ON GROTHENDIECK'S CONJECTURE ABOUT PRINCIPAL HOMOGENEOUS SPACES FOR SOME CLASSICAL ALGEBRAIC GROUPS

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Abstract. In the present paper we investigate the question about the injectivity of the map $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by the canonical inclusion of a local regular ring of geometric type R to its field of fractions K for a homotopy invariant functor $\mathfrak{F}$ with transfers satisfying some additional properties. As an application we get the proof of Special Unitary Case of Grothendieck's conjecture about principal homogeneous spaces and some other interesting examples.

In the present article we consider a regular local ring R with its field of fractions K. One of our main results is the following generalization of the theorem proved in the joint work [PS] of I. Panin and A. Suslin:

Theorem I. *Let $\mathcal{A}$ be an Azumaya algebra over a local regular ring R containing a field not necessary infinite. Then*

(a) *the homomorphism $R^*/\mathrm{Nrd}(\mathcal{A}^*) \longrightarrow K^*/\mathrm{Nrd}(\mathcal{A}_K^*)$ is injective;*

(b) *the canonical map $H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{SL}_{1,\mathcal{A}_K})$ on the first cohomology groups induced by the canonical inclusion $R \hookrightarrow K$ is injective.*

Another main result of the present article concerns the unitary variant of this theorem. Namely, we prove:

Theorem II. *Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with unitary involution (see subsection 3.1) over a local regular ring R containing an infinite field of characteristic $\neq 2$. Then*

(a) *the homomorphism $\mathrm{U}(C)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A})) \longrightarrow \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K))$ is injective;*

(b) *the kernel of the canonical map $H^1_{\text{ét}}(R, \mathrm{SU}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{SU}_{1,\mathcal{A}_K})$ is trivial.*

Theorems I and II prove a conjecture of A. Grothendieck [Gr] for Special Linear and for Special Unitary case (for details see section 3).

It turns out that the following two modifications of these theorems hold as well:

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Theorem III. *Let $\mathcal{A}$ be an Azumaya algebra over a local regular ring R containing a field not necessary infinite and d be some natural number. Then the homomorphism*

$$R^*/\mathrm{Nrd}(\mathcal{A}^*)(R^*)^d \longrightarrow K^*/\mathrm{Nrd}(\mathcal{A}_K^*)(K^*)^d$$

is injective.

Theorem IV. *Let $(\mathcal{A},\sigma)$ be an Azumaya algebra with unitary involution over a local regular ring R containing an infinite field of characteristic $\neq 2$ and d be some natural number. Then the homomorphism*

$$\mathrm{U}(C)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}))\mathrm{U}(C)^d \longrightarrow \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K))\mathrm{U}(C_K)^d$$

is injective.

Note that the method used in the proof of [PS] of Theorem I doesn't give us the way to prove Theorems II, III and IV.

Let R be a local regular ring of geometric type, i.e. R is the local ring of a point of a smooth affine variety A over a field k. Our main tool in the present work is the following result (for the proof see sections 1 and 2):

Theorem. *Let $\mathfrak{F}$ be the functor from the category of A-algebras to the category of abelian groups and let $\mathfrak{F}_R$ be its restriction to the category of R-algebras along the inclusion $A \hookrightarrow R$. If $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy all axioms from sections 1 and 2 below then the homomorphism $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by the canonical inclusion is injective.*

In particular, using the main result of [PO2] and Norm Principle for the Unitary group (section 4) we get the proof of Special Unitary case of Grothendieck's conjecture.

Finally, using a well-known theorem of D. Popescu [PO1] we generalize our results to the case when R is a local regular ring containing a field (section 5).

Our work was motivated by the paper of V. Voevodsky [Vo] and, mainly, by the paper of I. Panin and M. Ojanguren on one Grothendieck's conjecture for hermitian spaces [PO2]. The point was to offer a good axiomatization for the method they introduced.

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Agreements

All rings are assumed to be commutative with unit. By k we will mean an infinite field.

Let A and S be any rings. By an A-algebra S we will mean the pair (S,i), where $i : A \to S$ is a ring homomorphism. Sometimes, we will write just S, instead of the pair (S,i), keeping in mind the homomorphism i. We will denote an A-algebra S by $A \xrightarrow{i} S$.

By a morphism $f : (S,i) \to (S',i')$ between two A-algebras we will mean the commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & S \\ \Big\| & & \Big\downarrow f \\ A & \xrightarrow[\ i'\ ]{} & S' \end{array}$$

Let $\mathfrak{F}$ be a covariant functor on the category of A-algebras to the category of abelian groups. Let $A \xrightarrow{i} R$ be an A-algebra. By restriction of $\mathfrak{F}$ to the category of R-algebras along i we will call the functor denoted by $\mathfrak{F}_R$ and given as follows: $\mathfrak{F}_R(R \xrightarrow{t} T) = \mathfrak{F}(A \xrightarrow{i} R \xrightarrow{t} T)$ on objects and

$$\mathrm{Mor}(\mathfrak{F}_R(R \xrightarrow{t_1} T_1), \mathfrak{F}_R(R \xrightarrow{t_2} T_2)) = \mathrm{Mor}(\mathfrak{F}(A \xrightarrow{i} R \xrightarrow{t_1} T_1), \mathfrak{F}(A \xrightarrow{i} R \xrightarrow{t_2} T_2)),$$

on morphisms for any R-algebra T, T_1, T_2.

Further in the paper we will use the result of Grothendieck ([Ei], Corollary 18.17) which says that if we have a finite extension $A \hookrightarrow B$ of essentially smooth k-algebras then B is finitely generated projective as the A-module.

1. Constant Case

Let A be a smooth k-algebra. Let $R = A_{\mathfrak{p}}$ be the local ring at a prime ideal $\mathfrak{p}$. By $\mathfrak{m}_R$ we will denote the corresponding maximal ideal of R. Thus, we have the A-algebra $A \overset{i_R}{\hookrightarrow} R$, where i_R is the canonical inclusion.

Let $\mathfrak{F}$ be a covariant functor from the category of k-algebras to the category of abelian groups. By $\mathfrak{F}_R$ we denote its restriction to the category of R-algebras along $k \hookrightarrow A \overset{i_R}{\hookrightarrow} R$. Let functors $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy the following list of axioms:

Axiom for the functor $\mathfrak{F}$.

C. (continuity) For any A-algebra S essentially smooth over k and for any multiplicative system M in S the canonical map $\varinjlim_{g\in M} \mathfrak{F}(S_g) \to \mathfrak{F}(M^{-1}S)$ is an isomorphism, where $M^{-1}S$ is the localization of S with respect to M.

Axioms for the functor $\mathfrak{F}_R$.

For any R-algebra T finitely generated and projective as the R-module

TE. (existence) It is given a homomorphism $\mathrm{Tr}_R^T : \mathfrak{F}_R(T) \to \mathfrak{F}_R(R)$ called transfer map;

TA. (additivity) For every element $x \in \mathfrak{F}_R(R \times T)$ with $x_R = \mathrm{pr}_R^*(x) \in \mathfrak{F}_R(R)$ and $x_T = \mathrm{pr}_T^*(x) \in \mathfrak{F}_R(T)$ the relation $\mathrm{Tr}_R^{R\times T}(x) = x_R + \mathrm{Tr}_R^T(x_T)$ holds in $\mathfrak{F}_R(R)$, where pr_R^* and pr_T^* are induced by projections;

TB. (base changing and homotopy invariance) For any $R[t]$-algebra S finitely generated projective as the $R[t]$-module (thus, $S/(t)$ and $S/(t-1)$ are finitely generated projective as R-modules) the following diagram commutes:

$$\begin{array}{ccc}
\mathfrak{F}_R(S) & \xrightarrow{\mathrm{can}_0^*} & \mathfrak{F}_R(S/(t)) \\
{\scriptstyle \mathrm{can}_1^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_0} \\
\mathfrak{F}_R(S/(t-1)) & \xrightarrow[\mathrm{Tr}_1]{} & \mathfrak{F}_R(R)
\end{array}$$

where can_0^*, can_1^* are induced by the canonical projections and Tr_0, Tr_1 denote the corresponding transfer maps $\mathrm{Tr}_R^{S/(t)}$ and $\mathrm{Tr}_R^{S/(t-1)}$.

Let K be a field of fractions of the ring A. Then our aim is to prove:

Theorem (Constant case)**.** *Let $\mathfrak{F}$ be the functor on the category of k-algebras and let $\mathfrak{F}_R$ be its restriction to the category of R-algebras. If $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy axioms* **C**, **TE**, **TA**, **TB** *then the homomorphism $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by the canonical inclusion is injective.*

Remark 1. To get a better feeling for the axioms above observe that **TE**, **TA**, **TB** are the consequences of the following more strong conditions:

For any R-algebra S essentially smooth over k and for any S-algebras T_1, T_2 and T finitely generated projective as the S-modules

H. (homotopy invariance) The map $\mathfrak{F}_R(S) \to \mathfrak{F}_R(S[t])$ induced by the inclusion is an isomorphism;

TE′. (existence) It is given a homomorphism $\mathrm{Tr}_S^T : \mathfrak{F}_R(T) \to \mathfrak{F}_R(S)$ called transfer map;

TA′. (additivity) Let $T = T_1 \times T_2$. For every $x \in \mathfrak{F}_R(T)$, $x_1 = \mathrm{pr}_1^*(x) \in \mathfrak{F}_R(T_1)$ and $x_2 = \mathrm{pr}_2^*(x) \in \mathfrak{F}_R(T_2)$ the relation $\mathrm{Tr}_S^T(x) = \mathrm{Tr}_S^{T_1}(x_1) + \mathrm{Tr}_S^{T_2}(x_2)$ holds in $\mathfrak{F}_R(S)$, where $\mathrm{pr}_i^* : \mathfrak{F}_R(T_1 \times T_2) \to \mathfrak{F}_R(T_i)$ are induced by projections;

TB′. (base changing) For any extension U/S of an R-algebras with U essentially smooth over k the following diagram commutes:

$$\begin{CD} \mathfrak{F}_R(T) @>>> \mathfrak{F}_R(U \otimes_S T) \\ @V{\mathrm{Tr}_S^T}VV @VV{\mathrm{Tr}_U^{U\otimes_S T}}V \\ \mathfrak{F}_R(S) @>>> \mathfrak{F}_R(U) \end{CD}$$

Indeed, to show that the axiom **TB** is a consequence of the axioms **H** and **TB′** let look at two commutative diagrams arising when we apply **TB′** to the extension $S/R[t]$ and evaluations $i_0 : R[t] \overset{t\mapsto 0}{\to} R$, $i_1 : R[t] \overset{t\mapsto 1}{\to} R$ at 0 and 1 correspondingly:

$$\begin{CD} \mathfrak{F}_R(S) @>{\mathrm{can}_0^*}>> \mathfrak{F}_R(S/(t)) \\ @V{\mathrm{Tr}_{R[t]}^S}VV @VV{\mathrm{Tr}_0}V \\ \mathfrak{F}_R(R[t]) @>>{i_0^*}> \mathfrak{F}_R(R) \end{CD} \qquad \begin{CD} \mathfrak{F}_R(S) @>{\mathrm{can}_1^*}>> \mathfrak{F}_R(S/(t-1)) \\ @V{\mathrm{Tr}_{R[t]}^S}VV @VV{\mathrm{Tr}_1}V \\ \mathfrak{F}_R(R[t]) @>>{i_1^*}> \mathfrak{F}_R(R) \end{CD}$$

By axiom **H** the map $\mathfrak{F}_R(R) \to \mathfrak{F}_R(R[t])$ induced by the inclusion is an isomorphism. Since the compositions $R \hookrightarrow R[t] \overset{i_0}{\to} R$ and $R \hookrightarrow R[t] \overset{i_1}{\to} R$ are identities, evaluations i_0^* and i_1^* coincide.

Gluing together these two diagrams via the left-down part we get the required one from **TB**. $\square$

The rest of this section is organized as follows: 1.1 contains the proof of Specialization Lemma, 1.2 is devoted to a version of Quillen's Trick, and the last subsection 1.3 contains the proof of our theorem.

1.1. Specialization Lemma

Let R be a local regular ring of a smooth k-algebra A and let $R \overset{i}{\to} S$ be a given R-algebra with an element $f \in S$ such that:

S1. a) S is finite over $R[t]$ for some specially chosen $t \in S$; b) The quotient $S/(f)$ is finite over R;

S2. There is an augmentation map $\varepsilon : S \to R$ such that the composition $R \xrightarrow{i} S \xrightarrow{\varepsilon} R$ is the identity;

S3. a) S is essentially smooth over the field k; b) $S/\mathfrak{m}_R S$ is smooth over the residue field $R/\mathfrak{m}_R$ at the maximal ideal $\varepsilon^{-1}(\mathfrak{m}_R)/\mathfrak{m}_R S$.

Remark 2. To see that **S3.(b)** is described correctly observe that we have an obvious inclusion $\mathfrak{m}_R S \subset \varepsilon^{-1}(\mathfrak{m}_R)$. Since R is a local we conclude that the ideal $\varepsilon^{-1}(\mathfrak{m}_R)$ is the unique maximal ideal lying over the kernel of augmentation $I = \ker(\varepsilon)$. $\square$

Now we are going to prove:

Theorem (Specialization Lemma). *Let (R, S, f) be the triple satisfying conditions* **S1**, **S2** *and* **S3**. *Let $\mathfrak{G}$ be a covariant functor from the category of R-algebras to the category of abelian groups satisfying axioms* **TE**, **TA** *and* **TB**.

If the image in $\mathfrak{G}(S_f)$ of an element $\alpha_S \in \mathfrak{G}(S)$ is trivial then $\varepsilon^*(\alpha_S) = 0$ in $\mathfrak{G}(R)$.

Proof. To simplify the notation, for any element $\alpha_S \in \mathfrak{G}(S)$, we will denote by α_{S_f} its image under the map induced by the inclusion $S \hookrightarrow S_f$ and by α_0, α_1 and α_I – its images under the maps induced by the canonical projections can_0^*, can_1^* and can_I^* for the ideal I correspondingly.

