Decomposing some finitely generated groups into free products with amalgamation [1]

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Introduction

. We shall say that a group G is a non-trivial free product with amalgamation if $G = G_1 *_A G_2$, where $G_1 \neq A \neq G_2$ (see [1]). Wall [2] has posed the following question:

Which one-relator groups are non-trivial free products with amalgamation?

Let $G = \langle g_1, \dots, g_m \mid R_1 = \dots = R_n = 1 \rangle$ be a group with m generators and n relations such that $\operatorname{def} G = m - n \geq 2$. It is proved in [4] that G is a non-trivial free product with amalgamation. In particular, if G is a group with $m \geq 3$ generators and one relation, then G is a non-trivial free product with amalgamation. A case of groups with two generators and one relation is more complicated. For example, the free abelian group $G = \langle a, b \mid [a, b] = 1 \rangle$ of rank 2, where $[a, b] = aba^{-1}b^{-1}$, obviously, is not a non-trivial free product with amalgamation. Other examples are given by groups $G_n = \langle a, b \mid aba^{-1} = b^n \rangle$. For any n, the group G_n is solvable and using results from [3], it is easy to show that for $n \neq -1$ the group G_n is not decomposable into a non-trivial free product with amalgamation. The following conjecture was stated in [4].

Conjecture 1 *Let $G = \langle a, b \mid R^m(a, b) = 1 \rangle$, $m \geq 2$, be a group with two generators and one relation with torsion. Then G is a non-trivial free product with amalgamation.*

Zieschang [5] has studied the problem of decomposing into non-trivial free products with amalgamation for discontinuous groups of transformations of the plane. He has given a complete answer to the question when such a group is a non-trivial free product with amalgamation in all cases except for the groups $H_1 = \langle a, b \mid [a, b]^n = 1 \rangle$ and $H_2 = \langle a, b \mid a^2 = [a, b]^n = 1 \rangle$, $n \geq 2$. Rosenberger [6] has proved that the groups H_1 and H_2 are non-trivial free products with amalgamation if n is not a power of 2. In recent papers [7, 8] it was proved that H_1 is a non-trivial free product with amalgamation for arbitrary $n \geq 2$. The independent proof of this fact was given in [9, 10].

[1] This research was carried out with the financial support of the Institute of Mathematics of the National Academy of Sciences of Belarus in the framework of the National Programme of Fundamental Research “Mathematical Structures”.

In the present paper we study the more general case, namely, we consider so-called *generalized triangle groups*, that is, groups G having a presentation of the form

$$G = \langle a, b \mid a^m = b^n = R^l(a, b) = 1 \rangle,$$

where $l \geq 2$, $R(a, b)$ is a cyclically reduced word in the free group on a, b. Not all of these groups are decomposable into non-trivial free products with amalgamation. For example, Zieschang [5] has proved that *the ordinary triangle group*

$$T(m, n, l) = \langle a, b \mid a^m = b^n = (ab)^l = 1 \rangle,$$

where $m, n, l \geq 2$, is not a non-trivial free product with amalgamation. On the other hand, it was shown in [10] that the group $G = \langle a, b \mid a^{2m} = R^l(a, b) = 1 \rangle$, where $m \geq 1$, $l \geq 2$, is a free product with amalgamation. Theorems 2 and 3 of the present paper contain more general results about decomposing of the generalized triangle groups into non-trivial free products with amalgamation.

In Theorem 1 we prove that a finitely generated group Γ is a non-trivial free product with amalgamation if the dimension of some algebraic variety (so-called character variety of irreducible representations of the group Γ into $\mathrm{SL}_2(\mathbb{C})$) is more than 1. To formulate this result we recall some notations and facts from the geometric representation theory (see also [11, 12, 13, 14]).

Let $\Gamma = \langle g_1, \ldots, g_m \rangle$ be a finitely generated group and let $G \subset \mathrm{GL}_n(K)$ be a connected linear algebraic group defined over an algebraically closed field K of characteristic zero. Obviously, for each homomorphism $\rho : \Gamma \to G(K)$ the set of elements

$$(\rho(g_1), \ldots, \rho(g_m)) \in G(K) \times \cdots \times G(K)$$

satisfies all defining relations of Γ. So the correspondence $\rho \to (\rho(g_1), \ldots, \rho(g_m))$ is a bijection between the set $\mathrm{Hom}\,(\Gamma, G(K))$ and the set of K-points in some affine K-variety $R(\Gamma, G) \subset G^m$. The variety $R(\Gamma, G)$ is usually called the representation variety of the group Γ into the algebraic group G.

The group G acts on $R(\Gamma, G)$ in a natural way (by simultaneous conjugation of components) and its orbits are in one-to-one correspondence with the equivalence classes of representations of Γ. In a general case the orbits of the group G under this action are not necessarily closed and hence the variety of orbits (the geometric quotient) is not an algebraic variety. However, if G is a reductive group, then one can consider the categorical quotient $X(\Gamma, G) = R(\Gamma, G)/G$ (see [15]). Its points parametrize closed G-orbits. In the case $G = \mathrm{GL}_n(K)$ or $G = \mathrm{SL}_n(K)$ an orbit of G is closed if and only if the corresponding representation is completely reducible. Therefore in this case points of the variety $X(\Gamma, G)$ are in one-to-one correspondence with the equivalence classes of completely reducible representations of the group Γ into G or, in other words, with characters of representations of Γ into G.

