CORRESPONDENCES BETWEEN VALUED DIVISION ALGEBRAS AND GRADED DIVISION ALGEBRAS

Y.-S. Hwang[1] and A. R. Wadsworth[2]

Abstract. If D is a tame central division algebra over a Henselian valued field F, then the valuation on D yields an associated graded ring GD which is a graded division ring and is also central and graded simple over GF. After proving some properties of graded central simple algebras over a graded field (including a cohomological characterization of its graded Brauer group), it is proved that the map $[D] \mapsto [GD]_g$ yields an index-preserving isomorphism from the tame part of the Brauer group of F to the graded Brauer group of GF. This isomorphism is shown to be functorial with respect to field extensions and corestrictions, and using this it is shown that there is a correspondence between F-subalgebras of D (with center tame over F) and graded GF-subalgebras of GD.

Introduction

If D is a division ring finite-dimensional over its center F, and the field F has a Henselian valuation v, then v is known to extend uniquely to a valuation on D. The features associated with the valuation on D, especially the residue division algebra $\overline{D}$ and the value group Γ_D carry much information about the structure of D, and often can be used to settle questions such as decomposability, and which fields can be subfields of D. However, $\overline{D}$ and Γ_D do not determine D, and there are many subtleties in the way they interact.

Associated to the valuation on D there is a filtration of D by the principal fractional ideals of the valuation ring, which allows one to build an associated graded ring $GD = \bigoplus_{\gamma\in\Gamma_D} GD_\gamma$, where $GD_0 = \overline{D}$ and the grade group of GD is precisely the value group Γ_D of D. Furthermore, GD is a graded division ring, i.e., its homogeneous elements are all units. In addition, as shown in [B_2], the total ordering on Γ_D allows one to define a valuation on GD which extends to the ring of quotients QGD of GD, which is a division algebra. The valued division algebra QGD is usually not isomorphic to D, not even after Henselization, but we will see that their structures are closely related. The very presence of a valuation on QGD suggests that not so much is lost in the passage from D to its graded ring GD, even though GD appears to have a much simpler structure than D. We will show, in fact, that if D is tame then it is completely determined by GD, and its subalgebra structure is faithfully mirrored in that of GD.

Specifically, let $TBr(F)$ denote the tame part of the Brauer group of the Henselian field F, and let $GBr(GF)$ denote the graded Brauer group of the graded field GF determined by the valuation

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on F. We will show in Th. 5.3 that the map $[D] \mapsto [GD]_g$ gives a Schur-index-preserving group isomorphism $TBr(F) \to GBr(GF)$, which (see Cor. 5.7 and Th. 6.1) is functorial with respect to scalar extensions and corestrictions. The index-preserving and functorial properties allow us to deduce (see Th. 5.9) that if K is a tame valued field extension of F, and D and A are tame division algebras with center F, then K (resp. A) embeds in D iff GK (resp. GA) embeds in GD.

These results show that much of what is known about tame valued division algebras can be carried over readily to graded division algebras finite-dimensional over their centers, when the grade group is torsion-free. Beyond that, it lays the foundation for proving theorems about valued division algebras by first proving corresponding results in the relatively easier setting of graded division algebras. This approach has previously been applied successfully for wildly ramified valued division algebras by Tignol in [T].

This paper is organized as follows: Before considering connections between valued and graded division algebras, we develop the graded theory in the first three sections. In §1 we recall basic properties of graded division algebras and graded central simple algebras (GCSA's) over a graded field with torsion-free grade group, and point out the analogues of Wedderburn's theorem and the double centralizer theorem. We also prove a version of the Skolem-Noether theorem for GCSA's, which is somewhat delicate. In §2 we prove properties for graded division algebras which are analogous to known properties of tame valued division algebras. This is used in §3 to prove a cohomological characterization of the graded Brauer group $GBr(R)$ of a graded field R. In §4, we show how to get back and forth between tame valued division algebras and graded division algebras. If we start with a graded field R with totally ordered grade group Γ_R, then $GHR \cong_g R$ (graded isomorphism) canonically, where GHR is the graded field obtained from the valuation on the Henselian field HR obtained from the valuation on the quotient field of R determined by the grading on R. But, if we start with a Henselian valued field F, and take the Henselization HGF of the quotient field of the graded field GF (with respect to the valuation determined by the grading on GF), where GF is built from the valuation on F, then usually $HGF \not\cong F$. (These fields need not even have the same characteristic.) Nonetheless, we prove in Th. 4.4 that $TBr(HGF) \cong TBr(F)$. In §5 we prove the isomorphism $TBr(F) \cong GBr(GF)$ mentioned above, and the correspondences between tame subalgebras and graded subalgebras. Finally, the compatibility with the corestriction is given in §6.

§1 Graded division algebras and graded central simple algebras

We begin by setting up notation and recalling some results about graded division algebras and graded central simple algebras. Except for the graded Skolem-Noether theorem, Prop. 1.6, most of what we say in this section can be found in the literature somewhere (see especially [B_2], [CvO], [NvO]), though not always in the generality we need.

Let $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ be a graded ring. This means for us that A is an associative ring with 1, Γ is an abelian group, each A_γ is a subgroup of the additive group of A, and $A_\gamma \cdot A_\delta \subseteq A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. Because we are interested in the graded rings associated to valuation rings, *we will assume throughout that* Γ *is torsion-free.* We set

$$\Gamma_A = \{\gamma \in \Gamma \mid A_\gamma \neq (0)\}, \quad \text{the grade set of } A,$$

and

$$A^h = \bigcup_{\gamma \in \Gamma_A} A_\gamma, \quad \text{the set of homogeneous elements of } A.$$

For $a \in A^h$, $a \neq 0$, we write $deg(a) = \gamma$ if $a \in A_\gamma$. Each $c \in A$ is uniquely expressible as $c = \sum_{\gamma \in \Gamma_A} c_\gamma$ with each $c_\gamma \in A_\gamma$. The c_γ are called the homogeneous components of c. Let A^* denote the group of units of A. A subring S of A is a graded subring if $S = \bigoplus_{\gamma \in \Gamma_A} (S \cap A_\gamma)$ (iff for each $s \in S$, all the homogeneous components of s lie in S). Note that if S is a graded subring of A, then its centralizer $C_A(S)$ is also a graded subring of A. In particular, the center of A, $Z(A) = C_A(A)$ is a graded subring of A. A (left, right, or two-sided) ideal I of A is said to be *homogeneous* if $I = \bigoplus_{\gamma \in \Gamma_A} (I \cap A_\gamma)$ (iff I is generated as a left or ... ideal of A by homogeneous elements). Suppose $B = \bigoplus_{\gamma \in \Gamma'} B_\gamma$ is another graded ring, and suppose there is a torsion-free group Δ containing Γ and Γ' as subgroups. A graded ring homomorphism $f\colon A \to B$ is a ring homomorphism such that $f(A_\delta) \subseteq B_\delta$ for all $\delta \in \Delta$. (It is understood that $A = \bigoplus_{\delta \in \Delta} A_\delta$, where $A_\delta = (0)$ for $\delta \in \Delta - \Gamma_A$; likewise for B.) If, further, f is bijective, then f is a graded isomorphism, and we write $A \cong_g B$. We frequently abbreviate "graded" by *gr*. The graded ring A is said to be *graded simple* if $|A| > 1$ (i.e., $1_A \neq 0_A$) and the only homogeneous two-sided ideals of A are A and (0).

A graded left A-module M is a left A-module with a direct sum decomposition as abelian groups $M = \bigoplus_{\gamma \in \Gamma'} M_\gamma$, where Γ' is some torsion-free abelian group containing Γ, such that $A_\gamma \cdot M_\delta \subseteq M_{\gamma+\delta}$, for all $\gamma \in \Gamma_A$, $\delta \in \Gamma'$. Then Γ_M, M^h, and graded submodules are defined just as above for rings. We can make M into a graded A-module in other ways by shifting the grading: For any $\gamma \in \Gamma'$, the γ-shift of M, denoted $s_\gamma(M)$ is defined by

$$s_\gamma(M) = M \text{ as an } A\text{-module, and } s_\gamma(M)_\delta = M_{\gamma+\delta}, \qquad \text{for all } \delta \in \Gamma'.$$

So, $\Gamma_{s_\gamma(M)} = -\gamma + \Gamma_M$. Now, let $N = \bigoplus_{\gamma \in \Gamma''} N_\gamma$ be another graded left A-module, such that there is a torsion-free abelian group Δ containing Γ' and Γ'' as subgroups. A graded A-module homomorphism $f\colon M \to N$ is an A-module homomorphism such that $f(M_\delta) \subseteq N_\delta$ for all $\delta \in \Delta$. There is the corresponding notion of graded isomorphism, and when there is one between M and N we write $M \cong_g N$. Let $GHom_A(M, N)$ denote the group of graded A-module homomorphisms from M to N, so $GHom_A(M, N)$ is a subgroup of the group $Hom_A(M, N)$ of all A-module homomorphisms from M to N. For each $\delta \in \Delta$, we have a subgroup of $Hom_A(M,N)$ of δ-shifted homomorphisms

$$Hom_A(M,N)_\delta = \{f \in Hom_A(M,N) \mid f(M_\gamma) \subseteq N_{\gamma+\delta} \quad \text{for all } \gamma, \delta \in \Delta\}.$$

Of course, $Hom_A(M,N)_\delta = GHom_A(M, s_\delta(N)) = GHom_A(s_{-\delta}(M), N)$. Clearly, $\bigoplus_{\delta\in\Delta} Hom_A(M,N)_\delta$ is subgroup of $Hom_A(M,N)$; if M is a finitely-generated A-module, then

$$Hom_A(M,N) = \bigoplus_{\delta\in\Delta} Hom_A(M,N)_\delta$$

(cf. [NvO, Lemma I.6.1, p. 26]). Indeed, for $f \in Hom_A(M,N)$, $\delta \in \Delta$, define $f_\delta \in Hom_A(M,N)_\delta$ by, for $m = \sum_{\gamma\in\Delta} m_\gamma$ with $m_\gamma \in M_\gamma$, setting $f_\delta(m) = \sum_{\varepsilon\in\Delta} (f(m_{\varepsilon-\delta}))_\varepsilon$. When M is finitely-generated, all but finitely many $f_\delta = 0$, and $f = \sum_\delta f_\delta$. In particular, for any finitely-generated graded left A-module M, $End_A(M) = Hom_A(M,M)$ is a graded ring. When A acts on M on the left, we view $End_A(M)$ as acting on M on the right; so M is a graded A-$End_A(M)$-bimodule.

Now, let $M = \bigoplus_{\gamma\in\Gamma'} M_\gamma$ be a graded right A-module and $N = \bigoplus_{\gamma\in\Gamma''} N_\gamma$ a graded left A-module, with $\Gamma', \Gamma'' \subseteq \Delta$ for some torsion-free-abelian group Δ. Then, $M \otimes_A N$ has a natural grading as $Z(A)$-module given by

$$(M\otimes_A N)_\delta = \Big\{ \sum_i m_i \otimes n_i \mid m_i \in M^h,\ n_i \in N^h,\ deg(m_i) + deg(n_i) = \delta \Big\}, \quad \delta \in \Delta.$$

One can see that this gives a grading on $M \otimes_A N$ by observing that the corresponding grading on $M \otimes_{A_0} N$ is clearly well-defined, and the grading on $M \otimes_{A_0} N$ is inherited by $M \otimes_A N \cong (M \otimes_{A_0} N)/J$, since the subgroup J of $M \otimes_{A_0} N$ is generated by the homogeneous elements $\{ma \otimes n - m \otimes an \mid m \in M^h, n \in N^h, a \in A^h\}$.

For example, suppose F is a *graded free* right A-module of finite rank, i.e., F is graded right A-module which is free as an A-module with a finite base $\{b_1, \ldots, b_n\} \subseteq F^h$. Let $\delta_i = deg(b_i) \in \Gamma_F$. Of course, $End_A(F) \cong M_n(A)$ ($n \times n$ matrices over A) if we ignore the grading, and by convention $End_A(F)$ acts on F on the left. In this isomorphism the ij-matrix unit $E_{ij} \in M_n(A)$ corresponds to the map $e_{ij} \in End_A(F)$, defined by $e_{ij}(b_j) = b_i$ and $e_{ij}(b_k) = 0$, for $k \neq j$. Clearly, $e_{ij} \in End_A(F)_{\delta_i - \delta_j}$. So, when we take the grading into account, we find that

$$End_A(F) \cong_g M_n(A)(d), \quad \text{for } d = (\delta_1, \ldots, \delta_n), \tag{1.1}$$

where $M_n(A)(d)$ means $n \times n$ matrices over A but with the degree of the ij-entry shifted by $\delta_i - \delta_j$, i.e.,

$$M_n(A)(d) = \begin{pmatrix} s_{\delta_1-\delta_1}(A) & \ldots & s_{\delta_n-\delta_1}(A) \\ \cdot & \cdot & \cdot \\ s_{\delta_1-\delta_n}(A) & \ldots & s_{\delta_n-\delta_n}(A) \end{pmatrix}. \tag{1.2}$$

So, the ij-entry of $M_n(A)(d)$ is $s_{\delta_j-\delta_i}(A)$ (as $s_{\delta_j-\delta_i}(A)_{\delta_i-\delta_j} = A_0$). Thus, the ε-component of $M_n(A)(d)$ consists of matrices with ij-entry in $A_{\varepsilon+\delta_j-\delta_i}$.

For future reference, we point out a few elementary properties of these shifted graded matrix rings. Let A be any graded ring. Then,

(i) If $\pi \in S_n$ is any permutation, then

$$M_n(A)(\delta_1, \ldots, \delta_n) \cong_g M_n(A)(\delta_{\pi(1)}, \ldots, \delta_{\pi(n)}). \tag{1.3}$$

(ii) If $\gamma_1, \ldots, \gamma_n \in \Gamma_A$, with $\gamma_i = deg(a_i)$ for some unit $a_i \in A^h$, then

$$M_n(A)(\delta_1, \ldots, \delta_n) \cong_g M_n(A)(\delta_1 + \gamma_1, \ldots, \delta_n + \gamma_n). \tag{1.4}$$

(iii) If A is commutative, and $d = (\delta_1, \ldots, \delta_n)$, $e = (\varepsilon_1, \ldots, \varepsilon_m)$, then

$$M_n(A)(d) \otimes_A M_m(A)(e) \cong_g M_{mn}(A)(f), \tag{1.5}$$

where $f = \{\delta_i + \varepsilon_j \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$. (The order of the terms is immaterial, in view of (1.3).)

For, $M_n(A)(\delta_1, \ldots, \delta_n) \cong_g End_A(F)$, where F is a graded free graded right A-module with homogeneous base $b_1, \ldots, b_n$ with $deg(b_i) = \delta_i$. Since $b_{\pi(1)}, \ldots, b_{\pi(n)}$ is also a homogeneous base of F, we also have $M_n(A)(\delta_{\pi(1)}, \ldots, \delta_{\pi(n)}) \cong_g End_A(F)$, yielding (1.3). Likewise, if a_i is a homogeneous unit of A with $deg(a_i) = \gamma_i$, then $b_1 a_1, \ldots, b_n a_n$ is another homogeneous base of F, with $deg(b_i a_i) = \delta_i + \gamma_i$. So, $M_n(A)(\delta_1 + \gamma_1, \ldots, \delta_n + \gamma_n) \cong_g End_A(F)$, proving (1.4). Now, assuming A is commutative, let F' be another graded free A-module, with base $c_1, \ldots, c_m$ with $deg(c_j) = \varepsilon_j$. Then, $F \otimes_A F'$ is a free graded A-module with base $\{b_i \otimes c_j\}$, where $deg(b_i \otimes c_j) = \delta_i + \varepsilon_j$. So

$$M_n(A)(d) \otimes_A M_m(A)(e) \cong_g End_A(F) \otimes_A End_A(F') \cong_g End_A(F \otimes_A F') \cong_g M_{mn}(A)(f),$$

showing (1.5).

A graded ring $E = \bigoplus_{\gamma \in \Gamma_E} E_\gamma$ is called a *graded division ring* if every nonzero homogeneous element of E is a unit, and $1_E \neq 0_E$. Note that the grade set Γ_E is actually a group. Further, since Γ_E is torsion-free, it follows that E has no zero divisors and $E^* = E^h - \{0\}$. (This is easy to see by recalling that the torsion-free abelian group Γ_E can be given a total ordering compatible with the group operation. Thus, if $a \neq 0$, $a = a_\gamma +$ terms of higher degree and $b \neq 0$, $b = b_\delta +$ terms of higher degree, then $ab = a_\gamma b_\delta +$ terms of higher degree, so $ab \neq 0$.) Also, E_0 must be a division ring, and for each $\gamma \in \Gamma_E$, the group E_γ is a one-dimensional left and right vector space over E_0. Note further that every graded left (resp. right) E-module M is a graded free E-module (cf. [B_1, Th. 3, p. 29]). For, it is easy to check that a maximal homogeneous E-linearly independent subset of M is actually a base. We call M a *graded vector space* over E, and write $dim_E(M)$ for the rank of M as a graded free E-module. (This is well-defined, since one can apply the usual exchange argument to see that any two homogeneous bases of M have the same cardinality.) Note that if N is a graded submodule of M, then

$$dim_E(N) + dim_E(M/N) = dim_E(M). \tag{1.6}$$

Consequently, if $dim_E(M) < \infty$ and N is a proper submodule of M, then $dim_E(N) < dim_E(M)$. Let S be a graded subring of E such that S is also a graded division ring, and let $[E : S] = dim_S(E)$ (left dimension) and likewise $[E_0 : S_0] = dim_{S_0} E_0$ (left dimension). Note the easy but fundamental formula (cf. [B_2, p. 4278])

$$[E : S] = [E_0 : S_0] \cdot |\Gamma_E : \Gamma_S|. \tag{1.7}$$

This holds since if $\{a_i\}$ is a base of E_0 as left S_0-vector space and if $\{b_j\} \subseteq E^h - \{0\}$ is chosen so that $\{deg(b_j)\}$ is a set of coset representatives for Γ_S in Γ_E, then $\{a_i b_j\}$ is a homogeneous base of E as a left S-vector space.