Since **S1** and **S3** hold for the triple (R, S, f), by Geometric Presentation Lemma ([PO2], Lemma 10.1) for the given $f \in S$ and $t \in S$ we can find an element $t' \in S$ such that:

G1. S is finite over $R[t']$;

G2. There exists an ideal J coprime with I and with the property $I \cap J = (t')$;

G3. (f) is coprime with J and with $(t' - 1)$.

1. The last property **G3** means that we can factorize the canonical projections $S \xrightarrow{\mathrm{can}_J} S/J$ and $S \xrightarrow{\mathrm{can}_1} S/(t'-1)$ through the localization S_f.

Hence, if $\alpha_{S_f} = 0$ in $\mathfrak{G}(S_f)$ for some $\alpha_S \in \mathfrak{G}(S)$ then $\alpha_J = \mathrm{can}_J^*(\alpha_S) = 0$ and $\alpha_1 = \mathrm{can}_1^*(\alpha_S) = 0$.

2. By the theorem of Grothendieck (see the end of Agreements), since S is essentially smooth over k and S is finite over $R[t']$ by **G1**, S is finitely generated projective over $R[t']$.

Now look at the commutative diagram from **TB**:

$$\begin{array}{ccc} \mathfrak{G}(S) & \xrightarrow{\mathrm{can}_0^*} & \mathfrak{G}(S/(t')) \\ {\scriptstyle \mathrm{can}_1^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_0} \\ \mathfrak{G}(S/(t'-1)) & \xrightarrow[\mathrm{Tr}_1]{} & \mathfrak{G}(R) \end{array}$$

By the previous step we get $\mathrm{Tr}_0(\alpha_0) = \mathrm{Tr}_1(\alpha_1) = 0$.

3. By **G2**, there is the decomposition $S/(t') \cong R \times S/J$ via the projections $S/(t') \xrightarrow{\mathrm{can}_{0I}} S/I \xrightarrow{\bar\varepsilon} R$ and $S/(t') \xrightarrow{\mathrm{can}_{0J}} S/J$, where $\bar\varepsilon$ is an isomorphism taken from the obvious decomposition $\varepsilon : S \xrightarrow{\mathrm{can}_I} S/I \xrightarrow{\bar\varepsilon} R$.

Since $S/(t')$ is finitely generated projective over R, the quotient S/J is finitely generated projective over R, too. Therefore, by **TE** we have well-defined transfer map $\mathrm{Tr}_J : \mathfrak{G}(S/J) \to \mathfrak{G}(R)$.

By the additivity **TA** of the transfer and by the second step, we can write:

$$0 = \mathrm{Tr}_0(\alpha_0) = \bar{\varepsilon}^*(\mathrm{can}^*_{0I}(\alpha_0)) + \mathrm{Tr}_J(\mathrm{can}^*_{0J}(\alpha_0)).$$

Since $\mathrm{can}^*_{0I}(\alpha_0) = \alpha_I$ and $\mathrm{can}^*_{0J}(\alpha_0) = \alpha_J$, we can rewrite the last relation as:

$$0 = \bar{\varepsilon}^*(\alpha_I) + \mathrm{Tr}_J(\alpha_J).$$

By the first step $\alpha_J = 0$, thus, one gets $\varepsilon^*(\alpha_S) = \bar{\varepsilon}^*(\alpha_I) = 0$. This completes the proof of Specialization Lemma. $\square$

1.2. A version of Quillen's Trick

Let (A, R, f) be a triple, where A is a smooth d-dimensional k-algebra, R be a local regular ring at the prime ideal $\mathfrak{p}$ of A and $f \in \mathfrak{p}$ be some fixed regular element.

We would like to produce certain extension S of the ring R, such that properties **S1**, **S2**, **S3** of subsection 1.1 are satisfied.

We need the following lemma of Quillen (see [Qu], Lemma 5.12):

Theorem (Quillen's Lemma)**.** *Let A be a smooth finite type algebra of dimension d over a field k, let f be a regular element of A, and let $\mathfrak{I}$ be a finite subset of* Spec$A$*. Then there exist elements $x_1, \dots, x_d$ in A algebraically independent over k and such that if $P = k[x_1, \dots, x_{d-1}] \overset{q}{\hookrightarrow} A$, then* i) *$A/(f)$ is finite over P;* ii) *A is smooth over P at points of $\mathfrak{I}$; and* iii) *the inclusion q factors as $q : P \hookrightarrow P[x_d] \xrightarrow{q_1} A$, where q_1 is finite.*

Set $\mathfrak{I} = \{\mathfrak{p}\}$ and apply Quillen's Lemma to the given pair (A, f).

Look at the canonical tensor product diagram (we keep all notation coming from Quillen's Lemma):

$$\begin{array}{ccc} A & \xrightarrow{i_S} & A \otimes_P R \\ {\scriptstyle q}\uparrow & & \uparrow{\scriptstyle i} \\ P & \xrightarrow[r]{} & R \end{array}$$

where the map r is the composition $r : P \overset{q}{\hookrightarrow} A \overset{i_R}{\hookrightarrow} R$ and $i_S : a \mapsto a \otimes 1$, $i : r \mapsto 1 \otimes r$.

We are going to show that the triple $(R, S, f \otimes 1)$ with $S = A \otimes_P R$ satisfies properties **S1**, **S2**, **S3** of subsection 1.1:

- **S1.** a) Since A is finite over $P[x_d]$ via q_1, we get that $A \otimes_P R$ is finite over $P[x_d] \otimes_P R = R[x_d]$; b) Obviously, $S/(f \otimes 1) = A/(f) \otimes_P R$ is finite over R. Further we will identify f with $f \otimes 1$ via the map i_S.
- **S2.** Take the multiplication map $a \otimes r \mapsto ar$ as augmentation ε.
- **S3.** a) The fact that A is smooth over the ring P at $\mathfrak{p}$ is equivalent to the smoothness of the map $r : P \overset{q}{\hookrightarrow} A \overset{i_R}{\hookrightarrow} A_{\mathfrak{p}}$. Hence, r is smooth. Thus, i_S is smooth also and since A is smooth over k, so is S; b) By the property of smooth extensions, since A is smooth over the P at $\mathfrak{p}$, we have that S must be smooth over R at all primes lying above $\mathfrak{p}S$. Since $\varepsilon^{-1}(\mathfrak{m}_R)$ lies over $\mathfrak{p}S$, S is smooth over R at $\varepsilon^{-1}(\mathfrak{m}_R)$.

1.3. The Proof of Constant Case

Let $\mathfrak{F}$ be the functor on the category of k-algebras and let $\mathfrak{F}_R$ be its restriction to the category of R-algebras. By the hypothesis of the theorem $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy axioms **C**,

TE, **TA** and **TB**. We have to show that the map $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by the canonical inclusion is injective.

Let $\alpha' \in \mathfrak{F}(R)$ be such that its image α'_K in $\mathfrak{F}(K)$ is trivial.

Since $\mathfrak{F}$ is continuous (**C**), we may assume that α' came from an element α in $\mathfrak{F}(A_g)$, where A_g is the localization of A at some $g \in A \setminus \mathfrak{p}$, and the image of α in $\mathfrak{F}(K)$ is trivial. Thus, we can write $\alpha' = i_R^*(\alpha)$, where $i_R : A_g \hookrightarrow R$ is the canonical inclusion.

Observe that A_g is again smooth k-algebra and R is its local regular ring, therefore, we can replace A by A_g and consider the element α as lying in $\mathfrak{F}(A)$.

Using **C** again we get that there exists an element $f \in \mathfrak{p}$ such that the image α_f in $\mathfrak{F}(A_f)$ of the element α is trivial.

Hence, our problem has reduced to the following one:

Proposition. *For a given $\alpha \in \mathfrak{F}(A)$ and $f \in \mathfrak{p}$ such that the image α_f is trivial in $\mathfrak{F}(A_f)$ the element $\alpha' = i_R^*(\alpha)$ is trivial in $\mathfrak{F}(R)$.*

Proof. The proof consists of two steps. On the first step for any triple (A, R, f) we build up a 3×3 commutative diagram. On the second step by playing with this diagram for the specially chosen triple we will complete the proof.

1. Let $R \xrightarrow{i} S$ be the extension of the ring R built up by using Quillen's Trick for any triple (A, R, f) with $R = A_{\mathfrak{p}}$ and regular $f \in \mathfrak{p}$. In particular, there is the commutative diagram:

$$(*) \qquad \begin{array}{ccc} A & \xrightarrow{i_S} & S \\ \| & & \downarrow{\scriptstyle \varepsilon} \\ A & \xrightarrow[i_R]{} & R \end{array}$$

where $\varepsilon : S \to R$ is the augmentation from **S2** for the inclusion $i : R \to S$.

Let $\mathfrak{F}_R$ be the restriction of the functor $\mathfrak{F}$ to the category of R-algebras.

By the commutativity of the diagram

$$\begin{array}{ccccc} k & \longrightarrow & A & \xrightarrow{i_S} & S \\ \| & & & & \uparrow{\scriptstyle i} \\ k & \longrightarrow & A & \xrightarrow[i_R]{} & R \end{array}$$

we have the identities

$$\mathfrak{F}(S) = \mathfrak{F}(k \hookrightarrow A \xrightarrow{i_S} S) = \mathfrak{F}(k \hookrightarrow A \overset{i_R}{\hookrightarrow} R \xrightarrow{i} S) = \mathfrak{F}_R(R \xrightarrow{i} S) = \mathfrak{F}_R(S),$$

$\mathfrak{F}(S_f) = \mathfrak{F}_R(S_f)$ and $\mathfrak{F}(R) = \mathfrak{F}_R(R)$. Moreover, there is a natural identification of the functors $\mathfrak{F}$ and $\mathfrak{F}_R$ restricted to the category of S-algebras and the following diagram commutes:

$$(**) \qquad \begin{array}{ccc} \mathfrak{F}(S) & = & \mathfrak{F}_R(S) \\ {\scriptstyle \varepsilon^*}\downarrow & & \downarrow{\scriptstyle \varepsilon^*} \\ \mathfrak{F}(R) & = & \mathfrak{F}_R(R) \end{array}$$

Consider the 3×3 commutative diagram

$$(***) \qquad \begin{array}{ccccc}
\alpha_f \in \mathfrak{F}(A_f) & \longrightarrow & \mathfrak{F}(S_f) & = & \mathfrak{F}_R(S_f) \ni \alpha_{S_f} \\
\uparrow & & \uparrow & & \uparrow \\
\alpha \in \mathfrak{F}(A) & \xrightarrow{i_S^*} & \mathfrak{F}(S) & = & \mathfrak{F}_R(S) \ni \alpha_S \\
\| & & \downarrow{\scriptstyle \varepsilon^*} & & \downarrow{\scriptstyle \varepsilon^*} \\
\mathfrak{F}(A) & \xrightarrow[i_R^*]{} & \mathfrak{F}(R) & = & \mathfrak{F}_R(R) \ni \alpha'
\end{array}$$

where the upper squares are induced by the localization, the left-down square is induced by $(*)$ and the right-down square coincides with $(**)$.

2. Now let $\alpha \in \mathfrak{F}(A)$ and f be an element from $\mathfrak{p}$ such that $\alpha_f = 0$.

Consider the commutative diagram $(***)$ for the given triple (A, R, f) and consider the elements $\alpha' \in \mathfrak{F}_R(R)$, $\alpha_S \in \mathfrak{F}_R(S)$, $\alpha_f \in \mathfrak{F}(A_f)$, $\alpha_{S_f} \in \mathfrak{F}_R(S_f)$, where $\alpha' = i_R^*(\alpha)$, $\alpha_S = i_S^*(\alpha)$ and α_f, α_{S_f} are the images of the elements α, α_S under the maps induced by the canonical inclusions.

To finish our proof we apply Specialization Lemma (see 1.1) to the right column of our diagram.

By the very construction the triple (R, S, f) satisfies properties **S1**, **S2** and **S3** of subsection 1.1. And by the very assumption the functor $\mathfrak{F}_R$ satisfies axioms **TE**, **TA**, **TB**. So we are under the hypothesis of Specialization Lemma.

Since $\alpha_f = 0$ in $\mathfrak{F}(A_f)$, we get $\alpha_{S_f} = 0$ in $\mathfrak{F}_R(S_f)$. Using Specialization Lemma we conclude that $\varepsilon^*(\alpha_S) = 0$. By the commutativity of the diagram $\varepsilon^*(\alpha_S)$ coincides with α', thus, $\alpha' = 0$. This completes the proof of Constant Case. □

2. Non-Constant Case

Consider more general situation, namely, let the functors $\mathfrak{F}$ and $\mathfrak{F}_R$ (see section 1) are defined only on the category of A-algebras and satisfy axioms **C**, **TE**, **TA**, **TB**. And we still want to prove the injectivity of the homomorphism $\mathfrak{F}(R) \to \mathfrak{F}(K)$. Recall that A is smooth over k, $R = A_{\mathfrak{p}}$ is the local regular ring and K is its field of fractions.

In this case arguments used in the last part of the previous proof don't work because our functor is not defined on k-algebras. Moreover, the objects $\mathfrak{F}(S)$ and $\mathfrak{F}_R(S)$ are not isomorphic in general.