Throughout we shall consider only the case $G = \mathrm{SL}_2(K)$ and for brevity we shall denote $R(\Gamma), \mathrm{SL}_2(K)) = R(\Gamma)$, $X(\Gamma, \mathrm{SL}_2(K)) = X(\Gamma)$. One can find all used below information about varieties $R(\Gamma)$, $X(\Gamma)$ in [12, 16, 17, 18]. We set

$$R^s(\Gamma) = \{\rho \in R(\Gamma) \mid \rho \text{ is irreducible}\}, \qquad X^s(\Gamma) = \pi(R^s(\Gamma)),$$

where $\pi : R(\Gamma) \to X(\Gamma)$ is the canonical projection. It is shown in [12] that $R^s(\Gamma)$, $X^s(\Gamma)$ are open (in Zariski topology) subset of $R(\Gamma)$, $X(\Gamma)$ respectively. The aim of the present paper is to prove the following theorems.

THEOREM 1 *Let Γ be a finitely generated group such that* $\dim X^s(\Gamma) \geq 2$. *Then Γ is a non-trivial free product with amalgamation.*

THEOREM 2 *Let $k, m \in \mathbb{Z}$, $k, m \geq 2$, let $F_2 = \langle g, h \rangle$ be the free group of rank 2 with generators g, h, and let $R(g, h) = g^{u_1}h^{v_1} \dots g^{u_s}h^{v_s}$ be a cyclically reduced word in F_2 such that $u_i \neq 0$, $0 < v_i < k$, $s \geq 1$. Suppose that there exists $i \in \{1, \dots, s\}$ such that $|u_i| \geq 2$. Let p be a prime number such that $u_i p$ does not divide u_j for $j \neq i$ and let $0 \neq f \in \mathbb{Z}$. We set $n = u_i p f$. Then in the following cases the group*

$$\Gamma_n = \langle a, b \mid a^n = b^k = R^m(a, b) = 1 \rangle$$

is a non-trivial free product with amalgamation.

1) $m = 2$, p does not belong to some finite set of prime numbers S. The set S is completely determined by the exponent k and the word R.

2) $m = 3$ or $m = 2^l > 3$, $p \neq 2$;

3) $m > 3$ and $m \neq 2^l$.

Note that the condition $u_i p \nmid u_j$ for $j \neq i$ in Theorem 2 is carried out automatically if $u_i = \max_{1 \leq j \leq s} u_j \geq 2$ or $u_i \nmid u_j$ for each $j \neq i$.

THEOREM 3 *Let $\Gamma = \langle a, b \mid a^n = R^m(a, b) = 1 \rangle$, where $n = 0$ or $n \geq 2$, $m \geq 2$, $R(a, b) = a^{u_1}b^{v_1} \dots a^{u_s}b^{v_s}$, $s \geq 1$, $v_i \neq 0$, $0 < u_i < n$. Then Γ is a non-trivial free product with amalgamation.*

As a direct consequence of Theorem 3 we obtain the proof of Conjecture 1.

COROLLARY 1 *Let $\Gamma = \langle a, b \mid R^m(a, b) = 1 \rangle$, $m \geq 2$, be a group with two generators and one relation with torsion. Then Γ is a non-trivial free product with amalgamation.*

At the end of Section 2 we shall prove that the group Γ from Corollary 1 satisfies the assumptions of Theorem 1, that is, $\dim X_2^s(\Gamma) = 2$ so that we obtain an another proof of Conjecture 1.

COROLLARY 2 *Fuchsian groups $H_1 = \langle a, b \mid [a, b]^n = 1 \rangle$ and $H_2 = \langle a, b \mid a^2 = [a, b]^n = 1 \rangle$, $n \geq 2$, are non-trivial free products with amalgamation.*

1. The proof of Theorem 1

In what follows we shall denote the field of p-adic numbers by $\mathbb{Q}_p$, the ring of p-adic integers by $\mathbb{Z}_p$, the group of invertible elements in $\mathbb{Z}_p$ by $\mathbb{Z}_p^*$, the p-adic valuation by $|\ |_p$, the trace of a matrix A by $\operatorname{tr} A$, the identity 2×2 matrix by E.

We recall some facts about the character variety $X(\Gamma)$ of representations of a finitely generated group Γ into $\mathrm{SL}_2(\mathbb{C})$ (see [12]). For an arbitrary element $g \in \Gamma$ one can consider the regular function

$$\tau_g : R(\Gamma) \to \mathbb{C}, \qquad \tau_g(\rho) = \operatorname{tr} \rho(g).$$

Usually, τ_g is called *a Fricke character* of the element g. It is known that the $\mathbb{Z}$-algebra $T(\Gamma)$ generated by all functions τ_g, $g \in \Gamma$ is finitely generated. Moreover, if $\tau_{g_1}, \dots, \tau_{g_s}$ are the generators of $T(\Gamma)$ then the $\mathbb{C}$-algebra of $\mathrm{SL}_2(\mathbb{C})$-invariant regular functions $\mathbb{C}[R(\Gamma)]^{\mathrm{SL}_2(\mathbb{C})}$ is equal to $\mathbb{C}[\tau_{g_1}, \dots, \tau_{g_s}]$. Consider the morphism

$$\pi : R(\Gamma) \to \mathbb{A}^s, \qquad \pi(\rho) = (\tau_{g_1}(\rho), \dots, \tau_{g_s}(\rho)).$$

It is shown in [12] that the image $\pi(R(\Gamma))$ is closed in $\mathbb{A}^s$. Since $X(\Gamma)$ and $\pi(R(\Gamma))$ are biregularly isomorphic, then throughout we shall identify $X(\Gamma)$ and $\pi(R(\Gamma))$.