A commutative graded division ring is called a *graded field*. For example, if L is any field and Γ is any torsion-free abelian group, then the group ring $R = L[\Gamma]$ is a graded field with $R_0 = L$ and $\Gamma_R = \Gamma$. In fact, Γ is a free abelian group, then every graded field S with $\Gamma_S = \Gamma$ is a group ring (cf. [HW, Prop. 1.1]). However, there do exist graded fields which are not group rings (cf. [HW, Ex. 1.2]).

Let R be a graded field. A *graded R-algebra* A is graded ring which is an R-algebra such that the associated ring homomorphism $\varphi\colon R \to Z(A)$ is a gr-homomorphism. This φ is necessarily injective (assuming $1 \neq 0$ in A), as R is a graded field. We have A_0 is an R_0-algebra. Also, while Γ_A need not be a group, it is a union of cosets of the group Γ_R in some ambient torsion-free abelian group Γ'. We write

$$|\Gamma_A : \Gamma_R| = \text{the number of cosets of } \Gamma_R \text{ in } \Gamma_A$$

and

$$[A : R] = dim_R(A).$$

It is easy to check that

$$[A : R] \geq [A_0 : R_0] \cdot |\Gamma_A : \Gamma_R|, \tag{1.8}$$

but equality often does not hold (see Prop. 1.4 below).

For our graded field R, let Δ_R be the divisible hull of the torsion-free abelian group Γ_R, so

$$\Delta_R \cong \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_R,$$

and fix some $\mathbb{Q}$-vector space Δ' containing Δ_R with $dim_{\mathbb{Q}}(\Delta'/\Delta_R) = \infty$. Then (1.8) shows that if A is any finite dimensional graded R-algebra, then Γ_A is Γ_R-isomorphic to a subset of Δ'. Indeed, if Γ_A is a group (which occurs, e.g., whenever A is a graded division algebra) then, as Γ_A is torsion-free and Γ_A/Γ_R is torsion by (1.8), there is a unique group homomorphism $\Gamma_A \to \Delta_R$ which restricts to the identity on Γ_R. So we will assume henceforth that all graded R-algebras A satisfy $\Gamma_A \subseteq \Delta'$.

Note that if A and B are graded R-algebras, then $A \otimes_R B$ is also a graded R-algebra. If A' is a graded R-subalgebra of A and B' a graded R-subalgebra of B, then it is easy to check that

$$C_{A\otimes_R B}(A' \otimes_R B') = C_A(A') \otimes_R C_B(B'). \tag{1.9}$$

A graded algebra A over a graded field R is said to be a *graded central simple algebra* (GCSA) over R if A is a simple graded ring, $[A : R] < \infty$, and $Z(A) = R$. There is a theory of GCSA's over a graded field analogous to the theory of central simple algebras (CSA's) over a field, and we recall some basic properties here.

Proposition 1.1. *Let A be a GCSA over a graded field R, and let B be any graded R-algebra. If I is a homogeneous ideal of $A \otimes_R B$, then $I = A \otimes_R J$, where $J = I \cap B$, and J is a homogeneous ideal of B. Hence, if B is graded simple, then $A \otimes_R B$ is a GCSA over $Z(B)$.*

Of course, in Prop 1.1 we are identifying B with its gr-isomorphic copy $R \otimes_R B$ in $A \otimes_R B$. This proposition can be proved analogously to the ungraded result. One can first show the special case: if $I \cap B = (0)$, then $I = (0)$. The general result follows by applying the special case to $B' = B/J$ (after noting that J is homogeneous, so B'/J is graded); since $A \otimes_R B' \cong_g (A \otimes_R B)/(A \otimes_R J)$ and $(I/(A \otimes_R J)) \cap B' \cong (I \cap B)/J = (0)$, we obtain $I/(A \otimes_R J) = (0)$, as desired. Formula (1.9) shows $Z(A \otimes_R B) = Z(A) \otimes_R Z(B) \cong_g Z(B)$.

Corollary 1.2 (cf. [B$_2$, Prop. 5.1]). *Let A be an algebra over a graded field R. Then, A is a GCSA over R iff A is both an Azumaya algebra over R and also a graded R-algebra.*

PROOF. Suppose A is a GCSA over R. Then, A is a free R-module of finite rank, and the graded R-algebra homomorphism $A \otimes_R A^{\mathrm{op}} \to End_R(A)$ is injective, since the domain is graded simple by Prop. 1.1, and surjective by dimension count (using (1.6)). Hence, by [DI, Th. 3.4, p. 52], A is an Azumaya algebra over R. Conversely, suppose A is an Azumaya algebra over R such that A is also a graded R-algebra. We identify R with its gr-isomorphic copy in A. Since A is Azumaya over R, by [DI, Prop. 21, p. 47; Cor. 3.7, p. 54], A is a finitely-generated R-module, so $[A : R] < \infty$, and $Z(A) = R$, and every ideal I of A has the form $I = A(I \cap R)$. If I is a homogeneous ideal of A, then $I \cap R$ is a homogeneous ideal of R. Hence, A is graded simple since R is graded simple. □

A *graded central division algebra* (GCDA) over a graded field R is a GCSA E over R such that E is also a graded division ring. Observe that the usual matrix calculations show that for any GCDA E over R, any n, and any $d = (\delta_1, \ldots, \delta_n)$, $\delta_i \in \Delta'$, we have $M_n(E)(\delta)$ is a GCSA over R. Our next proposition is the graded Wedderburn theorem, which says that all GCSA's over R have this form.

Proposition 1.3. *Let A be a GCSA over a graded field R. Then,*

(a) There is a GCDA E over R such that $A \cong_g M_n(E)(d)$ for some $d = (\delta_1, \dots, \delta_n)$. Moreover, if $A \cong_g M_{n'}(E')(d')$ for some GCDA E' over R, then $n' = n$ and $E' \cong_g E$.

(b) Every graded left (or right) A-module is a direct sum of graded simple A-modules.

(c) If L is a minimal nonzero homogeneous left ideal of A and N is a graded simple A-module, then $N \cong_g s_\delta(L)$ i.e., N is the δ-shift of L for some δ. Hence,

$$dim_R(N) = n[E : R] = [A : R]/n. \tag{1.10}$$

This can be proved analogously to the usual Wedderburn theorem. Here is a sketch. Take a minimal nonzero homogeneous left ideal L of A (which exists as $[A : R] < \infty$), and let $E = End_A(L)$. Since L is a graded simple A-module (i.e., it has no nonzero proper graded A-submodule), the graded Schur's Lemma shows that E is a graded division ring, and $[E : R] \le [End_R(L) : R] < \infty$. Let $b_1, \dots, b_n$ be a homogeneous base of L as a graded free right E-module, so $L = b_1E \oplus \dots \oplus b_nE$. Then, $End_E(L) \cong_g M_n(E)(d)$, where $d = (deg(b_1), \dots, deg(b_n))$, as noted in (1.1) above. Rieffel's proof of Wedderburn's Theorem ([Ri], or see [L, Th. 5, p. 449]) can be applied here to see that the graded R-algebra homomorphism $A \to End_E(L)$ ($a \mapsto$ left multiplication by a) is an isomorphism, so $A \cong_g End_E(L) \cong_g M_n(E)(d)$. If N is any graded simple left A-module, there is a $b \in N^h - \{0\}$ with $L \cdot b \neq (0)$. Then the A-module homomorphism $\lambda\colon L \to N$, $\ell \mapsto \ell \cdot b$ is an isomorphism since $im(\lambda)$ and $ker(\lambda)$ are graded submodules. Since λ shifts degrees by $deg(b)$, we have $N \cong_g s_{-deg(b)}(L)$. This yields (c) and also the uniqueness part of (a). For, if $A \cong_g M_{n'}(E')(d')$ and L' is the set of first columns of elements of $M_{n'}(E')(d')$, then L' is a graded simple left $M_{n'}(E')(d')$-module with endomorphism ring E'. Since L' is graded simple when viewed as an A-module, $L' \cong_g s_\delta(L)$, for some δ, so $E = End_A(s_\delta(L)) \cong_g End_A(L') \cong_g E'$; then $n' = n$ by dimension count. Finally, for (b), since A is a sum of simple graded left A-modules (corresponding to the columns of $M_n(E)(d)$), every graded left A-module M is a sum $M = \sum N_i$ where the N_i are graded simple submodules of M. The usual argument shows that M is a direct sum of some subset of $\{N_i\}$.

For any GCSA A over R, we define the Schur index of A, $ind(A)$, analogously to the ungraded case: We have $A \cong M_n(E)(d)$ for a GCDA E over R; set

$$ind(A) = \sqrt{[E : R]}, \quad \text{a positive integer.}$$

The graded Wedderburn theorem yields a description of Γ_A and of A_0 for a GCSA. Let $A = M_n(E)(d)$ where E is a GCDA over R and $d = (\delta_1, \dots, \delta_n)$, $\delta_i \in \Delta'$. Let $\varepsilon_1 + \Gamma_E, \dots, \varepsilon_k + \Gamma_E$ be the distinct cosets of Γ_E of the form $\delta_i + \Gamma_E$, $1 \le i \le n$, and for each ε_ℓ let r_ℓ be the number of i with $\delta_i \equiv \varepsilon_\ell \,(mod\, \Gamma_E)$.

Proposition 1.4. *Let E be a graded division algebra, and let $A = M_n(E)(d)$, for $d = (\delta_1, \dots, \delta_n)$. Then,*

(a) $\Gamma_A = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} (\delta_i - \delta_j) + \Gamma_E$.

(b) $A_0 \cong M_{r_1}(E_0) \times \dots \times M_{r_k}(E_0)$, *with the r_i as defined above. In particular, A_0 is simple iff $k = 1$ iff $\Gamma_A = \Gamma_E$.*

Proof. (a) is immediate from the description of the grading on $M_n(E)(d)$ (see (1.2)). For (b), observe that by (1.3) and (1.4) above, $A \cong_g M_n(E)(e)$, where $e = (\varepsilon_1, \dots, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_2, \dots, \varepsilon_k, \dots, \varepsilon_k)$ with each ε_ℓ occurring r_ℓ times. In $M_n(E)(e)_0$ there is a contribution of E_0 in the ij-entry when the same ε_ℓ appears in the i-th and the j-th position in e. This accounts for all of $M_n(E)(e)_0$ since $0 \notin \varepsilon_\ell - \varepsilon_m + \Gamma_E$ when $\ell \neq m$. Thus, $A_0 \cong M_n(E)(e)_0 \cong M_{r_1}(E_0) \times \dots \times M_{r_k}(E_0)$. $\square$

The double centralizer theorem is also available in the graded context:

Proposition 1.5. *Let A be a GCSA over a graded field R and let B be a graded simple graded R-subalgebra of A. Set $C = C_A(B)$. Then,*

(a) *C is a graded simple R-subalgebra of A with $Z(C) = Z(B)$, and $C_A(C) = B$.*
(b) $[C : R] \cdot [B : R] = [A : R]$.
(c) *$B \otimes_{Z(B)} C \cong_g C_A(Z(B))$. In particular, if $Z(B) = R$, then $B \otimes_R C \cong_g A$.*

This is proved analogously to the ungraded version. We give a sketch. Let L be a minimal homogeneous left ideal of A, and let $E = End_A(L)$, so E is a GCDA over R and $A \cong_g End_E(L)$ as we saw in the discussion of Prop. 1.3. Let $T = B^{\mathrm{op}} \otimes_R E$, which is a GCSA over $Z(B)$ by Prop. 1.1, and view L as a graded right T-module. Now, $End_T(L) \subseteq End_E(L) \cong_g A$, and A acts faithfully on L. Hence $End_T(L)$ is gr-isomorphic to the set of elements of A acting on L compatibly with the B-action, i.e., $End_T(L) \cong_g C_A(B) = C$. Let N be a minimal homogeneous right ideal of T, and let $U = End_T(N)$, which is a GCDA over $Z(T)$. Then $L \cong_g s_{\delta_1}(N) \oplus \dots \oplus s_{\delta_k}(N)$ as graded T-modules by Prop. 1.3(b) and (c). Hence, $C \cong_g End_T(L) \cong_g End_T(s_{\delta_1}(N) \oplus \dots \oplus s_{\delta_k}(N)) \cong_g M_k(U)(d)$, where $d = (\delta_1, \dots, \delta_k)$, so C is graded simple with $Z(C) \cong_g Z(U) \cong_g Z(T) \cong_g Z(B)$. The formula in (b) follows from $[C : R] = k^2[U : R]$, $dim_R(L) = k\, dim_R(N)$, $[T : R] = [B : R] \cdot [E : R]$, together with (by (1.10)) $[A : R] \cdot [E : R] = dim_R(L)^2$ and $[T : Z(B)] \cdot [U : Z(B)] = dim_{Z(B)}(N)^2$. Then $C_A(C) = B$, since $C_A(C) \supseteq B$ and (b) shows $[C_A(C) : R] = [B : R]$. The graded homomorphism $B \otimes_{Z(B)} C \to C_A(Z(B))$ given by $b \otimes c \mapsto bc$ is injective as its domain is graded simple by Prop. 1.1, and surjective by dimension count.

There is a partial graded analogue to the Skolem-Noether theorem theorem. One would prefer to be able to conjugate a GCSA by a homogeneous unit, since then the grading is preserved. We will see that this is possible in some significant cases, but not always.

Proposition 1.6. *Let A be a GCSA over a graded field R, let B and B' be graded simple R-subalgebras of A, and let $C = C_A(B)$, $Z = Z(B)$, and $C' = C_A(B')$, $Z' = Z(B')$. Suppose there is a graded R-algebra isomorphism $\alpha \colon B \to B'$. Then,*

(a) There is $a \in A^*$ such that $\alpha(b) = aba^{-1}$ for all $b \in B$.

(b) The a of part (a) can be chosen to be homogeneous iff there is a gr-isomorphism $\gamma \colon C \to C'$ such that $\gamma|_Z = \alpha|_Z$.

(c) If C_0 is a division ring, then the a of part (a) can be chosen to be homogeneous.

In particular, every graded R-algebra automorphism of A is given by conjugation by a homogeneous unit of A.

Proof. The proof of part (a) is analogous to the ungraded theorem (cf. [R, pp. 103–104]): Let L, E, T, N be as in the proof above of the double centralizer theorem, so $T = B \otimes_R E^{\mathrm{op}}$. We make L into a graded left T-module in two ways, first by $(b \otimes e^{\mathrm{op}}) \cdot \ell = b\,\ell\,e$, and second by $(b \otimes e^{\mathrm{op}}) \cdot \ell = \alpha(b)\,\ell\,e$. Write L' for L with the second T-module action, while L unadorned denotes L with the first T-action. By Prop. 1.3, $L \cong_g s_{\delta_1}(N) \oplus \ldots \oplus s_{\delta_n}(N)$ and $L' \cong_g s_{\varepsilon_1}(N) \oplus \ldots \oplus s_{\varepsilon_m}(N)$ as graded T-modules, and $m = n$ by dimension count. Since each $s_\delta(N) = N$ as T-modules when we ignore the grading, we have $L' \cong L$ as ungraded T-modules. The ungraded argument as in [R] then shows there is $a \in A^*$ with

$$ab = \alpha(b)a \tag{1.11}$$

for all $b \in B$. Thus, $\alpha(b) = aba^{-1}$, for all $b \in B$, proving part (a). We proceed to the proof of (c). Let $c = a^{-1}$, and let $a = \sum a_\gamma$ and $c = \sum c_\delta$ be the homogeneous decompositions of a and c. Because α is a graded homomorphism, for each $b \in B^h$, formula (1.11) yields by comparing homogeneous components

$$a_\gamma b = \alpha(b) a_\gamma, \tag{1.12}$$

for each a_γ. Hence, (1.12) holds for all $b \in B$. Likewise, since $bc = c\alpha(b)$ for all $b \in B$, we find that $bc_\delta = c_\delta \alpha(b)$ for all c_δ. These equations show $c_\delta a_\gamma \in C$ for all a_γ and c_δ. So, the equation $1 = ca = \sum\limits_\delta \sum\limits_\gamma c_\delta a_\gamma$ has all its summands in C^h. Therefore, there must be a nonzero summand $c_\delta a_\gamma \in C_0$. When C_0 is a division ring, $c_\delta a_\gamma \in C_0^*$, so $a_\gamma \in A^*$. Then (1.12) shows $\alpha(b) = a_\gamma b a_\gamma^{-1}$ for all $b \in B$, proving (c).