To avoid this problem we will assume that the functors $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy the additional axiom:

E. (extension property) Given an R-algebra $R \xrightarrow{i} S$ essentially smooth over the field k, given an A-algebra $i_S : A \to S$ and an augmentation $\varepsilon : S \to R$ of i, such that the following diagram commutes:

$$\begin{array}{ccc}
A & \xrightarrow{i_S} & S \\
\| & & \downarrow{\scriptstyle \varepsilon} \\
A & \xrightarrow[i_R]{} & R
\end{array}$$

and given a multiplicative system M with respect to a finite set $\{\mathfrak{m}_i\}_{i\in\mathfrak{I}}$ of maximal ideals in S with the property $\varepsilon^{-1}(\mathfrak{m}_R) \subset \cup_{i\in\mathfrak{I}}\mathfrak{m}_i$

there exist

a) the localization S_g for a certain $g \in M$ with a finite etale extension $e : S_g \to \tilde{S}$;

b) an augmentation $\tilde{\varepsilon} : \tilde{S} \to R$ for the inclusion $R \xrightarrow{i} S_g \xrightarrow{e} \tilde{S}$ such that the following diagram commutes:

$$\begin{array}{ccc} S_g & \xrightarrow{e} & \tilde{S} \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\tilde{\varepsilon}} \\ R & = & R \end{array}$$

c) a natural transformation $\Phi : \mathfrak{F} \to \mathfrak{F}_R$ between two functors $\mathfrak{F}$ and $\mathfrak{F}_R$ restricted to the category of $\tilde{S}$-algebras along $A \xrightarrow{i_S} S_g \xrightarrow{e} \tilde{S}$ and $R \xrightarrow{i} S_g \xrightarrow{e} \tilde{S}$ correspondingly, such that the morphism $\Phi(\tilde{S} \xrightarrow{\tilde{\varepsilon}} R) : \mathfrak{F}(\tilde{S} \xrightarrow{\tilde{\varepsilon}} R) \longrightarrow \mathfrak{F}_R(\tilde{S} \xrightarrow{\tilde{\varepsilon}} R)$ is the identity.

In particular, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{F}(\tilde{S}) & \xrightarrow{\Phi(\tilde{S})} & \mathfrak{F}_R(\tilde{S}) \\ {\scriptstyle\tilde{\varepsilon}^*}\downarrow & & \downarrow{\scriptstyle\tilde{\varepsilon}^*} \\ \mathfrak{F}(R) & = & \mathfrak{F}_R(R) \end{array}$$

where R is considered as the $\tilde{S}$-algebra via the augmentation $\tilde{\varepsilon}$.

Remark 3. Since $\varepsilon^{-1}(\mathfrak{m}_R) \cap M = \varnothing$, we can extend our augmentation ε given on S to the augmentation given on the localization S_g, for any $g \in M$, i.e., we have the commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{i_g} & S_g \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ R & = & R \end{array}$$

with the canonical inclusion $S \overset{i_g}{\hookrightarrow} S_g$. Indeed, since R is local with the unique maximal ideal $\mathfrak{m}_R$, the image $\varepsilon(g)$ of any $g \in M$ is invertible in R. $\square$

Then our main result can be stated as follows:

Theorem (Non-Constant Case)**.** *Let $\mathfrak{F}$ be the functor on the category of A-algebras and let $\mathfrak{F}_R$ be its restriction to the category of R-algebras. If $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy axioms* **C**, **TE**, **TA**, **TB** *and* **E** *then the homomorphism $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by the canonical inclusion is injective.*

As in the proof of the Constant case (see the beginning of subsection 1.3), by using continuity **C**, we can reduce the problem about the injectivity of the map $\mathfrak{F}(R) \to \mathfrak{F}(K)$ to the following one:

Proposition. *For a given $\alpha \in \mathfrak{F}(A)$ and $f \in \mathfrak{p}$ such that the image α_f in $\mathfrak{F}(A_f)$ of α is trivial the element $i_R^*(\alpha)$ is trivial in $\mathfrak{F}(R)$.*

Proof. As in Constant Case the proof consists of two steps. On the first step for any triple (A, R, f) we build up a 3×5 commutative diagram. On the second step by playing with this diagram for the specially chosen triple we complete the proof of the proposition.

In contrast with Constant Case the first step is more complicated and can be subdivided as follows:

First we produce starting data to apply axiom **E**. In particular we construct an R-algebra S and a multiplicative system M. Second we construct a lot of elements $h \in M$ such that the extension S_h/R satisfies properties **S1**, **S2** and **S3** of subsection 1.1. Third for the specially chosen $h \in M$ we construct an extension $\tilde{S}_h/R$ satisfying conditions **S1**, **S2**, **S3** and such that the diagram $(**')$ below commutes. And finally we build up the 3×5 commutative diagram.

1. Let $R \xrightarrow{i} S$ be the extension built up by using Quillen's Trick for any triple (A, R, f). In particular, it means that there is a commutative diagram

$$(*) \qquad \begin{array}{ccc} A & \xrightarrow{i_S} & S \\ \| & & \downarrow{\scriptstyle\varepsilon} \\ A & \xrightarrow[i_R]{} & R \end{array}$$

where ε is the augmentation from **S2**, S is finite over $R[t]$ for specially chosen t in S and $S/(f)$ is finite over R by **S1**.

Let $I = \ker(\varepsilon)$, $J = (f) \cap I$ and $J' = (J \cap R[t])S$ (we identify $R[t]$ with its embedding in S). Observe, that $J' \subset J \subset I$.

Lemma 1. *The quotients S/J and S/J' are finite over R.*

Proof. Since $J = (f) \cap I$, there is the inclusion $S/J \hookrightarrow S/(f) \times S/I$ induced by projections. It is easy to see now that S/J is finite over R.

Denote $J_t = J \cap R[t]$ the ideal in $R[t]$. Then $S/J' = S/J_tS = S \otimes_{R[t]} (R[t]/J_t)$ and, since S is finite over $R[t]$, our quotient S/J_tS is finite over $R[t]/J_t$. But there is the inclusion $R[t]/J_t \hookrightarrow S/J$ induced by the given one $R[t] \hookrightarrow S$ and we have just proved that S/J is finite over R. Hence, $R[t]/J_t$ is finite over R and we have done. $\square$

Let $\{\mathfrak{m}_i\}_{i \in \mathfrak{I}}$ be the system of all maximal ideals lying over J'. It is finite because of the lemma. Let M be the corresponding multiplicative system. We see that $\varepsilon^{-1}(\mathfrak{m}_R)$ lies above $I \supset J'$, so it is one of the $\mathfrak{m}_i$, and $\varepsilon^{-1}(\mathfrak{m}_R) \subset \cup_{i \in \mathfrak{I}} \mathfrak{m}_i$.

2. Now we are interesting in the properties of localizations taken via the multiplicative system M.

Lemma 2. *For any localization S_g, $g \in M$,* i) *the quotient $S_g/(f)$ is finite over R;* ii) *there exists an extension S_h of S_g, $h \in M$, such that S_h has a structure of a finite algebra over $R[t']$ for some specially chosen $t' \in S_h$.*

Proof. i) Since $(f) \supset J$, it is enough to show that S_g/JS_g is finite over R. By construction of M we know that $g \in M$ is invertible in S/J, therefore, $S_g/JS_g = (S/J)_g = S/J$ is finite over R. ii) Indeed, it is the reformulation of Lemma 8.2 [PO2]. $\square$

Now show that the extension S_h/R satisfies properties **S1**, **S2** and **S3** of subsection 1.1.

The property **S1** follows from i) and ii). The existence of augmentation **S2** can be conclude from Remark 3. The property **S3** is the consequence of the fact that the localization of a smooth algebra is again smooth and the fact that the element h is coprime with the kernel of augmentation I.

3. Apply now axiom **E** to the data $(R \xrightarrow{i} S, A \xrightarrow{i_S} S, S \xrightarrow{\varepsilon} R, M)$. We get that there exists the localization S_g, $g \in M$, the finite etale extension $e : S_g \to \tilde{S}$, the augmentation $\tilde{\varepsilon} : \tilde{S} \to R$ and the natural transformation $\Phi : \mathfrak{F} \to \mathfrak{F}_R$ between two functors $\mathfrak{F}$ and $\mathfrak{F}_R$ restricted to the category of $\tilde{S}$-algebras satisfying properties **E.(b)** and **E.(c)**.

By the previous step, we can find an extension S_h, $h \in M$, of S_g with the property that S_h/R satisfies conditions **S1**, **S2**, **S3** of subsection 1.1.

Consider now the canonical tensor product diagram:

$$\begin{tikzcd} S_h \arrow[r, "i_1"] & S_h \otimes_{S_g} \tilde{S} \\ S_g \arrow[u] \arrow[r, "e"'] & \tilde{S} \arrow[u, "i_2"'] \end{tikzcd}$$

where $i_1 : s_h \mapsto s_h \otimes 1$ and $i_2 : \tilde{s} \mapsto 1 \otimes \tilde{s}$ are the canonical inclusions.

Set $\tilde{S}_h = S_h \otimes_{S_g} \tilde{S}$. We claim that the extension $\tilde{S}_h/R$ satisfies properties **S1**, **S2**, **S3**.

Indeed, we already know that the extension S_h/R does satisfy. Define the augmentation map as $\tilde{\varepsilon}_h : s_h \otimes \tilde{s} \mapsto \varepsilon(s_h)\tilde{\varepsilon}(\tilde{s})$, thus, we have checked **S2**. Since $\tilde{S}_h/S_h$ is the finite etale, the properties **S1** and **S3** hold.

Clearly, we have the commutative diagram:

$$(*') \qquad \begin{tikzcd} \tilde{S} \arrow[r, "i_2"] \arrow[d, "\tilde{\varepsilon}"'] & \tilde{S}_h \arrow[d, "\tilde{\varepsilon}_h"] \\ R \arrow[r, equal] & R \end{tikzcd}$$

In contrast with the extension S_h/R for the extension $\tilde{S}_h/R$ by **E.(c)** we have the commutative diagram

$$(**') \qquad \begin{tikzcd} \mathfrak{F}(\tilde{S}_h) \arrow[r, "\Phi(\tilde{S}_h)"] \arrow[d, "\tilde{\varepsilon}_h^*"'] & \mathfrak{F}_R(\tilde{S}_h) \arrow[d, "\tilde{\varepsilon}_h^*"] \\ \mathfrak{F}(R) \arrow[r, equal] & \mathfrak{F}_R(R) \end{tikzcd}$$

4. Consider the modification of the diagram $(***)$ built up by using steps 1-3 for any triple (A, R, f):

$$\begin{tikzcd} \mathfrak{F}(A_f) \arrow[r] & \mathfrak{F}(S_{gf}) \arrow[r] & \mathfrak{F}(\tilde{S}_f) \arrow[r] & \mathfrak{F}(\tilde{S}_{hf}) \arrow[r, "\Phi(\tilde{S}_{hf})"] & \mathfrak{F}_R(\tilde{S}_{hf}) \\ \mathfrak{F}(A) \arrow[u] \arrow[r, "i_S^* \circ i_g^*"] \arrow[d, equal] & \mathfrak{F}(S_g) \arrow[u] \arrow[r, "e^*"] \arrow[d, "\varepsilon^*"'] & \mathfrak{F}(\tilde{S}) \arrow[u] \arrow[r, "i_2^*"] \arrow[d, "\tilde{\varepsilon}^*"'] & \mathfrak{F}(\tilde{S}_h) \arrow[u] \arrow[r, "\Phi(\tilde{S}_h)"] \arrow[d, "\tilde{\varepsilon}_h^*"'] & \mathfrak{F}_R(\tilde{S}_h) \arrow[u] \arrow[d, "\tilde{\varepsilon}_h^*"] \\ \mathfrak{F}(A) \arrow[r, "i_R^*"'] & \mathfrak{F}(R) \arrow[r, equal] & \mathfrak{F}(R) \arrow[r, equal] & \mathfrak{F}(R) \arrow[r, equal] & \mathfrak{F}_R(R) \end{tikzcd}$$

where the lower squares are constructed as follows: the left side is induced by $(*)$ and by the commutative diagram from Remark 3; the next squares are induced by **E.(b)**, $(*')$ and $(**')$, correspondingly. Hence, we get the desired 3×5 commutative diagram.

5. Repeat now the arguments used in the step 2 of the proof of Constant Case (see the end of subsection 1.3):

Let $\alpha \in \mathfrak{F}(A)$ and f be an element from $\mathfrak{p}$ such that $\alpha_f = 0$.

Denote by $\alpha_{\tilde{S}_h}$ the image in $\mathfrak{F}_R(\tilde{S}_h)$ of α under the composition $i_S^* \circ i_g^* \circ e^* \circ i_2^* \circ \Phi(\tilde{S}_h)$. Since the image α_f in $\mathfrak{F}(A_f)$ of α is trivial, then by the commutativity of the diagram the image in $\mathfrak{F}_R(\tilde{S}_{hf})$ of the element $\alpha_{\tilde{S}_h}$ is trivial, too.

Hence, we can apply Specialization Lemma to the right column of our diagram and we get $\tilde{\varepsilon}_h^*(\alpha_{\tilde{S}_h}) = 0$. By the commutativity of the diagram $i_R^*(\alpha) = \tilde{\varepsilon}_h^*(\alpha_{\tilde{S}_h})$, thus, $i_R^*(\alpha) = 0$. And we have finished the proof of Non-Constant Case. □

3. Applications

We keep all notations and agreements used before. Let A be a smooth algebra over an infinite field k and let R be its local regular ring at some prime ideal $\mathfrak{p}$. Let K be a field of fractions of R.

In the present section we consider examples of some functors $\mathfrak{F}$ from the category of A-algebras to the category of abelian groups. Namely, the point is to show that such functor satisfies axioms **C**, **TE**, **TA**, **TB** and **E** from the previous sections. Then we can use our main theorem (see section 2)

Theorem (Non-Constant Case)**.** *Let $\mathfrak{F}$ be the functor on the category of A-algebras and let $\mathfrak{F}_R$ be its restriction to the category of R-algebras. If $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy axioms* **C***,* **TE***,* **TA***,* **TB** *and* **E** *then the homomorphism $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by the canonical inclusion is injective.*

Hence, we get the injectivity of the map $\mathfrak{F}(R) \to \mathfrak{F}(K)$ (see the (a) statement of the theorems below). Finally, playing with a few long exact cohomology sequences and using the injectivity above we get the injectivity on the first cohomology level (see the (b) statement of the theorems below).

As a consequence, we will get two cases of Grothendieck's Conjecture about principal homogeneous spaces [Gr] — the case of Special Linear group and the case of Special Unitary group.

