

GROUPS OF TYPE E_7 OVER ARBITRARY FIELDS

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ABSTRACT. Freudenthal triple systems come in two flavors, degenerate and nondegenerate. The best criterion for distinguishing between the two which is available in the literature is by descent. We provide an identity which is satisfied only by nondegenerate triple systems. We then use this to define algebraic structures whose automorphism groups produce all adjoint algebraic groups of type E_7 over an arbitrary field of characteristic $\neq 2, 3$.

As an application, we provide a construction of adjoint groups with Tits algebras of index 2. We use this construction to fully describe the degree one connecting homomorphism on Galois cohomology for all adjoint groups of type E_7 over a real-closed field.

A useful strategy for studying simple (affine) algebraic groups over arbitrary fields has been to describe such a group as the group of automorphisms of some algebraic object. We restrict our attention to fields of characteristic $\neq 2, 3$. The idea is that these algebraic objects are easier to study, and their properties correspond to properties of the group one is interested in. Weil described all groups of type A_n , B_n , C_n , 1D_n , and 2D_n in this manner in [Wei60]. Similar descriptions were soon found for groups of type F_4 (as automorphism groups of Albert algebras) and G_2 (as automorphism groups of Cayley algebras). Recently, groups of type 3D_4 and 6D_4 have been described in [KMRT98, §43] as groups of automorphisms of trialitarian central simple algebras. The remaining groups are those of types E_6 , E_7 , and E_8 .

As an attempt to provide an algebraic structure associated to groups of type E_7 , Freudenthal introduced a new kind of algebraic structure in [Fre54, §4], which was later studied axiomatically in [Mey68], [Bro69], and [Fer72]. These objects, called Freudenthal triple systems, come in two flavors: degenerate and nondegenerate. The automorphism groups of the nondegenerate ones provide all simply connected groups of type E_7 with trivial Tits algebras over an arbitrary field. In fact, more is true: they are precisely the G -torsors for G simply connected of type E_7 with trivial Tits algebras.

One issue that has not been addressed adequately in the study of triple systems is how to distinguish between the two kinds. A triple system consists of a 56-dimensional vector space endowed with a nondegenerate skew-symmetric bilinear form and a quartic form (see 1.1 for a complete definition), and we say that the triple system is nondegenerate precisely when this quartic form is irreducible when we extend scalars to a separable closure of the base field. This seems to be the best criterion available in the literature to distinguish between the two types. In Section 2, we show that one of the identities which nondegenerate triple

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systems are known to satisfy is not satisfied by degenerate ones, thus providing a criterion for differentiating between the two types which does not require enlarging the base field.

In Section 3, we define algebraic structures whose groups of automorphisms produce all groups of type E_7 up to isogeny. Thanks to the preceding result distinguishing between degenerate and nondegenerate triple systems, no Galois descent is needed for this definition. We call these structures gifts (short for *generalized Freudenthal triple systems*). They are triples (A, σ, π) such that A is a central simple F -algebra of degree 56, σ is a symplectic involution on A , and $\pi: A \rightarrow A$ is an F -linear map satisfying certain axioms (see 3.2 for a full definition). We also show an equivalence of categories between the category of gifts over an arbitrary field F and the category of adjoint (equivalently, simply connected) groups of type E_7 over F . A description of the flag (a.k.a. projective homogeneous) varieties of an arbitrary group of type E_7 is then easily derived, see Section 4.

We also give a construction of gifts with algebra component A of index 2 (which is equivalent to constructing groups of type E_7 with Tits algebra of index 2 and correspond to Tits' construction of analogous Lie algebras in [Tit66a], or see [Jac71, §10]) and we describe the involution σ explicitly in this case.

As a consequence of all this, in the final section we completely describe the connecting homomorphism on Galois cohomology in degree one for groups of type E_7 over a real-closed field. The interest in this connecting homomorphism arises from the Hasse Principle Conjecture II [BFP98], which generalizes the Hasse principle for number fields to fields F such that the cohomological dimension of $F(\sqrt{-1}) \leq 2$. This conjecture remains open for groups of type E_7 , and a possible ingredient in any proof would be a local-global principle for this connecting homomorphism (similar to what was done for the classical groups in [BFP98] and for triality groups in [Gar99b, 3.8]). Our result specifies the local values.

Notations and conventions. All fields that we consider will have characteristic $\neq 2, 3$. For a field F , we write F_s for its separable closure.

For g an element in a group G , we write $\text{Int}(g)$ for the automorphism of G given by $x \mapsto gxg^{-1}$.

For X a variety over a field F and K any field extension of F , we write $X(K)$ for the K -points of X .

When we say that an affine algebraic group (scheme) G is *simple*, we mean that it is absolutely almost simple in the usual sense (i.e., $G(F_s)$ has a finite center and no noncentral normal subgroups). For any simple algebraic group G over a field F , there is a unique minimal finite Galois field extension L of F such that G is of inner type over L (i.e., the absolute Galois group of L acts trivially on the Dynkin diagram of G). We call L the *inner extension* for G .

We write $\mathbb{G}_{m,F}$ for the algebraic group whose F -points are F^* .

We will also follow the usual conventions for Galois cohomology and write $H^i(F, G) := H^i(\text{Gal}(F_s/F), G(F_s))$ for G any algebraic group over F , and similarly for the cocycles $Z^1(F, G)$. For more information about Galois cohomology, see [Ser79] and [Ser94].

We follow the notation in [Lam73] for quadratic forms.

For $a, b \in F^*$, we write $(a, b)_F$ for the (associative) *quaternion F -algebra* generated by skew-commuting elements i and j such that $i^2 = a$ and $j^2 = b$, please see [Lam73] or [Dra83, §14] for more information.

For I a right ideal of in a central simple F -algebra A , we define the *rank* of I to be $(\dim_F I) / \deg A$. Thus when A is split, so that we may write $A \cong \text{End}_F(V)$ for some F -vector space V , $I = \text{Hom}_F(V, U)$ for some subspace U of V and the rank of I is precisely the dimension of U .

1. BACKGROUND ON TRIPLE SYSTEMS

Definition 1.1. (See, for example, [Fer72, p. 314] or [Gar99a, 3.1]) A (*simple*) *Freudenthal triple system* is a 3-tuple (V, b, t) such that V is a 56-dimensional vector space, b is a nondegenerate skew-symmetric bilinear form on V , and t is a trilinear product $t: V \times V \times V \rightarrow V$.

We define a 4-linear form $q(x, y, z, w) := b(x, t(y, z, w))$ for $x, y, z, w \in V$, and we require that

FTS1: q is symmetric,

FTS2: q is not identically zero, and

FTS3: $t(t(x, x, x), x, y) = b(y, x)t(x, x, x) + q(y, x, x, x)x$ for all $x, y \in V$.

We say that such a triple system is *nondegenerate* if the quartic form $v \mapsto q(v, v, v, v)$ on V is absolutely irreducible (i.e., irreducible over a separable closure of the base field) and *degenerate* otherwise.

