GENERIC SUBGROUPS OF LIE GROUPS

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Abstract. We investigate properties of generically choosen finitely generated subgroups of real Lie groups. In particular we ask whether discreteness is generic.

1. introduction

In this article we investigate generic subgroups of Lie groups. In particular we ask the following question:

> *Given a Lie group G and a natural number n, when is it true, that n generically choosen elements $g_1, \ldots, g_n$ will generate a discrete subgroup of G ?*

More precisely, given a topological group G we define a subset Δ_n of the n-fold product space $G^n = G \times \ldots \times G$ by

$$\Delta_k(G) = \{(g_1, \ldots, g_k) \in G^k : \langle g_1, \ldots, g_k \rangle \text{ is a discrete subgroup of } G\}$$

A complete description of these sets Δ_k is available only for rather special cases. It is easy to obtain such a description for abelian connected Lie groups. For semisimple Lie groups already the description of Δ_2 is rather complicated. A complete description has been achieved only for the case $SL(2, \mathbb{R})$. For $SL(2, \mathbb{C})$ there are partial results, in particular the famous inequality of Jørgensen.

In this article we do not look for a complete description of Δ_k. Here we concentrate on the question:

> *Given a Lie group G and a number $k \in \mathbb{N}$, under which conditions on G and k are Δ_k or $G^k \setminus \Delta_k$ are sets of measure zero (with respect to the product measure of the Haar measure of G).*

(The sets Δ_k are always measurable, see prop 2.)

For Δ_1 we have precise criteria answering these questions in dependance on the topological properties of the Cartan subgroups of G.

Theorem 1. *Let G be a connected real Lie group.*

Then the set Δ_1 has positive measure if and only if G admits a Cartan subgroup with non compact connected components.

1991 *Mathematics Subject Classification.* AMS Subject Classification: 22E40.

The set $G\setminus\Delta_1$ has positive measure if and only if G admits a Cartan subgroup with compact connected components.

If G contains both Cartan subgroups with compact connected components and Cartan subgroups with non compact components, then both Δ_1 and $G\setminus\Delta_1$ are sets of infinite measure.

We discuss compactness resp. non-compactness of Cartan subgroups in some detail, because the size of Δ_1 depends on these properties.

Let G be a connected real Lie group, R its radical and $S = G/R$. Cartan subgroups with compact connected components are necessarily compact (prop. 8). The existence of a compact Cartan subgroup implies that the center of S is finite (by prop. 7) and that G/G^∞ is compact (lemma 9).

For general k there is a strict dichotomy depending on whether S is compact or not.

Theorem 2. *Let G be a connected real Lie group, R ist radical, N its nilradical and $S = G/R$.*

If S is non-compact, then both Δ_k and $G^k\setminus\Delta_k$ are of infinite measure for all $k > 1$.

If S is compact, then there exists a natural number δ_G such that $G^k\setminus\Delta_k$ has measure zero for all $k \le \delta_G$ and Δ_k has measure zero for all $k > \delta_G$.

Furthermore the number δ_G has the following properties:

- $\delta_G = 0$ *if G is compact.*
- $\delta_G \le 1$ *unless G is nilpotent.*
- $\delta_G \le \dim G/G'$.

We also derive some related results concerning density of generic subgroups.

Theorem 3. *Let G be a connected semisimple linear algebraic group. Then for every $k \ge 2$ there is a subset $Z_k \subset G^k$ of measure zero such that for every $g = (g_1, \dots, g_k) \in G^k\setminus Z_k$ the group $\langle g_1, \dots, g_k\rangle$ generated by $g_1, \dots, g_k$ is Zariski dense in G.*

Theorem 4. *Let G be a connected semisimple real Lie group.*

There exists an open neighbourhood W of e in G and for every $k \ge 2$ a subset $Z_k \subset W^k$ of measure zero such that $\langle g_1, \dots, g_k\rangle$ is dense in G for all $(g_1, \dots, g_k) \in W^k\setminus Z_k$.

In particular, for every connected semisimple Lie group S there do exist two elements $g_1, g_2 \in S$ such that g_1 and g_2 generate a dense subgroup of S. This may be regarded as a Lie analog for a theorem in the theory of finite simple groups which states that every finite simple group is generated by two elements ([1]).

This paper is organized as follows: First we provide some examples, and introduce some basic facts and notations. We show that Δ_k is always measurable. We prove that under certain assumptions invariant sets of positive measure are automatically of infinite measure. Investigating Cartan subgroups we derive our above mentioned results on Δ_1. Subsequently we prove the theorem on Δ_k ($k \geq 2$) using a variety of different techniques ranging from Zassenhaus neighbourhoods over amenable groups to proximal elements.

1.1. **Proof of the main results.** The first two statements of Theorem 1 follow from prop. 5, prop. 9 and prop. 10. If G contains both Cartan subgroup with compact connected components and Cartan subgroups with non compact connected components, then prop. 6 implies that G/R is non-compact. This allows us to invoke cor. 1 in order to conclude that both Δ_1 and $G \setminus \Delta_1$ have infinite measure.

Theorem 2: For S non compact the statement follows from cor. 6 and cor. 7 combined with cor. 1. If S is compact and positive dimensional then $\Delta_k(G)$ is a set of measure zero for all $k \geq 2$ by prop. 11. Theorem 1 combined with prop. 6 implies that either $\Delta_1(G)$ or $G \setminus \Delta_1(G)$ is a set of measure zero for S compact and positive dimensional. If G is solvable, but not nilpotent, then the assertions of the theorem follows from prop. 6 combined with prop. 12. For nilpotent G the theorem follows from prop. 14 and prop. 15. Finally, the assertion that $\delta_G = 0$ for compact G follows from cor. 3.

Theorem 3 follows from prop. 16.

For every connected semisimple Lie group S the adjoint representation Ad has the property that $Ad(S)$ is linear algebraic and the kernel (which is the center of S) is discrete. Together with prop. 17 and lemma 16 this implies theorem 4.

2. Examples of Δ_k

For abelian and certain nilpotent connected Lie groups an explicit description is easy.

Example 1. *Let $G = (\mathbb{R}^d, +)$. Then Δ_k is the set of all $(v_1, \dots, v_k)$ such that $\dim_{\mathbb{Q}} \langle v_1, \dots, v_k \rangle_{\mathbb{Q}} = \dim_{\mathbb{R}} \langle v_1, \dots, v_k \rangle_{\mathbb{R}}$. In particular Δ_k is of measure zero for $k > d$ and $G^k \setminus \Delta_k$ is of measure zero for $k \leq d$.*

Example 2. *Let $G = (S^1)^n$ with $S^1 = \{z \in \mathbb{C}^* : |z| = 1\}$, let E denote the set of all roots of unity, i.e., $z \in E$ if $z^N = 1$ for some $N \in \mathbb{N}$ and* *let* $G_{tors} = E^n$. *Then* $\Delta_k = G^k_{tors}$ *for all* $k \in \mathbb{N}$ *and* Δ_k *has measure zero for all* $k \in \mathbb{N}$.