Now let describe our special functors and results we are interested in (detailed proofs one will find in the corresponding subsections below):

Linear Case. Let $\mathcal{A}$ be some Azumaya algebra over the given local regular ring R (for the definition see subsection 3.1). Let $\mathrm{Nrd} : \mathcal{A}^* \to R^*$ denotes the reduced norm homomorphism. For any R-algebra T let $\mathcal{A}_T = \mathcal{A} \otimes_R T$ be the extended Azumaya algebra over T.

Define the group scheme $\mathrm{SL}_{1,\mathcal{A}}$ related to the Azumaya R-algebra $\mathcal{A}$ as

$$\mathrm{SL}_{1,\mathcal{A}} : T \mapsto \mathrm{SL}(\mathcal{A}_T) = \{a \in \mathcal{A}_T \mid \mathrm{Nrd}(a) = 1\}.$$

Our aim is to show the result proved before in [PS]:

Theorem (Linear Case)**.** *Let $\mathcal{A}$ be an Azumaya algebra over a local regular ring R of geometric type. Then*

(a) *the homomorphism $R^*/\mathrm{Nrd}(\mathcal{A}^*) \longrightarrow K^*/\mathrm{Nrd}(\mathcal{A}_K^*)$ is injective;*

(b) *the canonical map $H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{SL}_{1,\mathcal{A}_K})$ on the first cohomology groups induced by the canonical inclusion is injective.*

Proof. See subsection 3.2.

Observe that the statement (a) of the theorem corresponds to the case when the functor $\mathfrak{F}$ is defined as $\mathfrak{F} : T \mapsto T^*/\mathrm{Nrd}(\mathcal{A}_T^*)$.

We recall that the conjecture of Grothendieck [Gr] states the triviality of the kernel of the canonical map

$$H^1_{\text{ét}}(R, G) \longrightarrow H^1_{\text{ét}}(K, G_K)$$

for any reductive flat group scheme G over a regular semilocal ring R. Thus, the statement (b) of the theorem above proves the following assertion:

Corollary (Special Linear Case)**.** *The Grothendieck's conjecture is true for the group scheme $G = \mathrm{SL}_{1,\mathcal{A}}$ related to an Azumaya algebra $\mathcal{A}$ over a local regular ring R of geometric type.*

In the same notation assume that there is the additional structure on our Azumaya algebra: Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with involution over R (for the definition see subsection 3.1). Further in the paper when the discussion is about the involution structure we assume that the characteristic of the base field k is different from 2.

We know that there can be three different types of involution: ortogonal, sympletic and unitary. It turns out that in the case of ortogonal and sympletic involution we can prove the same results as above by using well-known facts about quadratic forms (for the details see subsection 3.3). So the only interesting case for us is the unitary case:

Unitary Case. Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with unitary involution over R. It means that there is a tower $\mathcal{A}/C/R$, where C is the center of $\mathcal{A}$ and C/R is an etale quadratic extension over R with restricted involution σ. Therefore, C_K/K is a separable quadratic extension of the corresponding fields of fractions.

Let $\mathrm{U}(\mathcal{A}_T) = \{a \in \mathcal{A}_T \mid aa^\sigma = 1\}$ be the unitary group of an algebra $(\mathcal{A}_T, \sigma)$ for any R-algebra T. We define the group scheme $\mathrm{SU}_{1,\mathcal{A}}$ related to the Azumaya R-algebra $\mathcal{A}$ with unitary involution σ as

$$\mathrm{SU}_{1,\mathcal{A}} : T \mapsto \mathrm{SU}(\mathcal{A}_T) = \{a \in \mathcal{A}_T \mid aa^\sigma = 1,\ \mathrm{Nrd}(a) = 1\}$$

Our goal will be to show:

Theorem (Unitary Case)**.** *Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with unitary involution over a local regular ring R of geometric type. Then*

(a) *the homomorphism $\mathrm{U}(C)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A})) \longrightarrow \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K))$ is injective;*

(b) *the kernel of the canonical map $H^1_{\text{ét}}(R, \mathrm{SU}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{SU}_{1,\mathcal{A}_K})$ is trivial.*

Proof. See subsection 3.4.

Clearly, the statement (a) respects the functor $\mathfrak{F} : T \mapsto \mathrm{U}(C_T)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_T))$, where $C_T = C \otimes_R T$ is the center of the extended Azumaya algebra $\mathcal{A}_T$.

Thus, the statement (b) of the theorem above proves the following assertion:

Corollary (Special Unitary Case). *The Grothendieck's conjecture is true for the group scheme $G = \mathrm{SU}_{1,\mathcal{A}}$ related to an Azumaya R-algebra $\mathcal{A}$ with unitary involution over a local regular ring of geometric type.*

Torsion Cases. Next two theorems represent independent interest. One can look on it as on the modification of the corresponding Linear and Unitary Case.

For the functor $\mathfrak{F} : T \mapsto T^*/\mathrm{Nrd}(\mathcal{A}_T^*)(T^*)^d$, $d \in \mathbb{N}$, we have

Theorem (Linear Torsion Case). *Let $\mathcal{A}$ be an Azumaya algebra over a local regular ring R of geometric type and d be some natural number. Then the homomorphism*

$$R^*/\mathrm{Nrd}(\mathcal{A}^*)(R^*)^d \longrightarrow K^*/\mathrm{Nrd}(\mathcal{A}_K^*)(K^*)^d$$

is injective.

And for the functor $\mathfrak{F} : T \mapsto \mathrm{U}(C_T)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_T))\mathrm{U}(C_T)^d$, $d \in \mathbb{N}$, we have

Theorem (Unitary Torsion Case). *Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with unitary involution over a local regular ring R of geometric type and d be some natural number. Then the homomorphism*

$$\mathrm{U}(C)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}))\mathrm{U}(C)^d \longrightarrow \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K))\mathrm{U}(C_K)^d$$

is injective.

Proof. The torsion cases one can prove easily by following the proofs of the corresponding Linear and Unitary cases (see subsections 3.2 and 3.4).

3.1. The Properties of Azumaya algebras

Let us formulate some properties of Azumaya algebras which we are going to use. Mostly, one can find them in [Kn] and [PO2].

We recall that an Azumaya R-algebra over a ring R is an R-algebra $\mathcal{A}$ which satisfies the following two properties: it is finitely generated projective R-module and the canonical R-algebra homomorphism $\mathcal{A} \otimes_R \mathcal{A}^{op} \to \mathrm{End}_R(\mathcal{A})$ is an isomorphism. Clearly, for any R-algebra T, the scalar extension $\mathcal{A}_T = \mathcal{A} \otimes_R T$ of the algebra $\mathcal{A}$ is an Azumaya algebra over T.

We will use the reduced norm homomorphism $\mathrm{Nrd}_R : \mathcal{A}^* \to R^*$. This homomorphism respects the scalar extensions. In the case where R is a field and T is its algebraic closure, the reduced norm homomorphism Nrd_T can be identified with the usual determinant $\det : \mathrm{GL}_d(T) \to T^*$; this identification is induced by an T-algebra isomorphism between $\mathcal{A}_T$ and the matrix algebra $M_d(T)$ mentioned above.

By an Azumaya algebra with involution over a ring R we mean a pair $(\mathcal{A}, \sigma)$ consisting of an R-algebra $\mathcal{A}$ and an R-linear involution σ on $\mathcal{A}$ such that: i) $\mathcal{A}$ is an Azumaya algebra over its center C; ii) C is either R or an etale quadratic extension of R; and iii) $C^\sigma = R$. Observe that the involution σ commutes with the reduced norm map, i.e. $\mathrm{Nrd}_C(a^\sigma) = \mathrm{Nrd}_C(a)^\sigma$ for any $a \in \mathcal{A}^*$.

When $C = R$ we have involution of the first kind. We say that involution σ is of ortogonal (or sympletic) type if the dimension over R of the set of symmetric elements $\{a \in \mathcal{A} \mid a^\sigma = a\}$ of the algebra $\mathcal{A}$ equals $\frac{n(n+1)}{2}$ (or $\frac{n(n-1)}{2}$), where n is the degree of the Azumaya algebra $\mathcal{A}$ (see [Kn]).

In the case when C is quadratic etale over R we have unitary involution (involution of the second kind).

Now we fix a local regular ring $R = A_{\mathfrak{p}}$ of a smooth k-algebra A. Then we have the following properties:

A1. Since an Azumaya R-algebra $\mathcal{A}$ is given by the finite number of generators and relations we can find the localization A_g, $g \in A \setminus \mathfrak{p}$, such that the algebra $\mathcal{A}$ come from some Azumaya A_g-algebra $\mathcal{A}_g$, i.e. $\mathcal{A} = \mathcal{A}_g \otimes_{A_g} R$. On geometric language it means that we can extend an Azumaya algebra given at a point to some neighbourhood of this point.

By the same reasons as before we can state that if there is an isomorphism $\Psi : \mathcal{A} \to \mathcal{B}$ of Azumaya R-algebras then there exists the localization A_g, $g \in A \setminus \mathfrak{p}$, the Azumaya A_g-algebras $\mathcal{A}_g$ and $\mathcal{B}_g$, where $\mathcal{A} = \mathcal{A}_g \otimes_{A_g} R$ and $\mathcal{B} = \mathcal{B}_g \otimes_{A_g} R$, and the isomorphism $\Psi_g : \mathcal{A}_g \to \mathcal{B}_g$ of the Azumaya A_g-algebras such that $\Psi = \Psi_g \otimes_{A_g} \mathrm{id}_R$.

Observe that the arguments above also work in the case of Azumaya algebras with involutions and when R is a semilocal regular ring, i.e. R is the localization at some finite number of prime ideals of A.

A2. We also will need in the following reformulation of the Proposition 7.1 of [PO2]: Let $\mathcal{O}$ be some semilocal regular ring such that there is the inclusion $R \overset{i}{\hookrightarrow} \mathcal{O}$ with the augmentation ε. Let $(\mathcal{A}_{\mathcal{O}}, \sigma)$ and $(\mathcal{B}_{\mathcal{O}}, \tau)$ be two Azumaya algebras with involution over $\mathcal{O}$, of the same rank. Assume that there exists an isomorphism $\psi : (\mathcal{A}_R, \sigma) \to (\mathcal{B}_R, \tau)$. Then there exists a finite etale extension $e : \mathcal{O} \to \tilde{\mathcal{O}}$, an augmentation $\tilde{\varepsilon} : \tilde{\mathcal{O}} \to R$ of e over R and an isomorphism $\Psi : (\mathcal{A}_{\tilde{\mathcal{O}}}, \tilde{\sigma}) \to (\mathcal{B}_{\tilde{\mathcal{O}}}, \tilde{\tau})$ of extended Azumaya algebras with involutions over $\tilde{\mathcal{O}}$ such that the extension $\Psi \otimes_{\tilde{\mathcal{O}}} \mathrm{id}_R : (\mathcal{A}_{\tilde{\mathcal{O}}}, \tilde{\sigma}) \otimes_{\tilde{\mathcal{O}}} R \to (\mathcal{B}_{\tilde{\mathcal{O}}}, \tilde{\tau}) \otimes_{\tilde{\mathcal{O}}} R$ of Ψ via $\tilde{\varepsilon}$ coincides with ψ.

3.2. The Proof of Linear Case

Let $R = A_{\mathfrak{p}}$ be the local regular ring of the smooth k-algebra A and let $\mathcal{A}$ be the Azumaya algebra over R.

Let $\mathfrak{F}$ be the functor defined on the category of R-algebras as:

$$\mathfrak{F} : T \mapsto T^*/\mathrm{Nrd}(\mathcal{A}_T^*).$$

First of all, by property **A1**, we may assume that our Azumaya algebra $\mathcal{A}$ over the ring R come from some Azumaya algebra over the localization A_g, for some $g \in A \setminus \mathfrak{p}$. Thus, our functor is defined on the category of A_g-algebras.

Since A_g is smooth k-algebra and R is again its local regular ring, we may write A instead of A_g and nothing will be changed. Hence, we will write $\mathcal{A}_R$ instead of $\mathcal{A}$ meaning that $\mathcal{A}_R$ is the scalar extension of the Azumaya A-algebra $\mathcal{A}$ via the canonical inclusion $A \overset{i_R}{\hookrightarrow} R$. Thus, we may assume that our functor $\mathfrak{F}$ is defined on the category of A-algebras.

Show that the functor $\mathfrak{F}$ and its restriction $\mathfrak{F}_R$ to the category of R-algebras satisfy axioms **C**, **TE**, **TA**, **TB** and **E**:

C. Observe that the functor $G_m : T \mapsto T^*$ sending any A-algebra T to its group of units is continuous. Since $\mathcal{A}$ is finitely generated projective as the A-module, the functor of extension of scalars $T \mapsto \mathcal{A}_T$ is continuous. Thus, the functor $T \mapsto \mathrm{Nrd}(G_m(\mathcal{A}_T))$ is continuous, too, and we get the required. $\square$

E. Let we are under the hypothesis of the axiom **E** (see section 2): Let we have the R-algebra $R \xrightarrow{i} S$, the inclusion $i_S : A \to S$, the augmentation $\varepsilon : S \to R$ of i and the multiplicative system M.

For the functors $\mathfrak{F}$ and $\mathfrak{F}_R$ restricted to the category of S-algebras we may write:

$$\mathfrak{F} : T \mapsto T^*/\mathrm{Nrd}(\mathcal{A}_T^*) \qquad \text{and} \qquad \mathfrak{F}_R : T \mapsto T^*/\mathrm{Nrd}(\mathcal{B}_T^*),$$

where for any S-algebra T, $\mathcal{A}_T$ is the extension of scalars of $\mathcal{A}$ via the inclusion $A \xrightarrow{i_S} S \to T$ and $\mathcal{B}_T$ is the extension of scalars of $\mathcal{A}$ via $A \xrightarrow{i_R} R \xrightarrow{i} S \to T$ (in general, $\mathcal{A}_T$ and $\mathcal{B}_T$ are not isomorphic).