For part (b), observe first that if the a of part (a) is homogeneous, then a^{-1} is also homogeneous, so conjugation by a is a graded automorphism of A. Since this map sends B to B', it also sends C to C'. Hence, we can take γ to be the restriction to C of conjugation by a. Conversely, suppose there is $\gamma \colon C \to C'$ as in (b). Let $Y = C_A(Z)$ and $Y' = C_A(Z')$. Then, as $Y = BC \cong_g B \otimes_Z C$ by Prop. 1.5(c), we obtain a graded R-algebra isomorphism $\beta \colon Y \to Y'$ as the composition

$$Y \cong_g B \otimes_Z C \xrightarrow{\alpha \otimes \gamma} B' \otimes_Z C' \cong_g Y'.$$

Now, Y is graded simple and $C_A(Y) = Z$ by Prop. 1.5, and Z is a graded field. Therefore, we may apply parts (a) and (c) with Y, γ replacing B, α, to see that there is a homogeneous unit $a' \in A$ such that $\beta(y) = a' y a'^{-1}$ for all $y \in Y$. Since $\beta|_B = \alpha$, we can use a' for the a of part (a) for B, α, as desired.

The final assertion of the proposition follows by taking $B = A$ (so $C = R$) and invoking (a) and (c) (or (b)). $\square$

Remark 1.7. Note that for the C of Prop. 1.6, C_0 is a division ring iff C is a division ring, by Prop. 1.4(b).

Example 1.8. Let R be a graded field with $\Gamma_R = \mathbb{Z}$. Let $A = M_4(R)(0, 0, \frac{1}{2}, \frac{1}{2})$, $B_1 = C_1 = M_2(R)(0, \frac{1}{2})$, and $C_2 = M_2(R)(0, 0)$. Then, by (1.5), (1.3), (1.4) above,

$$B_1 \otimes_R C_1 \cong_g A \cong_g B_1 \otimes_R C_2.$$

Let B be the copy of B_1 in A given by the first graded isomorphism, and B' the copy of B_1 in A given by the second. Then $C_A(B) \cong_g C_1$ by (1.9), and $C_A(B') \cong_g C_2$. However, $C_1 \not\cong_g C_2$, e.g. since $\Gamma_{C_1} = \frac{1}{2}\mathbb{Z}$ while $\Gamma_{C_2} = \mathbb{Z}$. Thus, Prop. 1.6(a) and (b) show that although B' is obtainable from B by conjugating by some $a \in A^*$, there is no homogeneous such a. Furthermore, a graded R-isomorphism $B \to B'$ cannot be extended to a graded R-automorphism of A.

§2 Valuation-like properties of graded division algebras

Let $R = \bigoplus_{\gamma \in \Gamma_R} R_\gamma$ be a graded field (with Γ_R torsion-free, as we are always assuming), and let E be a GCDA over R. In this section we will describe some properties of E which are analogous to known properties for tame division algebras over a Henselian valued field. We will use them in §3 in proving a cohomological characterization of $GBr(R)$, see Prop. 3.3 below.

Before considering an arbitrary GCDA over R, we note a couple of extreme cases. First, a GCDA I over R is said to be *unramified* (or inertial) if $\Gamma_I = \Gamma_R$ (iff by (1.7), $[I : R] = [I_0 : R_0]$). In this situation the graded homomorphism $I_0 \otimes_{R_0} R \to I$ is actually an isomorphism, since it is clearly surjective, and a dimension comparison then shows it is also injective. Since $Z(I_0 \otimes_{R_0} R) = Z(I_0) \otimes_{R_0} R$, it follows that $Z(I_0) = R_0$. Thus, there is a one-to-one correspondence ($I \leftrightarrow I_0$) between isomorphism classes of unramified GCDA's over R and isomorphism classes of central division algebras (CDA's) over R_0. Also, if S is a graded R-subalgebra of I, then $\Gamma_S = \Gamma_R$, so $S = S_0 \otimes_R R$. Thus, graded R-subalgebras of I are in canonical one-to-one correspondence (not just up to isomorphism) with R_0-subalgebras of I_0.

At the other extreme, a GCDA T over R is said to be *totally ramified* if $T_0 = R_0$ (iff $|\Gamma_T : \Gamma_R| = [T : R]$, by (1.7)). In this case, there is a pairing $\gamma_T \colon T^* \times T^* \to R_0^*$ given by

$(s,t) \mapsto [s,t]$ $(= sts^{-1}t^{-1})$. (Recall that $T^* = T^h - \{0\}$.) The pairing is clearly skew-symmetric, and since the image of γ_T is central, the commutator identity $[s,tu] = [s,t]t[s,u]t^{-1}$ shows that γ_T is bimultiplicative as well. Because $\gamma_T(s,t) = 1$ if s or t is central, the pairing is actually well-defined on $T^*/R^* \times T^*/R^*$. But, as T is totally ramified $T^*/R^* \cong \Gamma_T/\Gamma_R$ which is finite; so every element of $im(\gamma_T)$ has finite order in R_0^*. Thus, γ_T induces a well-defined biadditive skew-symmetric pairing called the *canonical pairing*

$$\beta_T\colon \Gamma_T/\Gamma_R \times \Gamma_T/\Gamma_R \to \mu(R_0), \tag{2.1}$$

given by $(deg(s) + \Gamma_R, deg(t) + \Gamma_R) \mapsto sts^{-1}t^{-1}$, where $\mu(R_0)$ denotes the group of all roots of unity in the field R_0. (We will use the further notation: If F is a field and ℓ a positive integer, then $\mu_\ell(F)$ is the group of all ℓ-th roots of unity in F.)

Proposition 2.1 (cf. [B$_2$, Prop. 2.6]). *Let T be a totally ramified GCDA over R. Then the canonical pairing β_T of (2.1) is nondegenerate. The image of β_T is a cyclic subgroup of $\mu(R_0)$ of order equal to the exponent $exp(\Gamma_T/\Gamma_R)$. Hence, $char(R_0) \nmid |\Gamma_T/\Gamma_R|$.*

Proof. For $s \in T^*$, if $\gamma_T(s,t) = 1$ for all $t \in T^*$, then $s \in Z(T) = R$. This shows β_T is nondegenerate. Let $\ell = exp(\Gamma_T/\Gamma_R)$, i.e., the exponent of the finite abelian group Γ_T/Γ_R; then $im(\beta_T) \subseteq \mu_\ell(R_0)$ as β_T is biadditive. But, if we take any $\overline{\alpha} \in \Gamma_T/\Gamma_R$ of order ℓ, then the nondegeneracy of β_T forces the homomorphism $\beta_T(\overline{\alpha}, -)\colon \Gamma_T/\Gamma_R \to \mu_\ell(R_0)$ to be surjective, and forces $|\mu_\ell(R_0)| = \ell$. Hence, $im(\beta_T) = \mu_\ell(R_0)$, and $|im(\beta_T)| = \ell = exp(\Gamma_T/\Gamma_R)$. If $p = char(R_0)$, then as $\mu(R_0)$ has no p-torsion, we must have $p \nmid \ell$. $\square$

Remarks 2.2. (i) If Λ is any group with $\Gamma_R \subseteq \Lambda \subseteq \Gamma_T$, then $T_\Lambda = \bigoplus_{\lambda\in\Lambda} T_\lambda$ is a graded R-subalgebra of T with $\Gamma_{T_\Lambda} = \Lambda$; furthermore, since $T_0 = R_0$, every R-subalgebra of T has the form T_Λ for some Λ. Thus, the subgroups of Γ_T/Γ_R classify the R-subalgebras of T. Note also that $Z(T_\Lambda) = T_{\Lambda'}$, where $\Lambda'/\Gamma_R = \Lambda/\Gamma_R \cap (\Lambda/\Gamma_R)^\perp$, where $(\Lambda/\Gamma_R)^\perp$ denotes the orthogonal subgroup to Λ/Γ_R in Γ_T/Γ_R with respect to β_T.

(ii) Relative to the skew-symmetric nondegenerate biadditive pairing β_T there always exists a symplectic base for Γ_T/Γ_R (cf. [TW, Prop. 3.1]). This implies that the distinct invariant factors of the finite abelian group Γ_T/Γ_R each occur with even multiplicity. Also, the symplectic base allows one to decompose T into a tensor product of graded symbol algebras over R, analogous to the decomposition for tame totally ramified valued division algebras described in [TW, Prop. 4.2]. Furthermore, one can use the symplectic base to see that if $R_0^\ell = R_0$ where $\ell = exp(\Gamma_T/\Gamma_R)$, then T is determined up to isomorphism by Γ_T and β_T. Also, one can easily see that for any group Γ with $\Gamma_R \subseteq \Gamma \subseteq \Delta_R$, such that Γ/Γ_R is finite and all the distinct invariant factors of Γ/Γ_R occur with even multiplicity and $|\mu_\ell(R_0)| = \ell$, where $\ell = exp(\Gamma/\Gamma_R)$, there exists a nondegenerate skew-symmetric biadditive pairing $\beta\colon \Gamma/\Gamma_R \times \Gamma/\Gamma_R \to \mu_\ell(R_0)$; for any such Γ and β one can use a symplectic base for β on Γ/Γ_R to construct a totally ramified graded division algebra T over R such that $\Gamma_T = \Gamma$ and $\beta_T = \beta$.

Now, let E be any GCDA over the graded field R. Observe that there is a well-defined group homomorphism

$$\theta_E : \Gamma_E/\Gamma_R \to \mathcal{G}(Z(E_0)/R_0) \quad \text{given by} \quad deg(e) + \Gamma_R \mapsto (z \mapsto eze^{-1}), \tag{2.2}$$

for all $e \in E^*$ and $z \in Z(E_0)$, where $\mathcal{G}(Z(E_0)/R_0)$ denotes the Galois group of $Z(E_0)$ over R_0.

There are some graded R-subalgebras of E canonically determined by E_0: Let

$$\begin{aligned} Z &= Z(E_0) \cdot R \cong_g Z(E_0) \otimes_{R_0} R, \\ C &= C_E(Z), \\ I &= E_0 \cdot R \cong_g E_0 \otimes_{R_0} R, \\ T &= C_C(I) = C_E(I) = C_E(E_0). \end{aligned}$$

$$\begin{tikzcd}[column sep=small] & E \arrow[d, dash] & \\ & C = I \otimes_Z T \arrow[dl, dash] \arrow[dr, dash] & \\ I \arrow[dr, dash] & & T \arrow[dl, dash] \\ & Z \arrow[d, dash] & \\ & R & \end{tikzcd} \tag{2.3}$$

Diagram (2.3) shows the inclusion relations among these algebras. Note that Z is a graded field and C, I, and T are graded division algebras. Clearly, $\Gamma_Z = \Gamma_R$ and $Z_0 = Z(E_0)$. The double centralizer theorem, Prop. 1.5, shows that $Z = Z(C)$, and $[E : C] = [Z : R]$. Since $C = C_E(Z) = C_E(Z_0)$, the definition of θ_E shows that $\Gamma_C/\Gamma_R = ker(\theta_E)$; also clearly $C_0 = E_0$. As for I, we have $I_0 = E_0$ and $\Gamma_I = \Gamma_R$. Also, $Z(I) = Z(E_0 \otimes_{R_0} R) = Z(E_0) \otimes_{R_0} R = Z$, so I is unramified over its center Z. Turning to T, the double centralizer theorem shows $Z(T) = Z$ and $C \cong_g I \otimes_Z T$. Also, $T_0 = Z_0$ as T_0 centralizes I_0. Hence, T is totally ramified over its center Z. A dimension count using (1.7) shows $\Gamma_T/\Gamma_R = \Gamma_C/\Gamma_R = ker(\theta_E)$.

algebra	grade group $/\Gamma_R$	degree 0 component
E	Γ_E/Γ_R	E_0
C	$ker(\theta_E)$	E_0
T	$ker(\theta_E)$	$Z(E_0)$
I	(0)	E_0
Z	(0)	$Z(E_0)$
R	(0)	R_0

Proposition 2.3 (cf. [B_2, Prop. 2.4]). *For any GCDA E over a graded field R, the field $Z(E_0)$ is Galois over R_0 and the homomorphism θ_E of (2.2) maps Γ_E/Γ_R onto the Galois group $\mathcal{G}(Z(E_0)/R_0)$, so $\mathcal{G}(Z(E_0)/R_0)$ is abelian. Also, $char(R_0) \nmid |ker(\theta_E)|$.*

Proof. We give a different proof from the one in [B_2]. We use the information accumulated above about Z, C, I, T. Observe that

$$\begin{aligned} \big|\mathcal{G}(Z(E_0)/R_0)\big| \le [Z(E_0) : R_0] = [Z : R] = [E : C] = |\Gamma_E : \Gamma_C| \\ = \big|\Gamma_E/\Gamma_R : ker(\theta_E)\big| = |im(\theta_E)| \le \big|\mathcal{G}(Z(E_0)/R_0)\big|. \end{aligned}$$

Hence, equality holds throughout. This shows that θ_E is surjective and that $\left|\mathcal{G}(Z(E_0)/R_0)\right| = [Z(E_0) : R_0]$, hence $Z(E_0)$ is Galois over R_0. Because T is totally ramified, Prop. 2.1 shows that $char(R_0) = char(Z_0) \nmid |\Gamma_T/\Gamma_Z| = |ker(\theta_E)|$. $\square$

Remarks 2.4. (i) In [B₂, p. 4279] Boulagouaz defines a canonical pairing $C_E\colon ker(\theta_E) \times ker(\theta_E) \to \mu(Z(E_0))$. This pairing is just the pairing $\beta_T\colon \Gamma_T/\Gamma_Z \times \Gamma_T/\Gamma_Z \to \mu(Z_0)$ of (2.1) for the totally ramified T in E shown in (2.3). This pairing is canonically determined by E, since T is built canonically from E.

(ii) The graded subalgebras Z, C, I, T of E described here are analogous to valued subalgebras of a division algebra tame over a Henselian valued field, cf. [JW, §§1–2]. But notice that the subalgebras of E defined here are unique, not just unique up to isomorphism (as in the valued situation). Also the existence and properties of the subalgebras are easier to prove in the graded case than in the corresponding valued case.

(iii) There is a slight variation of the map θ_E of (2.2), which will appear in §4: If E is a graded division algebra over a graded field S with $[E : S] < \infty$ (so $S \subseteq Z(E)$, but possibly $S \neq Z(E)$), define

$$\theta_{E,S}\colon \Gamma_E/\Gamma_S \to \mathcal{G}(Z(E_0)/S_0) \quad \text{by} \quad deg(e) + \Gamma_S \mapsto (z \mapsto eze^{-1}) \tag{2.4}$$

for $e \in E^*$ and $z \in Z(E_0)$. This map is clearly well-defined.

§3 The graded Brauer group of a graded field

We can now consider the graded Brauer group of a graded field R. Define an equivalence relation $\sim_g$ on GCSA's over R by: $A \sim_g B$ if there are finitely-generated (hence graded free) graded R-modules M and N such that $A \otimes_R End_R(N) \cong_g B \otimes_R End_R(M)$ as graded R-algebras. So, $\sim_g$ is clearly an equivalence relation which is compatible with tensor products. Let $[A]_g$ denote the equivalence class of A with respect to $\sim_g$. Then, the graded Brauer group of R is defined to be

$$GBr(R) = \{[A]_g \mid A \text{ is a GCSA over } R\}.$$

(See [B₂, §5]; see also [CvO, III.4–IV.1] for the case $\Gamma_R = \mathbb{Z}$, but note that our $GBr(R)$ is their $UBr_g(R)$, see [CvO, p. 139], since we allow $\Gamma_A \supsetneqq \Gamma_R$.) The operation on $GBr(R)$ is induced by the tensor product, and, as we noted earlier for Cor. 1.2, $A \otimes_R A^{\mathrm{op}} \cong_g End_R(A)$. Thus, $GBr(R)$ is a group with identity element $[R]_g$ and $[A]_g^{-1} = [A^{\mathrm{op}}]_g$. Now, if E is any GCDA over R, L is any finitely-generated graded right E-vector space, and N is any finitely-generated graded R-vector space, then $End_E(N \otimes_R L) \cong_g End_R(N) \otimes_R End_E(L)$. It follows from this and the graded Wedderburn theorem, Prop. 1.3, that for GCSA's $A \cong_g M_n(E)(d)$ and $A' \cong_g M_{n'}(E')(d')$ with E, E' GCDA's over R, we have $A \sim_g A'$ iff $E \cong_g E'$. Thus, $GBr(R)$ classifies GCDA's over R up to graded isomorphism. Note that, unlike the case of central simple algebras over a field, we can have GCSA's A, B over R with $[A]_g = [B]_g$ and $[A:R] = [B:R]$, but $A \not\cong_g B$. This occurs when $A \cong_g M_n(E)(d)$ and $B \cong_g M_n(E)(d')$ with d and d' sufficiently different.

As we noted above for graded division algebras, the assumption that Γ_R is torsion-free implies that a graded field R is an integral domain. Let

$$QR = \text{the quotient field of } R.$$

Likewise, for any graded R-algebra B, let

$$QB = QR \otimes_R B,$$

an algebra over the field QR. Observe that as B is graded-free as a graded R-module, B is R-torsion-free, so the canonical map $B \to QB$ is injective; also

$$[QB : QR] = [B : R]. \tag{3.1}$$

Note in particular that if B is a graded division algebra over R with $[B:R] < \infty$, then since B has no zero divisors the same is true for QB; since also $[QB : QR] < \infty$ it follows that QB is a division ring.