Example 3. *Let* V *be a real vector space with antisymmetric bilinear form* $B(\cdot,\cdot)$. *The associated Heisenberg group* G *is* $V \times \mathbb{R}$ *as manifold with the group structure given by* $(v,x)\cdot(w,y) = (v+w, x+y+B(v,w))$. *Then the group generated by two elements* (v,x) *and* (w,y) *is*

$$\{(nv + mw, nx + my + (nm + k)B(v,w) : n,m,k \in \mathbb{Z}\}.$$

Hence $G^k \setminus \Delta_k$ *has measure zero for* $k = 1,2$. *On the other hand, if* $(v_i,x_i)_{i\in I}$ *is a family of elements in* G, *then the group generated by these elements contains* $2B(v_i,v_j)$ *for all* $i,j \in I$. *This implies that* Δ_k *is of measure zero for* $k > 2$.

Example 4. *Let* $\mathbf{g}$ *be the four dimensional nilpotent Lie algebra given by* $[X_1,X_2] = X_3$ *and* $[X_1,X_3] = X_4$.

Then Δ_k *is a set of measure zero for all* $k \geq 2$ *for the associated simply-connected Lie group.*

Based on free Lie algebras, for every $k \in \mathbb{N}$ it is possible to construct a nilpotent Lie group such that Δ_k is of positive measure.

For semisimple Lie groups there are many partial results.

Example 5. *There is a complete description of* Δ_2 *for* $SL(2,\mathbb{R})$, *obtained by J. Gilman (*[6]*).*

Example 6. *For* $SL_2(\mathbb{C})$ *there is the famous inequality of Jørgensen (*[8]*): If* $(A,B) \in \Delta_2$ *for* $G = SL_2(\mathbb{C}) \simeq \tilde{SO}(3,1)$, *then either the inequality*

$$|(tr\, A)^2 - 4| + |\, tr(ABA^{-1}B^{-1}) - 2| \geq 1$$

is fulfilled or A *and* B *generate a subgroup of very special kind, called "elementary" subgroup. It is easily verified that the set of all* (A,B) *generating an elementary subgroup is a set of measure zero. Thus Jørgensen's inequality implies that* $G^2 \setminus \Delta_2$ *is of positive measure for* $G = SL_2(\mathbb{C})$. *Results generalizing Jørgensen's inequality have been obtained for* $SO(n,1)$ *(n arbitrary), see* [3],[5],[11].

Example 7. *Let* G *be a compact Lie group. Every discrete subgroup of* G *is finite. Hence* $\Delta_1(G) = G_{tors} = \{g \in G : g^n = e\, \exists n\}$.

If G *is connected or nilpotent, then* $\mu(\Delta_1(G)) = \mu(G_{tors}) = 0$ *(see corollary 3 and lemma 5 below).*

If G *is neither connected nor nilpotent, then* G_{tors} *may be a set of positive measure. For instance, consider the compact solvable group*

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in S^1 \right\} \cup \left\{ \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix} : \lambda \in S^1 \right\}.$$

This compact group has two connected components and $g^2 = e$ for every element g in the connected component not containing the neutral element.

3. Infinity of volume

In order to show that certain sets have infinite volume with respect to Haar measure, we proceed as follows: We first show that these sets have positive measure. Then we apply a proposition to be deduced in this section which implies that often sets of positive measure in Lie groups have automatically infinite measure if they are invariant under conjugation.

Lemma 1. *Let V be a affine variety defined over $\mathbb{R}$, U a unipotent group acting on V morphically (also over $\mathbb{R}$) and μ a $U(\mathbb{R})$-invariant probability measure on $V(\mathbb{R})$ of Lebesgue measure class.*

Then μ is supported in the fixed point set $V^U(\mathbb{R})$.

Proof. If $V^U \neq V$, then there exists an $\mathbb{R}$-regular function f on V such $f \notin \mathbb{R}[V]^U$ but such that U stabilizes the vector space M spanned by f and $\mathbb{R}[V]^U$. For every one-parameter-subgroup α_t of U there is a U-invariant regular function $g = g_\alpha$ such that $\alpha_t^* f = f + tg$. Then f/g is an $\mathbb{R}$-regular map from $V_g = \{g \neq 0\}$ to $\mathbb{R}$ which is equivariant for $\mathbb{R}$ acting on V via α_t and on itself by addition. Now every U-invariant subset of V_g is mapped surjectively onto $\mathbb{R}$. Hence V_g can not carry any non-trivial U-invariant finite measure. Therefore μ must be supported inside the intersection of the zero sets of all the g_α. This intersection is again an affine U-variety, but of strictly smaller dimension that V (unless $V = V^U$). Arguing by induction, we may therefore deduce that the support of μ must be contained in V^U. □

Proposition 1. *Let G be a connected real Lie group and R its radical. Assume that there exists a probability measure μ on G which is invariant under G acting on itself by conjugation.*

Then G/R is compact or μ is concentrated on R.

Proof. Assume that G/R is not compact. Then there exists a real simple Lie group S_0 with trivial center and a surjective Lie group homomorphism $\tau : G \to S_0$. The measure μ on G induces a probablity measure μ_0 on S_0 via $\mu_0(A) = \mu(\tau^{-1}(A))$ for $A \subset S_0$. Note that S_0 is linear algebraic, because it is simple and center-free. Hence S_0 is an affine $\mathbb{R}$-variety. Thus we may invoke the above lemma and conclude that the support of μ_0 is concentrated at the intersection of the fixed point sets of all unipotent subgroups of S_0 acting on S_0 by conjugation. Since S_0 is simple and non-compact, it is generated by its unipotent subgroups. Hence this intersection is simply the center of S_0, i.e. μ_0 is concentrated at $\{e\}$. Thus the support of μ is contained in R. $\square$

Corollary 1. *Let G be a connected real Lie group G with radical R, $k \in \mathbb{N}$, μ the product measure of the Haar measures on G^k and $E \subset G^k$ a measurable set which is invariant under the diagonal G-action on G^k via conjugation. Assume that G/R is not compact.*

Then either $\mu(E) = 0$ or $\mu(E) = \infty$.

Proof. Assume the contrary, i.e., the existence of such an invariant set E with $0 < \mu(E) < +\infty$. Since E is invariant and of finite volume, it follows that conjugation by an element of G can not change the volume, i.e., G must be unimodular. Hence the Haar measure is invariant under conjugation.