We will check axiom **E** in four steps:

First, by using **A2** property we will produce the finite etale extension $\mathcal{O} \xrightarrow{e} \tilde{\mathcal{O}}$ of the localization $\mathcal{O} = M^{-1}S$ and the augmentation $\tilde{\varepsilon} : \tilde{\mathcal{O}} \to R$, such that the extended Azumaya $\tilde{\mathcal{O}}$-algebras $\mathcal{A}_{\tilde{\mathcal{O}}}$ and $\mathcal{B}_{\tilde{\mathcal{O}}}$ become equivalent via the isomorphism Ψ.

Secondly, we will show that the semilocal ring $\tilde{\mathcal{O}}$ is, indeed, the localization $M^{-1}\tilde{S}$ of some R-algebra $\tilde{S}$ which is finite etale over S.

After that, by using **A1** we will find the localization $\tilde{S}_g$, $g \in M$, of $\tilde{S}$ such that the extended Azumaya $\tilde{S}_g$-algebras $\mathcal{A}_{\tilde{S}_g}$ and $\mathcal{B}_{\tilde{S}_g}$ are still equivalent via the isomorphism Ψ_g with $\Psi = \Psi_g \otimes_{\tilde{S}_g} \tilde{\mathcal{O}}$. Considering the extension $e : S_g \to \tilde{S}_g$ and the augmentation $\tilde{\varepsilon} : \tilde{S}_g \to \tilde{\mathcal{O}} \xrightarrow{\tilde{\varepsilon}} R$ we get conditions (a) and (b) of axiom **E**.

We will end with showing condition (c) of axiom **E**. For this purpose by using the isomorphism $\Psi_g : \mathcal{A}_{\tilde{S}_g} \to \mathcal{B}_{\tilde{S}_g}$ we construct the natural equivalence Φ between functors $\mathfrak{F}$ and $\mathfrak{F}_R$ restricted to the category of $\tilde{S}_g$-algebras. It turns out that all necessary properties for condition (c) are satisfied.

1. We introduce the semilocal ring $\mathcal{O}$ as the localization $M^{-1}S$ of S. By Remark 3 we have the augmentation map $\varepsilon : \mathcal{O} \to R$ for the inclusion $R \xrightarrow{i} S \overset{i_{\mathcal{O}}}{\hookrightarrow} \mathcal{O}$ compatible with augmentation on S. Hence, there is the commutative diagram:

$$\begin{array}{ccccc}
A & \xrightarrow{i_S} & S & \xrightarrow{i_{\mathcal{O}}} & \mathcal{O} \\
\| & & {\scriptstyle\varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle\varepsilon} \\
A & \xrightarrow[i_R]{} & R & = & R
\end{array}$$

Consider the extended Azumaya $\mathcal{O}$-algebras $\mathcal{A}_{\mathcal{O}}$ and $\mathcal{B}_{\mathcal{O}}$. Since the extensions $\mathcal{A}_R$ and $\mathcal{B}_R$ of the algebras $\mathcal{A}_{\mathcal{O}}$ and $\mathcal{B}_{\mathcal{O}}$ via the augmentation ε coincide (see the diagram above), we are under the hypothesis of property **A2** (our ψ is the identity).

So there exists the finite etale extension $e : \mathcal{O} \to \tilde{\mathcal{O}}$, the augmentation $\tilde{\varepsilon} : \tilde{\mathcal{O}} \to R$ for the inclusion $R \xrightarrow{i} S \xrightarrow{i_{\mathcal{O}}} \mathcal{O} \xrightarrow{e} \tilde{\mathcal{O}}$ compatible with ε and the isomorphism $\Psi : \mathcal{A}_{\tilde{\mathcal{O}}} \to \mathcal{B}_{\tilde{\mathcal{O}}}$ of extended Azumaya algebras such that its extension $\Psi \otimes_{\tilde{\mathcal{O}}} \mathrm{id}_R$ via $\tilde{\varepsilon}$ is the identity.

2. By the properties of finite etale extensions there exists the localization S_h, $h \in M$, a finite etale extension $e : S_h \to \tilde{S}$ such that $\tilde{\mathcal{O}} = \mathcal{O} \otimes_{S_h} \tilde{S}$. Since $\mathcal{O} = M^{-1}S_h$, we have $\tilde{\mathcal{O}} = M^{-1}\tilde{S}$. To simplify the notation we will write S instead of S_h. Thus, we get the commutative diagram:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow[via\ M]{localization} & \tilde{\mathcal{O}} \\ {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle e} \\ S & \xrightarrow[via\ M]{localization} & \mathcal{O} \end{array}$$

3. By the properties of finite extensions for the direct system of localizations $\{S_g\}_{g\in M}$ with $\varinjlim_{g\in M} S_g = \mathcal{O}$ we have an induced direct system $\{\tilde{S}_g\}_{g\in M}$, where $\tilde{S}_g = S_g \otimes_S \tilde{S}$, and the canonical map $\varinjlim_{g\in M} \tilde{S}_g \to \tilde{\mathcal{O}}$ is an isomorphism. Thus, one has the diagram

$$\begin{array}{ccccc} \tilde{S} & \longrightarrow & \tilde{S}_g & \longrightarrow & \tilde{\mathcal{O}} \\ {\scriptstyle e}\uparrow & & {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle e} \\ S & \longrightarrow & S_g & \longrightarrow & \mathcal{O} \end{array}$$

where e is finite etale.

Since our Azumaya $\tilde{\mathcal{O}}$-algebras $\mathcal{A}_{\tilde{\mathcal{O}}}$ and $\mathcal{B}_{\tilde{\mathcal{O}}}$ are isomorphic via Ψ, applying **A1** to the case $A = \tilde{S}$ and $R = \tilde{\mathcal{O}}$ we get that there exists a localization $\tilde{S}_g$, $g \in M$, such that the Azumaya $\tilde{S}_g$-algebras $\mathcal{A}_{\tilde{S}_g}$ and $\mathcal{B}_{\tilde{S}_g}$ are isomorphic via Ψ_g.

Take the composition $\tilde{\varepsilon} : \tilde{S}_g \to \tilde{\mathcal{O}} \xrightarrow{\tilde{\varepsilon}} R$ to be the augmentation on $\tilde{S}_g$. Clearly, it is compatible with ε. Since the isomorphism Ψ is got from Ψ_g by extension of scalars $\tilde{S}_g \to \tilde{\mathcal{O}}$ and the extension of Ψ via $\tilde{\varepsilon} : \tilde{\mathcal{O}} \to R$ is the identity, the extension of Ψ_g via the augmentation $\tilde{\varepsilon} : \tilde{S}_g \to R$ is the identity as well.

To simplify the notations we replace $\tilde{S}_g$ by $\tilde{S}$ and Ψ_g by Ψ. So we have constructed:

a) the localization S_g, $g \in M$, and the finite etale extension $S_g \xrightarrow{e} \tilde{S}$;

b) the augmentation $\tilde{\varepsilon}$ on $\tilde{S}$ compatible with ε.

Thus, we get conditions (a) and (b) of axiom **E**.

4. The last step is to check **E.(c)**:

Since we have the isomorphism $\Psi : \mathcal{A}_{\tilde{S}} \to \mathcal{B}_{\tilde{S}}$ on Azumaya $\tilde{S}$-algebras, we have the isomorphism $\Psi(T) : \mathcal{A}_T \to \mathcal{B}_T$ of Azumaya T-algebras got by extension of scalars for any $\tilde{S}$-algebra T. By definition there is the commutative diagram:

$$\begin{array}{ccc} \mathcal{A}_T^* & \xrightarrow{\Psi(T)} & \mathcal{B}_T^* \\ {\scriptstyle \mathrm{Nrd}}\downarrow & & \downarrow{\scriptstyle \mathrm{Nrd}} \\ T^* = Z(\mathcal{A}_T)^* & \xrightarrow[\Psi_Z(T)]{} & Z(\mathcal{B}_T)^* = T^* \end{array}$$

where $\Psi_Z(T)$ is the restriction of $\Psi(T)$ to the groups of units T^* of the centers of the Azumaya algebras $\mathcal{A}_T$ and $\mathcal{B}_T$. Thus, there is the isomorphism on the quotients

$$\Phi(T) = \overline{\Psi_Z(T)} : T^*/\mathrm{Nrd}(\mathcal{A}_T^*) \to T^*/\mathrm{Nrd}(\mathcal{B}_T^*).$$

and $\Phi : \mathfrak{F} \to \mathfrak{F}_R$ is the natural equivalence of the functors $\mathfrak{F}$ and $\mathfrak{F}_R$ on the category of $\tilde{S}$-algebras.

Moreover, the extension of the isomorphism $\Phi(\tilde{S}) : \mathfrak{F}(\tilde{S}) \to \mathfrak{F}_R(\tilde{S})$ via the augmentation $\tilde{\varepsilon} : \tilde{S} \to R$ is the identity, i.e. $\Phi(R) = \mathrm{id}_{\mathfrak{F}(R)} : \mathfrak{F}(R) \to \mathfrak{F}_R(R)$.

Observe that in our concrete case the isomorphism $\Psi_Z(T)$ is the identity. In general, it is not true. For instance, when we have Azumaya algebras with unitary involutions (see subsections 3.1 and 3.4) the isomorphism on the centers may not coincide with the identity.

This completes the check of axiom **E**. $\square$

TE. Let T be an R-algebra finitely generated projective as the R-module. Thus, T is a semilocal ring. We define the transfer map

$$\mathrm{Tr}_R^T : T^*/\mathrm{Nrd}(\mathcal{A}_T^*) \longrightarrow R^*/\mathrm{Nrd}(\mathcal{A}_R^*)$$

to be the usual norm map $\mathrm{N}_R^T : T^* \to R^*$ on the quotients.

To see that it is well-defined we need in:

Lemma 3. *Let $\mathcal{A}$ be an Azumaya algebra over a semilocal ring R. Let T be an R-algebra finitely generated projective as the R-module and let $\mathrm{N}_R^T : T^* \to R^*$ be its norm map. Then we claim that:*

$$\mathrm{N}_R^T(\mathrm{Nrd}_T(\mathcal{A}_T^*)) \subset \mathrm{Nrd}_R(\mathcal{A}^*).$$

Proof. To prove this inclusion we are passing our reduced norms through K_1-groups.

By results of [Ba] (ch.V, Theorem 9.1) there is the commutative diagram:

$$\begin{tikzcd}
\mathcal{A}_T^* \arrow[r, "i_T"] \arrow[d, "\mathrm{Nrd}_T"'] & K_1(\mathcal{A}_T) \arrow[r, "\mathrm{N}^*"] \arrow[d, "\mathrm{Nrd}_T^*"'] & K_1(\mathcal{A}) \arrow[d, "\mathrm{Nrd}_R^*"] & \mathcal{A}^* \arrow[l, "i"', "surj"] \arrow[d, "\mathrm{Nrd}_R"] \\
T^* \arrow[r, equal] & T^* \arrow[r, "\mathrm{N}"'] & R^* \arrow[r, equal] & R^*
\end{tikzcd}$$

where N^* is the norm map on K_1-groups; i^* is the surjective homomorphism induced by inclusion $\mathcal{A}^* = \mathrm{GL}_1(\mathcal{A}) \hookrightarrow K_1(\mathcal{A})$; and Nrd^* is the reduced norm on K_1 of an Azumaya algebra. If $\mathcal{A}$ splits, i.e. $\mathcal{A} = M_d(R)$, we can write our reduced norm as the composition:

$$\mathrm{Nrd}^* : K_1(M_d(R)) \xrightarrow{\text{Morita equivalence}} K_1(R) \xrightarrow{\det} R^*.$$

We get the required inclusion since i is surjective. This completes the proof of Lemma 3 and the check of axiom **TE**. $\square$

TA. The additivity follows from the corresponding property of the norm map: If T_1 and T_2 are finitely generated projective over R then for any $(t_1, t_2) \in T_1 \times T_2$ we have $\mathrm{N}_R^{T_1 \times T_2}(t_1, t_2) = \mathrm{N}_R^{T_1}(t_1)\mathrm{N}_R^{T_2}(t_2)$. Since $\mathfrak{F}_R(T_1 \times T_2) = \mathfrak{F}_R(T_1) \times \mathfrak{F}_R(T_2)$, we get the required. $\square$

TB. Let S be any $R[t]$-algebra finitely generated projective as the $R[t]$-module. Observe that the functor $G_m : T \mapsto T^*$ with the usual norm in the role of the transfer map satisfies homotopy invariance **H** and base changing **TB′** properties, thus, it satisfies **TB** (see Remark 1).

To see the axiom **TB** look on the diagram induced by **TB** applying to G_m:

$$(*)\qquad \begin{array}{ccc} G_m(S) & \xrightarrow{\mathrm{can}_0^*} & G_m(S/(t)) \\ {\scriptstyle \mathrm{can}_1^*}\downarrow & & \downarrow{\scriptstyle \mathrm{N}_0} \\ G_m(S/(t-1)) & \xrightarrow[\mathrm{N}_1]{} & G_m(R) \end{array}$$

where can_i^*, $i=0,1$, are induced by the canonical projections, and N_i denote the corresponding norm map.

The following two diagrams induced by the canonical projection

$$\mathrm{pr} : G_m(T) = T^* \longrightarrow T^*/\mathrm{Nrd}(\mathcal{A}_T^*) = \mathfrak{F}_R(T)$$

are commutative as well:

$$\begin{array}{ccccc} G_m(S) & \xrightarrow{\mathrm{can}_i^*} & G_m(S/(t-i)) & \xrightarrow{\mathrm{N}_i} & G_m(R) \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathfrak{F}_R(S) & \xrightarrow[\mathrm{can}_i^*]{} & \mathfrak{F}_R(S/(t-i)) & \xrightarrow[\mathrm{Tr}_i]{} & \mathfrak{F}_R(R) \end{array}$$

where Tr_i, $i=0,1$, denote the corresponding transfer map.