The idea of the proof of Theorem 1 is to construct a representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{Q}_p)$ for some prime p such that the group $\rho(\Gamma)$ is dense in $\mathrm{SL}_2(\mathbb{Q}_p)$ in p-adic topology. After that Theorem 1 will follow from the following well known facts.

1) If H is a dense in p-adic topology subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ then H is a non-trivial free product with amalgamation (see [19]).

2) If $f : G_1 \to G_2$ is an epimorphism of groups and G_2 is a non-trivial free product with amalgamation then G_1 is a non-trivial free product with amalgamation.

LEMMA 1 *Let H be a subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Then H is dense in $\mathrm{SL}_2(\mathbb{Q}_p)$ in p-adic topology if and only if H is absolute irreducible (that is, irreducible over algebraic closure of $\mathbb{Q}_p$), unbounded, and non-discrete.*

Proof. If H is dense in $\mathrm{SL}_2(\mathbb{Q}_p)$ then obviously H is absolute irreducible, unbounded and non-discrete. Prove the inverse assertion. The unboundedness of H means that there exists an element $h \in H$ such that $|\operatorname{tr} h|_p > 1$. Indeed, otherwise the traces of all elements from H belong to $\mathbb{Z}_p$ whence the group H is conjugated to a subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ (see [20] or [12, lemma I.4.3]), that is, H is bounded. Let us show that the eigenvalues of the matrix h belong to $\mathbb{Q}_p$. Let $\operatorname{tr} h = p^{-s}\alpha$, where $\alpha \in \mathbb{Z}_p^*$, $s > 0$. Then the characteristic polynomial of h is of the form $f(y) = y^2 - p^{-s}\alpha y + 1$ and its discriminant is equal to $D = p^{-2s}\alpha^2 - 4 = p^{-2s}(\alpha^2 - 4p^{2s})$. Thus, D is a square in

$\mathbb{Q}_p$ whence the roots of $f(y)$ belong to $\mathbb{Q}_p$. So h is conjugate in $\mathrm{SL}_2(\mathbb{Q}_p)$ to a diagonal matrix of the form

$$\mathrm{diag}(\lambda, \lambda^{-1}), \quad \lambda = p^{-s}\gamma, \quad s \geq 0, \quad \gamma \in \mathbb{Z}_p^*. \tag{1}$$

Without loss of generality we can suppose considering if necessary a group conjugated to H that $h = \mathrm{diag}(\lambda, \lambda^{-1}) \in H$. Consider the following unipotent subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$:

$$U_1 = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \middle| a \in \mathbb{Q}_p \right\}, \qquad U_2 = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{Q}_p \right\}.$$

It is well known that U_1 and U_2 generate $\mathrm{SL}_2(\mathbb{Q}_p)$. Therefore it is sufficient to show that $U_1, U_2 \subset \overline{H}$, where $\overline{H}$ is the closure of H in p-adic topology. For example, let us prove that $U_1 \subset \overline{H}$. To this end, first we shall show that $\overline{H}$ contains a non-trivial unipotent element $u = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, $a \in \mathbb{Q}_p^*$. Let

$$\Gamma_j = \left\{ \begin{pmatrix} 1 + p^j a & p^j b \\ p^j c & 1 + p^j d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p) \middle| a, b, c, d \in \mathbb{Z}_p \right\}$$

be the principal congruence subgroup of a level j in $\mathrm{SL}_2(\mathbb{Q}_p)$. The groups Γ_j, $j \geq 1$, form a base for the neighborhoods of the identity in $\mathrm{SL}_2(\mathbb{Q}_p)$. Then non-discreteness of H implies that for each $j > 0$ there exists an element $E \neq x_j \in H \cap \Gamma_j$. Let

$$x_j = \begin{pmatrix} 1 + p^j a_j & p^j b_j \\ p^j c_j & 1 + p^j d_j \end{pmatrix},$$

where $a_j, b_j, c_j, d_j \in \mathbb{Z}_p$. Then $\lim_{j \to \infty} x_j = E$. Since H is an absolutely irreducible subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$, then without loss of generality we can suppose that almost all elements b_j, c_j (with the exception of a finite number) are not equal to 0 (otherwise one can conjugate all x_j by a suitable element y from H, and again we shall have $\lim_{j \to \infty} y x_j y^{-1} = E$). Let $c_j = p^{k_j} \varepsilon_j$, where $k_j \geq 0$, $\varepsilon_j \in \mathbb{Z}_p^*$, and let $t_j = [(k_j + j)/(2s)]$ be the integer part of the number $(k_j + j)/(2s)$, where s is defined in (1). Then we have $r_j = k_j + j - 2st_j \in \{0, 1, \ldots, 2s - 1\}$. Now consider the following sequence $\{x'_j\}$ of elements of H:

$$x'_j = h^{-t_j} x_j h^{t_j} = \begin{pmatrix} 1 + p^j a_j & p^{j+2st_j} \gamma^{-2t_j} b_j \\ p^{r_j} \gamma^{2t_j} \varepsilon_j & 1 + p^j d_j \end{pmatrix}. \tag{2}$$

The infinite sequence $\{p^{r_j} \gamma^{2t_j} \varepsilon_j\}$ is contained in $\mathbb{Z}_p$ hence it is bounded. Therefore one can choose from it the convergent subsequence

$$a_{j_m} = p^{r_{j_m}} \gamma^{2t_{j_m}} \varepsilon_{j_m}.$$

Since $|a_{j_m}|_p = p^{-r_{j_m}} \geq p^{-2s+1}$, then $\lim_{j_m \to \infty} a_{j_m} = a \neq 0$. Thus, it follows from (2) that

$$\lim_{j_m \to \infty} x'_{j_m} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = u \in \overline{H}.$$