Now, suppose A is a GCSA over R. Then as A is an Azumaya algebra over R (see Cor. 1.2), A determines a class $[A]$ in the (ungraded) Brauer group $Br(R)$; also QA is Azumaya over the field QR, i.e., QA is a central simple algebra over QR. Indeed QA is the classical ring of quotients of the prime p.i. ring A. There are canonical group homomorphisms

$$GBr(R) \to Br(R) \to Br(QR), \tag{3.2}$$

given by $[A]_g \mapsto [A]$ and $[C] \mapsto [QR \otimes_R C]$, and the composition is injective since if E is a GCDA over R, then QE is a CDA over QR of the same degree as E over R. So $GBr(R)$ injects into $Br(R)$. In general, $Br(R)$ and $Br(QR)$ may be much larger than $GBr(R)$(but not always, see [CvO, Th. IV.1.11, p. 139]). We will see below that if we give a total order to Γ_R, then there is a valued field extension of QR whose tame Brauer group coincides with $GBr(R)$.

From Prop. 1.1 it is clear that for any graded field extension S of R, there is a well-defined scalar extension group homomorphism

$$res_{S/R} \colon GBr(R) \to GBr(S) \quad \text{given by} \quad [A]_g \mapsto [S \otimes_R A]_g.$$

We have therefore a corresponding relative graded Brauer group

$$GBr(S/R) = ker(res_{S/R}).$$

We will give in Prop. 3.2 below a cohomological description of $GBr(S/R)$ when S is Galois over R, by adapting the usual crossed product construction.

Let $R \subseteq S$ be graded fields with $[S:R] < \infty$. Then QS $(= QR \otimes_R S$, as above) is the quotient field of S, and $[QS : QR] = [S : R]$. Recall from [HW, Th. 3.11] (or see [B_1, Th. 4, p. 33]) that S is *tame* over R (i.e., S_0 is separable over R_0 and $char(R_0) \nmid |\Gamma_S : \Gamma_R|$) iff QS is separable over QR, iff S is separable over R. Furthermore, every QR-automorphism of QS restricts to a graded R-automorphism of S (since S is the integral closure of R in QS and for every $s \in S^h$, $deg(s)$ is determined by its minimal polynomial over QR, cf. [HW, Cor. 2.5 (a), (d)]). Indeed, by [HW, Th. 3.11(b)], S is $\mathcal{G}$-Galois over R for some group $\mathcal{G}$ iff QS is Galois over QR, and when this occurs, $\mathcal{G}$ is canonically isomorphic to the Galois group $\mathcal{G}(QS/QR)$. There is thus no ambiguity in writing $\mathcal{G} = \mathcal{G}(S/R)$. Furthermore, the preceding comments show that every element of $\mathcal{G}(S/R)$ is a graded automorphism of S.

The results in [HW] quoted in the preceding paragraph were stated there with the added assumption that Γ_R is totally ordered. However, they are valid when one only assumes that Γ_R is torsion-free (our standing hypothesis here), since any torsion-free abelian group can be given a total ordering, and the quoted results are independent of the choice of total ordering on Γ_R. This remark applies a number of times below when we quote [HW].

We next construct graded crossed products. Assume the graded field S is Galois over R, and let $\mathcal{G} = \mathcal{G}(S/R)$. Then $S^*(= S^h - \{0\})$ is a $\mathcal{G}$-submodule of QS^*. We write $Z^i(\mathcal{G}, S^*)$, $B^i(\mathcal{G}, S^*)$, $H^i(\mathcal{G}, S^*)$ for the i-th cocycle group, i-th coboundary group, i-th cohomology group of $\mathcal{G}$ with coefficients in S^*. Take any $f \in Z^2(\mathcal{G}, S^*)$. We construct the crossed product algebra $(S/R, \mathcal{G}, f)$ in the usual way: Let $\{x_\sigma \mid \sigma \in \mathcal{G}\}$ be new symbols, and on the free S-module $\bigoplus_{\sigma \in \mathcal{G}} Sx_\sigma$, with base $\{x_\sigma \mid \sigma \in \mathcal{G}\}$, define multiplication by

$$(ax_\sigma)(bx_\tau) = a\sigma(b)f(\sigma,\tau)x_{\sigma\tau}, \tag{3.3}$$

for all $a, b \in S$, $\sigma, \tau \in \mathcal{G}$ (and extended distributively to all of $\bigoplus_\sigma Sx_\sigma$). It is well-known that $(S/R, \mathcal{G}, f)$ is an associative R-algebra since f is a 2-cocycle, and that it is an Azumaya algebra over R, since all $f(\sigma,\tau) \in S^*$. We now make it into a graded R-algebra.

Lemma 3.1. *There is a unique way of assigning degrees to the x_σ so that $(S/R, \mathcal{G}, f)$ is a graded R-algebra with grading extending the grading on S, such that the x_σ are all homogeneous. With this grading, $(S/R, \mathcal{G}, f)$ is a GCSA over R.*

Proof. Since $x_\sigma x_\tau = f(\sigma,\tau)x_{\sigma\tau}$, we need to assign degrees to the x_σ so that

$$deg(x_\sigma) + deg(x_\tau) = deg(x_{\sigma\tau}) + deg(f(\sigma,\tau)), \tag{3.4}$$

for all $\sigma, \tau \in \mathcal{G}$. Once this is done, define, for any $a \in S^h$, $deg(ax_\sigma) = deg(a) + deg(x_\sigma)$. Then formula (3.4) assures that the multiplication on $(S/R, \mathcal{G}, f)$ given in (3.3) is compatible with this assignment of degrees. We obtain a grading

$$(S/R, \mathcal{G}, f) = \bigoplus_{\delta \in \Delta_R} (S/R, \mathcal{G}, f)_\delta, \quad \text{where} \quad (S/R, \mathcal{G}, f)_\delta = \bigoplus_{\sigma \in \mathcal{G}} S_{\delta - deg(x_\sigma)} x_\sigma,$$

where $\Delta_R = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_R$. This makes $(S/R, \mathcal{G}, f)$ into a graded R-algebra. By arguing as in the ungraded case (or invoking Cor. 1.2, since the crossed product algebra is an Azumaya algebra over R), we see that A is a GCSA over R.

To find degrees for the x_σ satisfying (3.4), note that the degree map $deg\colon S^* \to \Delta_R$ is a $\mathcal{G}$-module homomorphism, with $\mathcal{G}$ acting trivially on Δ_R; there is an induced map $deg^*\colon Z^2(\mathcal{G}, S^*) \to Z^2(\mathcal{G}, \Delta_R)$. But $Z^2(\mathcal{G}, \Delta_R) = B^2(\mathcal{G}, \Delta_R)$, since $H^2(\mathcal{G}, \Delta_R) = 0$, as the $\mathbb{Q}$-vector space Δ_R is uniquely divisible. So, since $deg^*(f) \in B^2(\mathcal{G}, \Delta_R)$, there exists $\{b_\sigma \mid \sigma \in \mathcal{G}\} \subseteq \Delta_R$, such that

$$deg(f(\sigma, \tau)) = b_\sigma + b_\tau - b_{\sigma\tau}, \tag{3.5}$$

for all $\sigma, \tau \in \mathcal{G}$. Then, define $deg(x_\sigma) = b_\sigma$, and (3.4) holds, as desired. Note also that if we have another set $\{b'_\sigma \mid \sigma \in \mathcal{G}\} \subseteq \Delta_R$ satisfying $deg(f(\sigma,\tau)) = b'_\sigma + b'_\tau - b'_{\sigma\tau}$, then the map $\sigma \mapsto (b_\sigma - b'_\sigma)$ is a group homomorphism from the finite group $\mathcal{G}$ to the torsion-free group Δ_R; therefore this homomorphism must be trivial (i.e., $H^1(\mathcal{G}, \Delta_R) = 0$). So, the b_σ satisfying (3.5) are uniquely determined; hence, there is only one way to define $deg(x_\sigma)$ so that (3.4) holds. This gives the uniqueness asserted in the lemma. The values of $deg(x_\sigma)$ are given explicitly by the formula

$$deg(x_\sigma) = \frac{1}{|\mathcal{G}|} \sum_{\tau \in \mathcal{G}} deg(f(\sigma, \tau)) \in \Delta_R.$$

□

We call $(S/R, \mathcal{G}, f)$ with the grading of Lemma 3.1 a *graded crossed product algebra.*

Proposition 3.2. *Let S be a Galois graded field extension of a graded field R (with $[S:R] < \infty$), and let $\mathcal{G} = \mathcal{G}(S/R)$. Then,*

$$GBr(S/R) \cong H^2(\mathcal{G}, S^*).$$

Proof. Define a map $\psi\colon Z^2(\mathcal{G}, S^*) \to GBr(R)$ by $f \mapsto [(S/R, \mathcal{G}, f)]_g$, where the crossed product is given the grading of Lemma 3.1. We will show that ψ is a group homomorphism with $ker(\psi) = B^2(\mathcal{G}, S^*)$ and $im(\psi) = GBr(S/R)$. This will yield the desired isomorphism.

The following diagram is evidently commutative:

$$\begin{array}{ccccc} Z^2(\mathcal{G}, S^*) & \xrightarrow{\psi} & GBr(R) & \xrightarrow{res_{S/R}} & GBr(S) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(\mathcal{G}, QS^*) & \longrightarrow & Br(QR) & \longrightarrow & Br(QS) \end{array} \tag{3.6}$$

In this diagram, the bottom row is exact, and the middle and right vertical maps are injective, by the comments after (3.2). This shows that ψ is a group homomorphism (since the other maps in the left square are homomorphisms), and also $im(\psi) \subseteq GBr(S/R)$ and $B^2(\mathcal{G}, S^*) \subseteq ker(\psi)$.

To show that this last inclusion is an equality, take any $f \in ker(\psi)$. We may assume, after modifying f by a coboundary, that f is normalized. Hence, in $(S/R, \mathcal{G}, x_\sigma) = \bigoplus_{\sigma \in \mathcal{G}} Sx_\sigma$, x_e is the

1 ($e = id_S$ = identity element of $\mathcal{G}$) and the mapping $S \to (S/R, \mathcal{G}, f)$ given by $s \mapsto sx_e$ is a graded R-algebra monomorphism, so we identify S with its gr-isomorphic copy Sx_e in the crossed product. Let $n = [S : R] = |\mathcal{G}|$. Since $f \in ker(\psi)$, by Prop. 1.3 there is a graded R-algebra isomorphism $\alpha\colon (S/R, \mathcal{G}, f) \to End_R(M)$ for some graded R-vector space M. By dimension count, $dim_R(M) = n$. The copy of S in $(S/R, \mathcal{G}, f)$ acts on M by $s \cdot m = \alpha(s)(m)$. This action makes M into a graded S-vector space, necessarily of dimension 1. So $M = S \cdot m$ for any nonzero $m \in M^h$. Hence, as $End_R(M) = End_R(S \cdot m) \cong_g End_R(S)$, we may identify $(S/R, \mathcal{G}, f)$ with $End_R(S)$ so that $s \in S$ corresponds to left multiplication by s. For each $\sigma \in \mathcal{G} \subseteq End_R(S)_0$, let $s_\sigma = x_\sigma \sigma^{-1} \in End_R(S)^h$. Then, as $x_\sigma t x_\sigma^{-1} = \sigma(t) = \sigma \circ t \circ \sigma^{-1} \in End_R(S)$, for all $t \in S$, we have $s_\sigma \in C_{End_R(S)}(S)^h = S^h$. Since $s_\sigma \neq 0$, $s_\sigma \in S^*$. Furthermore, from $x_\sigma x_\tau = f(\sigma,\tau)x_{\sigma\tau}$, we obtain $s_\sigma \sigma(s_\tau)s_{\sigma\tau}^{-1} = f(\sigma,\tau)$, for all $\sigma, \tau \in \mathcal{G}$, proving that $f \in B^2(\mathcal{G}, S^*)$, as desired.

Finally, to see that $im(\psi) = GBr(S/R)$, take any GCSA A over R with $[A]_g \in GBr(S/R)$. Let $A' = A \otimes_R End_R(S)$, which is also a GCSA over R. After identifying S with its gr-isomorphic copy in $End_R(S)$, we have $A \otimes_R S$ is a graded R-subalgebra of A'. Since S splits A, we have $A \otimes_R S \cong_g M_m(S)(d)$ for some $d = (\delta_1, \ldots, \delta_m)$, where $m = \sqrt{[A:R]}$. Let B be a graded R-subalgebra of $A \otimes_R S$ such that $B \cong_g M_m(R)(d)$; let $C = C_{A'}(B)$. From $B \subseteq A \otimes_R S \subseteq A'$ we have, using (1.9), $C \supseteq C_{A'}(A \otimes_R S) = R \otimes_R C_{End_R(S)}(S) = R \otimes_R S \cong_g S$. Thus, we may view S as a graded R-subalgebra of C. The double centralizer theorem, Prop. 1.5, shows that C is a GCSA over R with $[C : R] = [A' : R]/[B : R] = [End_R(S) : R] = [S : R]^2$; hence, again by the double centralizer theorem, $C_C(S) = S$. Note also that in $GBr(R)$, $[A]_g = [A']_g = [B]_g + [C]_g = [C]_g$. Now, for each $\sigma \in \mathcal{G}$, the graded Skolem-Noether theorem, Prop. 1.6, shows that the graded R-algebra isomorphism $\sigma\colon S \to S$ is induced by conjugation by some $x_\sigma \in C^*$. Moreover, we may assume $x_\sigma \in C^h$ by Prop. 1.6(c), as $C_C(S)_0 = S_0$, which is a field. Set $f(\sigma,\tau) = x_\sigma x_\tau x_{\sigma\tau}^{-1} \in C_C(S)^h \cap C^* \subseteq S^h - \{0\} = S^*$. Then $f \in Z^2(\mathcal{G}, S^*)$. Also, the usual calculation shows that the sum $\sum\limits_{\sigma \in \mathcal{G}} Sx_\sigma$ in C is a direct sum; hence it is all of C, by dimension count. The multiplication in $\bigoplus\limits_{\sigma\in\mathcal{G}} Sx_\sigma$ is the same as that of $(S/R, \mathcal{G}, f)$, given in (3.3); so Lemma 3.1 shows $(S/R, \mathcal{G}, f) \cong_g C$. Thus, $\psi(f) = [C]_g = [A]_g$ in $GBr(S/R)$, completing the proof. $\square$

The description of $GBr(S/R)$ given in Prop. 3.2 leads to a corresponding cohomological description of all of $GBr(R)$. Recall from [HW, Prop. 3.7] that there is a maximal tame graded field extension Y of a graded field R. This Y is graded algebraic over R (though typically $[Y : R] = \infty$) and it contains a graded isomorphic copy of every tame graded field extension. We have that Y_0 is the separable closure of R_0 and Γ_Y/Γ_R is the prime-to-p primary component of the torsion group Δ_R/Γ_R, where $p = char(R_0)$. Moreover, QY ($= QR \otimes_R Y$ = quotient field of Y) is Galois over QR, and $\mathcal{G}(QY/QR)$ maps bijectively (by restriction to Y) to the group $\mathcal{G}(Y/R)$ of all R-algebra automorphisms of Y, and every such automorphism preserves the grading on Y. Therefore, $\mathcal{G}(Y/R)$ inherits from $\mathcal{G}(QY/QR)$ the structure of a profinite group, in which the closed normal subgroups of finite index correspond one-to-one to the finite-degree Galois graded field extensions of R in Y (cf. [HW, Th. 3.9, Th. 3.11]).

Proposition 3.3. *Let R be a graded field, and let Y be the maximal tame graded field extension of R. Then,*

$$GBr(R) \cong H^2(\mathcal{G}(Y/R), Y^*).$$

PROOF. Let $\mathcal{G} = \mathcal{G}(Y/R)$. Here $H^2(\mathcal{G}, Y^*)$ denotes the continuous cohomology group with respect to the discrete $\mathcal{G}$-module Y^*. Because $\mathcal{G} = \varprojlim \mathcal{G}(S/R)$ as S ranges over the finite degree Galois graded field extensions S of R in Y, we have, in light of Prop. 3.2,

$$GBr(Y/R) = \bigcup_S GBr(S/R) \cong \varinjlim_S H^2(\mathcal{G}(S/R), S^*) \cong H^2(\mathcal{G}, Y^*).$$

Thus, it remains only to prove that $GBr(R) = GBr(Y/R)$, which we do by showing that $GBr(Y) = 0$. For this, let E be any GCDA over Y, and form its graded Y-subalgebras Z, C, I, and T as in (2.3). Since Y_0 is separably closed and Z_0 is Galois over Y_0 by Prop. 2.3, we have $Z_0 = Y_0$, so $Z = Y$ and $C = E$. Moreover, as $Br(Y_0) = 0$, we have $E_0 = Z_0 = Y_0$, so $I = Y$, hence $E = T$, which is totally ramified over Y. Since $[E : Y] = [T : Y] = |\Gamma_T : \Gamma_Y|$ which is prime to $char(Y_0)$ by Prop. 2.1, and Δ_R/Γ_Y is $char(Y_0)$-primary, we must have $E = Y$. Hence, $GBr(Y) = 0$, as asserted. $\square$

§4 Valuations from gradings, and vice versa

We now consider the valuation which arises when the grade group of a graded field R is given a total ordering. Since Γ_R is assumed torsion-free there always exists a total ordering on Γ_R compatible with its group operation. There are typically many such total orderings. Fix one such on Γ_R, and denote it $\leq$. Then, for any torsion free abelian group Λ containing Γ_R as a subgroup such that Λ/Γ_R is torsion, there is a unique extension of $\leq$ to a total ordering on Λ.