Thus, since the projection pr is surjective, we get the required diagram commutes:

$$\begin{array}{ccc} \mathfrak{F}_R(S) & \xrightarrow{\mathrm{can}_0^*} & \mathfrak{F}_R(S/(t)) \\ {\scriptstyle \mathrm{can}_1^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_0} \\ \mathfrak{F}_R(S/(t-1)) & \xrightarrow[\mathrm{Tr}_1]{} & \mathfrak{F}_R(R) \end{array}$$

And we have checked that functors $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy axioms **C**, **TE**, **TA**, **TB**, **E**. □

Applying now our Theorem (Non-Constant Case) to the functors $\mathfrak{F}$ and $\mathfrak{F}_R$ we get the statement (a) of Linear Case:

Theorem. (a) *The map $R^*/\mathrm{Nrd}(\mathcal{A}^*) \longrightarrow K^*/\mathrm{Nrd}(\mathcal{A}_K^*)$ induced by the canonical inclusion is injective.*

To prove the statement (b) let look on the short exact sequence of smooth group schemes:

$$1 \longrightarrow \mathrm{SL}_{1,\mathcal{A}} \longrightarrow \mathrm{GL}_{1,\mathcal{A}} \xrightarrow{\mathrm{Nrd}} G_m \to 1,$$

where $\mathrm{GL}_{1,\mathcal{A}} : T \mapsto \mathcal{A}_T^*$ for any R-algebra T.

It induces the long exact cohomology sequence:

$$1 \longrightarrow \mathrm{SL}(\mathcal{A}) \longrightarrow \mathcal{A}^* \xrightarrow{\mathrm{Nrd}} R^* \longrightarrow H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(R, \mathrm{GL}_{1,\mathcal{A}}) \longrightarrow \cdots$$

Since R is local, the group $H^1_{\text{ét}}(R, \mathrm{GL}_{1,\mathcal{A}})$ is trivial (see [Kn]).

The inclusion $R \hookrightarrow K$ induces the natural map on our long cohomology sequence, so that the diagram

$$\begin{array}{ccccccccc} 1 \longrightarrow & \mathrm{SL}(\mathcal{A}) & \longrightarrow & \mathcal{A}^* & \xrightarrow{\mathrm{Nrd}} & R^* & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) & \longrightarrow 1 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 1 \longrightarrow & \mathrm{SL}(\mathcal{A}_K) & \longrightarrow & \mathcal{A}_K^* & \xrightarrow[\mathrm{Nrd}_K]{} & K^* & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{SL}_{1,\mathcal{A}_K}) & \longrightarrow 1 \end{array}$$

commutes.

Taking the cokernels from the left side we get:

$$\begin{array}{ccc} R^*/\mathrm{Nrd}(\mathcal{A}^*) & \xrightarrow{\cong} & H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) \\ \downarrow & & \downarrow \\ K^*/\mathrm{Nrd}(\mathcal{A}_K^*) & \xrightarrow[\cong]{} & H^1_{\text{ét}}(K, \mathrm{SL}_{1,\mathcal{A}_K}) \end{array}$$

Since the left vertical arrow is injective by (a), we get the statement (b):

Theorem. (b) *The map $H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{SL}_{1,\mathcal{A}_K})$ induced by the canonical inclusion is injective.*

This completes the proof of Linear Case. □

3.3. Ortogonal and Sympletic Cases

Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with ortogonal involution over R (for the definition see subsection 3.1).

Let $\mathrm{O}(\mathcal{A}_T) = \{a \in \mathcal{A}_T \mid aa^\sigma = 1\}$ be the ortogonal group of an algebra $(\mathcal{A}_T, \sigma)$ for any R-algebra T. We define the group scheme $\mathrm{SO}_{1,\mathcal{A}}$ related to the Azumaya R-algebra $\mathcal{A}$ with ortogonal involution σ as

$$\mathrm{SO}_{1,\mathcal{A}} : T \mapsto \mathrm{SO}(\mathcal{A}_T) = \{a \in \mathcal{A}_T \mid aa^\sigma = 1,\ \mathrm{Nrd}(a) = 1\}.$$

For the field of fractions K we have $\mathrm{O}(K) = \{x \in K \mid x^2 = 1\} = \{\pm 1\}$. Since R is the local ring of an affine variety over the field k, we get also $\mathrm{O}(R) = \{\pm 1\}$.

We would like to show the analogy of the statement (a) of Linear Case, i.e. the injectivity of the map

$$\mathrm{O}(R)/\mathrm{Nrd}(\mathrm{O}(\mathcal{A})) \longrightarrow \mathrm{O}(K)/\mathrm{Nrd}(\mathrm{O}(\mathcal{A}_K))$$

induced by the canonical inclusion.

First of all, note that our map is always surjective and it is not injective if and only if $\mathrm{Nrd}(\mathrm{O}(\mathcal{A})) = \{1\}$ and $\mathrm{Nrd}(\mathrm{O}(\mathcal{A}_K)) = \{\pm 1\}$.

Let $\mathrm{Nrd}(\mathrm{O}(\mathcal{A}_K)) = \{\pm 1\}$ then by theorem of Knezer [BI] we conclude that $\mathcal{A}_K$ splits, i.e. $\mathcal{A}_K$ is the matrix algebra over K. Moreover, by theorem of Grothendieck [BI], the algebra $\mathcal{A}$ splits, too. Hence, we get $\mathrm{Nrd}(\mathrm{O}(\mathcal{A})) = \{\pm 1\}$ and our map must be injective anyway, indeed, it is an isomorphism.

Now look on the short exact sequence of smooth group schemes:

$$1 \longrightarrow \mathrm{SO}_{1,\mathcal{A}} \longrightarrow \mathrm{O}_{1,\mathcal{A}} \xrightarrow{\mathrm{Nrd}} \mathrm{O} \to 1.$$

It induces the long exact cohomology sequence:

$$1 \longrightarrow \mathrm{SO}(\mathcal{A}) \longrightarrow \mathrm{O}(\mathcal{A}) \xrightarrow{\mathrm{Nrd}} \mathrm{O}(R) \longrightarrow H^1_{\text{ét}}(R, \mathrm{SO}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(R, \mathrm{O}_{1,\mathcal{A}}) \longrightarrow \cdots$$

The canonical inclusion $R \hookrightarrow K$ induces the natural map on our long cohomology sequence, so that the diagram

$$\begin{array}{ccccccccc} 1 \longrightarrow & \mathrm{SO}(\mathcal{A}) & \longrightarrow & \mathrm{O}(\mathcal{A}) & \xrightarrow{\mathrm{Nrd}} & \mathrm{O}(R) & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{SO}_{1,\mathcal{A}}) & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{O}_{1,\mathcal{A}}) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 \longrightarrow & \mathrm{SO}(\mathcal{A}_K) & \longrightarrow & \mathrm{O}(\mathcal{A}_K) & \xrightarrow[\mathrm{Nrd}_K]{} & \mathrm{O}(K) & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{SO}_{1,\mathcal{A}_K}) & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{O}_{1,\mathcal{A}_K}) \end{array}$$

commutes.

Taking the cokernels from the left side we get:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{O}(R)/\mathrm{Nrd}(\mathrm{O}(\mathcal{A})) & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{SO}_{1,\mathcal{A}}) & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{O}_{1,\mathcal{A}}) \\ & & \cong\big\downarrow & & \big\downarrow & & \big\downarrow \\ 1 & \longrightarrow & \mathrm{O}(K)/\mathrm{Nrd}(\mathrm{O}(\mathcal{A}_K)) & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{SO}_{1,\mathcal{A}_K}) & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{O}_{1,\mathcal{A}_K}) \end{array}$$

Since the left vertical arrow is the isomorphism and the right vertical arrow has the trivial kernel by the result of [PO2], we get the triviality of the kernel of the middle vertical arrow.

Thus, we have proved the following assertion:

Theorem (Special Ortogonal Case)**.** *The Grothendieck's conjecture is true for the group scheme $G = \mathrm{SO}_{1,\mathcal{A}}$ related to an Azumaya algebra $\mathcal{A}$ with ortogonal involution over a local regular ring R of geometric type.*

Special Sympletic Case of Grothendieck's conjecture about principal homogeneous spaces follows immediately from [PO2].

3.4. The Proof of Unitary Case

We keep all notations and definitions used in the subsection 3.1. The most arguments of our discussion here are taken from the proof of Linear Case (subsection 3.2). The only difference is that we have an additional structure of unitary involution on our Azumaya algebras.

Thus, let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with unitary involution over R.

Let $\mathfrak{F}$ be the functor defined on the category of R-algebras as:

$$\mathfrak{F} : T \mapsto \mathrm{U}(C_T)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_T)),$$

where $\mathrm{U}(C_T) = \{c \in C_T \mid cc^\sigma = 1\}$ is the unitary group of the center of $\mathcal{A}_T$.

By the same arguments as in subsection 3.2, we may assume that our functor is given on the category of A-algebras.

Now the problem is to check the axioms **C**, **TE**, **TA**, **TB** and **E**.

C, E. These axioms can be proved following exactly to the corresponding proofs in subsection 3.2.

TE. The main difficulty is to show the existence of the transfer map for some finitely generated projective extension T/R. Indeed, we would like to see the norm homomorphism in the role of the transfer map again but there is no way to show the inclusion

$$\mathrm{N}_C^{C_T}(\mathrm{Nrd}_{C_T}(\mathrm{U}(\mathcal{A}_T))) \subset \mathrm{Nrd}_C(\mathrm{U}(\mathcal{A}))$$

by using arguments with K_1 (there is no well-defined norm map for the unitary K_1).

The following important theorem gives us another possibility to do this:

Theorem (Norm Principle for the Unitary group). *Let T be a semilocal ring with infinite residue fields of characteristic different from 2. Let $(\mathcal{A}_T, \sigma)$ be an Azumaya algebra with unitary involution σ over T. Let C_T be the center of $\mathcal{A}_T$, so C_T/T is an etale quadratic extension with the restricted involution σ. Then the following equality holds:*

$$\mathrm{Nrd}_{C_T}(\mathrm{U}(\mathcal{A}_T)) = \mathrm{Nrd}_{C_T}(\mathcal{A}_T^*)^{1-\sigma},$$

where $c^{1-\sigma} = c(c^\sigma)^{-1}$, for any $c \in C_T^*$.

Proof. See section 4 below.

Since the norm commutes with the involution we get that

$$N_C^{C_T}(\mathrm{Nrd}_{C_T}(\mathrm{U}(\mathcal{A}_T))) = N_C^{C_T}(\mathrm{Nrd}_{C_T}(\mathcal{A}_T^*)^{1-\sigma}) =$$

$$N_C^{C_T}(\mathrm{Nrd}_{C_T}(\mathcal{A}_T^*))^{1-\sigma} \subset \mathrm{Nrd}_C(\mathcal{A}^*)^{1-\sigma} = \mathrm{Nrd}_C(\mathrm{U}(\mathcal{A})),$$

where the inclusion follows from the Lemma 3 applied to the Azumaya algebra $\mathcal{A}$ over the semilocal ring C and extension C_T/C.

Hence, we can take the norm homomorphism as the transfer map and we get **TE**. □

TA. Since we have the identity $\mathfrak{F}_R(T_1 \times T_2) = \mathfrak{F}_R(T_1) \times \mathfrak{F}_R(T_2)$, additivity axiom **TA** holds. □

TB. To prove **TB** we consider the proof of this axiom in subsection 3.2 but now the diagram $(*)$ is induced by the functor $\mathcal{R}^1_{C/R}(G_m) : T \mapsto \mathrm{U}(C_T)$ coming from the short exact sequence of group schemes:

$$1 \longrightarrow \mathcal{R}^1_{C/R}(G_m) \longrightarrow \mathcal{R}_{C/R}(G_m) \xrightarrow{\mathrm{N}_R^C} G_m \longrightarrow 1,$$

where $\mathcal{R}_{C/R}$ denotes Weil restriction.

We know that the functor $\mathcal{R}_{C/R}(G_m) : T \mapsto C_T^*$ satisfies **TB** and the involution is compatible with the norm map, hence, the kernel $\mathcal{R}^1_{C/R}(G_m)$ satisfies axiom **TB** as well. □

Summarizing our discussion we have proved that the functors $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy axioms **C**, **TE**, **TA**, **TB** and **E**.

Applying now our theorem (Non-Constant Case) to the functors $\mathfrak{F}$ and $\mathfrak{F}_R$ we get the statement (a) of Unitary Case:

Theorem. (a) *The map* $\mathrm{U}(C)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A})) \longrightarrow \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K))$ *induced by the canonical inclusion is injective.*

To show the statement (b) let look on the short exact sequence of smooth group schemes:

$$1 \longrightarrow \mathrm{SU}_{1,\mathcal{A}} \longrightarrow \mathrm{U}_{1,\mathcal{A}} \xrightarrow{\mathrm{Nrd}} \mathcal{R}^1_{C/R}(G_m) \longrightarrow 1,$$

where the $\mathrm{U}_{1,\mathcal{A}} : T \mapsto \mathrm{U}(A_T)$ for any R-algebra T.

It induces the long exact cohomology sequence:

$$1 \longrightarrow \mathrm{SU}(\mathcal{A}) \longrightarrow \mathrm{U}(\mathcal{A}) \xrightarrow{\mathrm{Nrd}} \mathrm{U}(C) \longrightarrow H^1_{\text{ét}}(R, \mathrm{SU}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(R, \mathrm{U}_{1,\mathcal{A}}) \longrightarrow \cdots$$

Observe that the set $H^1_{\text{ét}}(R, \mathrm{U}_{1,\mathcal{A}})$ is not trivial in general.

The canonical inclusion $R \hookrightarrow K$ induces the natural map on our long cohomology sequence, so that the diagram

$$\begin{array}{ccccccccc}
1 \longrightarrow & \mathrm{SU}(\mathcal{A}) & \longrightarrow & \mathrm{U}(\mathcal{A}) & \xrightarrow{\mathrm{Nrd}} & \mathrm{U}(C) & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{SU}_{1,\mathcal{A}}) & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{U}_{1,\mathcal{A}}) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 \longrightarrow & \mathrm{SU}(\mathcal{A}_K) & \longrightarrow & \mathrm{U}(\mathcal{A}_K) & \underset{\mathrm{Nrd}_K}{\longrightarrow} & \mathrm{U}(C_K) & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{SU}_{1,\mathcal{A}_K}) & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{U}_{1,\mathcal{A}_K})
\end{array}$$

commutes.