A similarity of triple systems is a map $f: (V, b, t) \xrightarrow{\sim} (V', b', t')$ defined by an F -vector space isomorphism $f: V \xrightarrow{\sim} V'$ such that $b'(f(x), f(y)) = \lambda b(x, y)$ and $t'(f(x), f(y), f(z)) = \lambda f(t(x, y, z))$ for all $x, y, z \in V$ and some $\lambda \in F^*$. If $\lambda = 1$ we say that f is an *isometry* or an *isomorphism*. For a triple system \mathfrak{M} , we write $\text{Inv}(\mathfrak{M})$ for the algebraic group whose F -points are the isometries of \mathfrak{M} .

Note that since b is nondegenerate, FTS1 implies that t is symmetric.

One can linearize FTS3 a little bit to get an equivalent axiom that will be of use later. Specifically, replacing x with $x + \lambda z$, expanding using linearity, and taking the coefficient of λ , one gets the equivalent formula

$$\mathbf{FTS3'} \quad t(t(x, x, z), z, y) + t(t(x, z, z), x, y) = zq(x, x, z, y) + xq(x, z, z, y) + b(y, z)t(x, x, z) + b(y, x)t(x, z, z).$$

Example 1.2. (Cf. [Bro69, p. 94], [Mey68, p. 172]) Let W be a 27-dimensional F -vector space endowed with a non-degenerate skew-symmetric bilinear form s and set

$$(1.3) \quad V := \begin{pmatrix} F & W \\ W & F \end{pmatrix}.$$

For

$$(1.4) \quad x := \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} \quad \text{and} \quad y := \begin{pmatrix} \gamma & k \\ k' & \delta \end{pmatrix}$$

set

$$b(x, y) := \alpha\delta - \beta\gamma + s(j, k') + s(j', k).$$

We define the *determinant map* $\det: V \longrightarrow F$ by

$$\det(x) := \alpha\beta - s(j, j')$$

and set

$$t(x, x, x) := 6 \det(x) \begin{pmatrix} -\alpha & j \\ -j' & \beta \end{pmatrix}.$$

Then (V, b, t) is a Freudenthal triple system. Since

$$q(x, x, x, x) := 12 \det(x)^2,$$

it is certainly degenerate, and we denote it by \mathfrak{M}_s . By [Bro69, §4] or [Mey68, §4], all degenerate triple systems are forms of one of these.

Example 1.5. For J an Albert F -algebra, there is a nondegenerate triple system denoted by $\mathfrak{M}(J)$ whose underlying F -vector space is $V = \begin{pmatrix} F & J \\ J & F \end{pmatrix}$. Explicit formulas for the b , t , and q for this triple system can be found in [Bro69, §3], [Mey68, §6], [Fer72, §1], and [Gar99a, 3.2]. For J^d the split Albert F -algebra, we set $\mathfrak{M}^d := \mathfrak{M}(J^d)$. It is called the *split* triple system because $\text{Inv}(\mathfrak{M}^d)$ is the split simply connected algebraic group of type E_7 over F [Gar99a, 3.5]. By [Bro69, §4] or [Mey68, §4] every nondegenerate triple system is a form of \mathfrak{M}^d .

Remark 1.6. Although it is not clear precisely what the structure of the automorphism group of a degenerate triple system is, a few simple observations can be made which make it appear to be not very interesting from the standpoint of simple algebraic groups.

Since by definition any element of $\text{Inv}(\mathfrak{M}_s)$ must preserve the quartic form q , it must also be a similarity of the quadratic form \det with multiplier ± 1 . This defines a map $\text{Inv}(\mathfrak{M}_s) \longrightarrow \mu_2$ which is surjective since $\varpi \in \text{Inv}(\mathfrak{M}_s)(F)$ maps to -1 , where

$$\varpi \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} := \begin{pmatrix} -\beta & j' \\ j & \alpha \end{pmatrix}.$$

So $\text{Inv}(\mathfrak{M}_s)$ is not connected.

Also, we can make some bounds on the dimension. Specifically, we define a map $f: \mathbb{G}_{m,F} \times W \times GL(W) \longrightarrow \text{Inv}(\mathfrak{M}_s)$ by

$$f(c, u, \phi) \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} := \begin{pmatrix} c\alpha & \phi(j) \\ \alpha u + \phi^\dagger(j') & \frac{1}{c}(\beta + s(\phi(j), u)) \end{pmatrix},$$

where $\phi^\dagger = \sigma(\phi)^{-1}$ for σ the involution on $\text{End}_F(W)$ which is adjoint to s . (So $s(\phi(w), \phi^\dagger(w')) = s(w, w')$ for all $w, w' \in W$.) Then f is an injection of varieties, but

$$f(c, u, \phi)f(d, v, \psi) = f(cd, du + \phi^\dagger(v), \phi\psi),$$

so it is not a group homomorphism. It does, however, restrict to be a morphism of algebraic groups on $\mathbb{G}_{m,F} \times \{0\} \times GL(W)$, so $\text{Inv}(\mathfrak{M}_s)$ contains a split torus of rank 28. This map f is also not surjective since for any $u \neq 0$, the map

$$\varpi f(c, u, \phi) \varpi^{-1} \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = \begin{pmatrix} \frac{1}{c}(\alpha - s(\phi(j'), u)) & \beta u + \phi^\dagger(j) \\ \phi(j') & \beta c \end{pmatrix}$$

is not in the image of f . For an upper bound, we observe that the identity component of $\text{Inv}(\mathfrak{M}_s)$ is contained in the isometry group of \det (which is of type D_{28} , hence is 1540-dimensional), and this containment is proper, as was remarked in [Fer72, p. 326]. Thus

$$757 < \dim \text{Inv}(\mathfrak{M}_s)^+ < 1540.$$

2. AN IDENTITY

For a Freudenthal triple system (V, b, t) over F , we define an F -vector space map $p : V \otimes_F V \longrightarrow \text{End}_F(V)$ given by

$$(2.1) \quad p(u \otimes v)w := t(u, v, w) - b(w, u)v - b(w, v)u.$$

In the case where the triple system is nondegenerate, Freudenthal [Fre54, 4.2] also defined a map $V \otimes_F V \longrightarrow \text{End}_F(V)$ which he denoted by \times . The obvious computation shows that his map is related to our map p by

$$(2.2) \quad 8v \times v' = p(v \otimes v').$$

Theorem 2.3. *Let $\mathfrak{M} := (V, b, t)$ be a Freudenthal triple system with map p as given above. Then \mathfrak{M} is nondegenerate if and only if it satisfies the identity*

$$(2.4) \quad \text{tr}(p(x \otimes x)p(y \otimes y)) = 24(q(x, x, y, y) - 2b(y, x)^2)$$

for all $x, y \in V$, where tr is the usual trace form on $\text{End}_F(V)$.

To simplify some of our formulas, we define the *weighted determinant*, $\text{wdet} : \mathfrak{M}_s \longrightarrow F$, to be given by

$$\text{wdet}(x) := 3\alpha\beta - s(j, j')$$

for x and y as in (1.4). We also bilinearize the determinant to define

$$\det(x, y) := \det(x + y) - \det(x) - \det(y).$$

Proof of Theorem 2.3: If \mathfrak{M} is nondegenerate, then the conclusion is [Fre63, 31.3.1] or it can be easily derived from [Mey68, 7.1]. So we may assume that $\mathfrak{M} = \mathfrak{M}_s$, the degenerate triple system from Example 1.2, and show that it doesn't satisfy (2.4).