Now let $\pi_1 : G^k \to G$ denote the projection onto the first factor. We define a probability measure η on G by $\eta(X) = \mu(\pi^{-1}(X) \cap E)$. Note that $R \neq G$ and therefore $\mu(R \times G^{k-1}) = 0$, implying $\eta(R) = 0$ Thus η is a probability measure on G which is invariant under conjugation and such that $\eta(R) = 0$. This contradicts the preceding proposition. $\square$

4. Preparations

In this article, F_n always denotes the free group with n elements ξ_i. There is a natural map $\alpha : F_n \times G^n \to G$ defined as follows: To every element $x = (g_1, \dots, g_k) \in G^k$ we associated a group homomorphism $\phi_x : F_n \to G$ induced by $\phi_x : \xi_i \mapsto g_i$. We define $\alpha(\xi, x) = \phi_x(\xi)$. Furthermore, for every $R \in F_n$ we define a continuous map $\zeta_R : G^k \to G$ via $\zeta_R(x) = \phi_x(R)$.

Proposition 2. *Let G be a topological group fulfilling the second axiom of countability.*

Then $\Delta_k(G)$ is measurable (with respect to the Borel algebra generated by the topology) for all $k \in \mathbb{N}$.

Proof. Note that $x = (g_1, \dots, g_k) \notin \Delta_k$ iff the subgroup generated by the g_i is not discrete and that this condition is equivalent to the property that e is not an isolated point in the group generated by the g_i.

Let $(V_j)_{j \in J}$ be a neighbourhood basis of the topology of G at e. By assumption J may be choosen to be countable.

Then

$$G^k \setminus \Delta_k = \cap_{j \in J} \cup_{R \in F_n} \zeta_R^{-1}(V_j \setminus \{e\})$$

Since both J and F_n are countable, it follows that Δ_k is measurable. $\square$

Subgroups of discrete groups are again discrete. Hence $\Delta_{k+l} \subset \Delta_k \times G^l$ for all $k, l \in \mathbb{N}$, implying the observation stated below.

Observation. *If Δ_k is of measure zero, then Δ_l is of measure zero for all $l \geq k$.*

Real Lie groups are intrinsically analytic, i.e., they admit a structure as a real analytic manifold such that the maps defining the group structure are real analytic. Furthermore, every continuous group homomorphism between real analytic Lie groups is automatically real analytic. This is useful for our purposes, since the identity principle for real analytic maps has the following consequence:

Lemma 2. *Let $f : M \to N$ be a real analytic map between connected real analytic manifolds. Assume that f has maximal rank somewhere.*

Then $f^{-1}(S)$ is of measure zero for every subset $S \subset N$ of measure zero. (where the measure is defined with respect to (otherwise arbitrary) everywhere positive volume forms on M and N.)

Corollary 2. *Let G be a connected Lie group, S a subset of measure zero and*

$$\hat{S} = \{g \in G : g^n \in S \ \exists n \in \mathbb{N}\}.$$

Then $\hat{S}$ is a set of measure zero.

Proof. For every natural number n the map $\phi_n : g \mapsto g^n = g \cdot \ldots \cdot g$ is a real analytic map which has maximal rank at e. Thus $\hat{S} = \cup_{n \in \mathbb{N}} \phi_n^{-1}(S)$ is a countable union of sets of measure zero and therefore itself a set of measure zero. $\square$

Corollary 3. *Let G be a connected Lie group.*

Then $G_{tors} = \{g \in G : g^n = e \ \exists n\}$ is a set of measure zero.

Lemma 3. *Let H be a connected nilpotent Lie group. Then there exists a unique maximal compact subgroup $K \subset H$. Furthermore K is central in H.*

Proof. Let K be a maximal compact subgroup. The adjoint representation of K on $Lie(H)$ is completely reducible. Combined with $Ad(g)$ being unipotent for all $g \in H$ this implies that K is central in H. Uniqueness of K follows from K being central, because maximal compact subgroups in a connected Lie group are all conjugate. $\square$

Lemma 4. *Let G be a connected Lie group and H a connected normal compact nilpotent Lie subgroup.*

Then H is central in G.

Proof. The complete reducibility of representations of compact groups implies that there exists an $Ad(H)$-stable vector subspace $V \subset Lie\, G$ such that $Lie\, G = V \oplus Lie\, H$ as a vector space. Since V is $Ad(H)$-stable, it is clear that $[V, Lie\, H] \subset V$. On the other hand, $[V, Lie\, H] \subset Lie\, H$, because H is normal. Thus $[V, Lie\, H] = \{0\}$.

Complete reducibility of representations of compact groups can also be used to deduce that a compact connected nilpotent Lie group is necessarily commutative. Together with $[V, Lie\, H] = \{0\}$ this implies that H is central. □

Lemma 5. *Let K be a compact nilpotent Lie group and K_{tors} its set of torsion elements.*

Then K_{tors} is a set of measure zero.

Proof. For every every natural number $n \geq 2$ the set

$$\{g \in K : g^n = e\}$$

is a closed real analytic subset of K. Therefore either K_{tors} is a set of measure zero, or K_{tors} contains a whole connected component of K.

Let C denote the connected component of the center of K^0. Since K is nilpotent, this is a positive dimensional group, i.e. $C \simeq (S^1)^g$ with $g > 0$. Since C is central in K^0, there is an action of the finite group K/K^0 on C. Again using the fact that K is nilpotent it is clear that this action must be trivial, i.e., C is central in K. Let $\alpha \in C \setminus C_{tors}$. Now for any $k \in K$ both elements k and $k\alpha$ are contained in the same connected component of K and they cannot be simultaneously torsion elements. Therefore no connected component of K is contained in K_{tors}. □

5. Relations

We start by introducing some notation.

Definition. *Let G be a group and k a natural number. An element $R \in F_k$ is called a* relation *for $v = (g_1, \ldots, g_k)$ if $\zeta_R(v) = e$.*

An element $R \in F_k$ is called a general relation *for G if $\zeta_R \equiv e$.*

The set of all general relations for G is denoted by $R_k(G) = R_k$.

For example, $\xi_1\xi_2\xi_1^{-1}\xi_2^{-1}$ is a general relation for every commutative group. Actually, a group G is commutative if and only if $R_k = F_k'$ for all k where F_k' denotes the commutator group of F_k. Similarily, properties like being m-step nilpotent or m-step solvable can be translated in conditions on $R_k(G)$.

Lemma 6. *Let G be a group and k a natural number. Then $R_k(G)$ is a normal subgroup of F_k.*

Proof. If $A, B \in F_k$, then $\zeta_{ABA^{-1}}(x) = \zeta_A(x)\zeta_B(x)\zeta_{A^{-1}}(x)$. Hence $\zeta_{ABA^{-1}} \equiv e$ if and only if $\zeta_A \equiv e$.

Therefore $R_k(G)$ is normal in F_k. □

We will show that generic k-tuples $(g_1, \dots, g_k)$ have no relations except the general relations of the ambient Lie group.

Definition. *Let G be a group. Then we define Σ_k as the set of all $(g_1, \dots, g_k) \in G^k$ for which there are more relations than the general relations of the group G.*

Proposition 3. *Let G be a connected Lie group and $k \in \mathbb{N}$.*

Then Σ_k is a subset of G^k of measure zero.