Conjugating u by a suitable power of h, we can suppose that $a \notin \mathbb{Z}_p$. Furthermore, $u^n = \begin{pmatrix} 1 & 0 \\ an & 1 \end{pmatrix}$ for arbitrary $n \in \mathbb{Z}$. Since the closure of $a\mathbb{Z}$ in $\mathbb{Q}_p$ contains $\mathbb{Z}_p$, then $\overline{H}$ contains a group $U_1' = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \middle| x \in \mathbb{Z}_p \right\}$. Furthermore, let $u_1 = \begin{pmatrix} 1 & 0 \\ p^{-r}\beta & 1 \end{pmatrix} \in U_1$ be an element such that $r > 0$ and $\beta \in \mathbb{Z}_p^*$. Choose an integer m such that $2sm - r \geq 0$. Then $h^m u_1 h^{-m} \in U_1'$ whence $u_1 \in \overline{H}$. Thus, we have proved that $U_1 \subset \overline{H}$. Similarly, one can proved that $U_2 \subset \overline{H}$. Hence $\overline{H} = \mathrm{SL}_2(\mathbb{Q}_p)$. Lemma 1 is proved.

LEMMA 2 *Let X,Y be irreducible $\mathbb{Q}$-defined affine varieties, let* $\dim Y \geq 1$, *and let $f : X \to Y$ be a dominant $\mathbb{Q}$-defined regular morphism. Then there exists a prime number $p \neq 2$ and a point $x \in X(\mathbb{Q}_p)$ such that not all coordinates of the image $f(x) \in Y(\mathbb{Q}_p)$ belong to a ring $\mathbb{Z}_p$.*

Proof. Let K be the algebraic closure of $\mathbb{Q}$. Let D be an arbitrary irreducible curve in $Y(K)$, and let L be an arbitrary irreducible curve such that $L \subset f^{-1}(D)$ and $f(L)$ is dense in D. Let $\overline{D}$ and $\overline{L}$ be the projective closure of D and L respectively, and let $\widetilde{L}$ be the smooth projective model of $\overline{L}$. The regular morphism $f : L \to D$ determines a rational morphism $\widetilde{f} : \widetilde{L} \to \overline{D}$. Since any rational morphism from a smooth curve to a projective variety is regular and the image of a projective variety under regular map is closed (see [21]), then $\widetilde{f}$ is a regular surjective morphism. Let $v \in \overline{D} \backslash D$ be a point at infinity on $\overline{D}$ and let $w \in \widetilde{f}^{-1}(v)$. The coordinates of points v and w generate the finite extension $K_1/\mathbb{Q}$. By Chebotarev's density theorem there exist infinitely many prime numbers p such that $K_1 \subset \mathbb{Q}_p$. Choose one of such p. Then $w \in \widetilde{L}(\mathbb{Q}_p)$, $v \in \overline{D}(\mathbb{Q}_p)$. Since w is a non-singular point on $\widetilde{L}$, then w has p-adic neighborhood $W \subset \widetilde{L}(\mathbb{Q}_p)$ such that W is homeomorphic to an area in $\mathbb{Q}_p$ (see [21, chapter II]). This means that there exists an infinite sequence of elements $w_i \in W$ such that $w_i \in L(\mathbb{Q}_p)$ and $\lim_{i\to\infty} w_i = w$ in p-adic topology. Then by continuity of $\widetilde{f}$ we have $\lim_{i\to\infty} \widetilde{f}(w_i) = v$. Since $v \in \overline{D}(\mathbb{Q}_p)$ is a point at infinity, then the sequence of elements $f(w_i) = \widetilde{f}(w_i) \in D(\mathbb{Q}_p)$ is not bounded. This means that there exists i such that not all of the coordinates of the point $f(w_i)$ belong to $\mathbb{Z}_p$. Lemma 2 is proved.

The proof of Theorem 1. Let $g_1 \ldots, g_s \in \Gamma$ be elements such that the functions $\tau_{g_1}, \ldots, \tau_{g_s}$ generate the ring $T(\Gamma)$. Then the projection $\pi : R(\Gamma) \to X(\Gamma)$ is defined by the formula $\pi(\rho) = (\tau_{g_1}(\rho), \ldots, \tau_{g_s}(\rho))$. Since by assumption of Theorem 1 we have $\dim X^s(\Gamma) \geq 2$, then there exists an irreducible component Z of the variety $X(\Gamma)$ such that $\dim Z \geq 2$ and $U = Z \cap X^s(\Gamma) \neq \varnothing$. Let $p_i : Z \to \mathbb{A}^1$ be the projection defined by the formula $p_i(z_1, \ldots, z_s) = z_i$. Since $\dim Z \geq 2$, then there exists i such that the projection p_i is dominant, that is $p_i(U)$ is dense in $\mathbb{A}_1$. Therefore there exists an integer $n > 2$ such that $n \in p_i(U)$. Let $Y = p_i^{-1}(n) \subset Z$. Then by the Dimension Theorem $\dim Y \geq \dim Z - 1 \geq 1$ and $Y \cap U \neq \varnothing$. Furthermore, let X be an irreducible component of $\pi^{-1}(Y)$ such that $\pi(X))$ is dense in Y. Applying Lemma 2 to varieties X, Y and morphism π, we obtain that there exists a prime p such that $X(\mathbb{Q}_p)$ contains a representation ρ with the following property: ρ is irreducible and not all coordinates of the point $\pi(\rho)$ belong to $\mathbb{Z}_p$. The last means that there exists j such that $\tau_{g_j}(\rho) = \operatorname{tr}\rho(g_j) \notin \mathbb{Z}_p$. Hence the group $\rho(\Gamma)$ is an unbounded subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Moreover, it follows from the construction of the representation ρ that $\tau_{g_i}(\rho) = \operatorname{tr}\rho(g_i) = n > 2$. Thus, the cyclic subgroup of $\rho(\Gamma)$ generated by $\rho(h_i)$ is infinite and bounded. Hence, $\rho(\Gamma)$ is a non-discrete subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. It follows from Lemma 1 that $\rho(\Gamma)$ is dense in $\mathrm{SL}_2(\mathbb{Q}_p)$ in p-adic topology whence $\rho(\Gamma)$ (and consequently Γ) is a non-trivial free product with amalgamation. Theorem 1 is proved.

