ON A CONJECTURE OF LE BRUYN

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Abstract. Given a generic field extension F/k of degree $n > 3$ (i.e. the Galois group of the normal closure of F is isomorphic to the symmetric group S_n), we prove that the norm torus, defined as the kernel of the norm map $N\colon R_{F/k}(\mathbb{G}_{\ textm}) \to \mathbb{G}_{\mathrm{m}}$, is not rational over k.

Given an arbitrary field k, we call a separable extension F/k of degree n *generic* if the Galois group $G = \mathrm{Gal}(L/k)$ of the normal closure L of F over k is isomorphic to the symmetric group S_n. We consider the norm map $N\colon F^* \to k^*$. The kernel of N can be regarded as the set of k-points of an affine algebraic k-variety T called *norm torus*. Using the Weil symbol of restriction of scalars, we write T as the kernel of $R_{F/k}(\mathbb{G}_{\mathrm{m}}) \to \mathbb{G}_{\mathrm{m},k}$ where $\mathbb{G}_{\mathrm{m}}$ stands for the multiplicative group. If the extension F/k is generic, the norm torus is also called generic and is denoted by $T_{F/k}$, or just T_n if it does not lead to any confusion.

In [LB], assuming $n > 3$, Le Bruyn proves that the generic norm torus T_n is non-rational over k whenever n is prime, and states a conjecture that T_n is never k-rational except, possibly, for $n = 6$. Our goal is to prove the above conjecture (including the case $n = 6$). Recall that T is called *stably rational* if there is a rational variety T' such that $T \times T'$ is rational.

Theorem. *With the above notation, $T_n (n > 3)$ is never stably rational over k.*

Remark. The result might look a little bit surprising in view of good arithmetic properties of generic norm tori: in particular, if k is a number field, they are known to satisfy weak approximation property and their principal homogeneous spaces satisfy the Hasse principle. Moreover, for the case when n is prime, T_n is known to be a direct factor of a rational variety [CT/S2]. Note that the result cannot be ameliorated in the sense that for $n = 2$ or 3 the torus T_n is of dimensio n 1 or 2 and hence rational [V], 4.73, 4.74.

The proof follows from the lemmas below. Throughout we denote by $M_n = \mathrm{Hom}(T_n, \mathbb{G}_{\mathrm{m}})$ the group of rational characters of T_n viewed as a G-module. By definition, there is an exact sequence of G-modules

$$0 \to \mathbb{Z} \to P_n \to M_n \to 0 \tag{1}$$

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where $P_n = \mathbb{Z}[G/H]$ is a permutation module, $G = S_n$, $H = \mathrm{Gal}(L/F)$ is isomorphic to S_{n-1}. The following lemma is the key one.

Lemma 1. *Let $n = rs$ with arbitrary $r, s > 1$, and let F/k be a generic extension of degree n. If $T_{F/k} = T_n$ is stably rational over k, there is a generic extension K/E of degree r such that $T_{K/E} = T_r$ is stably rational over E.*

Proof. Take a subgroup $U = S_r \subset S_n$ embedded in such a way that P_n restricted to U is a direct sum $\underbrace{P_r \oplus \cdots \oplus P_r}_{s \text{ times}}$. (This simply means that we partition $\{1, \ldots, n\}$ into s disjoint subset s of cardinality r and let U act in a standard way on each of these subsets.) We then regard (1) as a sequence of U-modules and notice that M_n restricted to U splits into a direct sum:

$$(M_n)|_U = M_r \oplus \underbrace{P_r \oplus \cdots \oplus P_r}_{(s-1) \text{ times}}. \tag{2}$$

In the language of tori, (2) reads as follows: let $E = L^U$ be the fixed subfield of U, then the E-torus $T_E = T \times_k E$ is isomorphic to a direct product of $T_r = \ker[R_{K/E}(\mathbb{G}_{\mathrm{m}}) \to \mathbb{G}_{\mathrm{m},E}]$ and a quasi-split torus $S = \prod_{i=1}^{s-1} R_{K/E}(\mathbb{G}_{\mathrm{m}})$ where K/E is a generic extension of degree r, $K = L^V$, $V \subset U$, $V \cong S_{r-1}$. By assumption, T is stably rational over k, hence T_E is stably rational over E. Since any quasi-split torus is rational, we are done. $\square$

Lemma 2 (Le Bruyn). *If $n > 3$ is a prime number, T_n is not stably rational.*

Proof. See [LB]. $\square$

Before stating the next lemma, we recall that the group

$$Ш^2_\omega(G, M) = \ker[H^2(G, M) \to \prod_{g \in G} H^2(\langle g \rangle, M)]$$