We first compute the value of the left side of (2.4). For x and y as in (1.4), we can directly calculate the action of $p(x \otimes x)p(y \otimes y)$ on each of the four entries of our matrix as in (1.3). Since we are interested in the trace of this operator, we only record the projection onto the entry that we are looking at.

$$(2.5) \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mapsto 4[\text{wdet}(x)\text{wdet}(y) - 4\alpha\delta s(j, k')]$$

$$(2.6) \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mapsto 4 [\text{wdet}(x) \text{wdet}(y) + 4\beta\gamma s(j', k)]$$

$$(2.7) \quad \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \mapsto 4 \left[\begin{array}{c} \det(x) \det(y) m + 4(\alpha\delta - s(k, j')) s(k', m) j \\ + 2 \det(x) s(k', m) k + 2 \det(y) s(j', m) j \end{array} \right]$$

$$(2.8) \quad \begin{pmatrix} 0 & 0 \\ m' & 0 \end{pmatrix} \mapsto 4 \left[\begin{array}{c} \det(x) \det(y) m' - 4(\beta\gamma - s(j, k')) s(k, m') j' \\ - 2 \det(x) s(k, m') k' - 2 \det(y) s(j, m') j' \end{array} \right]$$

Since s is nondegenerate, it induces an identification of V with its dual, V^* , by sending $x \in V$ to the map $v \mapsto s(x, v)$. We may also identify $V \otimes_F V^*$ with $\text{End}_F(V)$, and combining these two identifications provides an isomorphism $\varphi_s: V \otimes_F V \xrightarrow{\sim} \text{End}_F(V)$ given by

$$(2.9) \quad \varphi_s(x \otimes y)w = xs(y, w),$$

cf. [KMRT98, 5.1]. One has $\text{tr}(\varphi_s(x \otimes y)) = s(y, x)$.

With that notation, the terms in the brackets of (2.7) and (2.8) with coefficient 4 give the maps

$$(\alpha\delta - s(k, j'))\varphi_s(j \otimes k') \text{ and } -(\beta\gamma - s(j, k'))\varphi_s(j' \otimes k)$$

on W , which have traces

$$(\alpha\delta - s(k, j'))s(k', j) \text{ and } (\beta\gamma - s(j, k'))s(j', k)$$

respectively. Similarly, the terms with coefficient 2 give the maps

$$\det(x)\varphi_s(k \otimes k') + \det(y)\varphi_s(j \otimes j') \text{ and } -\det(x)\varphi_s(k' \otimes k) - \det(y)\varphi_s(j' \otimes j)$$

whose sum has trace

$$2[\det(x)s(k', k) + \det(y)s(j', j)].$$

Adding all of this up, we see that

$$(2.10) \quad \begin{aligned} \frac{1}{8}\text{tr}(p(x \otimes x)p(y \otimes y)) &= \text{wdet}(x) \text{wdet}(y) + 2\beta\gamma s(j', k) - 2\alpha\delta s(j, k') \\ &\quad + 27 \det(x) \det(y) \\ &\quad + 2[(\beta\gamma - s(j, k'))s(j', k) + (\alpha\delta - s(k, j'))s(k', j)] \\ &\quad + 2[\det(x)s(k', k) + \det(y)s(j', j)] \\ &= 3q(x, x, y, y) - b(y, x)^2 + 20 \det(x) \det(y) - 5 \det(x, y)^2 \end{aligned}$$

Taking the difference of the left side of (2.10) and one-eighth of the left side of (2.4), we get the quartic polynomial

$$(2.11) \quad 5b(y, x)^2 + 20 \det(x) \det(y) - 5 \det(x, y)^2.$$

We plug

$$x := \begin{pmatrix} 0 & j \\ j' & 0 \end{pmatrix} \text{ and } y := \begin{pmatrix} 0 & k \\ k' & 0 \end{pmatrix}$$

into (2.11), where we have chosen $j, j', k,$ and k' such that

$$s(j, k') = s(j', k) = 0 \text{ and } s(j, j') = s(k, k') = 1.$$

Then $b(y, x) = 0$ and $\det(x) = \det(y) = -1$. Since $\det(x + y) = 2$, we have $\det(x, y) = 0$. Thus the value of (2.11) is 20 and (2.4) does not hold for degenerate triple systems. \square

It really was important that we allow $x \neq y$ in (2.4), for (2.10) shows that all triple systems satisfy the identity

$$(2.12) \quad \text{tr}(p(x \otimes x)^2) = 24q(x, x, x, x).$$

3. GIFTS

In this section we will define an object we call a gift, such that every adjoint group of type E_7 is the automorphism group of some gift. We must first have some preliminary definitions.

Suppose that (A, σ) is a central simple algebra with a symplectic involution σ . The sandwich map

$$\text{Sand}: A \otimes_F A \longrightarrow \text{End}_F(A)$$

defined by

$$\text{Sand}(a \otimes b)(x) = axb \text{ for } a, b, x \in A$$

is an isomorphism of F -vector spaces by [KMRT98, 3.4]. Following [KMRT98, §8.B], we define an involution σ_2 on $A \otimes_F A$ which is defined implicitly by the equation

$$\text{Sand}(\sigma_2(u))(x) = \text{Sand}(u)(\sigma(x)) \text{ for } u \in A \otimes A, x \in A.$$

Suppose now that A is split. Then $A \cong \text{End}_F(V)$ for some F -vector space V , and σ is the adjoint involution for some nondegenerate skew-symmetric bilinear form b on V (i.e., $b(fx, y) = b(x, \sigma(f)y)$ for all $f \in \text{End}_F(V)$). As in (2.9), we have an identification $\varphi_b: V \otimes_F V \xrightarrow{\sim} \text{End}_F(V)$, and by a straightforward computation (or see [KMRT98, 8.6]), σ_2 is given by

$$(3.1) \quad \sigma_2(\varphi_b(x_1 \otimes x_2) \otimes \varphi_b(x_3 \otimes x_4)) = -\varphi_b(x_1 \otimes x_3) \otimes \varphi_b(x_2 \otimes x_4)$$

for $x_1, x_2, x_3, x_4 \in V$.

Finally, for $f: A \longrightarrow A$ an F -linear map, we define $\widehat{f}: A \otimes_F A \longrightarrow A$ by

$$\widehat{f}(a \otimes b) = f(a)b.$$

Definition 3.2. A *gift* \mathfrak{G} over a field F is a triple (A, σ, π) such that A is a central simple F -algebra of degree 56, σ is a symplectic involution on A , and $\pi: (A, \sigma) \longrightarrow (A, \sigma)$ is an F -vector space map such that

- G1:** $\sigma\pi(a) = \pi\sigma(a) = -\pi(a)$,
- G2:** $a\pi(a) \neq 2a^2$ for some $a \in \text{Skew}(A, \sigma)$,
- G3:** $\pi(\pi(a)a) = 0$ for all $a \in \text{Skew}(A, \sigma)$,
- G4:** $\widehat{\pi} - \widehat{\sigma} - \widehat{Id} = -(\widehat{\pi} - \widehat{\sigma} - \widehat{Id})\sigma_2$, and
- G5:** $\text{Trd}_A(\pi(a)\pi(a')) = -24 \text{Trd}_A(\pi(a)a')$ for all $a, a' \in (A, \sigma)$.