2. Some auxiliary results

In this section we shall prove several auxiliary results used in the proofs of Theorems 2 and 3. Throughout we shall denote the ring of algebraic integers in $\mathbb{C}$ by $\mathcal{O}$, the group of units in $\mathcal{O}$ by $\mathcal{O}^*$, the free group of a rank 2 with generators g and h by $F_2 =< g, h >$, the greatest common divisor of integers a and b by (a, b). If $K \supset L$ is a finite extension of fields and $x \in K$, then we shall denote the norm of x by $N_{K/L}(x)$. The following lemma characterizes elements of the finite order in $\mathrm{SL}_2(\mathbb{C})$.

LEMMA 3 *Let* $2 < m \in \mathbb{Z}$ *and* $\pm E \neq X \in \mathrm{SL}_2(\mathbb{C})$. *Then* $X^m = E$ *if and only if* $\operatorname{tr} X = \varepsilon + \varepsilon^{-1}$, *where* $\varepsilon^m = 1$, $\varepsilon \neq \pm 1$ *(in other words,* $\operatorname{tr} X = 2\cos(2r\pi/m)$ *for some* $r \in \{1, \dots, m-1\}$*). In particular, if* $\operatorname{tr} X = 0$, *then* $X^2 = -E$.

Proof. If $X^m = E$, then the assertion is obvious. If $\operatorname{tr} X = \varepsilon + \varepsilon^{-1}$, then $\varepsilon, \varepsilon^{-1}$ are the eigenvalues of the matrix X. Hence, X is conjugated with the matrix $\operatorname{diag}(\varepsilon, \varepsilon^{-1})$, that is, $X^m = E$ as required.

Obviously, the representation variety $R(F_2)$ of the free group $F_2 = \langle g, h \rangle$ is equal to $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. It is known that the ring $T(F_2)$ is generated by the functions $\tau_g, \tau_h, \tau_{gh}$ (see [12, 16, 17]). For an element $u \in F_2$ the function τ_u is usually called a Fricke character of the element u.

LEMMA 4 *For all* $\alpha, \beta, \gamma \in \mathbb{C}$ *there exist matrices* $A, B \in \mathrm{SL}_2(\mathbb{C})$ *such that* $\tau_g(A, B) = \operatorname{tr} A = \alpha$, $\quad \tau_h(A, B) = trB = \beta$, $\quad \tau_{gh}(A, B) = \operatorname{tr} AB = \gamma$.

This lemma can be easily proved by straightforward computations.

Lemma 4 implies that $X(F_2) = \pi(R(F_2)) = \mathbb{A}^3$. Moreover, the functions $\tau_g, \tau_h, \tau_{gh}$ are algebraically independent over $\mathbb{C}$ and for all $u \in F_2$ we have

$$\tau_u = Q_u(\tau_g, \tau_h, \tau_{gh}),$$

where $Q_u \in \mathbb{Z}[x, y, z]$ is a uniquely determined polynomial with integer coefficients. The polynomial Q_u is usually called the Fricke polynomial of the element u. The following relations for Fricke characters follow from the relations between the traces of arbitrary matrices in $\mathrm{SL}_2(\mathbb{C})$:

$$1)\tau_{u^{-1}} = \tau_u; \quad 2)\tau_{uv} = \tau_{vu}; \quad 3)\tau_{vuv^{-1}} = \tau_u; \quad 4)\tau_{uv} = \tau_u\tau_v - \tau_{uv^{-1}}. \tag{3}$$

Furthermore, we require the more detailed information on the Fricke polynomials (see [22]). Consider polynomials $P_n(\lambda)$ satisfying the initial conditions $P_{-1}(\lambda) = 0$, $P_0(\lambda) = 1$ and the recurrence relation

$$P_n(\lambda) = \lambda P_{n-1}(\lambda) - P_{n-2}(\lambda).$$

If $n < 0$, then we set $P_n(\lambda) = -P_{|n|-2}(\lambda)$. The degree of the polynomial $P_n(\lambda)$ is equal to n if $n > 0$ and to $|n| - 2$ if $n < 0$. It is easy to verify by induction on n that

$$P_n(2\cos(\varphi)) = \frac{\sin((n+1)\varphi)}{\sin(\varphi)}. \tag{4}$$

It follows from (4) that the polynomial $P_n(\lambda)$, $n \geq 1$, has n zeros described by the formula

$$\lambda_{n,k} = 2\cos(\tfrac{k\pi}{n+1}), \qquad k = 1, 2, \ldots, n. \tag{5}$$