Proof. Let $S = F_k \setminus R_k(G)$. Then $\zeta_A : G^k \to G$ is a non-constant real analytic map for every $A \in S$. It follows that

$$\Sigma_k = \cup_{A \in S}\, \zeta_A^{-1}(e)$$

is a set of measure zero, because S is a countable set and $\zeta_A^{-1}(e)$ is of measure zero for every $A \in S$. □

Thus a generic finitely generated subgroup of a connected Lie group fulfills no relations except the general relations of the ambient Lie group. It is therefore useful to determine the general relations for Lie groups.

Lemma 7. *Let G be a connected Lie group. Then G is commutative resp. nilpotent resp. solvable if and only if every subgroup with two generators has the respective property.*

Proof. The statement is trivial concerning commutativity. In respect to solvability it follows from the "Tits alternative" (see [15]). Thus we only have to show, that given a non-nilpotent connected Lie group G, there exists a subgroup with 2 generators which is not nilpotent. Since G is not nilpotent, Ado's theorem implies that there is an element v in the Lie algebra $Lie(G)$ such that $ad(v)$ is not nilpotent. Let W be an irreducible $ad(v)$-sub module of $Lie(G)$ on which $ad(v)$ is not trivial. Then either W is real one-dimensional and $ad(v)(w) = \lambda w$ for $w \in W$ with $\lambda \in \mathbb{R} \setminus \{0\}$ or V is real two-dimensional. In both cases it is easy to check that $\exp(v)$ and $\exp(w)$ generate a non-nilpotent group for every $w \in W$. □

Corollary 4. *A connected Lie group G is solvable resp. nilpotent resp. commutative if and only if $F_2/R_2(G)$ has the respective property.*

In contrast, for non-solvable Lie groups there are no general relations.

Proposition 4. *Let G be a connected Lie group and assume that G is not solvable. Then $R_k(G) = \{e\}$ for all $k \in \mathbb{N}$.*

Proof. Consider the adjoint representation $Ad : G \to GL(Lie\, G)$. Since G is a central extension of $Ad(G)$ by the center of G, non-solvability of G implies that $Ad(G)$ is also non-solvable. Due to "Tits-alternative" it follows that for every $k \in \mathbb{N}$ the group $Ad(G)$ contains a free subgroup with k generators. This subgroup can be lifted to a subgroup of G (because of its freeness). It follows that $R_k(G) = \{e\}$. □

Lemma 8. *Let G be a positive-dimensional Lie group. Then $R_k(G)$ is contained in the commutator group F'_k of F_k for every k.*

Proof. The Lie group G must contain a one-parameter subgroup. Such a one-parameter subgroup is isomorphic to $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$. Both contain subgroups isomorphic to $\mathbb{Z}^k$ for every k. □

Corollary 5. *Let G be a positive-dimensional abelian connected Lie group.*

Then $R_k(G) = F'_k$ for every $k \in \mathbb{N}$.

6. Cartan subgroups

We recall the notion of Cartan subalgebras and Cartan subgroups for arbitrary Lie groups. As standard references we use [2] and [13].

Definition. *Let $\mathbf{g}$ be a finite-dimensional Lie algebra over a field k. A Lie subalgebra $\mathbf{h}$ is called* Cartan subalgebra *if it is nilpotent and equals its own normalizer (i.e. $[x, a] \in \mathbf{h}\ \forall a \in h$ implies $x \in \mathbf{h}$.)*

A Cartan subgroup *H of a group G is a maximal nilpotent subgroup such that $N_G(I)/I$ is finite for every normal subgroup I of H with H/I finite.*

Since Cartan subgroups are maximal nilpotent, and nilpotency is inherited by the closure of a subgroup in a topological group, it is clear that Cartan subgroups are closed for any topology compatible with the group structure. For instance, considering Zariski topology it follows that in an algebraic group every Cartan subgroup is an algebraic subgroup.

Every Lie algebra contains Cartan subalgebras. More precisely, an element x in a Lie algebra $\mathbf{g}$ is called *regular* if the multiplicity of 0 as root of the characteristic polynomial of $ad(x)$ is minimal. Regular elements form a dense open subset of the Lie algebra. For every regular element x the vector subspace

$$\{v \in \mathbf{g} : ad(x)^n(v) = 0 \ \exists n\}$$

is a Cartan subalgebra. Conversely, every Cartan subalgebra arises in this way.

For a connected Lie group G every Cartan subgroup is a closed Lie subgroup such that the corresponding Lie subalgebra of $Lie(G)$ is a Cartan subalgebra. Conversely, given a Cartan subalgebra of the Lie algebra of a Lie group G, there always exists a Cartan subgroup H of G whose Lie algebra is the given Lie subalgebra of $Lie(G)$.

There is also a notion of *regular elements* for Lie groups: An element g in a Lie group G is called regular, if the multiplicity of 1 as root of the characteristic polynomial of $Ad(g)$ is minimal. This definition implies immediately that the set of all non regular elements in a connected Lie group constitutes a nowhere dense real analytic subset. In particular this is a set of measure zero. For a regular element g in a connected Lie group G the weight space of 1

$$\mathbf{g}_1 = \{v \in Lie\, G : (Ad(g) - I)^N(v) = 0 \ \exists N\}$$

is a Cartan subalgebra of $Lie\, G$ and the corresponding Cartan subgroup of G can be described as the set of all elements $x \in G$ such that $Ad(x)$ stabilizes $\mathbf{g}_1$ and preserves the weight space decomposition of $Lie\, G$ with respect to the $ad(\mathbf{g}_1)$-action on $Lie\, G$ (see [12]). Evidently this Cartan subgroup contains the element g with which we started. Thus every regular element in a connected Lie group is contained in a Cartan subgroup. This implies the subsequent statement.

Proposition 5. *Let G be a connected Lie group and W the union of all Cartan subgroups of G. Then $G \setminus W$ is a set of measure zero (with respect to the Haar measure of G).*

Lemma 9. *Let G be a connected Lie group. Let G^k be the k-th derived group (i.e. $G^0 = G$ and $G^{k+1} = [G, G^k]$ for all $k \in \mathbb{N}$) and $G^\infty = \cap_k G^k$.*

Then every Cartan subgroup of G maps surjectively on G/G^∞.

Proof. This is an immediate consequence of the fact for every Cartan subgroup H the associated Lie algebra $Lie(H)$ can be described as

$$\{v \in \mathbf{g} : ad(x)^n(v) = 0 \ \exists n\}$$

for some regular element $x \in G$. □

Cartan subgroups behave nice under central extensions.

Lemma 10. *Let G be a Lie group, C a connected central subgroup of G and H an arbitrary subgroup of G.*

Then H is a Cartan subgroup of G if and only if the following two conditions are fulfilled:

1. *$C \subset H$ and*
2. *H/C is a Cartan subgroup of G/C.*

Proof. Centrality of C implies that H is maximal nilpotent if and only if $C \subset H$ and H/C is maximal nilpotent in G/C. Connectedness of C implies that $C \subset I$ for every subgroup of finite index I of H provided $C \subset H$. Hence the assertion is an easy consequence of the definition. □

Proposition 6. *Let G be a connected Lie group and R its radical. Assume that G/R is compact.*

Then all the Cartan subgroups of G are conjugate.