Such a strange definition demands an example.

Example 3.3. Suppose that $\mathfrak{M} = (V, b, t)$ is a nondegenerate Freudenthal triple system over F . Let $\text{End}(\mathfrak{M}) := (\text{End}_F(V), \sigma, \pi)$ where σ is the involution on $\text{End}_F(V)$ adjoint to b . Using the identification $\varphi_b: V \otimes V \longrightarrow \text{End}_F(V)$, we define $\pi: A \longrightarrow A$ by $\pi := p\varphi_b^{-1}$, where p is as in (2.1).

We show that $\text{End}(\mathfrak{M})$ is a gift. A quick computation shows that

$$-b(\pi(\varphi_b(x \otimes y))z, w) = b(z, \pi(\varphi_b(x \otimes y))w) = -b(\pi(\varphi_b(y \otimes x))z, w),$$

which demonstrates G1, since $\sigma\varphi_b(x \otimes y) = -\varphi_b(y \otimes x)$.

Suppose that G2 fails. Then for $v \in V$, we set $a := \varphi_b(v \otimes v)$ and observe that $a^2 = 0$, so

$$0 = \varphi_b(v \otimes v)\pi(\varphi_b(v \otimes v))v = q(v, v, v, v)v.$$

Since this holds for all $v \in V$, q is identically zero, contradicting FTS2. Thus G2 holds.

Since elements a like in the preceding paragraph span $\text{Skew}(A, \sigma)$, in order to prove G3 we may show that

$$(3.4) \quad \pi(\pi(a)a' + \pi(a')a)y = 0,$$

where

$$(3.5) \quad a = \varphi_b(x \otimes x) \text{ and } a' = \varphi_b(z \otimes z).$$

A direct expansion of the left hand side of (3.4) shows that it is equivalent to FTS3'.

Using just the bilinearity and skew-symmetry of b and the trilinearity of t , G4 is equivalent to

$$t(x, y, x') = t(x, x', y) \text{ for all } x, x', y \in V.$$

Thus G4 holds by FTS1.

Finally, consider G5. If a is symmetric, then by G1 $\pi(a) = 0$ and the identity holds. If a' is symmetric then the left-hand side of G5 is again zero by G1. Since σ and Trd_A commute, we have

$$\text{Trd}_A(\pi(a)a') = \sigma(\text{Trd}_A(\pi(a)a')) = -\text{Trd}_A(a'\pi(a)) = -\text{Trd}_A(\pi(a)a'),$$

so the right-hand side of G5 is also zero. Consequently, by the bilinearity of G5, we may assume that a and a' are skew-symmetric, and we may further assume that a and a' are as given in (3.5). Then G5 reduces to (2.4).

It turns out that this construction produces all Freudenthal triple systems with the central simple algebra component split.

Lemma 3.6. *Suppose that $\mathfrak{G} = (A, \sigma, \pi)$ is a gift over F . Then $\mathfrak{G} \cong \text{End}(\mathfrak{M})$ for some nondegenerate Freudenthal triple system over F if and only if A is split.*

Proof: One direction is done by Example 3.3, so suppose that (A, σ, π) is a gift with A split. Then we may write $A \cong \text{End}_F(V)$ for some 56-dimensional F -vector space V such that V is endowed with a nondegenerate skew-symmetric bilinear form b and σ is the involution on A which is adjoint to b . We define $t: V \times V \times V \longrightarrow V$ by

$$t(x, y, w) := \pi(\varphi_b(x \otimes y))w + b(w, x)y + b(w, y)x.$$

Observe that t is trilinear. We define a 4-linear form q on V as in FTS2.

The proof that FTS3' implies G3 in Example 3.3 reverses to show that G3 implies FTS3'. Similarly, G4 implies that $t(x, y, z) = t(x, z, y)$ for all $x, y, x' \in V$, so $q(w, x, y, z) = q(w, x, z, y)$. G1 implies that $q(w, x, y, z) = q(z, x, y, w) = q(w, y, x, z)$. Since the permutations (34), (14), and (23) generate \mathcal{S}_4 (= the symmetric group on four letters) q is symmetric.

Next, suppose that FTS2 fails, so that q is identically zero. Then since b is nondegenerate, t is also zero. Then for $v, v', z \in V$ and $a := \varphi_b(v \otimes v)$ and $a' := \varphi_b(v' \otimes v')$,

$$(a\pi(a') + a'\pi(a))z = 2(b(v, v')b(v', z)v + b(v', v)b(v, z)) = 2(aa' + a'a)z.$$

Since elements of the same form as a and a' span $\text{Skew}(A, \sigma)$, this implies that G2 fails, which is a contradiction. Thus FTS2 holds and (V, b, t) is a Freudenthal triple system.

Finally, writing out G5 in terms of V gives (2.4), which shows that (V, b, t) is nondegenerate. \square

Remark 3.7. The astute reader will have noticed that our definition of $\text{End}(\mathfrak{M})$ almost works if \mathfrak{M} is degenerate, in that the only problem is that the resulting (A, σ, π) doesn't satisfy G5. That example and the proof of 3.6 make it clear that if we remove the axiom G5 from the definition of a gift, then we would get an analog to Lemma 3.6 where the Freudenthal triple system is possibly degenerate.

Remark 3.8. Observe that in the isomorphism $\mathfrak{G} \cong \text{End}(\mathfrak{M})$ from the preceding lemma, \mathfrak{M} is only determined up to similarity. To wit, for a triple system $\mathfrak{M} = (V, b, t)$ and $\lambda \in F^*$, we define a similar structure $\mathfrak{M}_\lambda = (V, \lambda b, \lambda t)$. Then \mathfrak{M}_λ is also a triple system and is degenerate if and only if \mathfrak{M} is. We observe that $\text{End}(\mathfrak{M}) = \text{End}(\mathfrak{M}_\lambda)$. The only possible difficulty would be if the π produced by \mathfrak{M}_λ , which we shall denote by π_λ , is different from the π produced by \mathfrak{M} . However, we see that

$$\begin{aligned} \pi_\lambda(\varphi_{\lambda b}(x \otimes y))w &= \lambda t(x, y, w) - \lambda b(w, x)y - \lambda b(w, y)x \\ &= \lambda \pi(\varphi_b(x \otimes y))w \\ &= \pi(\varphi_{\lambda b}(x \otimes y))w. \end{aligned}$$

Isometries and derivations.

Definition 3.9. An *isometry* of a gift $\mathfrak{G} := (A, \sigma, \pi)$ is an element $f \in A$ such that $\sigma(f)f = 1$ and $\pi(faf^{-1}) = f\pi(a)f^{-1}$ for all $a \in A$ (i.e., $\text{Int}(f)$ is an automorphism of \mathfrak{G}). We set $\text{Iso}(\mathfrak{G})$ to be the algebraic group whose F -points are the group of isometries of \mathfrak{G} .

A *derivation* of \mathfrak{G} is an element $f \in \text{Skew}(A, \sigma)$ such that

$$\mathbf{GD}: \pi(fa) - \pi(af) = f\pi(a) - \pi(a)f \text{ for all } a \in A.$$

We define $\text{Der}(\mathfrak{G})$ to be the vector space of derivations of \mathfrak{G} .