Taking the cokernels from the left side we get:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathrm{U}(C)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A})) & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{SU}_{1,\mathcal{A}}) & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{U}_{1,\mathcal{A}}) \\
& & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
1 & \longrightarrow & \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K)) & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{SU}_{1,\mathcal{A}_K}) & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{U}_{1,\mathcal{A}_K})
\end{array}$$

Since the left vertical arrow is injective by (a) and the right vertical arrow has the trivial kernel by the main result of [PO2], we get the triviality of the kernel of the middle vertical arrow, i.e. we get:

Theorem. (b) *The kernel of the map $H^1_{\text{ét}}(R, \mathrm{SU}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{SU}_{1,\mathcal{A}_K})$ induced by the canonical inclusion is trivial.*

This completes the proof of Unitary Case. $\square$

4. Norm Principle for the Unitary group

In this section we will prove the following theorem (see section 3 for the definitions):

Theorem (Norm Principle for the Unitary group)**.** *Let R be a semilocal ring with infinite residue fields of characteristic different from 2. Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with unitary involution σ over R. Let C be a center of $\mathcal{A}$, so C is the etale quadratic R-algebra with the standard involution σ. Then the following equality holds:*

$$\mathrm{Nrd}_C(\mathrm{U}(\mathcal{A})) = \mathrm{Nrd}_C(\mathcal{A}^*)^{1-\sigma},$$

where $c^{1-\sigma} = c(c^\sigma)^{-1}$, for any $c \in C^*$.

The constant case, i.e., when R is a field, was proved by A. Merkurjev in [Me] (the elementary proof of this result can be found in [BP]).

To simplify the proof we will assume that R is a local ring with maximal ideal $\mathfrak{m}$ and with infinite residue field k. Indeed, one can easily extend all arguments below to the semilocal case.

The proof consists of three steps:

First, we are going to prove the inclusion $\mathrm{U}(\mathcal{A}) \subset {\mathcal{A}^*}^{1-\sigma}$ (see subsection 4.1). Thus, taking reduced norms, we will get $\mathrm{Nrd}(\mathrm{U}(\mathcal{A})) \subset \mathrm{Nrd}({\mathcal{A}^*}^{1-\sigma})$.

Afterwards, we will show the inclusion $\mathrm{Nrd}(V^{1-\sigma}) \subset \mathrm{Nrd}(\mathrm{U}(\mathcal{A}))$ (see subsection 4.2), where V is an open subset in $\mathcal{A}^*$, in some specially chosen topology.

For this we will introduce some kind of Zariski topology on the free R-module $\mathcal{A}$. Then by using tricky arguments with the intersection of two free submodules of $\mathcal{A}$, we get that almost all invertible elements have their reduced norms in the $\mathrm{Nrd}(\mathrm{U}(\mathcal{A}))$. The precise description of this procedure is a bit technical and is contained in subsection 4.3.

We will finish with 'approximation' lemma which gives us the way to prove the inclusion $\mathrm{Nrd}({\mathcal{A}^*}^{1-\sigma}) \subset \mathrm{Nrd}(\mathrm{U}(\mathcal{A}))$.

4.1. The Proof of the Inclusion $\mathrm{U}(\mathcal{A}) \subset \mathcal{A}^{*1-\sigma}$

Let make some remarks and definitions on the structure of considered rings and unitary groups.

Consider the quotients $\bar{C} = C/\mathfrak{m}C$ and $\bar{R} = k$. By definition $\bar{C}$ is the etale quadratic algebra over k with the standard involution σ. Therefore, $\bar{C}$ is the quadratic field extension $k(\sqrt{s})$, $s \in k^*$, of k or it is the product of two fields $k \times k$.

Recall that $\mathrm{U}(C) = \{c \in C \mid \mathrm{N}(c) = cc^\sigma = 1\}$, where $\mathrm{N} : C^* \to R^*$ is the norm map. And similar let $\mathrm{U}(\bar{C}) = \{\bar{c} \in \bar{C} \mid \bar{\mathrm{N}}(\bar{c}) = \bar{c}\bar{c}^\sigma = 1\}$, where $\bar{\mathrm{N}} : \bar{C}^* \to k^*$ is the norm map for the etale quadratic k-algebra $\bar{C}$. The canonical projection $C \longrightarrow \bar{C}$ induces a map $\mathrm{U}(C) \to \mathrm{U}(\bar{C})$ and we denote by $\overline{\mathrm{U}(C)}$ its image. Observe that $\overline{\mathrm{U}(C)} = \mathrm{U}(\bar{C})$. In fact by Hilbert 90 for local rings [Se] one has $\overline{\mathrm{U}(C)} = \overline{C^{*1-\sigma}} = \bar{C}^{*1-\sigma} = \mathrm{U}(\bar{C})$.

Now we are going to prove the inclusion $\mathrm{U}(\mathcal{A}) \subset \mathcal{A}^{*1-\sigma}$: The main idea here is the same as in the proof of Hilbert 90.

Let $a \in \mathrm{U}(\mathcal{A})$. Write $b_c = c + c^\sigma a$, for every $c \in C^*$. If one of these b_c is invertible, then $ab_c^\sigma = a(c^\sigma + ca^\sigma) = b_c$, hence, $a = b_c(b_c^\sigma)^{-1} \in \mathcal{A}^{*1-\sigma}$.

Denote $l = c(c^\sigma)^{-1} \in \mathrm{U}(C)$, then $b_c = c^\sigma(l + a)$. Thus, our aim is to show that for every $a \in \mathrm{U}(\mathcal{A})$ there exists $l \in \mathrm{U}(C)$ such that $(l + a) \in \mathcal{A}^*$.

The condition $(l + a) \in \mathcal{A}^*$ is equivalent to the condition $\mathrm{Nrd}(l + a) = \mathrm{chr}_a(-l) \in C^*$, where $\mathrm{chr}_a(t) \in C[t]$ is the reduced characteristic polynomial of a.

Since $\mathfrak{m}C$ is the radical ideal of C, the element $\mathrm{chr}_a(-l)$ is invertible in C if and only if its reduction $\overline{\mathrm{chr}_a(-l)}$ modulo $\mathfrak{m}C$ is invertible in $\bar{C}$.

Thus, our aim is to prove:

Lemma 4. *For every $a \in \mathrm{U}(\mathcal{A})$ there exists $\bar{l} \in \mathrm{U}(\bar{C})$ such that $\overline{\mathrm{chr}_a}(-\bar{l}) \in \bar{C}^*$, where $\overline{\mathrm{chr}_a}(t) \in \bar{C}[t]$ denotes the reduction modulo $\mathfrak{m}C$ of the reduced characteristic polynomial $\mathrm{chr}_a(t)$.*

Proof. Assume that $\overline{\mathrm{chr}_a}(-\bar{l})$ is non-invertible for every $\bar{l} \in \mathrm{U}(\bar{C})$.

In case when $\bar{C}$ is the field it means that $\overline{\mathrm{chr}_a}(-\bar{l}) = 0$ for every $\bar{l} \in \mathrm{U}(\bar{C})$. Hence, every element $\bar{l} \in \mathrm{U}(\bar{C})$ is the root of the polynomial $\overline{\mathrm{chr}_a}(t)$. Since the number of roots is finite we get that the unitary group $\mathrm{U}(\bar{C})$ must be finite.

Consider now the case when $\bar{C} = k \times k$ is the product of two fields. Since the involution σ in this case is just the permutation map $(x, y) \mapsto (y, x)$ we get that the unitary group $\mathrm{U}(\bar{C})$ consists of the points of the type (x, x^{-1}), $x \in k^*$.

Clearly, $\bar{C}[t] = k[t] \times k[t]$. Therefore, a polynomial $f \in \bar{C}[t]$ is in fact just a pair (f', f'') of polynomials over $k[t]$. In particular $\overline{\mathrm{chr}_a}(t)$ is just a pair of polynomials $\mathrm{chr}'(t)$ and $\mathrm{chr}''(t)$ over k.

By assumption the element $\overline{\mathrm{chr}_a}(-\bar{l}) \in \bar{C}$ is non-invertible for every $\bar{l} \in \mathrm{U}(\bar{C})$. Thus, either $\mathrm{chr}'(x) = 0$ or $\mathrm{chr}''(x^{-1}) = 0$ for every $\bar{l} = (x, x^{-1}) \in \mathrm{U}(\bar{C})$.

Since the number of solutions of the equation $\mathrm{chr}'(x)\mathrm{chr}''(x^{-1}) = 0$ is finite, the group $\mathrm{U}(\bar{C})$ is finite as well.

On the other hand we have the isomorphism $\mathrm{U}(\bar{C}) \longrightarrow k^*$, $(x, x^{-1}) \mapsto x$, and the group k^* is infinite. Thus, we get contradiction. This completes the proof of Lemma 4 and the Inclusion. $\square$

4.2. The Proof of the Equality $\mathrm{Nrd}(\mathcal{A}^{*1-\sigma}) = \mathrm{Nrd}(\mathrm{U}(\mathcal{A}))$

Topology on R^n. We introduce topology on R^n by lifting the Zariski topology given on the affine space k^n via the map $R^n \stackrel{\mathrm{can}}{\to} k^n$ induced by the canonical projection. Thus, the subbase of this topology consists of the preimages $\mathrm{can}^{-1}(V_f)$ of the main open subsets $V_f = \{x \in k^n \mid f(x) \neq 0\}$, where $f \in k[t_1, t_2, \dots, t_n]$.

Two important and obvious properties of this topology are: i) every finite system of open subsets has nonempty intersection; and ii) any polynomial map $g : R^l \to R^m$ is continuous.

For instance, to see ii) it is enough to look at the commutative diagram:

$$\begin{tikzcd} R^l \arrow[r, "g"] \arrow[d, "\mathrm{can}"'] & R^m \arrow[d, "\mathrm{can}"] \\ k^l \arrow[r, "\bar{g}"'] & k^m \end{tikzcd}$$

where $\bar{g}$ denotes the quotient of the polynomial map g. Since the vertical arrows and the down arrow are continuous, the upper arrow is continuous as well.

In the same way we get another important examples of continuous maps:

1. Let A be an R-algebra and a free R-module of rank n. Then the regular representation $R^n = A \stackrel{rpr}{\to} \mathrm{End}_R A = M_n(R) = R^{n^2}$ is continuous.
2. Multiplication map $M_n(R) \times R^n \stackrel{m}{\to} R^n$, $m : (M, x) \mapsto Mx$, is continuous.
3. Let $\mathrm{GL}_n(R)$ be the group of invertible R-matrices. Then $\mathrm{GL}_n(R)$ is open in $M_n(R)$ and the inverse map $\mathrm{GL}_n(R) \stackrel{inv}{\to} \mathrm{GL}_n(R)$, $inv : M \mapsto M^{-1}$, is continuous in the induced topology on $\mathrm{GL}_n(R)$.

The Proof of the Inclusion of $\mathrm{Nrd}(\mathcal{A}^{*1-\sigma}) \subset \mathrm{Nrd}(\mathrm{U}(A))$. Consider the Azumaya algebra $\mathcal{A}$ as the free R-module of rank $2m$ with the topology constructed above. Further we will always identify $\mathcal{A}$ with R^{2m}.

Define two free submodules of $\mathcal{A}$ of rank m:

$$\mathcal{A}_+ = \{x \in \mathcal{A} \mid x = x^\sigma\} \quad \text{and} \quad \mathcal{A}_- = \{x \in \mathcal{A} \mid x = -x^\sigma\}.$$

It is easy to see that $\mathcal{A}$ is the direct sum of the R-modules $\mathcal{A}_-$ and $\mathcal{A}_+$. Moreover, we have $\mathcal{A}_- = \mathcal{A}_+\sqrt{b}$ and $\mathcal{A}_+ = \mathcal{A}_-\frac{\sqrt{b}}{b}$.

Consider now the intersection $a\mathcal{A}_- \cap (R \cdot 1_{\mathcal{A}} \oplus \mathcal{A}_-)$, for some chosen $a \in \mathcal{A}^*$. We claim that (we will prove this in the subsection 4.3) for almost all a there exists an invertible element in this intersection.

The last means that there exists a non-empty open $V \subset \mathcal{A}^*$ such that for every $a \in V$ there exist $r \in R$, $u \in \mathcal{A}_-$ and invertible element $v \in \mathcal{A}_-$ with the property $r + u = av$.

Take $a \in V$. By the property above we may write $a = (r + u)v^{-1}$ and

$$\mathrm{Nrd}(a^{1-\sigma}) = \mathrm{Nrd}(-(r+u)v^{-1}(r-u)^{-1}v) = \mathrm{Nrd}(-1)\mathrm{Nrd}(\tfrac{r+u}{r-u}),$$

but the elements $\frac{r+u}{r-u}$ and -1 lie in the unitary group $\mathrm{U}(\mathcal{A})$, thus, we get that

$$\mathrm{Nrd}(a)^{1-\sigma} \in \mathrm{Nrd}(\mathrm{U}(\mathcal{A})).$$

The following 'approximation' lemma finishes our proof:

Lemma 5. *Let $V \subset \mathcal{A}^*$ is the open subset of $\mathcal{A}^*$, then for every $a \in \mathcal{A}^*$ there exist v_1, $v_2 \in V$ such that $a = v_1 v_2$.*

Proof. By (3) the subset V^{-1} is open in $\mathcal{A}^*$, thus, aV^{-1} is open in $\mathcal{A}^*$. Since the intersection $V \cap aV^{-1}$ is nonempty, there are v_1, $v_2 \in V$, such that $v_1 = av_2^{-1}$ and we have proved the lemma. □

Now let $a \in \mathcal{A}^*$, $a = v_1 v_2$, where v_1, $v_2 \in V$ and $\mathrm{Nrd}(V)^{1-\sigma} \subset \mathrm{Nrd}(\mathrm{U}(\mathcal{A}))$, then

$$\mathrm{Nrd}(a^{1-\sigma}) = \mathrm{Nrd}(v_1)^{1-\sigma}\mathrm{Nrd}(v_2)^{1-\sigma} \in \mathrm{Nrd}(\mathrm{U}(\mathcal{A})).$$

This completes the proof of Equality.