Moreover, it is easy to verify by induction that for $n \geq 0$ we have

$$\begin{aligned} P_{2n}(\lambda) &= \lambda^{2n} + \cdots + (-1)^n \\ P_{2n-1}(\lambda) &= \lambda(\lambda^{2n-2} + \cdots + (-1)^{n-1}n). \end{aligned} \tag{6}$$

Furthermore, let $w = g^{\alpha_1}h^{\beta_1}\ldots g^{\alpha_s}h^{\beta_s}$ be a cyclically reduced word in F_2 and let $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$. Let us treat the Fricke polynomial $Q_w(x, y, z)$ as a polynomial in z. Let

$$Q_w(x, y, z) = M_n(x, y)z^n + M_{n-1}(x, y)z^{n-1} + \ldots + M_0(x, y)$$

Lemma 5 ([22]) *The degree of the Fricke polynomial $Q_w(x, y, z)$ with respect to z is equal to s, that is, the number of blocks of the form $g^{\alpha_i}h^{\beta_i}$ in w. The leading coefficient $M_s(x, y)$ of the polynomial $Q_w(x, y, z)$ has the following form*

$$M_s(x, y) = \prod_{i=1}^{s} P_{\alpha_i-1}(x))P_{\beta_i-1}(y). \tag{7}$$

The following lemma plays an important role in the proof of Theorems 2 and 3.

LEMMA 6 *Let* $\Gamma = \langle a, b \mid a^n = R^m(a,b) = 1 \rangle$, *where* $n = 0$ *or* $n \geq 2$, $m \geq 2$, $R(a,b)$ *is a cyclically reduced containing* b *word in the free group on* a *and* b. *Assume that there exists matrices* $A, B \in \mathrm{SL}_2(\mathbb{C})$ *such that* $\operatorname{tr} A = \alpha = 2\cos(t\pi/n)$ *for some* $t \in \{1, \dots, n-1\}$ *and* $\operatorname{tr} R(A,B) = Q_R(\alpha, y, z) = c$, *where* Q_R *is the Fricke polynomial of an element* $R(g,h) \in F_2$, $c = 2\cos(r\pi/m)$ *for some* $r \in \{1, \dots, m-1\}$, $y = \operatorname{tr} B$, $z = \operatorname{tr} AB$. *Let* $H =< A, B >$ *be the group generated by matrices* A *and* B. *Assume that the following two conditions hold:*

1) *there exists a unipotent (or finite-order) element* $W \in H$ *of the form* $W = A^{\alpha_1}B^{\beta_1}\dots A^{\alpha_s}B^{\beta_s}$, *where* $\alpha_i, \beta_i \neq 0$ *for* $i = 1, \dots, s$, *such that* $l = \sum_{i=1}^{s} \beta_i \neq 0$.

2) *There exists an element* $h \in H$ *such that* $\operatorname{tr} h \notin \mathcal{O}$.

Then the group Γ *is a non-trivial free product with amalgamation.*

If instead of the condition 1) the following condition holds

1′) B *has finite order, that is,* $\operatorname{tr} B = 2\cos(k_1\pi/k)$ *for some* $k \geq 2$, $(k_1, k) = 1$,

then the group $\Gamma_1 = \langle a, b \mid a^n = b^{kv} = R^m(a,b) = 1 \rangle$ *is a non-trivial free product with amalgamation for any integer* v.

The proof of this lemma is based on the Bass' classification of finitely generated subgroups in $\mathrm{SL}_2(\mathbb{C})$ [23].

PROPOSITION 1 ([23]) *Let* $H \subset \mathrm{GL}_2(\mathbb{C})$ *be a finitely generated subgroup. Then one of the following cases must occur:*

1) *there exists an epimorphism* $f : H \to \mathbb{Z}$ *such that* $f(u) = 0$ *for all unipotent elements* $u \in H$;

2) $\operatorname{tr} h \in \mathcal{O}$ *for any element* $h \in H$;

3) H *is a non-trivial free product with amalgamation.*

Proof of lemma 6. It is easy to see that the group H does not satisfy the conditions 1), 2) of Proposition 1. Indeed, assume that $f : H \to \mathbb{Z}$ is an epimorphism such that $f(z) = 0$ for all unipotent elements $z \in H$. Then $f(A) = 0$ because $A^{2n} = E$ by Lemma 3. Furthermore, $f(u) = lf(B) = 0$ whence $f(B) = 0$ because by assumption u is either unipotent or has finite order and $l \neq 0$. Thus, $f(H) = \{0\}$ which is a contradiction. By assumption H does not satisfy condition 2) of Proposition 1. Hence H is a non-trivial free product with amalgamation, that is, $H = H_1 *_F H_2$, where $H_1 \neq F \neq H_2$. Let $\overline{A}$, $\overline{B}$, $\overline{H}$, $\overline{H}_1$, $\overline{H}_2$, $\overline{F}$ be the images of A, B, H, H_1, H_2, F in $\mathrm{PSL}_2(\mathbb{C})$, respectively. If $-E \notin H$, then groups H and $\overline{H}$ are isomorphic. If $-E \in H$, then $-E$ belongs to the centre of H hence $-E \in F$. In all of these cases, $\overline{H}_1 \neq \overline{F} \neq \overline{H}_2$ and hence $\overline{H} = \overline{H}_1 *_{\overline{F}} \overline{H}_2$ is a non-trivial free product with amalgamation. The conditions $\operatorname{tr} A = \alpha$ and $Q_R(\alpha, y, z) = c$ imply by Lemma 3 that $A^{2n} = R^{2m}(A,B) = E$. Hence $\overline{A}^n = R^m(\overline{A}, \overline{B}) = 1$ in $\mathrm{PSL}_2(\mathbb{C})$. Thus, $\overline{H}$ is an epimorphic image of Γ therefore Γ is a non-trivial free product with amalgamation too.