Since any automorphism of \mathfrak{G} is also an isomorphism of A , it must be of the form $\text{Int}(f)$ for some $f \in A^*$. Thus there is a surjection $\text{Iso}(\mathfrak{G}) \longrightarrow \text{Aut}(\mathfrak{G})$. When A is split (for example, when the ground field is separably closed), then $\mathfrak{G} \cong \text{End}(\mathfrak{M})$ for some nondegenerate triple system \mathfrak{M} by 3.6, hence $\text{Iso}(\mathfrak{G}) \cong \text{Inv}(\mathfrak{M})$, so over any field $\text{Iso}(\mathfrak{G})$ is simple simply

connected of type E_7 . Since the surjection $\text{Iso}(\mathfrak{G}) \longrightarrow \text{Aut}(\mathfrak{G})$ is that induced by restriction from $GL(A) \longrightarrow \text{Aut}^+(A)$, the first map is even a central isogeny, and so $\text{Aut}(\mathfrak{G})$ is adjoint of type E_7 .

It is easy to see that $\text{Der}(\mathfrak{G})$ is actually a Lie sub-algebra of $\text{Skew}(A, \sigma)$, where the bracket is the usual commutator. In fact, by formal differentiation as in [Bor91, 3.21] or [Jac59, §4], $\text{Der}(\mathfrak{G})$ is the Lie algebra of $\text{Iso}(\mathfrak{G})$.

Proposition 3.10. *For $\mathfrak{G} := (A, \sigma, \pi)$ a gift, $\text{im } \pi = \text{Der}(\mathfrak{G})$.*

Proof: Since $\text{im } \pi$ and $\text{Der}(\mathfrak{G})$ are both vector subspaces of A , it is equivalent to prove this over a separable closure. Thus we may assume that A is split, so that $\mathfrak{G} = \text{End}(\mathfrak{M})$ for some nondegenerate triple system $\mathfrak{M} := (V, b, t)$ over F by Lemma 3.6, and we use the notation (e.g., φ_b) from the proof of that lemma.

Consider the vector subspace D of $\text{Skew}(A, \sigma)$ consisting of elements d such that $dt(v, v, v) = 3t(dv, v, v)$ for all $v \in V$. (These are known as the derivations of \mathfrak{M} .) The obvious computation shows that $\text{im } \pi \subseteq D$, which one can find in [Mey68, p. 166, Lem. 1.3]. Conversely, the reverse containment holds by [Mey68, p. 185, S. 8.3]. (He has an ‘‘extra’’ hypothesis that the characteristic of F is not 5 because he is also considering triple systems of dimensions 14 and 32, but that is irrelevant for our purposes.)

For $d \in \text{Skew}(A, \sigma)$, consider

$$(3.11) \quad \pi(da) - \pi(ad) - d\pi(a) + \pi(a)d,$$

and suppose first that a is symmetric. Then $-\pi(ad) = \pi\sigma(ad) = -\pi(da)$ by G1 and $\pi(a) = 0$ so the whole of (3.11) is zero.

Now $\text{Skew}(A, \sigma)$ is spanned by elements of the form $a = \varphi_b(x \otimes x)$ for $x \in V$, and for such an a , (3.11) applied to x is equal to

$$3t(dx, x, x) - dt(x, x, x)$$

just by expanding out using the definition of π and cancelling the terms. Thus $d \in A$ is in D if and only if d satisfies GD for all $a \in \text{Skew}(A, \sigma)$ if and only if d satisfies GD in general, so $\text{Der}(\mathfrak{G}) = D = \text{im } \pi$. \square

A category equivalence. We will now show that there is an equivalence of categories between the category of adjoint groups of type E_7 over F and the category of gifts over F , where both categories have isomorphisms for morphisms (i.e., they are groupoids). We use the notation and vocabulary of [KMRT98, §26] with impunity.

Theorem 3.12. *The automorphism group of a gift defined over a field F is an adjoint group of type E_7 over F . This provides an equivalence between the groupoid of gifts over F and the groupoid of adjoint groups of type E_7 over F .*

Proof: Let $\mathcal{C}_{\mathfrak{G}}(F)$ denote the groupoid of nondegenerate gifts over F and let $\mathcal{C}_{E_7}(F)$ denote the groupoid of adjoint groups of type E_7 over F . Let

$$S(F): \mathcal{C}_{\mathfrak{G}}(F) \longrightarrow \mathcal{C}_{E_7}(F)$$

be the functor induced by the map on objects given by $\mathfrak{G} \mapsto \text{Aut}(\mathfrak{G})$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_{\mathfrak{G}}(F) & \xrightarrow{S(F)} & \mathcal{C}_{E_7}(F) \\ i \downarrow & & \downarrow j \\ \mathcal{C}_{\mathfrak{G}}(F_s) & \xrightarrow{S(F_s)} & \mathcal{C}_{E_7}(F_s) \end{array}$$

where i, j are the obvious scalar extension maps. They are both “ Γ -embeddings” for Γ the Galois group of F_s over F , in that there is a Γ -action on the morphisms in the category over F_s with fixed points the morphisms coming from F . Since the diagram is commutative and is compatible with the Γ -action on the morphisms in the categories over F_s , $S(F_s)$ is said to be a “ Γ -extension” of $S(F)$.

Since any nondegenerate triple system is a form of \mathfrak{M}^d (1.5), every nondegenerate gift is a form of $\text{End}(\mathfrak{M}^d)$ by Lemma 3.6. Thus $\mathcal{C}_{\mathfrak{G}}(F_s)$ is connected. Since $\mathcal{C}_{E_7}(F_s)$ is also connected and any object in either category has automorphism group the split adjoint group of type E_7 , $S(F_s)$ is an equivalence of groupoids. (For G adjoint of type E_7 , $\text{Aut}(G) \cong G$ by [Tit66b, 1.5.6].)

By [KMRT98, 26.2] we only need to show that i satisfies the descent condition, i.e., that 1-cocycles in the automorphism group of some fixed element of $\mathcal{C}_{\mathfrak{G}}(F_s)$ define objects in $\mathcal{C}_{\mathfrak{G}}(F)$. Let (A, σ, π) be a gift over F and set

$$W := \text{Hom}_F(A \otimes_F A, A) \oplus \text{Hom}_F(A, A) \oplus \text{Hom}_F(A, \text{Skew}(A, \sigma)).$$

Here $(m, \sigma, \pi) \in W$ gives the structure of (A, σ, π) . The rest of the argument is as in 26.9, 26.12, 26.14, 26.15, 26.18, or 26.19 of [KMRT98]. We let ρ denote the natural map $GL(A)(F_s) \rightarrow GL(W)(F_s)$ and observe that elements of the orbit of w under $\text{im } \rho(F_s)$ define all possible gifts over F_s and the objects of $\mathcal{C}_{\mathfrak{G}}(F)$ can be identified with the set of all $w' \in W$ such that w' is in the orbit of w over F_s . Then i satisfies the descent condition by [KMRT98, 26.4]. \square

4. APPLICATIONS TO FLAG VARIETIES

A *Brown algebra* is a 56-dimensional central simple structurable algebra with involution $(B, -)$ such that the space of skew-symmetric elements is one-dimensional. In characteristic 5, a different definition is needed at the moment due to insufficiently strong classification results in that characteristic. See [Gar99a, §2] for a full definition and 5.2 for examples.