Let E be any GCDA over R. So, the fixed total ordering on Γ_R extends uniquely to a total ordering on Γ_E, again denoted $\leq$. A key observation in [B_2] (see also [HvO, Prop. 3.1] when $\Gamma_E \cong \mathbb{Z}$) is that the ordering on Γ_E induces a valuation on QE: One first defines $v\colon E - \{0\} \to \Gamma_E$ by, for $a = \sum_{\gamma \in \Gamma_E} a_\gamma$

$$v(a) = min\{\gamma \in \Gamma_E \mid a_\gamma \neq 0\}.$$

Clearly, for all $a, b \in E - \{0\}$ we have (i) $v(ab) = v(a) + v(b)$ as Γ_E is totally ordered, and (ii) $v(a+b) \geq min(v(a), v(b))$ (if $b \neq -a$). The function v has an extension to QE^*, also denoted $v\colon QE^* \to \Gamma_E$ given by $v(ab^{-1}) = v(a) - v(b)$. (This is well-defined by property (i).) Then, properties (i) and (ii) hold for all $a, b \in QE^*$. (This is very easy to verify, since every element a of QE is expressible as $a = er^{-1}$ with $e \in E$ and $r \in R - \{0\}$.) So, v is a valuation on QE. Clearly, for the value group of v on QE, denoted Γ_{QE}, we have

$$\Gamma_{QE} = \Gamma_E. \tag{4.1}$$

Also, for the residue division ring, denoted $\overline{QE}$, of the valuation ring of v on QE, we have

$$\overline{QE} \cong E_0. \tag{4.2}$$

(For, if $c \in QE^*$ with $v(c) = 0$, then $c = ab^{-1}$ with $a = \sum_{\gamma \in \Gamma_E} a_\gamma$, $b = \sum_{\gamma \in \Gamma_E} b_\gamma \in E - \{0\}$ with $v(a) = v(b) = \delta$, say. Then c has the same image in $\overline{QE}$ as $a_\delta b_\delta^{-1} \in E_0$.) The valuation on QE restricts to a valuation on its center QR, which clearly coincides with the valuation determined by the grading on R.

The properties of graded division algebras correspond most closely to those of valued vision over a Henselian field, as we will see. We can obtain such a division algebra from the GCDA E over R by Henselizing: Let HR denote "the" Henselization of the field QR with respect to the restriction of v to QR (cf. [E, p. 131, (17.11)]). So, HR is a separable algebraic field extension of QR which is uniquely determined up to isomorphism, and there is a Henselian valuation on HR extending v on QR, with residue field and value group satisfying

$$\overline{HR} \cong \overline{QR} \quad \text{and} \quad \Gamma_{HR} = \Gamma_{QR}. \tag{4.3}$$

Define

$$HE = HR \otimes_{QR} QE = HR \otimes_R E. \tag{4.4}$$

Because QE has a valuation extending v on QR, Morandi's Henselization theorem [M_1, Th. 2] shows that HE is a division ring (with center HR), and its unique valuation extending the Henselian valuation on HR restricts to v on QE. Furthermore, for the residue division algebra $\overline{HE}$ and value group Γ_{HE} of the valuation on HE we have (using (4.2) and (4.1)),

$$\overline{HE} \cong \overline{QE} \cong E_0 \quad \text{and} \quad \Gamma_{HE} = \Gamma_{QE} = \Gamma_E. \tag{4.5}$$

Moreover, by (1.7) together with (4.5) for E and for R, we have,

$$[HE : HR] = [QE : QR] = [E : R] = [\overline{HE} : \overline{HR}] \left|\Gamma_{HE} : \Gamma_{HR}\right| \tag{4.6}$$

so the valuation on HE is defectless over HR (likewise QE is defectless over QR). It follows easily from Prop. 2.3 that HE is also tame over HR (as described in Prop. 4.3 below). We will see in Th. 5.1 below that the map $[E]_g \mapsto [HE]$ is an isomorphism from $GBr(R)$ to the tame part of the Brauer group of HR.

Remark 4.1. If A is a GCSA over the graded field R, then we have corresponding CSA's $QA = QR \otimes_R A$ over QR (so QA is the Artinian ring of quotients of the prime p.i. ring A) and $HA = HR \otimes_R A$ over HR. We can define a valuation-like function on $QA - \{0\}$, but when A is not a graded division algebra, we generally obtain a ring in QA that could not be reasonably called a valuation ring. Specifically, define $w\colon QA - \{0\} \to \Gamma_A$ by first defining $w(a) = min\{\gamma \mid a_\gamma \neq 0\}$ for $a = \sum a_\gamma \in A - \{0\}$, and then for $c = ar^{-1} \in QA - \{0\}$ with $a \in A - \{0\}$ and $r \in R - \{0\}$, defining

$w(c) = w(a) - w(r)$. This is a well-defined function satisfying $w(cd) \geq w(c) + w(d)$ (if $cd \neq 0$) and $w(c+d) \geq min(w(c), w(d))$ (if $d \neq -c$) for all $c, d \in QA - \{0\}$. This function yields a subring V_{QA} of QA given by $V_{QA} = \{a \in QA - \{0\} \mid w(a) \geq 0\} \cup \{0\}$, but V_{QA} need not be a valuation ring, not even in the sense of Dubrovin (see after Th. 5.3 below). For example, suppose F is a field, t an indeterminate over F, and $R = F[t, t^{-1}] = \bigoplus_{i \in \mathbb{Z}} R_i$ where $R_i = t^i F$. So, R is a graded field with $R_0 = F$ and $\Gamma_R = \mathbb{Z}$ with its usual ordering. We have $QR = F(t)$, and the valuation on QR induced by the grading on R is the t-adic valuation ring $V_{QR} = F[t]_{(t)} = \{fg^{-1} \mid f, g \in F[t],\ g(0) \neq 0\}$. Let $M_{QR} = tV_{QR}$, the maximal ideal of V_{QR}, so the residue field is $\overline{QR} = V_{QR}/M_{QR} \cong F$. Now, let $A = M_2(R)(d)$ where $d = (0, \frac{1}{2})$. Then the ring V_{QA} obtained from w on $QA - \{0\}$ is $V_{QA} = \begin{pmatrix} V_{QR} & M_{QR} \\ V_{QR} & V_{QR} \end{pmatrix}$, with Jacobson radical $J(V_{QA}) = \begin{pmatrix} M_{QR} & M_{QR} \\ V_{QR} & M_{QR} \end{pmatrix}$. ($J(V_{QA})$ is also the ideal of elements of positive w "value" in V_{QA}.) So, $V_{QA}/J(V_{QA}) \cong \overline{QR} \oplus \overline{QR}$, which is semisimple, but not simple. So, V_{QA} is not a Dubrovin valuation ring, and is also not a maximal order in QA. In fact, every Dubrovin valuation ring of QA contracting to V_{QR} in QR is isomorphic to $M_2(V_{QR})$.

We now turn to valued division algebras, and the graded division algebras derived from them. This will lead to consideration of the tame part of the Brauer group of a Henselian valued field F, whose central division algebras all have associated graded division algebras with center the graded field GF. For the rest of this section we will take the first steps toward proving an isomorphism between the tame part of the Brauer group of the Henselian field F and the graded Brauer group of GF; the proof of this will finally be completed in §5 (see Th. 5.3).

Let D be a division ring, and suppose there is a valuation $v : D^* \to \Gamma$ on D. That is, Γ is a totally ordered abelian group, and v satisfies, for all $a, b \in D^*$,

(i) $v(ab) = v(a) + v(b)$;

(ii) $v(a + b) \geq min(v(a), v(b))$ if $b \neq -a$.

Let $V_D = \{a \in D^* \mid v(a) \geq 0\} \cup \{0\}$, the valuation ring of v; $M_D = \{a \in D^* \mid v(a) > 0\} \cup \{0\}$ the unique maximal left ideal (and maximal right ideal) of V_D; $U_D = V_D^* = V_D - M_D$; $\overline{D} = V_D/M_D$, the residue division ring of v on D; and $\Gamma_D = im(v)$, the value group. There will be no ambiguity in indexing these objects by D, since we will never consider more than one valuation on a given division ring. Let $\overline{p} = char(\overline{D})$.

The filtration of fractional ideals of V_D determined by v on D yields an associated graded ring GD. Specifically, for $\gamma \in \Gamma_D$, let

$$W^{\gamma} = \{d \in D^* \mid v(d) \geq \gamma\} \cup \{0\} \quad \text{and} \quad W^{>\gamma} = \{d \in D^* \mid v(d) > \gamma\} \cup \{0\}\,;$$

so $W^{>\gamma}$ is a subgroup of the additive group W^{γ}. Then set

$$GD = \bigoplus_{\gamma \in \Gamma_D} GD_{\gamma}, \quad \text{where } GD_{\gamma} = W^{\gamma}/W^{>\gamma}.$$

Because $W^{>\gamma}W^{\delta} + W^{\gamma}W^{>\delta} \subseteq W^{>\gamma+\delta}$, for all $\gamma, \delta \in \Gamma_D$, the multiplication on D induces a well-defined multiplication on GD, making it into a graded ring. Moreover, property (i) of the valuation assures that GD is a graded division ring. Clearly $GD_0 = \overline{D}$ and the grade group is $\Gamma_{GD} = \Gamma_D$. It is the basic theme of this paper that much of the structure of D is well reflected in GD.

Now, let $F = Z(D)$, and suppose $[D : F] < \infty$, so D is a *central division algebra* (CDA) over F. The restriction $v|_F$ of the valuation on D is a valuation on F (with associated structures $V_F, M_F, U_F, \overline{F}, \Gamma_F$), which induces a corresponding graded field GF. Clearly, GD is a graded GF-algebra, with

$$[GD : GF] = [\overline{D} : \overline{F}]|\Gamma_D : \Gamma_F| = [D : F]/\delta_{D/F}, \tag{4.7}$$

by (1.7) above, where $\delta_{D/F}$ is the defect of D over F with respect to the valuation. By Morandi's Ostrowski theorem for valued division algebras, [M$_1$, Th. 3], $\delta_{D/F}$ is a nonnegative power of $\overline{p}$ if $\overline{p} > 0$, while $\delta_{D/F} = 1$ if $\overline{p} = 0$. Let $Z = Z(GD)$, a graded GF-subalgebra of GD. Even though $F = Z(D)$, this Z may be strictly larger than GF; the following result of Boulagouaz shows when this occurs. Recall (see [JW, (1.6)]) that there is a canonical homomorphism

$$\theta_D \colon \Gamma_D/\Gamma_F \to \mathcal{G}(Z(\overline{D})/\overline{F}) \tag{4.8}$$

given by, for any $d \in D^*$ and any $z \in V_D$ with $\overline{z} \in Z(\overline{D})$, $\theta(v(d) + \Gamma_F)(\overline{z}) = \overline{dzd^{-1}}$. Note that the following diagram is evidently commutative with horizontal maps the obvious isomorphisms,

$$\begin{array}{ccc} \Gamma_D/\Gamma_F & \longrightarrow & \Gamma_{GD}/\Gamma_{GF} \\ {\scriptstyle \theta_D}\big\downarrow & & {\scriptstyle \theta_{GD,GF}}\big\downarrow \\ \mathcal{G}(Z(\overline{D})/\overline{F}) & \longrightarrow & \mathcal{G}(Z(GD_0)/GF_0) \end{array} \tag{4.9}$$

where $\theta_{GD,GF}$ is the map of (2.4).

Proposition 4.2. *With D and F as above, for $Z = Z(GD)$,*

(a) *Z_0 is the purely inseparable closure of $\overline{F}$ in $Z(\overline{D})$;*
(b) *Γ_Z/Γ_F is the $\overline{p}$-primary part of $\ker(\theta_D)$.*

Proof. See [B$_2$, Prop. 3.1, Th. 3.4]. $\square$

Now, let F be a field with Henselian valuation. That is, v has a unique extension to each field $L \supseteq F$ with L algebraic over F. Likewise, as is well known (see [S, Th. 9, p. 53] or [W$_1$, Th.], v has a unique extension to each CDA D over F (given by $v(d) = \frac{1}{\sqrt{[D:F]}} v(Nrd(d)) \in \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_F$, for all $d \in D^*$). We will focus on tame division algebras D (defined below), which all have the property that GD is a GCDA over the graded field GF. But first we recall some terminology connected with Henselian valued fields.

For our Henselian field F, let F_{sep} denote the separable closure of F (in some fixed algebraic closure F_{alg} of F). Let F_{nr} be the inertia field of F_{sep} over F, with respect to the unique extension of the Henselian valuation v to F_{sep}. Then, (see [E, (19.12), (19.8)(b)]) $\Gamma_{F_{nr}} = \Gamma_F$, $\overline{F_{nr}} \cong (\overline{F})_{sep}$, and F_{nr} is Galois over F with $\mathcal{G}(F_{nr}/F) \cong \mathcal{G}((\overline{F})_{sep}/\overline{F})$. Moreover, for any field L with $F \subseteq L \subseteq F_{sep}$

and $[L:F]<\infty$, we have (by [E, (19.14)] and an application of Hensel's Lemma) $L \subseteq F_{nr}$ iff L is unramified over F (i.e., $[\overline{L}:\overline{F}]=[L:F]$ and $\overline{L}$ is separable over $\overline{F}$). Note that F_{nr} is the compositum of all finite degree unramified field extensions of F in F_{sep}; F_{nr} is called the *maximal unramified extension* of F. Let F_t be the ramification field of F_{sep} over F. Then (see [E, (20.17)]) $\overline{F_t} \cong (\overline{F})_{sep}$ and Γ_{F_t}/Γ_F is the prime-to-$\overline{p}$ primary part of Δ_F/Γ_F, where $\Delta_F = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_F$. That is, if $\overline{p}=0$ then $\Gamma_{F_t}=\Delta_F$; if $\overline{p}\neq 0$, then Γ_{F_t}/Γ_F is $\overline{p}$-torsion-free and Δ_F/Γ_{F_t} is $\overline{p}$-primary torsion. Also, F_t is Galois over F, by [E, (21.2)]. Further, for any field L with $F\subseteq L\subseteq F_{sep}$ and $[L:F]<\infty$, we have $L\subseteq F_t$ iff L is tame (= tamely ramified and defectless) over F (i.e., $[L:F]=[\overline{L}:\overline{F}]|\Gamma_L:\Gamma_F|$, $\overline{L}$ is separable over $\overline{F}$, and $\overline{p}\nmid|\Gamma_L:\Gamma_F|$). Here, "only if" follows by [E, (20.20), (19.10)(b), (19.14)], and "if" from [E, (20.18)], since when L is tame over F, $L\cdot F_{nr}$ is tame and totally ramified over F_{nr}. Note that F_t is the compositum of all the finite-degree tame extension fields of F in F_{sep}; F_t is called the *maximal tame extension* of F. We have $F\subseteq F_{nr}\subseteq F_t\subseteq F_{sep}$, and if $\overline{p}=0$, then $F_t=F_{sep}$.

The Henselian valuation on F yields certain distinguished subgroups of its Brauer group $Br(F)$, discussed in [JW] and denoted as follows:

$$\begin{aligned} IBr(F) &= \{[D] \mid D \text{ is a CDA over } F \text{ with } [\overline{D}:\overline{F}]=[D:F] \\ &\qquad \text{and } Z(\overline{D})=\overline{F}\} \cong Br(\overline{F}), \text{ the \textit{inertial} part of } Br(F); \\ SBr(F) &= \{[D] \mid D \text{ is a CDA over } F,\ [\overline{D}:\overline{F}]\mid\Gamma_D:\Gamma_F| = [D:F], \\ &\qquad Z(\overline{D}) \text{ is separable over } \overline{F}, \text{and } \theta_D \text{ is injective}\} \\ &= Br(F_{nr}/F), \text{ the \textit{inertially split} part of } Br(F); \\ TBr(F) &= Br(F_t/F), \text{ the \textit{tame} part of } Br(F) \text{ (further described} \\ &\qquad \text{in Prop. 4.3 below).} \end{aligned}$$

So, $IBr(F)\subseteq SBr(F)\subseteq TBr(F)\subseteq Br(F)$ and if $\overline{p}=0$, then $TBr(F)=Br(F)$. Our focus will be on the *tame* CDA's D over F (i.e., those for which $[D]\in TBr(F)$), since they are the ones for which the associated graded division rings GD carry the most complete information about D (see, e.g., Th. 5.9 below). Tame CDA's over F were defined in a different way in [JW], but the next proposition shows that the definitions are equivalent.