Proof. We may consider Lie algebras instead of Lie groups, since there is a one-to-one correspondance between Cartan subgroups of G and Cartan subalgebras of $Lie\,G$. Let $Lie(H_i)$ (with $i = 1, 2$) be Cartan subalgebras of $Lie\,G$. Then $(Lie(H_i) + Lie\,R)/\,Lie\,R$ are Cartan subalgebras of $Lie\,G/\,Lie\,H$ ([4]) and they are conjugate, because G/R is compact. Thus there is no loss in generality in assuming $Lie\,I = Lie\,H_1 + Lie\,G = Lie\,H_2 + Lie\,R$. Now $Lie\,H_i \subset Lie\,I \subset Lie\,G$ implies that both $Lie\,H_i$ are Cartan subalgebras in $Lie\,I$. On the other hand $Lie\,I$ is solvable. Therefore the Cartan subalgebras of $Lie\,I$ are conjugate ([2], VII. $3, thm. 3). Hence $Lie\,H_1$ and $Lie\,H_2$ are conjugate. □

Lemma 11. *Let G be a connected Lie group and H a Cartan subgroup.*

Then $\cup_{g \in G}\, gH^0g^{-1}$ is a subset of G of positive measure.

Proof. By [2], Ch.VII, §4, lemme 3 and prop. 7. this set contains an open set. □

Lemma 12. *Let G be a connected Lie group and H a Lie subgroup. Assume that* $\dim N_G(H^0) - \dim H > 0$.

Then

$$A = \cup_{g \in G}\, gHg^{-1}$$

is a set of measure zero.

Proof. Note that $N_G(H^0) = \{g \in G : Ad(g)(Lie\,H) = Lie\,H\}$. Hence $N = N_G(H^0)$ is closed (even if H is not). Then $\pi : G \to G/N$ is a locally trivial fiber bundle. Let $(U_i)_i$ be a countable family of open subsets of G/N which cover all of G/N and such that there are sections

$\sigma_i : U_i \to G$. Let M be the disjoint union of all U_i. Then $M \times H$ is a manifold with countably many connected components such that $\dim M < \dim G$ and such that there exists a surjective differentiable map $f : M \to G$ with $f(M) = A$, namely $f(u,h) = \sigma_i(u)h\sigma_i(h)^{-1}$ for $u \in U_i$ and $h \in H$. It follows that A has measure zero. □

7. Compactness of Cartan subgroups

Proposition 7. *Let S be a connected semisimple Lie group and H a Cartan subgroup. Assume that the center of S is infinite.*

Then the connected components of H are not compact.

Proof. Let Z denote the center of S, let $S_0 = S/Z$ and let K_0 be a maximal compact subgroup of S_0. Then $K_0 = A \cdot U$ where U is semisimple, A is central in K_0, and $A \cap U$ is finite. Now K_0 is a deformation retract of S_0, hence $\pi_1(S_0) \simeq \pi_1(K_0)$. As a compact semisimple group, U has a finite fundamental group. Therefore the image of the group homomorphism $i_* : \pi_1(A) \to \pi_1(S_0)$ induced by the inclusion map $i : A \to S_0$ is of finite index in $\pi_1(S_0)$. Since $\rho : S \to S_0$ is an infinite covering, it follows that there is a 1-torus $S^1 \simeq A_1 \subset A$ such that $\rho^{-1}(A_1)$ has a connected component isomorphic to $\mathbb{R}$.

Now, if $H \subset S$ would be a Cartan subgroup with compact connected component, then $\rho(H)$ would have compact connected components, too. In this case $\rho(H)$ would have to be a compact Cartan subgroup of S_0 and be conjugate to a Cartan subgroup of K_0. Hence $\rho(H)$ would contain a conjugate of A and therefore H would contain a closed subgroup isomorphic to $(\mathbb{R}, +)$. Thus there can not exist a Cartan subgroup of S with compact connected components. □

Proposition 8. *Let G be a connected Lie group and H a Cartan subgroup.*

Assume that H has compact connected components.

Then H is compact.

Proof. Every compact connected normal nilpotent Lie subgroup of G is central (lemma 4) and therefore contained in every Cartan subgroup (lemma 10). Hence we may divide G by any such subgroup and therefore without loss of generality assume that G contains no compact connected normal nilpotent Lie subgroup.

Let Z denote the center of G. Its connected component is contained in H, hence Z^0 is compact and by the assumption we just made we derive $Z^0 = \{e\}$, i.e., Z is discrete.

Let N be the nilradical of G and C a maximal compact subgroup of N. Connectedness of N implies connectedness of C. Hence lemma 3 implies that C is characteristic in N and therefore normal in G. Thus $C = \{e\}$, since we assumed that G contains no compact normal connected nilpotent Lie subgroup. It follows that N is simply-connected. Since every Cartan subgroup maps surjectively on G/G', compactness of a connected component of a Cartan subgroup implies that $G/G' \simeq R/(G' \cap R)$ is compact. Hence $N \subset G'$. Since $G' \cap R \subset N$ for every connected Lie group G, it follows that $N = G' \cap R$.

We will now discuss the adjoint representation of G. Its kernel is the center Z of G. Since the center is discrete, the fibers of ad are discrete. Now Ad maps $G' \cap R$ to a unipotent subgroup of $GL(Lie\, G)$. Thus $Ad(N)$ is simply-connected and closed in $GL(Lie\, G)$. This implies that $Z \cap N = \{e\}$ and that ZN is closed in G. Using compactness of R/N it follows that $R \cap Z$ is finite.

Now let us consider the projection $\pi : G \to G/R$. The Lie algebra of $\pi(H)$ is a Cartan subalgebra of $Lie(G/R)$ ([4]). Hence the semisimple Lie group G/R contains a Cartan subgroup with compact connected component. By the preceding proposition it follows that the center of G/R is finite. Therefore $\pi(Z)$ is finite and consequently Z itself is finite.

Because H is a Cartan subgroup and $\ker(Ad) = Z$ is central, it is clear that $Ad(H)$ is a maximal nilpotent subgroup of $Ad(G)$.