Furthermore, if we replace the condition 1) to 1′), then again the group $\overline{H}$ is a non-trivial free product with amalgamation. Moreover, we have $\overline{A}^n = \overline{B}^k = R^m(\overline{A}, \overline{B}) = 1$ in $\mathrm{PSL}_2(\mathbb{C})$. Hence $\overline{H}$ is an epimorphic image of Γ_1. Thus, Γ_1 is a non-trivial free product with amalgamation. Lemma 6 is proved.

LEMMA 7 1) *Let $r, s \in \mathbb{Z}$, where $s \geq 3$ and $(r, s) = 1$. Then* $\cos(r\pi/s) \notin \mathcal{O}$.

2) *Let $s \in \mathbb{Z}$, $s \geq 1$, and $r \not\equiv 0 \pmod{2s+1}$. Then* $2\cos(r\pi/(2s+1)) \in \mathcal{O}^*$.

3) *Let $u \in \mathbb{Z}$, $|u| \geq 1$, and let p be a prime number. Set*

$$x_r = 2\cos(\tfrac{r\pi}{2pu}), \quad y_r = 2\sin(\tfrac{r\pi}{2pu}), \quad K_r = \mathbb{Q}(x_r), \quad L_r = \mathbb{Q}(y_r).$$

Then there exist $r, r_1 \not\equiv 0 \pmod p$ such that p divides $N_{K_r/\mathbb{Q}}(x_r)$ and $N_{L_{r_1}/\mathbb{Q}}(y_{r_1})$. In particular, $x_r, y_{r_1} \notin \mathcal{O}^*$.

4) *Let $u, c \in \mathbb{Z}$, $|u| \geq 2$, $c \neq 0$, and let p be a prime number not dividing c. Set $x_0 = -2\cos(\pi/u)$, $x_r = 2\cos(r\pi/(pu))$. Then there exists $r \not\equiv 0 \pmod p$ such that $c/(x_r - x_0) \notin \mathcal{O}$.*

5) *Let $p > 2$ be a prime number. Then for each $r \not\equiv 0 \pmod p$ and $s \geq 1$ we have* $\sin(r\pi/p^s) \notin \mathcal{O}^*$.

6) *Let $t \geq 1$. Then for each odd r we have $2sin(r\pi/2^t) \notin \mathcal{O}^*$.*

Proof. 1) Assume that $\cos(r\pi/s) \in \mathcal{O}$. Then for each $d \in \mathbb{Z}$ we have $\cos(dr\pi/s) \in \mathcal{O}$. Since by assumption $(r, s) = 1$, then for each integer l there exists d such that $dr \equiv l \pmod s$. Hence for each integer l we have $\cos(l\pi/s) \in \mathcal{O}$. By (5) the polynomial $P_{s-1}(\lambda)$ has the roots $2\cos(l\pi/s)$, $l = 1, \ldots, s-1$. Then the polynomial $P_{s-1}(2\lambda)$ has the roots $\cos(l\pi/s)$, $l = 1, \ldots, s-1$. If $s = 2s_1 + 1$ is odd, then by (6) we have $P_{2s_1}(2\lambda) = 2^{2s_1}\lambda^{2s_1} + \cdots + (-1)^{s_1}$. Since $1/2^{2s_1} \notin \mathbb{Z}$, then $P_{2s_1}(2\lambda)$ has a root not belonging to $\mathcal{O}$, that is, there exists l such that $\cos(l\pi/s) \notin \mathcal{O}$. If $s = 2s_1$ is even, then (6) implies that $P_{2s_1-1}(2\lambda) = 2\lambda(2^{2s_1-2}\lambda^{2s_1-2} + \cdots + (-1)^{s_1-1}s_1)$. By assumption $s \geq 3$ hence $s_1 \geq 2$. Then $s_1/2^{2s_1-2} \notin \mathbb{Z}$ and $P_{2s_1-1}(2\lambda)$ has a root not belonging to $\mathcal{O}$.

2) By (5), (6) the number $2\cos(r\pi/(2s+1))$ is a root of the polynomial $P_{2s}(\lambda) = \lambda^{2s} + \cdots + (-1)^s$ and therefore belongs to $\mathcal{O}^*$.

3) Since $y_r = 2\cos((pu - r)\pi/(2pu)) = x_{pu-r}$, then it is sufficient to prove the assertion for x_r. Let $u = p^f u'$, where $f \geq 0$, $p \nmid u'$, and let $r = r_1 u'$, where $p \nmid r_1$. Then $x_r = 2\cos(r_1\pi/(2p^{f+1}))$. By (5), (6) the polynomial

$$P_{2p^{f+1}-1}(\lambda) = \lambda(\lambda^{2p^{f+1}-2} + \cdots + (-1)^{p^{f+1}-1}p^{f+1})$$

has the roots $2\cos(r'\pi/(2p^{f+1}))$, $r' = 1, \ldots, 2p^{f+1} - 1$, and the polynomial