The relevant point is that given a Brown F -algebra \mathcal{B} , one can produce a nondegenerate Freudenthal triple system $\mathfrak{M} := (V, b, t)$ in a relatively natural way, see [AF84, §2] or [Gar99a, §4]. This triple system is determined only up to similarity (i.e., for every $\lambda \in F^*$, $(V, \lambda b, \lambda t)$ is also a triple system associated to \mathcal{B} , and these are all of them). Also, this construction produces *all* nondegenerate Freudenthal triple systems over F by [Gar99a, 4.14].

We define a gift $\text{End}(\mathcal{B})$ by setting $\text{End}(\mathcal{B}) := \text{End}(\mathfrak{M})$. Although \mathfrak{M} is only determined up to similarity by \mathcal{B} , $\text{End}(\mathcal{B})$ is still well-defined, as observed in Remark 3.8.

Definition 4.1. An *inner ideal* of a gift $\mathfrak{G} = (A, \sigma, \pi)$ is a right ideal I of A such that $\pi(I\sigma(I)) \subseteq I$. A *singular ideal* of G is a right ideal of A such that $\pi(I\sigma(I)) = 0$.

If A is split so that $G \cong \text{End}(\mathfrak{M})$ for some triple system $\mathfrak{M} = (V, b, t)$, there is a bijection between subspaces U of V and right ideals $\text{Hom}_F(V, U)$ of $\text{End}_F(V)$. In this bijection, inner ideals of \mathfrak{M} (i.e., those subspaces U such that $t(U, U, V) \subseteq U$) correspond to inner ideals of \mathfrak{G} , and the same holds for singular ideals where a singular ideal of \mathfrak{M} is a subspace U such that $t(u, u, v) = 2b(v, u)u$ for all $u \in U$ and $v \in V$, see [Gar99a, 6.11].

Since all inner ideals in a nondegenerate Freudenthal triple system are totally isotropic with respect to the skew-symmetric bilinear form [Fer72, 2.4], any singular or inner ideal in a gift \mathfrak{G} is isotropic, i.e., $\sigma(I)I = 0$.

Now all of the flag varieties (a.k.a. homogeneous projective varieties) for an arbitrary group $\text{Aut}(\mathfrak{G})$ of type E_7 can easily be described in terms of the singular and rank 12 inner ideals of the gift \mathfrak{G} , by translating the corresponding description for the flag varieties for such groups with trivial Tits algebras from [Gar99a, §7]. Specifically, one simply takes the statement of [Gar99a, 7.4] and replaces every instance of “ n -dimensional” with “rank n ” as well as replacing $\text{Inv}(\mathcal{B})$ with $\text{Aut}(\mathfrak{G})$.

5. A CONSTRUCTION

The rest of this section is taken up with a construction which produces groups of type E_7 with Tits algebras of index 2, and in particular all gifts (hence all groups of type E_7) over a real-closed field. (Generally, we say that a field F is *real* if -1 is not a sum of squares in F and that it is *real-closed* if it is real and no algebraic extension is real, please see [Lam73, Ch. 9, §§1, 2] for more information.) Since one of these groups is anisotropic, this is the first construction of an anisotropic group of type E_7 directly in terms of this 56-dimensional form. (Groups of this type have been implicitly constructed as the automorphism groups of Lie algebras given by the Tits construction.)

A diversion to symplectic involutions. Let $Q := (a, b)_F$ be a quaternion algebra over F . Our construction will involve taking a 56-dimensional skew-symmetric bilinear form from a triple system over F and twisting it to get a symplectic involution of σ on $M_{28}(Q)$. However, if our triple system is of the form $\mathfrak{M}(J)$ for some J , then the skew-symmetric form is of a very special kind, namely it is obtainable from a quadratic form q (specifically, $\langle 1 \rangle \perp T$ where T is the trace on the Jordan algebra), and we use this fact to get an explicit description of σ .

More generally, suppose that (V, q) is a nondegenerate quadratic space over F with associated symmetric bilinear form b_q such that $b_q(x, x) = q(x)$. Set $W := V \oplus V$. For $w_i = (v_i, v'_i) \in W$ for $i = 1, 2$, define a skew-symmetric bilinear form s by

$$s(w_1, w_2) := b_q(v_1, v'_2) - b_q(v_2, v'_1).$$

Then s is nondegenerate since q is. Now consider the map $\psi_c : W \rightarrow W$ given by $\psi_c(v, v') = (cv', (b/c)v)$. This is a similarity of s with multiplier b . Let $K := F(\sqrt{a})$

and let ι be the unique F -linear automorphism of K . Then $\text{Int}(\psi_c) \otimes \iota$ is an automorphism of $(\text{End}_F(W), \sigma_s) \otimes_F K$ whose fixed subalgebra is isomorphic to (A, σ) where A is Brauer-equivalent to Q and of degree $\dim_F W$ and σ is some symplectic involution.

Lemma 5.1. *Suppose that for $i = 1, 2$, (V_i, q_i) is a nondegenerate quadratic space over F , (W_i, s_i) are the associated symplectic spaces as described above, and for $c_i \in F^*$, (A, σ) is defined by descent from $K := F(\sqrt{a})$ as above by $\text{Int}(\psi_{c_1} \oplus \psi_{c_2})$. Then if $q_1 = \langle \alpha_1, \dots, \alpha_m \rangle$ and $q_2 = \langle \beta_1, \dots, \beta_n \rangle$, the involution σ is adjoint to the hermitian form*

$$\langle \alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \rangle$$

over Q .

Of course, this generalizes directly to an orthogonal sum of any finite number of such skew-symmetric spaces.

Proof: Without loss of generality, we may assume that $m = n = 1$ and write $q_1 = \langle \alpha \rangle$ and $q_2 = \langle \beta \rangle$. We first describe an isomorphism

$$f: (Q \otimes_F K) \otimes_F M_2(F) \longrightarrow M_2(K) \otimes_F M_2(F).$$

We set i and j to be the skew-commuting generators for Q such that $i^2 = a$ and $j^2 = b$. Then we fix a square root of a in K and let E_{ij} be the matrix whose only nonzero entry is a 1 in the (i, j) -position. We define f by

$$f(i \otimes E_{rs}) := \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\frac{c_r}{c_s} \sqrt{a} \end{pmatrix} \otimes E_{rs}$$

and

$$f(j \otimes E_{rs}) := \begin{pmatrix} 0 & c_s \\ b/c_r & 0 \end{pmatrix} \otimes E_{rs}.$$

For $C_i := \begin{pmatrix} 0 & c_i \\ b/c_i & 0 \end{pmatrix}$ and $g := \text{Int} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$, $g \otimes \iota$ is an ι -semilinear algebra automorphism of $M_4(F) \otimes_F K$ of order 2 and is the specified descent. Also, $f^{-1}(g \otimes \iota)f$ fixes $M_2(Q) \otimes 1$ elementwise as an F -subalgebra of $M_2(Q) \otimes_F K$. Thus $g \otimes \iota = f(\text{Id} \otimes \iota)f^{-1}$.