The group $U = Ad(G' \cap R) = Ad(N)$ is a unipotent subgroup of $GL(Lie\, G)$, hence algebraic. Let A denote the normalizer of U in $GL(Lie\, G)$. Then A is algebraic and $U \subset Ad(G) \subset A$. The quotient group A/U is algebraic and the projection $\rho : A \to A/U$ is an morphism of algebraic groups. Now $\rho(Ad(G))$ is compact and therefore algebraic. It follows that $Ad(G) = \rho^{-1}(\rho(Ad(G)))$ is algebraic, too. The Zariski closure of the nilpotent group $Ad(H)$ in $Ad(G)$ is likewise nilpotent. But $Ad(H)$ is maximal nilpotent. Hence $Ad(H)$ is an algebraic subgroup of $GL(Lie\, G)$. Together with the finiteness of Z this implies that H has only finitely many connected components. Thus compactness of the connected components of H implies compactness of H. □

Remark. *For a Cartan subgroup H with non-compact connected components it is possible that H has infinitely many connected components. For instance $G_0 = SL(2, \mathbb{R})$ contains a Cartan subgroup H isomorphic to the multiplicative group $\mathbb{R}^*$. If $\pi : G \to G_0$ denotes the universal covering, then $\pi^{-1}(H)$ is a Cartan subgroup of G which has infinitely many connected components, because H is simply connected while $\pi_1(G_0) \simeq \mathbb{Z}$ is infinite.*

8. Δ_1 AND THE STRUCTURE OF CARTAN SUBGROUPS

Proposition 9. *Let G be a connected Lie group, H a Cartan subgroup with non compact connected components, and $\Omega = \cup_{g \in G} gHg^{-1}$.*

Then $\mu(\Omega \cap \Delta_1) > 0$ and $\mu(\Omega \setminus \Delta_1) = 0$.

Proof. Let K be a maximal compact subgroup of H^0. Then K is normal in H^0 (lemma 3) and $S = \cup_{g \in G} gKg^{-1}$ is a set of measure zero (lemma 12). Note that $\overline{\{g^n : n \in \mathbb{Z}\}}$ is compact for all $g \notin \Delta_1$. This implies that for every $g \in \Omega \setminus \Delta_1$ there exists a natural number such that $g^n \in S$. Now cor. 2 implies that $\Omega \setminus \Delta_1$ has measure zero. On the other hand Ω has positive measure by lemma 11. Thus $\Omega \cap \Delta_1$ has positive measure. □

Proposition 10. *Let G be a connected Lie group, H a Cartan subgroup with compact connected components and $\Omega = \cup_{g \in G} gHg^{-1}$.*

Then $\mu(\Omega \cap \Delta_1) = 0$ and $\mu(\Omega \setminus \Delta_1) > 0$.

Proof. From prop. 8 it follows that H is compact. Hence for every $g \in \Omega$ the generated group $\{g^n : n \in \mathbb{Z}\}$ is contained in a compact subgroup of G. Thus $\Omega \cap \Delta_1 \subset G_{tors}$ implying that $\Omega \cap \Delta_1$ has measure zero (corollary 3). □

9. ZASSENHAUS NEIGHBOURHOODS

We recall the existence of "Zassenhaus neighbourhoods".

Definition. *Let G be a Lie group. An open neighbourhood U of the neutral element e is called* Zassenhaus neighbourhood *if the following assertion is true:*

For every discrete subgroup Γ of G the intersection $\Gamma \cap U$ is contained in a connected nilpotent Lie subgroup of G.

Theorem 5 (Zassenhaus, see [18],[9])**.** *Every Lie group contains Zassenhaus neighbourhoods.*

This has the following consequence:

Corollary 6. *Let G be a connected Lie group which is not nilpotent and $n \geq 2$. Then there exists an open neighbourhood W_k of $(e, \ldots, e)$ in G^k and a subset $\Sigma_k \subset W_k$ of measure zero such that $\langle x \rangle$ is not discrete for any $x = (g_1, \ldots, g_k) \in W_k \setminus \Sigma_k$.*

Proof. Let U be a Zassenhaus neighbourhood and $W_k = U \times \ldots \times U$. Let Σ_k be defined as in def. 5. Then $\langle x \rangle$ is not nilpotent for $x \in W_k \setminus \Sigma_k$, since $\langle x \rangle \simeq F_k / R_k(G)$ for $x \notin \Sigma_k$ and $F_k / R_k(G)$ is not nilpotent for a non-nilpotent Lie group G. On the other hand, U being a Zassenhaus neighbourhood implies that $\langle x \rangle$ is nilpotent for $x \in W_k \cap \Delta_k$. Hence $W_k \setminus \Sigma_k \subset G^k \setminus \Delta_k$. □

10. Amenable Lie groups

A topological group is called *amenable* if it admits a "left invariant mean" (see [7] for more details of this definition and basic properties of amenable topological groups). Compact and solvable topological groups are amenable as well as extensions of solvable by compact topological groups. Closed subgroups of amenable groups are amenable. Free discrete groups are not amenable.

Proposition 11. *Let G be a connected Lie group, R its radical and assume that G/R is compact, but not trivial.*
Then Δ_k is of measure zero for all $k \geq 2$.

Proof. Since G is an extension of a solvable group by a compact one, it is clear that G and every closed subgroup of G must be amenable. On the other hand $R_k(G) = \{e\}$, because G is not solvable. Since free groups are not amenable, it follows that $\langle x \rangle$ can not be closed in G for $x \notin \Sigma_k$. Therefore Δ_k is contained in Σ_k and has measure zero. □

11. Solvable Lie groups

Proposition 12. *Let G be a connected solvable Lie group. Assume that G is not nilpotent.*
Then Δ_k is of measure zero for all $k \geq 2$.

Proof. Let Γ be a non-nilpotent discrete subgroup of G. Then its commutator group Γ' is contained in the commutator group G' of G. Let N denote the universal cover group of G', $\pi : N \to G'$ the natural projection, and $\Gamma_1 = \pi^{-1}(\Gamma')$. Since N is nilpotent and simply-connected, the exponential map $exp : Lie(N) \to N$ is a diffeomorphism. It is known that for every discrete subgroup $\Gamma_1 \subset N$ the preimage $\exp^{-1}(\Gamma_1) \subset Lie\, N$ spans a finite-dimensional $\mathbb{Q}$-vector W subspace of $Lie\, N$. The Γ-action on G by conjugation preserves Γ'. Hence there is an induced action on N and $Lie(N) = Lie(G')$, for simplicity denoted by Ad. Evidently $Ad(\Gamma)$ stabilizes W. Since Γ is not nilpotent, the $Ad(\Gamma)$-action on W can not be unipotent, i.e., there must be an element $\gamma \in \Gamma$ such that $Ad(\gamma)$ considered as $\mathbb{Q}$-linear endomorphism of the $\mathbb{Q}$-vector space W has a non zero eigenvalue λ. This number λ is contained in an algebraic extension field of $\mathbb{Q}$. As a consequence one of the eigenvalues of $Ad(\gamma)$ considered as a $\mathbb{R}$-linear transformation of $Lie(N)$ must be a non-zero algebraic number. However, the set of all algebraic numbers in $\mathbb{C}$ is countable, hence this set has measure zero.

From this fact it is easily deduced that a generic finitely generated subgroup is not discrete. □

12. Proximal elements

Proximal elements where utilized by Tits in proving what is now commonly called the "Tits alternative". Their usage is based on a freeness condition. As explained in [17] this criterion can be modified to check for freeness and discreteness.