Proposition 4.3. *Let D be a CDA over a Henselian field F, with char$(\overline{F})=\overline{p}$. Then, the following properties are equivalent:*

(i) *D is tame (i.e., $[D]\in TBr(F)$);*
(ii) *$[D_{\overline{p}}]\in SBr(F)$, where $D_{\overline{p}}$ is the $\overline{p}$-primary component of D;*
(iii) *$[\overline{D}:\overline{F}]|\Gamma_D:\Gamma_F|=[D:F]$, $Z(\overline{D})$ is separable over $\overline{F}$, and $\overline{p}\nmid \ker(\theta_D)$;*
(iv) *D has a maximal subfield which is tame over F;*
(v) *$[GD:GF]=[D:F]$ and $Z(GD)=GF$.*

Proof. (i) $\Rightarrow$ (ii) If $[D]\in TBr(F)$, then $[D_{\overline{p}}]\in TBr(F)$, since $[D_{\overline{p}}]=[D^{\otimes n}]$ for some n. (If $\overline{p}=0$, it is understood that $D_{\overline{p}} = F$.) Let $B = D_{\overline{p}} \otimes_F F_{nr}$. Since F_t splits B, there is a splitting field L of B with $F_{nr} \subseteq L \subseteq F_t$ and $[L : F_{nr}] < \infty$. Since L is tame over F_{nr} and F_{nr} is separably closed, $[L : F_{nr}] = |\Gamma_L : \Gamma_{F_{nr}}|$, which is prime to $\overline{p}$. Because $gcd([B : F_{nr}], [L : F_{nr}]) = 1$ and L splits B, B must already be split, proving (ii).

(ii) $\Leftrightarrow$ (iii) was proved in [JW, Lemma 6.1].

(iii) $\Leftrightarrow$ (v) Let $Z = Z(GD)$, a graded GF-subalgebra of GD. Then, $Z = GF$ iff $Z_0 = \overline{F}$ and $\Gamma_Z = \Gamma_F$. Therefore, (iii) $\Leftrightarrow$ (v) follows immediately from (4.7) and Prop. 4.2, since $Z(\overline{D})$ is normal over $\overline{F}$, by [JW, Prop. 1.7].

(ii) $\Rightarrow$ (iv) Assume $[D_{\overline{p}}] \in SBr(F)$. Then, $D_{\overline{p}}$ has a maximal subfield K with K unramified over F, by [JW, Lemma 5.1]. Let M be any maximal subfield of D', the prime-to-$\overline{p}$ component of D. Since $\overline{p} \nmid [M : F]$, M is tame over F, by "Ostrowski's theorem," which says that $[M : F]/([\overline{M} : \overline{F}]|\Gamma_M : \Gamma_F|)$ equals a nonnegative power of $\overline{p}$ if $\overline{p} > 0$, and equals 1 if $\overline{p} = 0$, (by [E, (20.21)] applied to N over M and N over F, where N is the normal closure of M over F). Since $K \subseteq F_t$ and $M \subseteq F_t$, the compositum $K \cdot M$ is also tame over F. Clearly, $K \cdot M$ splits $D \cong D_{\overline{p}} \otimes_F D'$ and $[K \cdot M : F] \leq [K : F][M : F] = deg(D)$, so $K \cdot M$ is a maximal subfield of D.

(iv) $\Rightarrow$ (i) is clear, since any maximal subfield of D is a splitting field. $\square$

Now, let F be a Henselian field, with its associated graded field GF, and let QGF be the quotient field of GF. The total ordering on Γ_F gives us a total ordering on $\Gamma_{GF} = \Gamma_F$, which induces a valuation on QGF, as described at the beginning of this section. Let HGF be the Henselization of QGF with respect to this valuation. Then, HGF need not be isomorphic to F (they need not even have the same characteristic). But we have shown in [HW, Prop. 5.1, Th. 5.2] that for the maximal tame extensions $HG(F_t) \cong (HGF)_t$, and the canonical map of Galois groups $\mathcal{G}(F_t/F) \to \mathcal{G}((HGF)_t/HGF)$ is an isomorphism. We also have homomorphisms of multiplicative groups $F_t^* \to G(F_t)^*$ and $G(F_t)^* \to HG(F_t)^* \to (HGF)_t^*$; these maps compose to give a group homomorphism $F_t^* \to (HGF)_t^*$ which is clearly compatible with the respective Galois group actions. Therefore, there is a homomorphism of continuous cohomology groups $H^i(\mathcal{G}(F_t/F), F_t^*) \to H^i(\mathcal{G}((HGF)_t/HGF), (HGF)_t^*)$. In particular, for $i = 2$, we obtain a group homomorphism

$$\gamma\colon TBr(F) \to TBr(HGF).$$

Theorem 4.4. *Let F be a Henselian valued field. Then, the map $\gamma\colon TBr(F) \to TBr(HGF)$ just defined is an isomorphism.*

Proof. We do this by stages. For the maximal unramified extension F_{nr} of F, it is easy to check that $HG(F_{nr}) \cong (HGF)_{nr}$ canonically, and that γ maps $SBr(F)$ into $SBr(HGF)$. We first show that this map $SBr(F) \to SBr(HGF)$ is an isomorphism. We have $SBr(F) \cong H^2(\overline{\mathcal{G}}, F_{nr}^*)$, where $\overline{\mathcal{G}} = \mathcal{G}(\overline{F}_{sep}/\overline{F}) \cong \mathcal{G}(F_{nr}/F) \cong \mathcal{G}((HGF)_{nr}/HGF)$. Let $\Gamma = \Gamma_{F_{nr}} = \Gamma_F = \Gamma_{HGF}$. We have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2(\overline{\mathcal{G}}, U_{F_{nr}}) & \longrightarrow & H^2(\overline{\mathcal{G}}, F_{nr}^*) & \longrightarrow & H^2(\overline{\mathcal{G}}, \Gamma) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^2(\overline{\mathcal{G}}, U_{(HGF)_{nr}}) & \longrightarrow & H^2(\overline{\mathcal{G}}, (HGF)_{nr}^*) & \longrightarrow & H^2(\overline{\mathcal{G}}, \Gamma) & \longrightarrow & 0 \end{array} \tag{4.10}$$

The first row of (4.10) arises from the short exact sequence of $\overline{\mathcal{G}}$-modules

$$1 \to U_{F_{nr}} \to F_{nr}^* \to \Gamma \to 1$$

induced by the valuation on F_{nr}, and the second row arises likewise from $(HGF)_{nr}$. It was shown in [JW, (5.4)] that the rows of (4.10) are exact, since F and HGF are Henselian. It was also shown that

$$IBr(F) \cong H^2(\overline{\mathcal{G}}, U_{F_{nr}}) \cong H^2(\overline{\mathcal{G}}, \overline{F_{nr}}^*) \cong Br(\overline{F}),$$

where the middle isomorphism arises from the projection $U_{F_{nr}} \to \overline{F_{nr}}^*$. It follows from this and the isomorphism $\overline{F_{nr}} \cong \overline{(HGF)_{nr}}$ that the left vertical map in (4.10) is an isomorphism. Since the right vertical map is the identity, the 5-lemma shows that the middle vertical map in (4.10) is also an isomorphism. Thus,

$$SBr(F) \cong H^2(\overline{\mathcal{G}}, F_{nr}^*) \cong H^2(\overline{\mathcal{G}}, (HGF)_{nr}^*) \cong SBr(HGF),$$

and the composition of these isomorphisms coincides with γ on $SBr(F)$.

Now, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & SBr(F) & \longrightarrow & TBr(F) & \longrightarrow & TBr(F_{nr}) \\ & & \downarrow & & \gamma\downarrow & & \downarrow \\ 0 & \longrightarrow & SBr(HGF) & \longrightarrow & TBr(HGF) & \longrightarrow & TBr((HGF)_{nr}) \end{array} \tag{4.11}$$

We just showed that the left vertical map in (4.11) is an isomorphism. To analyze the right vertical map, first note that $TBr(F_{nr})$ has no $\overline{p}$-torsion, since $[F_t : F_{nr}]$ is prime to $\overline{p}$ as a supernatural number. Take any positive integer n such that n is prime to $\overline{p}$ (if $\overline{p} = 0$, this means any positive integer at all). Since the n-th power map $F_t^* \to F_t^*$ is surjective, we have for the n-torsion in the tame Brauer group, ${}_nTBr(F_{nr}) \cong H^2(\mathcal{G}', \mu_n(F_t))$, where $\mathcal{G}' = \mathcal{G}(F_t/F_{nr}) \cong \mathcal{G}((HGF)_t/(HGF)_{nr})$. (Note that $|\mu_n(F_t)| = n$ since F_t is Henselian and $|\mu_n(\overline{F_t})| = n$, which holds as $\overline{F_t}$ is separably closed.) Because $\mu_n(F_t)$ maps isomorphically onto $\mu_n((HGF)_t)$, the map ${}_nTBr(F_{nr}) \to {}_nTBr((HGF_{nr}))$ is an isomorphism. Since this is true for all n prime to $\overline{p}$, the right vertical map in (4.11) is an isomorphism.

The scalar extension map $TBr(F) \to TBr(F_{nr})$ is in general not onto. In fact, we claim that the image of $TBr(F) \to TBr(F_{nr})$ equals $\bigcup\limits_m {}_mTBr(F_{nr})$ as m ranges over the positive integers prime to $\overline{p}$ such that $|\mu_m(\overline{F})| = m$. For, by [TW, Th. 4.3, Prop. 4.2] every class in $TBr(F_{nr})$ is represented by a division algebra D which is tame and totally ramified over F, and D is determined up to isomorphism by its associated nondegenerate symplectic pairing $\beta_D\colon \Gamma_D/\Gamma_{F_{nr}} \times \Gamma_D/\Gamma_{F_{nr}} \to \mu(\overline{F_{nr}})$ (given by $(\delta + \Gamma_{F_{nr}}, \varepsilon + \Gamma_{F_{nr}}) \mapsto \overline{(ded^{-1}e^{-1})}$ for any $d, e \in D^*$ with $v(d) = \delta$ and $v(e) = \varepsilon$). Moreover, if $\ell = exp(\Gamma_D/\Gamma_F)$ then ℓ is prime to $\overline{p}$, $im(\beta_D) = \mu_\ell(\overline{F_{nr}})$, and $[D]$ has exponent ℓ in $TBr(F_{nr})$. If $|\mu_\ell(F)| = \ell$ (which occurs iff $|\mu_\ell(\overline{F})| = \ell$ as F is Henselian), then one can easily construct a tame totally ramified division algebra over F as a tensor product of symbol algebras which will have the same value group and pairing as D; this algebra will map to $[D]$ in $TBr(F_{nr})$. Suppose, on the other hand, that $|\mu_\ell(\overline{F})| < \ell$. By [JW, Lemma 6.2], any tame central division algebra E over F is representable as $E \sim S \otimes_F T$ in $TBr(F)$ where S is inertially split and T is a tame and totally ramified division algebra over F. Then in $TBr(F_{nr})$, $E \otimes_F F_{nr} \sim T \otimes_F F_{nr}$, and $T \otimes_F F_{nr}$ has the same value group and pairing as T, so the image of the pairing must lie in $\mu(\overline{F})$. So, $T \otimes_F F_{nr} \ncong D$, since their canonical pairings have different images. Thus, when $|\mu_\ell(F)| < \ell$, then $[D]$ cannot lie in the image of $TBr(F)$, proving the claim. Since $\overline{F} \cong \overline{HGF}$, so they have the same roots of unity, the claim shows the right vertical map of (4.11) restricts to an isomorphism of the images of the maps $TBr(F) \to TBr(F_{nr})$ and $TBr(HGF) \to TBr((HGF)_{nr})$. Thus, the 5-lemma can be applied to diagram (4.11) to see that γ is an isomorphism. $\square$

Corollary 4.5. *Suppose F is a Henselian valued field, and K is a finite degree tame Galois extension field of F. Take any 2-cocycle $f \in Z^2(\mathcal{G}(K/F), K^*)$ such that $f(\sigma,\tau) \in 1 + M_K$ for all $\sigma, \tau \in \mathcal{G}(K/F)$. Then the crossed product algebra $(K/F, \mathcal{G}(K/F), f)$ is split.*

PROOF. The group homomorphism $F_t^* \to G(F_t)^* \to (HGF)_t^*$ has kernel $1 + M_{F_t}$. Hence, $\gamma[(K/F, \mathcal{G}(K/F), f)] = 1$ in $TBr(HGF)$. So, Th. 4.4 shows that $(K/F, \mathcal{G}(K/F), f)$ must be split. $\square$

§5 Isomorphisms between GBr and TBr

Let R be a graded field with Γ_R totally ordered. We have seen that for any GCSA A over R there is a CSA $HA = HR \otimes_R A$ over the Henselian field HR. The map $[A]_g \mapsto [HA]$ gives a well-defined group homomorphism $GBr(R) \to Br(HR)$, since it is the composition of the forgetful homomorphism $GBr(R) \to Br(R)$ with the scalar extension map $Br(R) \to Br(HR)$. Also, $A \cong_g M_n(E)(d)$, where E is a GCDA over R with $[A]_g = [E]_g$, and by (4.6), Prop. 2.3, and Prop. 4.3, HE is a tame CDA over HR with $[HE : HR] = [E : R]$. Thus, our map to $Br(HR)$ actually lands in $TBr(HR)$, and we have an index-preserving group homomorphism $\beta\colon GBr(R) \to TBr(HR)$. Now, the Henselian field HR has an associated graded field GHR, and by Prop. 4.3 there is a map $\delta\colon TBr(HR) \to GBr(GHR)$ taking $[D] \mapsto [GD]_g$ for any tame CDA D over HR. Also, let Y be the maximal tame graded field extension of R, so we have isomorphisms $(HR)_t \cong HY$ and $GHY \cong G((HR)_t) \cong (GHR)_t$, by [HW, Prop. 5.1]. Let $\mathcal{G} = \mathcal{G}(Y/R)$, which we identify with $\mathcal{G}((HR)_t/HR)$ and $\mathcal{G}(GHY/GHR)$ in view of [HW, Prop. 5.1]. The continuous $\mathcal{G}$-module homomorphisms $Y^* \to (HR)_t^*$ and $(HR)_t^* \to (GHY)^*$ lead to homomorphisms $\beta'\colon H^2(\mathcal{G}, Y^*) \to H^2(\mathcal{G}, (HR)_t^*)$ and $\delta'\colon H^2(\mathcal{G}, (HR)_t^*) \to H^2(\mathcal{G}, GHY^*)$. These maps fit together into a diagram

$$\begin{array}{ccccc} GBr(R) & \xrightarrow{\beta} & TBr(HR) & \xrightarrow{\delta} & GBr(GHR) \\ \uparrow & & \uparrow & & \uparrow \\ H^2(\mathcal{G}, Y^*) & \xrightarrow{\beta'} & H^2(\mathcal{G}, (HR)_t^*) & \xrightarrow{\delta'} & H^2(\mathcal{G}, GHY^*) \end{array} \tag{5.1}$$

Theorem 5.1. *For any graded field R with Γ_R totally ordered, diagram (5.1) is commutative, and all the maps in it are group isomorphisms. The maps β and δ are index-preserving. Also, $\delta \circ \beta$ and $\delta' \circ \beta'$ coincide with the isomorphisms arising from the canonical isomorphism $GHR \cong_g R$.*

Proof. The vertical maps in (5.1) are the isomorphisms of Prop. 3.3 for the graded fields R, GHR, and the standard Brauer group isomorphism for the valued field HR. It is clear from the definitions that $\gamma \circ \beta$ and $\gamma' \circ \beta'$ are the isomorphisms arising from the canonical map $R \cong_g GHR$. Hence, the outer rectangle in (5.1) is commutative, and β and β' are injective. Also, it is clear from the definitions that the left square in (5.1) is commutative, and that β is a group homomorphism (since it is essentially a scalar extension map). We will show below that β' is onto. Assume this for now. Then from the commutative left square, β is onto, hence an isomorphism. So, $\delta = (\delta \circ \beta) \circ \beta^{-1}$ is a group isomorphism. (Note that it is not at all apparent from the definition or from direct calculations that δ is even a group homomorphism.) Likewise, $\delta' = (\delta' \circ \beta') \circ \beta'^{-1}$ is an isomorphism. The commutativity of the right square in (5.1) follows from the commutativity of the outer rectangle and the left square. We noted above that β is index-preserving. Because $\delta \circ \beta$ is also index-preserving, so must be δ.

It remains to verify the surjectivity of β'. For this, consider the maps

$$H^2(\mathcal{G}, Y^*) \xrightarrow{\beta'} H^2(\mathcal{G}, (HR)_t^*) \xrightarrow{\delta'} H^2(\mathcal{G}, GHY^*) \xrightarrow{\beta''} H^2(\mathcal{G}, (HGHR)_t^*),$$

where β', δ' are as in (5.1) and β'' corresponds to β' when we start with GHR instead of R as ground graded field. The canonical gr-isomorphism $GHR \cong_g R$ induces an isomorphism $HGHR \cong HR$ of Henselian valued fields, and the map $\beta'' \circ \delta'$ is the corresponding isomorphism of cohomology groups. (It is also the isomorphism given by Th. 4.4.) Since $\beta'' \circ \delta'$ is onto, so is β''. But with respect to the gr-isomorphism $GHR \cong_g R$, β'' corresponds to β', so β' must also be onto. $\square$

Remark 5.2. There is a variation of Th. 5.1 which does not involve Henselization. For this, let R be a graded field (with Γ_R torsion-free) and let Y be the maximal tame graded field extension of R, as in [HW, Prop. 3.7]. Give some total ordering to Γ_R, use this to define a valuation on the quotient field QR of R, and let GQR be the associated graded field for the filtration on QR arising from the valuation. Let $QY = QR \otimes_R Y$, which is the quotient field of Y. Then, there are index-preserving group isomorphisms $\widetilde{\beta}$ and $\widetilde{\delta}$ in a diagram

$$GBr(R) \xrightarrow{\tilde{\beta}} Br(QY/QR) \xrightarrow{\tilde{\delta}} GBr(GQR) \tag{5.2}$$

such that $\widetilde{\delta}\circ\widetilde{\beta}$ coincides with the isomorphism arising from the canonical graded isomorphism $GQR \cong_g R$. Also, there is commutative diagram like (5.1) (where $\mathcal{G} = \mathcal{G}(QY/QR)$) in which all the maps are isomorphisms, and the middle column is $H^2(\mathcal{G}, QY) \to Br(QY/QR)$. Here, $\widetilde{\beta}$ is the map $[E]_g \mapsto [E \otimes_R QR]$, and $\widetilde{\delta}$ will be described below.