(where M stands for the character module of an algebraic torus T defined over k and split over L, $G = \mathrm{Gal}(L/k)$), is a birational invariant of T. To be more precise, this group is zero whenever T is stably rational over k. Here is another useful description of the above invariant: consider a flasque resolution of M, i.e. an exact sequence of G-modules

$$0 \to M \to S \to N \to 0$$

where S is a permutation module and N is a flasque module (the latter means that $H^{-1}(G', N) = 0$ for all subgroups $G' \subseteq G$), then $Ш^2_\omega(G, M) \cong H^1(G, N)$. See [V], 4.61, [CT/S1] for more details.

Lemma 3. *If n is a square, T_n is not stably rational.*

Proof. Let $n = m^2$. Take a subgroup $U = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \subset S_n$ embedded in such a way that the module P from sequence (1) viewed as a U-module is isomorphic to $\mathbb{Z}[U]$. In other words, we choose U generated by

$$\sigma = (1 \quad 2 \ldots m)(m+1 \quad m+2 \ldots 2m) \ldots (n-m+1 \quad n-m+2 \ldots n),$$
$$\tau = (1 \quad m+1 \ldots n-m+1)(2 \quad m+2 \ldots n-m+2) \ldots (m \quad 2m \ldots m^2).$$

Then M_n, regarded as a U-module, is none other than $\hat{J} = \mathbb{Z}[U]/\mathbb{Z}$, the character module of the norm torus $J = \ker[R_{L/E}(\mathbb{G}_{\mathrm{m}}) \to \mathbb{G}_{\mathrm{m},E}]$ where $E = L^U$. It is well known that J is not rational over E because $Ш^2_\omega(U, \hat{J}) = \mathbb{Z}/p\mathbb{Z}$. Since $T_n \times_k E = J$, we conclude that T_n cannot be stably rational over k. □

Corollary (Saltman, Snider). *If n is divisible by a square, T_n is not stably rational.*

Proof. Combine Lemma 1 and Lemma 3. □

Lemma 4. *The torus T_6 is not stably rational.*

Proof. Take $U = \mathbb{Z}/2 \times \mathbb{Z}/2 \subset S_6$ generated by (12)(34) and (34)(56). We observe that U coincides with the Sylow 2-subgroup of the alternating group A_4 embedded into S_6 via its action on the edges of tetrahedron. Let $M = M_6$ be the module of characters of T_6 defined by sequence (1) with $G = S_6$, $H = S_5$. It is known that $Ш^2_\omega(A_4, M) = \mathbb{Z}/2\mathbb{Z}$ ([D/P], Lemma 13). This implies $Ш^2_\omega(U, M) \neq 0$. Indeed, assume the contrary. Then, since any Sylow 3-subgroup V of A_4 is cyclic, one has $Ш^2_\omega(V, M) = 0$, and vanishing of $Ш^2_\omega(U, M)$ would imply vanishing of $Ш^2_\omega(A_4, M)$ (one may apply the above interpretation of $Ш^2_\omega(G, M)$ as $H^1(G, N)$ to the case $G = A_6$ and use the fact that the restriction to a Sylow p-subgroup is injective on the p-component of H^1). □

Remark. Of course, one may give a more direct proof of Lemma 4 without referring to [D/P], either by a straightforward computation of $Ш^2_\omega(U, M)$ (which goes much simpler than for A_4), or by constructing an exact sequence of U-m odules

$$0 \to M_a \to M \to \mathbb{Z} \oplus \mathbb{Z} \to 0$$

with M_a the character module of an anisotropic torus T_a which, in our case, turns out to be $\mathbb{Z}[U]/\mathbb{Z}$; by [V], 4.22, the latter exact sequence induces a birational equivalence of tori $T_n \sim T_a \times \mathbb{G}^2_{\mathrm{m}}$, whence the result.

Proof of the Theorem. We are now ready to prove the Theorem. Indeed, the above Corollary reduces the problem to the case when n is square-free, and Lemmas 1 and 2 englobe all n having a prime divisor greater than 3. We thus have to apply Lemma 4 for the only remaining case $n = 6$. □

Concluding remark. Our theorem can (and should) be viewed in a broader context. Namely, one can extend it to generic tori in (almost absolutely) simple groups. Indeed, the above result corresponds to the case of an inner form of a simply connected group of type A_{n-1}. Such a generalization to the other types of inner and outer forms of simply connected and adjoint groups is the subject of our forthcoming paper.

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