4.3. The Proof of Existence of Invertible Element

We are fixing the basis of $\mathcal{A}$ over R:

$$\mathcal{A} = \mathcal{A}_{\_} \oplus \mathcal{A}_{+} = \{\sqrt{b} \cdot 1_{\mathcal{A}}, \sqrt{b}e_2, \sqrt{b}e_3, \dots, \sqrt{b}e_m\} \oplus \{1_{\mathcal{A}}, e_2, e_3, \dots, e_m\}.$$

The element $\sqrt{b} \in \mathcal{A}_{\_}$ in this basis is represented by the matrix $\begin{pmatrix} 0_m & E_m \\ bE_m & 0_m \end{pmatrix}$.

Consider two continuous maps $M_{2m}(R) \overset{\mathrm{pr}_1}{\to} M_{m-1}(R)$ and $M_{2m}(R) \overset{\mathrm{pr}_2}{\to} M_{m-1}(R)$, where pr_1 and pr_2 are the projections:

$$M = (m_{ij})_{i=1,\dots,2m}^{j=1,\dots,2m} \xrightarrow{\mathrm{pr}_1} (m_{ij})_{i=m+2,\dots,2m}^{j=2,\dots,m} = N,$$

$$M = (m_{ij})_{i=1,\dots,2m}^{j=1,\dots,2m} \xrightarrow{pr_2} (m_{ij})_{i=m+2,\dots,2m}^{j=1} = \bar{c}.$$

In other words, pr_1 sends any matrix M to its left-down corner $(m-1) \times m$ without the first column denoted by $\bar{c}$, i.e., to the $(m-1) \times (m-1)$ matrix N. And pr_2 sends the matrix M to the column $\bar{c}$.

Consider now continuous map $\psi : \mathcal{A} \overset{rpr}{\to} M_{2m}(R) \overset{\mathrm{pr}_1}{\to} M_{m-1}(R)$, where rpr is the representation of $\mathcal{A}$ as the R-module in our fixed basis.

Let V_1 be the preimage of the open subset $\mathrm{GL}_{m-1}(R) \subset M_{m-1}(R)$ under the map ψ.

The intersection $V_2 = V_1 \cap \mathcal{A}^*$ is the open subset in $\mathcal{A}^*$ (the subset $\mathcal{A}^*$ is open in $\mathcal{A}$).

Define the map ω as the composition:

$$\omega : V_2 \overset{(\rho,-pr_2)}{\longrightarrow} \mathrm{GL}_{m-1}(R) \times R^{m-1} \xrightarrow{m} R^{m-1} \xrightarrow{i} R^{2m} = \mathcal{A},$$

where the map ρ is the composition $\rho : a \overset{\psi}{\mapsto} N \overset{inv}{\mapsto} N^{-1}$, $m : (N, \bar{c}) \mapsto N\bar{c} = \bar{d}$ is the multiplication and $i : \bar{d} \mapsto (1, \bar{d}, 0, \dots, 0)^t = v$ is the inclusion.

Clearly, ω is continuous, so the preimage $V = \omega^{-1}(\mathcal{A}^*)$ of the open subset $\mathcal{A}^* \subset \mathcal{A}$ is open, too. Since $\sqrt{b} \in V$, the subset V is not empty.

We claim that for any $a \in V$ the product av, where $v = \omega(a)$, is the required invertible element. Indeed, by the very construction, $v \in \mathcal{A}_{\_}$ and av is invertible. Consider the product av in our fixed basis:

$$rpr(a)\omega(a) = \begin{pmatrix} a_{1,1} & \dots & a_{1,m} & \dots \\ \vdots & & \vdots & \\ a_{m+1,1} & \dots & a_{m+1,m} & \dots \\ \bar{c} & & N & \dots \end{pmatrix} \begin{pmatrix} 1 \\ N^{-1}(-\bar{c}) \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_{m+1} \\ \bar{c} + NN^{-1}(-\bar{c}) \end{pmatrix} =$$

$$= \begin{pmatrix} u_1 \\ \vdots \\ u_{m+1} \\ 0 \end{pmatrix} \in \mathcal{A}_{\_} \oplus R \cdot 1_{\mathcal{A}}, \quad \text{and we have finished.}$$

5. The Case of a Local Regular Ring containing a Field

In this section we generalize all theorems of section 3 to the case of a local regular ring containing a field. Let R be a local regular ring containing a field k not necessary infinite and let K be its field of fractions.

Theorem I. *Let $\mathcal{A}$ be an Azumaya algebra over a local regular ring R containing a field not necessary infinite. Then*
 (a) *the homomorphism $R^*/\mathrm{Nrd}(\mathcal{A}^*) \longrightarrow K^*/\mathrm{Nrd}(\mathcal{A}_K^*)$ is injective;*
 (b) *the canonical map $H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{SL}_{1,\mathcal{A}_K})$ on the first cohomology groups induced by the canonical inclusion is injective.*

Theorem II. *Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with unitary involution (see subsection 3.1) over a local regular ring R containing an infinite field of characteristic $\neq 2$. Then*
 (a) *the homomorphism $\mathrm{U}(C)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A})) \longrightarrow \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K))$ is injective;*
 (b) *the kernel of the canonical map $H^1_{\text{ét}}(R, \mathrm{SU}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{SU}_{1,\mathcal{A}_K})$ is trivial.*

Theorem III. *Let $\mathcal{A}$ be an Azumaya algebra over a local regular ring R containing a field not necessary infinite and d be some natural number. Then the homomorphism*

$$R^*/\mathrm{Nrd}(\mathcal{A}^*)(R^*)^d \longrightarrow K^*/\mathrm{Nrd}(\mathcal{A}_K^*)(K^*)^d$$

is injective.

Theorem IV. *Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with unitary involution over a local regular ring R containing an infinite field of characteristic $\neq 2$ and d be some natural number. Then the homomorphism*

$$\mathrm{U}(C)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}))\mathrm{U}(C)^d \longrightarrow \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K))\mathrm{U}(C_K)^d$$

is injective.

We consider in details only the proof of Theorem I. Other theorems are proved similarly.

Let $\mathcal{A}$ be an Azumaya algebra over R. Let $\mathfrak{F}$ be the functor (subsection 3.2) defined on the category of R-algebras as:

$$\mathfrak{F} : T \mapsto T^*/\mathrm{Nrd}(\mathcal{A}_T^*).$$

Our aim is to show

Theorem I. (a) *The map $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by the canonical inclusion is injective.*

Proof. First, we consider the case when k is infinite field (compare with Proof of Theorem B, section 8, [PO1]). Then we prove the theorem for any finite field k (compare with section 4, [PO1]).

1. Assume k is infinite. By Popescu's theorem ([PO1], section 7) R is a filtered direct limit of essentially smooth local k-algebras, i.e, $R = \varinjlim R_i$, where R_i is a local regular ring of an affine smooth variety over k.

Since an Azumaya R-algebra $\mathcal{A}$ is given by the finite number of generators and relations we can find the index j such that the algebra $\mathcal{A}$ come from some Azumaya algebra $\mathcal{A}_j$ over R_j, i.e. $\mathcal{A} = \mathcal{A}_j \otimes_{R_j} R$.

Indeed, let all relations between generators $e_1, e_2, \dots, e_n$ of $\mathcal{A}$ over R come from $R_{j'}$ for some index j'. Thus, we get some $R_{j'}$-algebra $\mathcal{A}_{j'}$ with the same generators and relations, i.e. $\mathcal{A} = \mathcal{A}_{j'} \otimes_{R_{j'}} R$. Then we can find an index $j \geq j'$ such that the determinant $\det(\mathrm{Tr}(e_k e_l)_{k,l})$ is invertible in R_j, where $\mathrm{Tr} : \mathcal{A}_j \to R_j$ is the trace map. Hence, the R_j-algebra $\mathcal{A}_j$ becomes an Azumaya algebra over R_j.

Fix the index j. Thus, we may assume that the functor $\mathfrak{F}$ is given on the category of R_j-algebras. We may replace the filtered direct system of the R_i by the subsystem of all R_i with $i \geq j$. Clearly we have $R = \varinjlim_{i \geq j} R_i$.

Let $\alpha \in \mathfrak{F}(R)$ be such that its image α_K in $\mathfrak{F}(K)$ is trivial. Clearly, for a suitable $f \in R$ the image α_f in $\mathfrak{F}(R_f)$ of the element α is trivial.

For a suitable index $k \geq j$ choose a lift f_k of f in R_k. Replacing the filtered direct system of the R_i, $i \geq j$, by the subsystem of all R_i with $i \geq k$ we still have $R = \varinjlim_{i \geq k} R_i$. We put , for every $i \geq k$, $f_i = \phi_{ik}(f_k)$ where the $\phi_{ik} : R_k \to R_i$ are the transition homomorphisms. It is easy to see that $\varinjlim_{i \geq k} (R_i)_{f_i} = R_f$.

By the very definition the functor $\mathfrak{F}$ commutes with filtered direct limits (compare with the proof of axiom **C** in subsection 3.2), i.e. the canonical maps $\varinjlim_{i \geq k} \mathfrak{F}(R_i) \to \mathfrak{F}(R)$ and $\varinjlim_{i \geq k} \mathfrak{F}((R_i)_{f_i}) \to \mathfrak{F}(R_f)$ are isomorphisms. Thus, we have

$$\varinjlim_{i \geq k} \ker[\mathfrak{F}(R_i) \to \mathfrak{F}((R_i)_{f_i})] = \ker[\mathfrak{F}(R) \to \mathfrak{F}(R_f)].$$

By Theorem (a) of Linear Case (section 3) for any i the map $\mathfrak{F}(R_i) \to \mathfrak{F}(K_i)$ induced by the canonical inclusion of local ring R_i to its field of fractions K_i is injective. Since the inclusion $R_i \hookrightarrow K_i$, $i \geq k$, is factorized through $(R_i)_{f_i}$ the map $\mathfrak{F}(R_i) \to \mathfrak{F}((R_i)_{f_i})$ is injective as well.

Thus, we get that the left side of the relation above is trivial. By the very assumption the element $\alpha \in \mathfrak{F}(R)$ lies on the right side, therefore, $\alpha = 0$.

Hence, the map $\mathfrak{F}(R) \to \mathfrak{F}(K)$ is injective.

2. Assume that k is finite field of characteristic p. Suppose that the injectivity above holds for essentially smooth local algebras over any infinite field.

Let p^m be the cardinality of the residue field $R/\mathfrak{m}$ and s be any integer greater than 1 and prime with m. For any natural i let k_i be the field of degree s^i over k (in some algebraic closure of k).

Since degrees of k_i and $R/\mathfrak{m}$ over k are coprime, $k_i \otimes_k (R/\mathfrak{m})$ is a field for any i. Thus, $R_i = k_i \otimes_k R$ is essentially smooth local k_i-algebra for any i. Let K_i be its field of fractions.

Let $\tilde{k}$ be the union of all k_i. Clearly, $\tilde{R} = \tilde{k} \otimes_k R$ is essentially smooth local algebra over the infinite field $\tilde{k}$. Denote by $\tilde{K}$ its field of fractions.

Consider commutative diagram:

$$\begin{array}{ccccc}
\mathfrak{F}(R) & \longrightarrow & \mathfrak{F}(R_i) & \longrightarrow & \mathfrak{F}(\tilde{R}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{F}(K) & \longrightarrow & \mathfrak{F}(K_i) & \longrightarrow & \mathfrak{F}(\tilde{K})
\end{array}$$

Let $\alpha \in \mathfrak{F}(R)$ be such that $\alpha_K = 0$ in $\mathfrak{F}(K)$. Denote by α_i, $\tilde{\alpha}$ images of α in $\mathfrak{F}(R_i)$ and $\mathfrak{F}(\tilde{R})$, correspondingly. Thus, one has $\tilde{\alpha}_K = 0$.

By assumption the map $\mathfrak{F}(\tilde{R}) \to \mathfrak{F}(\tilde{K})$ is injective, therefore, $\tilde{\alpha} = 0$. Since the functor $\mathfrak{F}$ commutes with filtered direct limits, we can find an index j such that $\alpha_i = 0$ for any $i \geq j$.

Consider essentially smooth local k_j-algebra R_j. By definition it is finite and free over R, therefore, there is the transfer map $\mathrm{Tr} : \mathfrak{F}(R_j) \to \mathfrak{F}(R)$ (for the construction see section 3.2, the proof of **TE**). Clearly, the composition $\mathfrak{F}(R) \to \mathfrak{F}(R_j) \stackrel{\mathrm{Tr}}{\to} \mathfrak{F}(R)$ is the multiplication by s^i the degree of k_i over k. Thus, one has $s^i\alpha = 0$ for any integer s prime to m and for any integer i. It means that $\alpha = 0$ in $\mathfrak{F}(R)$. And we have finished. $\square$

Theorem I. (b) *The map $H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{SL}_{1,\mathcal{A}_K})$ induced by the canonical inclusion is injective.*

Proof. To prove this statement one has to repeat the corresponding arguments given in the proof of statement (b) of Linear Case (see the end of subsection 3.2).

In particular to construct the commutative diagram

$$\begin{array}{ccccccccc} 1 \longrightarrow & \mathrm{SL}(\mathcal{A}) & \longrightarrow & \mathcal{A}^* & \stackrel{\mathrm{Nrd}}{\longrightarrow} & R^* & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) & \longrightarrow 1 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 1 \longrightarrow & \mathrm{SL}(\mathcal{A}_K) & \longrightarrow & \mathcal{A}_K^* & \underset{\mathrm{Nrd}_K}{\longrightarrow} & K^* & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{SL}_{1,\mathcal{A}_K}) & \longrightarrow 1 \end{array}$$

we don't use the fact that R is essentially smooth over k. Indeed, it is enough for R to be just a local ring.

Thus, nothing will be changed and we have finished $\square$

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