The properties asserted for the maps in (5.2) can be seen as follows: Let HR be the Henselization of QR with respect to our valuation on QR, and consider the diagram

$$\begin{tikzcd} GBr(R) \arrow[r, "\beta"] \arrow[d, equal] & TBr(HR) \arrow[r, "\delta"] & GBr(GHR) \\ GBr(R) \arrow[r, "\tilde{\beta}"] & Br(QY/QR) \arrow[r, "\tilde{\delta}"] \arrow[u, "\varepsilon"'] & GBr(GQR), \arrow[u, "c"'] \end{tikzcd} \tag{5.3}$$

where β, δ are the maps of (5.1), ε is the scalar extension map, and c is the isomorphism arising from the canonical gr-isomorphisms $GQR \cong_g R \cong_g GHR$. The left square of (5.3) is clearly commutative. Hence, $\widetilde{\beta}$ is injective and index-preserving, since this is true for β by Th. 5.1. Moreover, $\widetilde{\beta}$ is onto, since the corresponding homological map is onto, by arguing just as in the last paragraph of the proof of Th. 5.1. Hence, ε is an index-preserving isomorphism (and indeed maps into $TBr(HR)$, not just into $Br(HR)$). Take any CDA D over QR with D split by QY. Since $\widetilde{\beta}$ is onto and index-preserving, there is a GCDA E over R with $D \cong E \otimes_R QR = QE$ so the valuation on QE induced by the grading on E yields a valuation on D extending the one on QR; this valuation on D is uniquely determined, by [W$_1$, Th.], and it is a tame valuation (in the sense of Prop. 4.3(iii)) by Prop. 2.3. The map $\widetilde{\delta}$: $Br(QR/QR) \to GBr(GQR)$ can now be defined by $[D] \mapsto [GD]_g$, where GD is the associated graded division ring arising from the valuation on D. It is clear that the right square in (5.3) is commutative. Hence, $\widetilde{\delta}$ is an index-preserving isomorphism, since this is true for ε, δ, and c.

We next prove our main result, which is a theorem like 5.1, but starting with a Henselian field instead of with a graded field. Fix a field F with Henselian valuation v, and let GF be the associated graded field. As in Th. 4.4, let HGF be the Henselization of QGF with respect to the valuation induced by the grading on GF using the total ordering on Γ_{GF} corresponding to the ordering on Γ_F. There is a map α: $TBr(F) \to GBr(GF)$ mapping $[D] \mapsto [GD]_g$, for any tame CDA D over F. There is also a group homomorphism β: $GBr(GF) \to TBr(HGF)$ given by $[A]_g \mapsto [A \otimes_{GF} HGF]$. Let $\mathcal{G} = \mathcal{G}(F_t/F) \cong \mathcal{G}(G(F_t)/GF) \cong \mathcal{G}((HGF)_t/HGF)$ (see before Th. 4.4 and [HW, Prop. 5.1]). We have homomorphisms α': $H^2(\mathcal{G}, F_t^*) \to H^2(\mathcal{G}, G(F_t)^*)$ and β': $H^2(\mathcal{G}, G(F_t)^*) \to H^2(\mathcal{G}, (HGF)_t)$ induced by the $\mathcal{G}$-module homomorphisms $F_t^* \to G(F_t)^*$ and $G(F_t)^* \to (HGF)_t^*$. These fit into a diagram:

$$\begin{tikzcd} TBr(F) \arrow[r, "\alpha"] & GBr(GF) \arrow[r, "\beta"] & TBr(HGF) \\ H^2(\mathcal{G}, F_t^*) \arrow[r, "\alpha'"] \arrow[u] & H^2(\mathcal{G}, G(F_t)^*) \arrow[r, "\beta'"] \arrow[u] & H^2(\mathcal{G}, (HGF)_t^*) \arrow[u] \end{tikzcd} \tag{5.4}$$

Theorem 5.3. *For any Henselian valued field F, diagram (5.4) is commutative, and all its maps are group isomorphisms. Furthermore, α and β are index-preserving.*

Before proving Th. 5.3, we recall some facts about Dubrovin valuation rings. If A is a CSA over a field L, then a subring B of A is called a *Dubrovin valuation ring* of A if B has an ideal J such that B/J is simple Artinian, and for each $a \in A - B$ there exist $b_1, b_2 \in B$ such that $b_1 a, ab_2 \in B - J$, cf. [D$_1$], [D$_2$], [W$_2$]. It is known that J is the Jacobson radical of B, that B is a prime ring which is a left and right order in A, that $J \cap L$ is a valuation ring of L, and that the two-sided ideals of B are linearly ordered by inclusion. Let $st(B) = \{a \in A^* \mid aBa^{-1} = B\}$, and let $\Gamma_B = st(B)/B^*$, the "value group" of B. For each $\delta = sB^* \in \Gamma_B$ there is an associated fractional ideal (= B-B sub-bimodule of A), $I^\delta = Bs = sB$. The ideals I^δ are all the fractional ideals of B which are cyclic as left and as right B-modules. One has $I^\delta \cdot I^{\delta'} = I^{\delta'} \cdot I^\delta = I^{\delta+\delta'}$. The abelian group Γ_B is given a total ordering by $\delta \le \delta'$ iff $I^\delta \supseteq I^{\delta'}$. If we set $I^{>\delta} = \bigcup_{\delta' > \delta} I^{\delta'}$ then we may form the associated graded ring of B with respect to the filtration by fractional ideals I^δ,

$$GB = \bigoplus_{\delta \in \Gamma_B} GB_\delta, \quad \text{where } GB_\delta = I^\delta / I^{>\delta}.$$

If V is a given valuation ring of L, Dubrovin's existence theorem ([D$_2$, §3, Th. 2], see [BG, Th. 3.8] for another proof) says that there is a Dubrovin valuation ring B of A with $B \cap L = V$. Moreover, the conjugacy theorem ([W$_2$, Th. A], with a more direct proof given in [G, Th. 3.3]) says that if B' is another Dubrovin valuation ring of A with $B' \cap L = V$, then $B' = aBa^{-1}$, for some $a \in A^*$; so clearly $GB' \cong_g GB$. Since we are interested in ground fields L with a fixed valuation v on L, we will write GA for GB, for any Dubrovin valuation ring B of A with $B \cap L = V_L$. So, GA is a graded GL-algebra, which is well-defined up to gr-isomorphism, with $\Gamma_{GA} = \Gamma_B$ and $GA_0 = B/J$. The notation GA is consistent with our previous usage. For, if D is a valued CDA over L with $V_D \cap L = V_L$, then V_D is the unique Dubrovin valuation ring of D contracting to V_L in L, and then GD as just defined (i.e., $G(V_D)$) is exactly the graded division ring GD defined in §4 above.

Also, for any natural number n, if B is a Dubrovin valuation ring of A, then $M_n(B)$ is a Dubrovin valuation ring of $M_n(A)$ with $M_n(B) \cap L = B \cap L$ and $\Gamma_{M_n(B)} \cong \Gamma_B$ canonically (cf. [W$_2$, Cor. 3.5]). Consequently, $G(M_n(A)) \cong_g M_n(GA)$.

One convenient way of building Dubrovin valuation rings is by using Morandi's value functions introduced in [M$_2$]: Given a CSA A over L, suppose Γ is a totally ordered abelian group and $w \colon A - \{0\} \to \Gamma$ is a function such that for all $a, b \in A - \{0\}$,

(i) $w(a+b) \ge min(w(a), w(b))$, if $b \ne -a$;
(ii) $w(ab) \ge w(a) + w(b)$;
(iii) B_w/J_w is a simple Artinian ring, where $B_w = \{a \in A - \{0\} \mid w(a) \ge 0\} \cup \{0\}$ (a ring), and $J_w = \{a \in A - \{0\} \mid w(a) > 0\} \cup \{0\}$ (an ideal of B_w);
(iv) $im(w) = w(st(w))$, where $st(w) = \{a \in A^* \mid w(a^{-1}) = -w(a)\}$.

Then, by [M$_2$, Th. 2.4, Cor. 2.5], B_w is a Dubrovin valuation ring with B_w integral over $B_w \cap L$. Further, $st(B_w) = st(w)$ and $\Gamma_{B_w} \cong im(w)$. Also, for $\delta \in \Gamma_{B_w}$, $I^\delta = \{a \in A - \{0\} \mid w(a) \geq \delta\} \cup \{0\}$ and $I^{>\delta} = \{a \in A - \{0\} \mid w(a) > \delta\} \cup \{0\}$.

Lemma 5.4. *Let F be a Henselian field, let K be a tame finite degree Galois extension field of F, and set $\mathcal{H} = \mathcal{G}(K/F)$. Let $f \in Z^2(\mathcal{H}, K^*)$ be a normalized 2-cocycle, and let $A = (K/F, \mathcal{H}, f) = \bigoplus_{\sigma \in \mathcal{H}} Kx_\sigma$. There is a unique function $w\colon \{x_\sigma \mid \sigma \in \mathcal{H}\} \to \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_F$ such that, for all $\sigma, \tau \in \mathcal{H}$,*

$$w(x_\sigma) + w(x_\tau) = v(f(\sigma, \tau)) + w(x_{\sigma\tau}) \tag{5.5}$$

(where v denotes the valuation on K extending the given valuation on F). Extend w to $A - \{0\}$ by defining

$$w\Big(\sum_{\sigma \in \mathcal{H}} c_\sigma x_\sigma\Big) = min\{v(c_\sigma) + w(x_\sigma) \mid c_\sigma \neq 0\}. \tag{5.6}$$

Suppose B_w/J_w is simple Artinian (for B_w, J_w defined as in (iii) above). Then $GA \cong_g (GK/GF, \mathcal{H}, \overline{f})$, where $\overline{f}(\sigma, \tau) =$ image of $f(\sigma, \tau)$ in GK^h.

Proof. Let $\Delta_F = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_F \supseteq \Gamma_K$. The valuation $v\colon K^* \to \Delta_F$ is $\mathcal{H}$-equivariant (with $\mathcal{H}$ acting trivially on Δ_F), so it induces a map in cohomology $v^*\colon H^2(\mathcal{H}, K^*) \to H^2(\mathcal{H}, \Delta_F)$. Because $\mathcal{H}$ is finite and Δ_F is uniquely divisible, $H^2(\mathcal{H}, \Delta_F) = H^1(\mathcal{H}, \Delta_F) = (0)$. The existence of $\{w(x_\sigma)\}$ satisfying (5.5) is a restatement of $v^*[f] = 0$, and the uniqueness of $\{w(x_\sigma)\}$ follows from $H^1(\mathcal{H}, \Delta_F) = (0)$. See the proof of Lemma 3.1.

Now consider the function w on $\Lambda - \{0\}$ defined by (5.6). We check that w satisfies conditions (i)–(iv) above for a value function. (i) is clear, and (ii) follows because by (5.5), for any $c, d \in K^*$, $\sigma, \tau \in \mathcal{H}$, $w((cx_\sigma)(dx_\tau)) = w(cx_\sigma) + w(dx_\tau)$. Property (iii) holds by hypothesis. The previous equation shows $\{cx_\sigma \mid c \in K^*,\ \sigma \in \mathcal{H}\} \subseteq st(w)$, so (iv) holds. Therefore, B_w is a Dubrovin valuation ring of A with $B_w \cap F = V_F$, so $GA = GB_w$.

We show that GB_w is the desired crossed product. Since w restricts to v on K, GA contains GK as a graded subring. Further, if y_σ is the image of x_σ in GB_w, then $y_\sigma y_\tau = \overline{f}(\sigma, \tau) y_{\sigma\tau}$; also, each $y_\sigma \in (GB_w)^*$ and $y_\sigma c y_\sigma^{-1} = \sigma(c)$ for any $c \in GK^h$, hence for any $c \in GK$. Because K is tame over F, the elements of $\mathcal{H}$ induce distinct graded automorphisms of GK, and GK is $\mathcal{H}$-Galois over GF. The sum $\sum_{\sigma \in \mathcal{H}} GKy_\sigma$ in GB_w is actually a direct sum, as one can see by the usual argument (conjugating a homogeneous sum $\sum c_\sigma y_\sigma = 0$ with the minimal number of nonzero $c_\sigma \in GK$ by an element of $GK^h - \{0\}$, and subtracting to get a contradiction). Finally, to see $\sum GKy_\sigma$ is all of GB_w, take any $\delta \in \Gamma_{GB_w} = \Gamma_{B_w}$ and any $b \in (GB_w)_\delta$, $b \neq 0$. Then, b is the image of some $a = \sum_{\sigma \in \mathcal{H}} c_\sigma x_\sigma$, $c_\sigma \in K$, with $w(a) = \delta$. So, each $w(c_\sigma x_\sigma) \geq \delta$ and some $w(c_\tau x_\tau) = \delta$. If we let $a' = \sum_{\sigma \in S} c_\sigma x_\sigma$, where $S = \{\sigma \in \mathcal{H} \mid w(c_\sigma x_\sigma) = \delta\}$, then $a' \equiv a \,(\mathrm{mod}\, I^{>\delta})$, so if we let bar denote the image in GA, $b = \overline{a} = \overline{a'} = \sum_{\sigma \in S} \overline{c_\sigma x_\sigma} = \sum_{\sigma \in S} \overline{c_\sigma} y_\sigma$. Thus, we have $GA = GB_w = \sum_{\sigma \in \mathcal{H}} GKy_\sigma = \bigoplus_{\sigma \in \mathcal{H}} GKy_\sigma = (GK/GF, \mathcal{H}, \overline{f})$, as desired. $\square$

PROOF OF TH. 5.3. The vertical maps in (5.4) are isomorphisms (by Prop. 3.3 for the middle map). We know from Th. 5.1 that the right square of (5.4) is commutative and that β and β' are isomorphisms with β index-preserving. Also, $\beta' \circ \alpha'$ is the isomorphism γ of Th. 4.4. Hence, $\alpha' = \beta'^{-1} \circ (\beta' \circ \alpha')$ is also an isomorphism. Since α maps tame CDA's over F to GCDA's over GF, it is index-preserving. It remains only to prove that the left square of (5.4) is commutative. (Note that it is not apparent at this point even that α is a group homomorphism. We would like to complete the proof by invoking the fact that the δ of (5.1) is an isomorphism but we do not know how to carry out such an argument.)

Take any finite tame Galois extension field K of F, set $\mathcal{H} = \mathcal{G}(K/F)$, and take any $f \in Z^2(\mathcal{H}, K^*)$. If f satisfies the hypothesis of Lemma 5.4 (i.e., B_w/J_w is simple Artinian), then that lemma shows the left square of (5.4) is commutative for the image of f in $H^2(\mathcal{G}, F_t^*)$. Note the following two cases where this applies. First, whenever K is unramified over F, it was shown in [MW, Th. 2.3] that the hypotheses of Lemma 5.4 hold for any $f \in Z^2(\mathcal{H}, K^*)$. Second, if T is any division algebra tame and totally ramified over F, then (as T is isomorphic to a tensor product of symbol algebras, see [Dr, Th. 1]) T has a maximal subfield L which is Galois (and necessarily tame and totally ramified) over F. So T is a crossed product, say $T = \bigoplus_{\sigma \in \mathcal{G}(L/F)} Lz_\sigma$, with $z_\sigma z_\tau = g(\sigma, \tau) z_{\sigma\tau}$. Then, for the unique w of (5.5), we must have $w(z_\sigma) = v(z_\sigma)$, where v is the valuation on T. Furthermore, the $v(z_\sigma)$ are distinct mod Γ_L. For, if $v(z_\sigma) \equiv v(z_\tau) \ (mod \Gamma_L)$, then $v(z_\rho) \in \Gamma_L$ for $\rho = \sigma\tau^{-1}$. Since the canonical pairing β_T on Γ_T/Γ_F is trivial on Γ_L/Γ_F as L is commutative, we have $1 = \overline{z_\rho \ell z_\rho^{-1} \ell^{-1}} = \overline{\rho(\ell)\ell^{-1}} \in \mu(\overline{F})$ for all $\ell \in L^*$. From the nondegenerate pairing $\Gamma_L/\Gamma_F \times \mathcal{G}(L/F) \to \mu(\overline{F})$ for a tame totally ramified Galois field extension, see [TW, Prop. 1.4(i)], it follows that $\rho = id_L$; so $\sigma = \tau$ whenever $v(z_\sigma) \equiv v(z_\tau) \ (mod \Gamma_L)$. Hence, the function w on $T - \{0\}$ defined by (5.6) from $\{w(z_\sigma)\}$ coincides with v on all of $T - \{0\}$. So, $B_w = V_T$, $J_w = M_T$, and $B_w/J_w = \overline{T} \cong \overline{F}$, so the hypotheses of Lemma 5.4 are satisfied for the cocycle g.