Observation. *Let X be a topological space, G a topological group acting continuously on X.*

Assume that there exist families of open subsets $(V_i^+)_{i\in I}$, $(V_i^-)_{i\in I}$ of X with $i \in \{1, \dots, k\}$ such that the closures of all these open sets are compact and mutually disjoint. Furthermore let $p \in X$ be an element not contained in the closure of any of these open sets.

Let W denote the set of all k-tuples $x = (g_1, \dots, g_k) \in G^k$ such that

$$g_i(p) \in V_i^+ \tag{1}$$

$$g_i^{-1} \in V_i^- \tag{2}$$

$$g_i(\bar{V}_j^+ \cup \bar{V}_j^-) \subset V_i^+ \tag{3}$$

$$g_i^{-1}(\bar{V}_j^+ \cup \bar{V}_j^-) \subset V_i^- \tag{4}$$

$$g_i(\bar{V}_i^+) \subset V_i^+ \tag{5}$$

$$g_i^{-1}(\bar{V}_i^-) \subset V_i^- \tag{6}$$

for all $i, j \in I$, $i \neq j$.

Then

1. *W is an open subset in G^k,*
2. *For every $x = (g_1, \dots, g_k) \in G^k$ the group $\langle g_1, \dots, g_k\rangle$ is a free and discrete subgroup of G.*

The openness of W follows form the simply fact that

$$\{g \in G : g(K) \subset \Omega\}$$

is open in G for every compact subset $K \subset X$ and every open subset $\Omega \subset X$.

Furthermore the construction ensures that $\gamma(p)$ is contained in the union A of the closures of the sets U_i^+ and U_i^- with i running through I for every non-trivial expression of the form

$$\gamma = g_{i_1}^{n_1} \cdot \ldots \cdot g_{i_N}^{n_N}.$$

It follows that no such expression is trivial (because $\gamma(p) \in A \not\ni p$) and no sequence of such expressions converges in G to e (because no sequence in A converges to p, since A is closed and $p \notin A$). Therefore the generated subgroup of G is free and discrete.

The following existence result is taken from an earlier paper of the author, see [17].

Proposition 13. *Let S be a connected non compact semisimple linear algebraic group (defined over $\mathbb{R}$ or $\mathbb{C}$).*

Then for every $k \in \mathbb{N}$ there exists an action of S on a projective space X, a point $p \in X$ and families V_i^+ and V_i^- as required in the above observation such that this open set W is non empty.

Corollary 7. *Let G be a connected Lie group and $k \in \mathbb{N}$. Assume that G contains a non compact semisimple Lie subgroup.*

Then there exists an open subset U of G^k such that $u_1, \dots, u_k$ generate a free discrete subgroup of G for all $u = (u_1, \dots, u_k) \in G^k$.

Proof. Let R denote the radical. Then $\pi : G \to G/R$ restricted to S induces a group homomorphism $S \to \pi(S)$ with discrete fibers. Compact semisimple Lie groups have finite fundamental groups, therefore non-compactness of S implies that G/R is non compact. It follows that $Ad(G/R)$ is a non compact semisimple linear algebraic group. Thus we can apply the preceding proposition to $Ad(G/R)$. □

13. Nilpotent Lie groups

Proposition 14. *Let G be a connected nilpotent Lie group and assume that Δ_k has positive measure. Then $G^k \setminus \Delta_k$ has measure zero.*

Proof. Let K be a maximal compact subgroup of G. Then K is central in G (lemma 3) and G/K is a simply-connected nilpotent Lie group. Hence G/K admits a unique structure as a real unipotent linear algebraic group.

Let $\Lambda = F_k/R_k(G)$ and Δ_k^* be the set of all $(g_1, \dots, g_k) \in G^k$ such that the group generated by the g_i is isomorphic to Λ and such that in addition the group generated by the g_i has trivial intersection with K. Then $\Delta_k^* \setminus \Delta_k$ is a set of measure zero. If Δ_k has positive measure, then Δ_k^* is not empty, and there is an embedding i of Λ into $U = G/K$ as a discrete subgroup. By Malcev theory there is a real algebraic subgroup V of U such that $i(\Lambda) = V(\mathbb{Z})$. Furthermore every group homomorphism from $\Lambda \simeq V(\mathbb{Z})$ to U is induced by an algebraic group homomorphism of V to U which in turn corresponds to a Lie algebra homomorphism from $Lie\,V$ to $Lie\,U$. Thus we obtain amap $\eta : F_k \to F_k/R_k(G) = \Lambda \to Lie\,V$. Choose a finite set $\alpha_1, \dots, \alpha_d \in F_k$ such that the $\eta(\alpha_i)$ constitute a vector space basis of $Lie\,V$ (This can be done, since $i(\Lambda)$ is cocompact in V.).

Now for every $g = (g_1, \dots, g_k) \in G^k$, we obtain a group homomorphism from $\Lambda = F_k/R_k(G)$ to U which is induced from a Lie algebra homomorphism from $Lie\, V$ to $Lie\, U$. Thus the subgroup of U generated by the g_iK in $U = G/K$ is discrete and isomorphic to Λ if and only the associated Lie algebra homomorphism is injective. The latter is condition expressible in determinants of images of the $\eta(\alpha_i)$. Therefore the projection of Δ_k in $(G/K)^k$ contains the complement of a closed real algebraic subvariety of $(G/K)^k$. It follows that $G^k \setminus \Delta_k$ has measure zero. ☐

Proposition 15. *Let G be a connected nilpotent Lie group and assume that Δ_k has positive measure.*

Then $\dim G/G' \geq k$.

Proof. We may divide G by its maximal compact subgroup C (which is normal in G, see lemma 3) and thereby assume that G is simply-connected. In this case G carries the structure of a real unipotent group in a natural way.

By assumption there is a discrete subgroup $\Gamma \subset G$ with $\Gamma \simeq F_k/R_k(G)$. Then $rank_{\mathbb{Z}}(\Gamma/\Gamma') \geq k$ by cor. 5. Using "Malcev theory" ([10]) it follows that there is a connected Lie subgroup $H \subset G$ such that H/Γ is compact and $\dim H/H' = rank_{\mathbb{Z}}(\Gamma/\Gamma') \geq k$.

Now G/G' is a real vector space, and, due to the genericity of Γ we may assume that the real vector subspace of G/G' spanned by the image of Γ is of real dimension $\min\{\dim G/G', k\}$.

Assume $\dim G/G' > k$. Then the image of Γ generates G/G' as a real vector space. It follows that H maps surjectively onto G/G'. However, for a subgroup H of a nilpotent group G the equality $HG' = G$ already implies $H = G$. Thus

$$\dim H/H' = \dim G/G' > \underset{\mathbb{Z}}{rank}(\Gamma/\Gamma') = \dim H/H'$$

which is absurd. ☐

14. Density results

Lemma 13. *Let S be a connected semisimple linear algebraic group. Let Ω denote the set of all elements $g \in S$ such that the Zariski-closure of $\{g^n : n \in \mathbb{Z}\}$ is a Cartan subgroup of S.*

Then $S \setminus \Omega$ is a set of measure zero.