Now the involution on $M_4(F)$ which is adjoint to $s_1 \oplus s_2$ is precisely

$$\text{Int} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \circ \text{Int} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \circ t$$

where $A = \alpha \cdot \text{Id}_2$, $B = \beta \cdot \text{Id}_2$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and t is the transpose [KMRT98, p. 24]. But

$$f(\gamma \otimes \iota)f^{-1} = \text{Int} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \circ t,$$

so we are done. \square

Let Q be a quaternion algebra over F with unique symplectic involution γ and h a γ -hermitian form on some Q -vector space V . There is a useful invariant of h which we call

a *quadratic trace form* over F . It is the quadratic form q on V regarded as a $(4 \dim_Q(V))$ -dimensional vector space over F defined by $q(x) := h(x, x)$, see [Sch85, p. 352] or [Jac40, p. 266]. This gives an injection from the Witt group of γ -hermitian forms on Q to the Witt group of quadratic forms over F [Sch85, 10.1.7]. If h is diagonal so that $h = \langle \alpha_1, \dots, \alpha_n \rangle$ for $\alpha_i \in F^*$, then $q \cong \langle \alpha_1, \dots, \alpha_n \rangle \otimes \text{Nrd}_Q$.

The construction. Our construction will need to use a specific kind of Brown algebra explicitly.

Example 5.2. ([Gar99a, 2.3, 2.4], cf. [All90, 1.9]) The principal examples of Brown F -algebras are denoted by $\mathcal{B}(J, \Delta)$ for J an Albert F -algebra and Δ a quadratic étale F -algebra. We set $\mathcal{B}(J, F \times F)$ to be the F -vector space $\left(\begin{smallmatrix} F & J \\ J & F \end{smallmatrix} \right)$ with multiplication given by

$$\begin{pmatrix} \alpha_1 & j_1 \\ j'_1 & \beta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & j_2 \\ j'_2 & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \alpha_2 + T(j_1, j'_2) & \alpha_1 j_2 + \beta_2 j_1 + j'_1 \times j'_2 \\ \alpha_2 j'_1 + \beta_1 j'_2 + j_1 \times j_2 & \beta_1 \beta_2 + T(j_2, j'_1) \end{pmatrix},$$

where T is the bilinear trace form on J and \times is the Freudenthal cross product (see [KMRT98, §38] for more information about these maps). The involution $-$ on $\mathcal{B}(J, F \times F)$ is given by

$$\overline{\begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix}} = \begin{pmatrix} \beta & j \\ j' & \alpha \end{pmatrix}.$$

The map ϖ defined by

$$\varpi \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = \begin{pmatrix} \beta & j' \\ j & \alpha \end{pmatrix}$$

is an automorphism of $\mathcal{B}(J, F \times F)$ as an algebra with involution. For Δ a quadratic field extension of F and ι the nontrivial F -automorphism of Δ , we define $\mathcal{B}(J, \Delta)$ to be the F -subspace of $\mathcal{B}(J, F \times F) \otimes_F \Delta$ fixed by the ι -semilinear automorphism $\varpi \otimes \iota$. Note that this construction is compatible with scalar extension in that for any field extension K of F , $\mathcal{B}(J, \Delta) \otimes_F K \cong \mathcal{B}(J \otimes_F K, \Delta \otimes_F K)$.

Construction 5.3. *Suppose that K is a quadratic field extension of F , Q is a quaternion algebra over F which is split by K , and J is an Albert F -algebra which is also split by K . Then there is a gift $\mathfrak{G} := (M_4(Q), \sigma, \pi)$ such that*

- (1) *Iso(\mathfrak{G}) is split by K .*
- (2) *For γ the unique symplectic involution on Q , the involution σ is adjoint to the γ -hermitian form $\langle 1 \rangle \perp T$ for T the trace on J .*
- (3) *If Q is split, then $\mathfrak{G} \cong \text{End}(\mathcal{B}(J, K))$.*

Proof: Consider the Brown algebras $\mathcal{B} := \mathcal{B}(J, K)$ and $\mathcal{B}^q := \mathcal{B}(J^d, K)$. Their spaces of skew-symmetric elements are both spanned by some element s_0 such that $F(s_0) \cong K$, and they are isomorphic as algebras with involution over K , so \mathcal{B} corresponds to a 1-cocycle γ in $H^1(K/F, \text{Aut}^+(\mathcal{B}^q))$, which must be given by

$$\gamma_\iota \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = \begin{pmatrix} \alpha & \psi(j) \\ \psi^\dagger(j') & \beta \end{pmatrix}$$

for some norm isometry $\psi \in \text{Inv}(J^d)(K)$. Since γ is a 1-cocycle, $\gamma_i \iota \gamma_i = \text{Id}_{\mathcal{B}}$, so we note that $\psi \iota \psi^\dagger \iota = \text{Id}_{J^d}$.

Since Q is split by K , it is isomorphic to a quaternion algebra $(a, b)_F$ for some $a, b \in F^*$ such that $K = F(\sqrt{a})$. Consider $t \in \text{End}_F(\mathcal{B}^a)$ given by

$$t \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = \begin{pmatrix} \alpha/b & b\psi(j) \\ \psi^\dagger(j') & b^2\beta \end{pmatrix}.$$

By [Gar99a, 5.10], \mathfrak{M}^d is the unique triple system associated to \mathcal{B}^a . The map t is a similarity of \mathfrak{M}^d with multiplier b , so $\text{Int}(t)$ is an automorphism of the gift associated to $\text{End}(\mathcal{B}^a)$, which is the split gift \mathfrak{G}^d . Also,

$$t \iota t \iota(x) = bx$$

for all $x \in \mathcal{B}^a$. Thus setting $\eta_i = \text{Int}(t)$ defines a 1-cocycle $\eta \in Z^1(K/F, \text{Aut}(\mathfrak{G}^d))$, which defines a nondegenerate gift (A, σ, π) over F which is split over K .

The map $H^1(K/F, \text{Aut}(\mathfrak{G}^d)) \rightarrow H^2(K/F, \mu_2)$ sends the class of \mathfrak{G} to the class of A (which is the Tits algebra associated to $\text{Iso}(\mathfrak{G})$), where we are identifying $H^2(K/F, \mu_2)$ and the subgroup of the Brauer group of F consisting of algebras which are split by K . The image of η under this map is the 2-cocycle f given by $f_{i,i} = b$. This 2-cocycle determines the quaternion algebra Q by [Spr59, pp. 250, 251] or [GTW97, Lem. 3.5].

Part (2) follows directly from Lemma 5.1, and part (3) is clear. \square

Example 5.4. Let Q be a nonsplit quaternion algebra over a field F with splitting field K a quadratic extension of F and set \mathfrak{G} to be the gift over F constructed in 5.3 from K , Q , and the split Albert F -algebra J^d . Then \mathfrak{G} contains a rank 6 maximal singular ideal corresponding to the 6-dimensional maximal singular ideal in $\mathcal{B}(J^d, \Delta)$ given in [Gar99a, 7.5], so by the description of the flag varieties from Section 4 the non-end vertex of the length 2 arm of the Dynkin diagram of $\text{Iso}(\mathfrak{G})$ is circled. Since the Tits algebra of $\text{Iso}(\mathfrak{G})$ is Brauer-equivalent to Q and hence nonsplit, the end vertex of the long arm is not circled, so $\text{Iso}(\mathfrak{G})$ must be of type $E_{7,4}^9$ in the notation of [Tit66b, p. 59].