Now, let $\psi\colon H^2(\mathcal{G}, F_t^*) \to TBr(F)$ be the left vertical map of (5.4) (an isomorphism), and take any $[A] \in TBr(F)$. By [JW, Lemma 6.2], $[A] = [S] + [T]$ in $TBr(F)$, for some CDA's S and T with S inertially split and T tame and totally ramified over F. By the preceding paragraph, the left square of (5.4) is commutative for $\psi^{-1}[S]$ and $\psi^{-1}[T]$. But it was shown by Boulagouaz in [B_2, Prop. 9.4], using value functions and [MW, Th. 2.1] that $\alpha[A] = \alpha[S] + \alpha[T]$. Consequently, the left square of (5.4) is commutative for $\psi^{-1}[A]$, hence for all of $H^2(\mathcal{G}, F_t^*)$, as ψ^{-1} is onto. This completes the proof. □

Remark 5.5. One interesting fact that Th. 5.3 makes clear is that if F is any Henselian valued field, then the structure of $TBr(F)$ is independent of the ordering on Γ_F (though it certainly depends on Γ_F as an abstract group). For, $TBr(F) \cong GBr(GF)$, and the graded Brauer group is independent of the ordering on Γ_F.

Remark 5.6. It was proved in [B_2, Th. 10.3] that if F is a Henselian field and D is a tame CDA over F, then $exp(GD) = exp(D)$, where $exp(GD)$ denotes the order of $[GD]$ in the abelian group $GBr(GF)$. Observe that this follows immediately from the α of (5.4) being a monomorphism, which we proved in Th. 5.3.

Corollary 5.7. *Let $F \subseteq K$ be Henselian fields (with the valuation on K extending the one on F). Then, there is a commutative diagram:*

$$\begin{array}{ccccc} TBr(F) & \xrightarrow{\alpha_F} & GBr(GF) & \xrightarrow{\beta_F} & TBr(HGF) \\ \downarrow & & \downarrow & & \downarrow \\ TBr(K) & \xrightarrow{\alpha_K} & GBr(GK) & \xrightarrow{\beta_K} & TBr(HGK) \end{array} \tag{5.7}$$

Proof. The vertical maps in (5.7) are the canonical scalar extension maps. It is clear from the description of K_t as the ramification field for the Galois extension K_s/K that $F_t \subseteq K_t$. So, the scalar extension maps $Br(F) \to Br(K)$ maps $TBr(F) \to TBr(K)$. Likewise, since we are assuming the Henselization HGK of QGK has been chosen to contain HGF, we have $(HGF)_t \subseteq (HGK)_t$, so $TBr(HGF)$ maps to $TBr(HGK)$. The right inner square in (5.7) is commutative because β_F and β_K are essentially scalar extension maps. Also, it is clear from the homological definition of the tame Brauer group, and Th. 5.3, that the outer rectangle in (5.7) is commutative. Hence, by Th. 5.3 and the fact that β_F and β_K are isomorphisms, it follows that the left inner square in (5.7) is also commutative. $\square$

Corollary 5.8. *Let $F \subseteq K$ Henselian fields as in Cor. 5.7, and let A be a tame CSA over F. Then, K splits A iff GK splits GA, iff HGK splits HGA.*

Proof. This is immediate from the commutativity of diagram (5.7), since the horizontal maps are isomorphisms by Th. 5.3. $\square$

Theorem 5.9. *Let D be a tame CDA over a Henselian field F, and let A be a tame CDA over a field $K \supseteq F$ such that $[K : F] < \infty$ and K is defectless over F. Then A is F-isomorphic to an F-subalgebra of D iff GA is GF-gr-isomorphic to a graded GF-subalgebra of GD, iff HGA is HGF-isomorphic to an HGF-subalgebra of HGD.*

Proof. Let $k = [K : F]$ and let $a^2 = [A : K]$. Recall (see, e.g., [MiW, Prop. 2.1(b)]) that $ind(D)/ind(D \otimes_F A^{\mathrm{op}}) \le ak$, with equality holding iff A embeds F-isomorphically into D.

Clearly, any F-monomorphism of A into D induces a graded GF-monomorphism of GA into GD. Suppose next that GA is graded GF-isomorphic to a subalgebra of GD. Since K is defectless over F, $GK \otimes_{GF} HGF = HGK$. Also, HGF is flat over GF, since it is a direct sum of copies of the localization QGF of GF. Hence HGA, which equals $GA \otimes_{GK} HGK \cong GA \otimes_{GF} HGF$, embeds in $HGD = GD \otimes_{GF} HGF$.

Now, suppose HGA embeds in HGD over HGF. Since $HGK = Z(HGA)$ and also $[HGA : HGK] = [A : K] = a^2$ and $[HGK : HGF] = [K : F] = k$ (as K/F is defectless), the formula in the first paragraph gives

$$ind(HGD)/ind(HGD \otimes_{HGF} (HGA)^{\rm op}) = ak. \tag{5.8}$$

Let $\gamma_F = \beta_F \circ \alpha_F$, where β_F and α_F are the maps of (5.4) for F; likewise, let $\gamma_K = \beta_K \circ \alpha_K$. Also, let res denote the scalar extension map. Then, in $TBr(HGK)$ we have

$$\begin{aligned}[HGD \otimes_{HGF} (HGA)^{\rm op}] &= [HGD \otimes_{HGF} HGK] + [(HGA)^{\rm op}] \\ &= res_{HGK/HGF} \circ \gamma_F[D] + \gamma_K[A^{\rm op}] \\ &= \gamma_K[D \otimes_F K] + \gamma_K[A^{\rm op}] = \gamma_K[D \otimes_F A^{\rm op}],\end{aligned}$$

where the second equality uses Th. 5.3 twice and the third equality is by Cor. 5.7. Since γ_K and γ_F are index-preserving by Th. 5.3, this yields

$$ind(D)/ind(D \otimes_F A^{\rm op}) = ind(HGD)/ind(HGD \otimes_{HGF} (HGA)^{\rm op}) = ak.$$

It follows by the first paragraph that there is an F-algebra monomorphism of A into D. $\square$

Remark 5.10. It follows from Th. 5.9 that results on defectless subfields of division algebras over Henselian fields can be carried over completely to results on graded subfields of graded division algebras. This applies, for example, to the work of Morandi and Sethuraman in [MS] on Kummer subfields, and that of Brussel in [Br] on totally ramified subfields, as well as many of the results in [JW]. Also, there are of course analogues to Cor. 5.7, 5.8, Th. 5.9 where we start with a graded field instead of a Henselian field, and use Th. 5.1 instead of Th. 5.3.

§6 Compatibility with corestriction

We have shown that the maps between tame Brauer groups of Henselian valued fields and graded Brauer groups of graded fields are compatible with scalar extensions. We now show that they are compatible with the corestriction.

Let $R \subseteq S$ be graded fields (with Γ_R torsion-free, as always), with $[S : R] = k < \infty$ and S tame over R. Then, by [HW, Th. 3.9, Th. 3.11] there is a graded field $U \supseteq S$ with $[U : R] < \infty$ and U Galois over R. Let $\mathcal{G} = \mathcal{G}(U/R) = \mathcal{G}(QU/QR)$, and let $\mathcal{H} = \mathcal{G}(QU/QS) \subseteq \mathcal{G}$. Then, as $S = U \cap QS$ (see [HW, Cor. 2.5(b)]), for the elements of U fixed by $\mathcal{H}$, we have $U^{\mathcal{H}} = U \cap QU^{\mathcal{H}} = S$; so by [Gr, Th. 2.2, p. 7], U is $\mathcal{H}$-Galois over S, and $|\mathcal{G} : \mathcal{H}| = [QS : QR] = [S : R] = k$, by (3.1). Note that if N is any finite-dimensional graded U-vector space, and $\mathcal{G}$ acts on N by graded R-automorphisms and the action is semilinear (i.e. $\sigma(un) = \sigma(u) \cdot \sigma(n)$ for all $\sigma \in \mathcal{G}$, $u \in U$, $n \in N$), then

$$\begin{aligned}&N^{\mathcal{G}} \text{ is a graded } R\text{-vector space with } dim_R(N^{\mathcal{G}}) = dim_U(N), \text{ and the map} \\ &U \otimes_R N^{\mathcal{G}} \to N \text{ given by } u \otimes n \mapsto u \cdot n \text{ is a graded } U\text{-vector space isomorphism.}\end{aligned} \tag{6.1}$$

That $N^{\mathcal{G}}$ is graded is clear; the other assertions in (6.1) follow by noting that $QR \otimes_R N^{\mathcal{G}} = (QU \otimes_U N)^{\mathcal{G}}$, and applying the corresponding properties for semilinear group actions on vector spaces (since $\mathcal{G} = \mathcal{G}(QU/QR)$).

We first describe the corestriction of a finite dimensional graded S-vector space M. Let $\tau_1, \ldots, \tau_k$ be a set of representatives of the left cosets of $\mathcal{H}$ in $\mathcal{G}$. The left action of $\mathcal{G}$ on the cosets $\{\tau_1\mathcal{H}_1 \ldots \tau_n\mathcal{H}\}$ associates to each $\sigma \in \mathcal{G}$ a permutation $\widetilde{\sigma}$ of $\{1, 2, \ldots, k\}$ defined by $\sigma\tau_i\mathcal{H} = \tau_{\widetilde{\sigma}(i)}\mathcal{H}$. For each i, let $M_i = U \otimes_{S,\tau_i} M$, i.e., the scalar extension of M from S to U, with U treated as an S-algebra via $\tau_i \colon S \to U$. That is, M_i satisfies the middle linearity rule $u \otimes sm = u\tau_i(s) \otimes m$, for all $u \in U$, $s \in S$, $m \in M$. Since τ_i is a gr-isomorphism of U, M_i is a graded U-vector space. Note that for each $\sigma \in \mathcal{G}$ there is a (well-defined!) σ-semilinear graded R-vector space isomorphism $\overline{\sigma} : M_i \to M_{\widetilde{\sigma}(i)}$ given by $u \otimes m \mapsto \sigma(u) \otimes m$; clearly, $\overline{\rho\sigma} = \overline{\rho} \circ \overline{\sigma}$ for $\rho, \sigma \in \mathcal{G}$. Then, observe that there is a graded semilinear action of $\mathcal{G}$ on $M_1 \otimes_U \ldots \otimes_U M_k$ given by $\sigma(m_1 \otimes \ldots \otimes m_k) = n_1 \otimes \ldots \otimes n_k$, where $n_{\tau(i)} = \overline{\sigma}(m_i)$ for each i. Define $cor_{S/R}(M)$ to be $(M_1 \otimes_U \ldots \otimes_U M_k)^{\mathcal{G}}$. So, $cor_{S/R}(M)$ is a graded R-vector space, and (6.1) shows

$$dim_R(cor_{S/R}(M)) = [S : R]\, dim_S(M) \quad \text{and} \quad U \otimes_R cor_{S/R}(M) \cong_g M_1 \otimes_U \ldots \otimes_U M_k. \tag{6.2}$$

Now, if A is a GCSA over S, then $cor_{S/R}(A)$, as just defined but with A replacing M, is a graded R-algebra since $\mathcal{G}$ acts on $A_1 \otimes_U \ldots \otimes_U A_k$ (where $A_i = U \otimes_{S,\tau_i} A$) by algebra automorphisms. Furthermore, the graded U-algebra isomorphism $U \otimes_R cor_{S/R}(A) \cong_g A_1 \otimes_U \ldots \otimes_U A_k$ shows that $cor_{S/R}(A)$ must be a GCSA over R. The construction of $cor_{S/R}(A)$ is clearly independent of the choice of coset representatives τ_i of $\mathcal{H}$ in $\mathcal{G}$, and, by the usual argument, is also independent (up to graded isomorphism) of the choice of Galois graded field extension U of R containing S. If B is another GCSA over S, then clearly $cor_{S/R}(A) \otimes_R cor_{S/R}(B) \cong_g cor_{S/R}(A \otimes_S B)$. Also, if M is a finite dimensional graded S-vector space, then clearly $U \otimes_{S,\tau_i} End_S(M) \cong_g End_U(U \otimes_{S,\tau_i} M)$, with compatible $\mathcal{G}$-actions. It follows easily that $cor_{S/R}(End_S(M)) \cong_g End_R(cor_{S/R}(M))$. Thus, the corestriction $cor_{S/R}$ yields a well-defined group homomorphism $cor_{S/R} \colon GBr(S) \to GBr(R)$. It is clear from the definitions and a dimension count that

$$QR \otimes_R cor_{S/R}(A) \cong cor_{QS/QR}(QS \otimes_S A). \tag{6.3}$$

Likewise, for any total ordering on Γ_S (with corresponding ordering on $\Gamma_R \subseteq \Gamma_S$), if HS is the Henselization of QS with respect to the valuation on QS induced by the ordering on Γ_S, then

$$HR \otimes_R cor_{S/R}(A) \cong cor_{HS/HR}(HS \otimes_S A). \tag{6.4}$$

It follows from (6.3) and the injectivity of $GBr(U/R) \to Br(QU/QR)$ that we have a commutative diagram

$$\begin{array}{ccc} H^2(\mathcal{H}, U^*) & \xrightarrow{cor} & H^2(\mathcal{G}, U^*) \\ \cong \downarrow & & \cong \downarrow \\ GBr(U/S) & \xrightarrow[cor_{S/R}]{} & GBr(U/R) \end{array} \tag{6.5}$$

where the top map in (6.5) is the cohomological corestriction.

Theorem 6.1. *Let F be a field with Henselian valuation, and K a finite degree tame field extension of F. Then, there is a commutative diagram,*

$$\begin{array}{ccccc} TBr(K) & \xrightarrow{\alpha_K} & GBr(GK) & \xrightarrow{\beta_K} & TBr(HGK) \\ \big\downarrow{\scriptstyle cor_{K/F}} & & \big\downarrow{\scriptstyle cor_{GK/GF}} & & \big\downarrow{\scriptstyle cor_{HGK/HGF}} \\ TBr(F) & \xrightarrow{\alpha_K} & GBr(GF) & \xrightarrow{\beta_K} & TBr(HGF) \end{array} \tag{6.6}$$

where α, β are the maps of (5.4).

Proof. Let $\gamma_F = \beta_F \circ \alpha_F$ and $\gamma_K = \beta_k \circ \alpha_K$, which are isomorphisms by Th. 5.3. The outer rectangle of (6.6) is commutative since by Th. 5.3 γ_F and γ_K correspond to maps in cohomology, the cohomological corestriction is functorial with respect to $\mathcal{G}$-module homomorphisms, and the cohomological corestriction is consistent with the algebra corestriction. The right inner square of (6.6) is commutative by (6.4). Since $\alpha_K = \beta_K \circ \gamma_K^{-1}$ and $\alpha_F = \beta_F \circ \gamma_F^{-1}$, it follows that the left inner square of (6.6) is also commutative. $\square$

Proposition 6.2. *Let $R \subset S$ be a finite degree tame extension of graded fields (with Γ_R torsion-free). Let E be a GCDA over S, and let A be the underlying GCDA of $cor_{S/R}(E)$ (i.e., $[A]_g = [cor_{S/R}(E)]_g$ in $GBr(R)$). Then,*

$$\Gamma_A \subseteq \Gamma_{cor_{S/R}(E)} \subseteq \Gamma_E \tag{6.7}$$

and

$$Z(A_0) \subseteq N^{1/k}, \tag{6.8}$$

where N is the normal closure of $Z(E_0)$ over R_0 and $k = \exp(ker(\theta_E))$, where θ_E is the map of (2.2) .

Proof. The inclusions in (6.7) are evident from the definitions. For (6.8), let $\leq$ be any total ordering on Γ_R, and HR the Henselization of QR with respect to the valuation on QR determined by the ordering on Γ_R. Then HA $(= HR \otimes_R A)$ is a CDA over HR (see after (4.4) above), and $[HA] = [cor_{HS/HR}(HE)]$ in $Br(HR)$ by (6.4). Since $\overline{HE} \cong E_0$ and $\overline{HR} \cong R_0$ and the map θ_{HE} of (4.8) corresponds to θ_E (so $ker(\theta_{HE}) \cong ker(\theta_E)$), it follows by [H, Th. 18] that $Z(\overline{HA}) \subseteq N^{1/k}$. Then (6.8) follows as $A_0 \cong \overline{HA}$. $\square$

The value of k given in Prop. 6.2 can be improved by taking into account $[S_0 : R_0]$ and which roots of unity lie in R_0. See [H, Th. 18] for the full statement.

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Department of Mathematics
College of Science
Korea University
5–1, Anam-dong, Sungbuk-ku
Seoul 136–701
Korea
e-mail: `yhwang@semi.korea.ac.kr`

Department of Mathematics, 0112
University of California, San Diego
9500 Gilman Drive
La Jolla, California 92093-0112
USA
e-mail: `arwadsworth@ucsd.edu`