Proof. For every semisimple element g of S the Zariski closure of $\{g^n : n \in \mathbb{Z}\}$ is a commutative reductive algebraic group. Cartan subgroups of semisimple linear algebraic groups are connected commutative reductive algebraic groups. Commutative reductive algebraic groups have only countably many algebraic subgroups. Every connected commutative reductive algebraic subgroup of S is conjugate to a subgroup of a Cartan subgroup of S. Now fix a Cartan subgroup T of S. If g is a semisimple element of S such that the Zariski closure of $\{g^n : n \in \mathbb{Z}\}$ is not a Cartan subgroup, then a power g^n is conjugate to an element in a proper algebraic subgroup of T. Now lemma 2 combined with lemma 12 implies the statement together with the fact that the set of all non-semisimple elements of S is contained in an algebraic subvariety of S. $\square$

Lemma 14. *Let S be a connected semisimple linear algebraic group and T a Cartan subgroup. Then there exists a subset C_T of measure zero such that for every $g \in S \setminus C_T$ the group generated by T and g is Zariski dense in S.*

Proof. It is well-known for every non-zero weight the weight space of $Ad(T)$ acting on $Lie\, S$ is one-dimensional. This implies that every Lie subalgebra of $Lie\, S$ containing $Lie\, T$ is a direct sum of $Ad(T)$ weight spaces. It follows that there exist only finitely many connected Lie subgroups of S containing T.

A semisimple Lie group has no normal subgroups except for the products of its simple factors. For this reason a connected Lie subgroup H of S containing T is not normal in S (unless $H = S$). Hence $N_S(H) \neq S$ for such H. Therefore

$$C_T = \cup\{N_S(H^0) : \ H^0 \text{ connected}, \ T \subset H^0\}$$

is a finite union of proper submanifolds of S and thus a set of measure zero. Now C_T contains every closed Lie subgroup H with $T \subset H \subset S$. Hence for every $g \in G \setminus C_T$ the subgroup of S generated by T and g is Zariski dense (in fact dense) in S. $\square$

Proposition 16. *Let S be a connected semisimple linear algebraic group. Then there exists a subset $W \subset S \times S$ such that $S \times S \setminus W$ is a set of measure zero and for every $(g_1, g_1) \in W$ the subgroup of S generated by g_1 and g_2 is Zariski dense in S.*

Proof. Let T be a Cartan subgroup, N its normalizer in S and Ω and C_T as in the preceding lemmata. Let $\pi : G \to G/N$ denote the natural projection and let $\sigma : G/N \to G$ be a measurable section. Let T_0 be the set of all elements in T which are not contained in any proper algebraic subgroup of T. Then there is a measurable bijection $\phi : G/N \times T_0 \to \Omega$ given by

$$\phi : (x, t) \mapsto \sigma(x) \cdot t \cdot \sigma(x)^{-1}.$$

Let $\xi : \Omega \to G/N$ be defined by the composition of ϕ^{-1} with the projection on the factor G/N. Define

$$W = \{(g_1, g_2) \in S \times S : g_2 \notin \xi(g_1) \cdot C_T \cdot \xi(g_1)^{-1}\}.$$

Then $S \times S \setminus W$ is of measure zero, because C_T is of measure zero in S and ξ is a measurable map. Furthermore $(g_1, g_2) \in W$ implies that $\xi(g_1)g_1\xi(g_1)^{-1}$ generates a Zariski dense subgroup of T and $\xi(g_1)g_2\xi(g_1)^{-1} \notin C_T$. Then $\xi(g_1)g_1\xi(g_1)^{-1}$ and $\xi(g_1)g_2\xi(g_1)^{-1}$ generate a Zariski dense subgroup of S. Since

$$x \mapsto \xi(g_1) \cdot x \cdot \xi(g_1)^{-1}$$

is an automorphism of S as algebraic group, this is equivalent to the assertion that g_1 and g_2 generate a Zariski dense subgroup of S. □

Lemma 15. *Let G be a semisimple linear algebraic group. Let $G = \Pi_{i \in I} G_i$ be the representation of G as product of its simple algebraic subgroups. Assume that $H \subset G$ is a subgroup of G which is dense in the Zariski topology and such that $\pi_J(H)$ is not discrete for any subset $J \subseteq I$ where π_J denotes the natural projection $\pi_J : G \to \Pi_{i \in J} Ad(G_i)$.*

Then H is dense in G (in its Hausdorff topology).

Proof. Let $\bar{H}$ denote the closure of H with respect to the Hausdorff topology. By assumption H is not discrete. Hence $\dim \bar{H} > 0$. The connected component $\bar{H}^0$ is normalized by H. However, the normalizer of $\bar{H}^0$ equals the set of all $g \in G$ for which $Ad(g)$ stabilizes the vector subspace $Lie(\bar{H})$ of $Lie\, G$. For this reason, the normalizer of $\bar{H}^0$ in G is an algebraic subgroup of G. Since H is contained in this normalizer, it follows that the normalizer equals the whole group G, i.e., $\bar{H}^0$ is normal in G. Thus $\bar{H}^0 = \Pi_{i \in I \setminus K} G_i$ for some subset $K \subset I$. But this implies that there is a morphism with finite fibers from $\bar{H}/\bar{H}^0$ to $\Pi_{i \in K} Ad(G_i)$. Since $\bar{H}/\bar{H}^0$ is discrete, it follows that K must be empty. This implies $\bar{H} = G$. □

Proposition 17. *Let S be a connected semisimple linear algebraic group. Then there exists an open neighbourhood W of (e, e) in $S \times S$ and a set of measure zero $\Lambda \subset S \times S$ such that g_1, g_2 generate a dense subgroup of S for all $(g_1, g_2) \in W \setminus \Lambda$.*

Proof. Choose an open neighbourhood V of e in S in such a way that $\pi_J(W)$ is contained in a Zassenhaus neighbourhood of S_J for all $J \subset I$. Let Λ_1 be the set of all (g_1, g_2) such that there exists an $i \in I$ such that $\pi_1(g_1)$, $\pi_i(g_2)$ do not generate a free group in $Ad(S_i)$. Let Λ_2 be the set of all (g_1, g_2) such that the generated subgroup is not Zariski dense. Let $\Lambda = \Lambda_1 \cup \Lambda_2$. □

Lemma 16. *Let G be a connected semisimple Lie group, Z a discrete central subgroup and H an arbitrary subgroup in S.*

Then H is dense in S if and only if ZH is dense in S.

Proof. The closure $\bar{H}$ is evidently normalized by H and therefore normalized by ZH and its closure $\overline{ZH} = S$, i.e., $\bar{H}$ is a closed normal subgroup of S. Now $S/\bar{H}$ is a semisimple Lie group, but the image of Z in $S/\bar{H}$ is dense and central. A topological group with a dense central subgroup is necessarily commutative, because the center is closed. Thus we arrive at a contradiction unless $\bar{H} = S$. $\square$

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