6. GROUPS OF TYPE E_7 OVER REAL-CLOSED FIELDS

Construction 5.3 produces all four isogeny classes of groups of type E_7 over a real-closed field. The two forms with nontrivial Tits algebra are given by taking $K = R(\sqrt{-1})$, $Q = (-1, -1)_R$, and $J = \mathfrak{H}_3(\mathfrak{C}, 1)$ for \mathfrak{C} the two different Cayley algebras over \mathbb{R} . (For a definition and a general discussion of Cayley algebras, please see [Sch66, Ch. III, §4] or [KMRT98, §33.C].) Taking \mathfrak{C} to be the Cayley division algebra, gives the anisotropic or compact form. Since we have an explicit description of the involution in the associated gift, we can use this to describe the connecting homomorphism in degree one of the Galois cohomology of such a group.

We have a short exact sequence

$$(6.1) \quad 1 \longrightarrow \mu_2 \longrightarrow \text{Iso}(\mathfrak{G}) \xrightarrow{\text{Int}} \text{Aut}(\mathfrak{G}) \longrightarrow 1$$

where we have identified μ_2 with the center of $\text{Iso}(\mathfrak{G})$. We want to describe the image of the induced connecting homomorphism

$$(6.2) \quad \partial: \text{Aut}(\mathfrak{G})(F) \longrightarrow H^1(F, \mu_2) \cong F^{*2}.$$

By the Skolem-Noether Theorem, everything in $\text{Aut}(\mathfrak{G})(F)$ is of the form $\text{Int}(f)$ for some $f \in A^*$. Since $\text{Int}(f)$ is an isomorphism of (A, σ) , $\sigma(f)f = \mu \in F^*$, and ∂ is given by $\partial(\text{Int}(f)) = \mu F^*/F^{*2}$.

Application 6.3. *For $\mathfrak{G} = (A, \sigma, \pi)$ a nondegenerate gift over a real-closed field R , the following are equivalent:*

- (1) A is split.
- (2) σ is hyperbolic.
- (3) The map $\partial: \text{Aut}(\mathfrak{G})(R) \longrightarrow H^1(R, \mu_2)$ from (6.2) is surjective.

Point (3) is interesting because of connections with the Hasse Principle Conjecture II, see the introduction.

Of course, $H^1(R, \mu_2) \cong R^*/R^{*2} \cong \mathbb{Z}_2$, so if the map is not surjective it is trivial.

Proof: (1) \implies (3): Since A is split, $\mathfrak{G} \cong \text{End}(\mathfrak{M})$ for a nondegenerate triple system \mathfrak{M} , and since R is real-closed, $\mathfrak{M} \cong \mathfrak{M}(J)$ for some reduced Albert R -algebra J [Fer76, 5.1]. Since J is reduced, for any $\mu \in R^*$, there is a norm similarity φ of J with multiplier μ [Gar99a, 1.6] and the map

$$\begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} \mapsto \begin{pmatrix} \mu^{-2}\alpha & \mu\varphi(j) \\ \varphi^\dagger(j') & \mu^3\beta \end{pmatrix}$$

is a similarity of $\mathfrak{M}(J)$ with multiplier μ .

(3) \implies (2) purely because (A, σ) is a central simple algebra with symplectic involution, see the following Lemma 6.5.

(2) \implies (1): As mentioned above, \mathfrak{G} is obtained from Construction 5.3 with $J = \mathfrak{H}_3(\mathfrak{C}, 1)$ for some Cayley R -algebra \mathfrak{C} . Then by 5.3(2), σ is adjoint to a hermitian form with quadratic trace form $\text{Nrd}_Q \otimes (\langle 1 \rangle \perp T)$, which is hyperbolic since σ is. By [KMRT98, p. 533],

$$T = \langle 1, 1, 1 \rangle \otimes (\langle 1 \rangle \perp \mathfrak{n}),$$

for \mathfrak{n} the quadratic norm form on \mathfrak{C} . Thus the two possibilities for T are $\langle 1, 1, 1 \rangle \perp 12\mathcal{H}$ and $27\langle 1 \rangle$ where \mathcal{H} denotes a hyperbolic plane, both of which are not hyperbolic. Since R is real-closed, the Witt ring is an integral domain (isomorphic to \mathbb{Z} by [Sch85, 2.4.8]), and so Nrd_Q must be hyperbolic and Q must be split. \square

Taking $\mathfrak{G} := (A, \sigma, \pi)$ to be the gift constructed in Example 5.4 with $F = \mathbb{R}$, $K = \mathbb{C}$, and $Q = \mathbb{H}$ provides an example of a gift over \mathbb{R} with A nonsplit and σ isotropic, but necessarily non-hyperbolic. (The involution is isotropic because the gift is nondegenerate and contains a singular ideal, see the comment following Definition 4.1.)

Example 6.4. That (2) implies (1) in Application 6.3 is special to the field being real-closed. For example, consider any field F which contains a square root of -1 and has a

nonsplit quaternion algebra Q . Then for K any quadratic subfield of Q , let $(M_4(Q), \sigma, \pi)$ be the gift constructed in 5.3 from K , Q , and $J = \mathfrak{H}_3(\mathfrak{C}, 1)$ for \mathfrak{C} the split Cayley F -algebra. The involution σ corresponds to a hermitian form with quadratic trace form

$$\mathrm{Nrd}_Q \otimes (\langle 1, 1, 1 \rangle \perp 12\mathcal{H}),$$

where \mathcal{H} denotes a hyperbolic plane. Since F contains a square root of -1 , this form is hyperbolic over F , so σ is hyperbolic over F .

We close with the promised lemma about symplectic involutions.

For (A, σ) a central simple F -algebra with involution, we write $G(A, \sigma)$ for the group of *multipliers of similitudes* of (A, σ) , i.e., for the set of all $\lambda \in F^*$ such that there is some $f \in A^*$ with $\sigma(f)f = \lambda$.

Lemma 6.5. *Let (A, σ) be a central simple algebra with symplectic involution over a euclidean field E (i.e., E is formally real and $|E^*/E^{*2}| = 2$). Then $G(A, \sigma)$ is all of E^* if and only if σ is hyperbolic.*

Proof: If σ is hyperbolic, then $G(A, \sigma) = E^*$ over any field since there is at most one hyperbolic form of a given dimension over any skew field.

Suppose now that $G(A, \sigma) = E^*$. If A is split, then σ is hyperbolic (regardless of $G(A, \sigma)$) and so we are done. So we may assume that A is nonsplit, and since it supports a symplectic involution it must have index 2. Then $A \cong \mathrm{End}_Q(V)$ for the unique nonsplit quaternion algebra Q over E and σ is adjoint to some γ -hermitian form h on V where γ is the unique symplectic involution on Q . Let q be the quadratic trace form of h on V as described above. It is clear from the definition of q that $G(A, \sigma) \subseteq G(q)$, where

$$G(q) = \{\lambda \in E^* \mid \lambda q \cong q\}.$$

Then $G(q) = E^*$. Since E is euclidean, q must be hyperbolic. Since the map on Witt groups is an injection, h is hyperbolic, hence σ is as well. \square

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