0^*. Thus, γ_T induces a well-defined biadditive skew-symmetric pairing called the *canonical pairing*

$$\beta_T \colon \Gamma_T/\Gamma_R \times \Gamma_T/\Gamma_R \to \mu(R_0), \tag{2.1}$$

given by $(deg(s) + \Gamma_R, deg(t) + \Gamma_R) \mapsto sts^{-1}t^{-1}$, where $\mu(R_0)$ denotes the group of all roots of unity in the field R_0. (We will use the further notation: If F is a field and ℓ a positive integer, then $\mu_\ell(F)$ is the group of all ℓ-th roots of unity in F.)

Proposition 2.1 (cf. [B_2, Prop. 2.6])**.** *Let T be a totally ramified GCDA over R. Then the canonical pairing β_T of (2.1) is nondegenerate. The image of β_T is a cyclic subgroup of $\mu(R_0)$ of order equal to the exponent $exp(\Gamma_T/\Gamma_R)$. Hence, $char(R_0) \nmid |\Gamma_T/\Gamma_R|$.*

Proof. For $s \in T^*$, if $\gamma_T(s,t) = 1$ for all $t \in T^*$, then $s \in Z(T) = R$. This shows β_T is nondegenerate. Let $\ell = exp(\Gamma_T/\Gamma_R)$, i.e., the exponent of the finite abelian group Γ_T/Γ_R; then $im(\beta_T) \subseteq \mu_\ell(R_0)$ as β_T is biadditive. But, if we take any $\overline{\alpha} \in \Gamma_T/\Gamma_R$ of order ℓ, then the nondegeneracy of β_T forces the homomorphism $\beta_T(\overline{\alpha}, -) \colon \Gamma_T/\Gamma_R \to \mu_\ell(R_0)$ to be surjective, and forces $|\mu_\ell(R_0)| = \ell$. Hence, $im(\beta_T) = \mu_\ell(R_0)$, and $|im(\beta_T)| = \ell = exp(\Gamma_T/\Gamma_R)$. If $p = char(R_0)$, then as $\mu(R_0)$ has no p-torsion, we must have $p \nmid \ell$. $\square$

Remarks 2.2. (i) If Λ is any group with $\Gamma_R \subseteq \Lambda \subseteq \Gamma_T$, then $T_\Lambda = \bigoplus_{\lambda \in \Lambda} T_\lambda$ is a graded R-subalgebra of T with $\Gamma_{T_\Lambda} = \Lambda$; furthermore, since $T_0 = R_0$, every R-subalgebra of T has the form T_Λ for some Λ. Thus, the subgroups of Γ_T/Γ_R classify the R-subalgebras of T. Note also that $Z(T_\Lambda) = T_{\Lambda'}$, where $\Lambda'/\Gamma_R = \Lambda/\Gamma_R \cap (\Lambda/\Gamma_R)^\perp$, where $(\Lambda/\Gamma_R)^\perp$ denotes the orthogonal subgroup to Λ/Γ_R in Γ_T/Γ_R with respect to β_T.

(ii) Relative to the skew-symmetric nondegenerate biadditive pairing β_T there always exists a symplectic base for Γ_T/Γ_R (cf. [TW, Prop. 3.1]). This implies that the distinct invariant factors of the finite abelian group Γ_T/Γ_R each occur with even multiplicity. Also, the symplectic base allows one to decompose T into a tensor product of graded symbol algebras over R, analogous to the decomposition for tame totally ramified valued division algebras described in [TW, Prop. 4.2]. Furthermore, one can use the symplectic base to see that if $R_0^\ell = R_0$ where $\ell = exp(\Gamma_T/\Gamma_R)$, then T is determined up to isomorphism by Γ_T and β_T. Also, one can easily see that for any group Γ with $\Gamma_R \subseteq \Gamma \subseteq \Delta_R$, such that Γ/Γ_R is finite and all the distinct invariant factors of Γ/Γ_R occur with even multiplicity and $|\mu_\ell(R_0)| = \ell$, where $\ell = exp(\Gamma/\Gamma_R)$, there exists a nondegenerate skew-symmetric biadditive pairing $\beta \colon \Gamma/\Gamma_R \times \Gamma/\Gamma_R \to \mu_\ell(R_0)$; for any such Γ and β one can use a symplectic base for β on Γ/Γ_R to construct a totally ramified graded division algebra T over R such that $\Gamma_T = \Gamma$ and $\beta_T = \beta$.