$$P_{2p^f-1}(\lambda) = \lambda(\lambda^{2p^f-2} + \cdots + (-1)^{p^f-1}p^f)$$

has the roots $2\cos(r'\pi/(2p^f))$, $r' = 1, \ldots, 2p^f - 1$. Hence the polynomial $P_{2p^f-1}(\lambda)$ divides the polynomial $P_{2p^{f+1}-1}(\lambda)$, that is,

$$P_{2p^{f+1}-1}(\lambda) = P_{2p^f-1}(\lambda)F(\lambda), \tag{8}$$

where, as it is easy to see, $F(\lambda)$ is the polynomial of degree $2(p^{f+1} - p^f)$ with the leading coefficient 1 and the constant term p. The roots of $F(\lambda)$ are the numbers $2\cos(r'\pi/(2p^{f+1}))$, $r' \not\equiv 0 \,(\mathrm{mod}\, p)$. Let K be the splitting field of $F(\lambda)$. Obviously, there exists $r' \not\equiv 0 \,(\mathrm{mod}\, p)$ such that $N_{K/\mathbb{Q}}(2\cos(r'\pi/(2p^{f+1}))) = \pm p$ as required.

4) Note that

$$x_r - x_0 = 2\cos(\tfrac{r\pi}{pu}) + 2\cos(\tfrac{\pi}{u}) = \left(2\cos(\tfrac{(r+p)\pi}{2pu})\right)\left(2\cos(\tfrac{(r-p)\pi}{2pu})\right).$$

Therefore it is sufficient to show that for some $r \not\equiv 0 \,(\mathrm{mod}\, p)$ we have $c/\alpha_r \notin \mathcal{O}$, where $\alpha_r = 2\cos((r+p)\pi/(2pu))$. Let $K_r = \mathbb{Q}(\alpha_r)$. By item 3) proven above, there exists $r \not\equiv 0$ $(\mathrm{mod}\, p)$ such that p divides $N_{K_r/\mathbb{Q}}(\alpha_r)$. Then

$$N_{K_r/\mathbb{Q}}\left(\frac{c}{\alpha_r}\right) = \frac{c}{N_{K_r/\mathbb{Q}}(\alpha_r)} \notin \mathbb{Z}$$

because by assumption $p \nmid c$. Hence $c/\alpha_r \notin \mathcal{O}$, as required.

5) Note that $1/\sin(r\pi/p^s) = 2/(2\cos((p^s - 2r)\pi/(2p^s)))$. It follows from the proof of item 4) that there exists $r_0 \not\equiv 0 \,(\mathrm{mod}\, p)$ such that

$$\frac{2}{2\cos((p^s - 2r_0)\pi/(2p^s))} \notin \mathcal{O}.$$

Now we shall show that for every $r \not\equiv 0 \,(\mathrm{mod}\, p)$ we have $\sin(r\pi/p^s) \notin \mathcal{O}$. Assume the contrary. Let for some r, $(r,p) = 1$, we have $1/\sin(r\pi/p^s) \in \mathcal{O}$. Since $p \nmid (p^s - 2r_0)$, then there exists d such that $r \equiv d(p^s - 2r_0) \,(\mathrm{mod}\, p^s)$. Then by (4) we have

$$P_d\left(2\cos(\tfrac{(p^s-2r_0)\pi}{p^s})\right) = \frac{\sin(d(p^s - 2r_0)\pi/p^s)}{\sin((p^s - 2r_0)\pi/p^s)} = \pm\frac{\sin(r\pi/p^s)}{\sin((p^s - 2r_0)\pi/p^s)}.$$

So we immediately obtain that

$$\frac{1}{\sin((p^s - 2r_0)\pi/p^s)} = \pm\frac{1}{\sin(r\pi/p^s)}P_d\left(2\cos(\tfrac{(p^s-2r_0)\pi}{p^s})\right) \in \mathcal{O}$$

which is a contradiction.

6) For $t = 1$ the assertion is obvious. Suppose that $t > 1$. By item 3) there exists an odd r_0 such that $2\sin(r_0\pi/2^t) \notin \mathcal{O}^*$. Prove that for each odd r we have $2\sin(r\pi/2^t) \notin \mathcal{O}^*$. Assume the contrary. Let for some odd r we have $2\sin(r\pi/2^t) \in \mathcal{O}^*$. Obviously, there exists an integer d such that $r \equiv dr_0 \,(\mathrm{mod}\, 2^t)$. Then by (4) we have

$$P_d\left(2\cos(\tfrac{r_0\pi}{2^t})\right) = \pm\frac{2\sin(r\pi/2^t)}{2\sin(r_0\pi/2^t)}.$$

The last equality implies that $2\sin(r_0\pi/2^t) \in \mathcal{O}^*$ which is a contradiction. Lemma 7 is proved.

LEMMA 8 1) *Let* $s, t \geq 0$. *Then*

$$P_s(\lambda)P_t(\lambda) = \sum_{i=0}^{t} P_{s-t+2i}(\lambda). \tag{9}$$

2) *The polynomial* $P_s(\lambda) - P_{s-1}(\lambda)$ *has the roots* $\lambda_r = 2\cos((2r+1)\pi/(2s+1))$, $r \in \{0, 1, \ldots, s-1\}$.

3) *If* $\gamma = 2\cos(2r\pi/(2s+1))$, *where* $s \geq 1$, $r \in \{1\ldots, s\}$, $(r, 2s+1) = 1$, *then* $P_s(\gamma) - P_{s-1}(\gamma) \notin \mathcal{O}^*$.

4) *If* $\gamma = 2\cos((2r+1)\pi/(2s))$, *where* $s \geq 2$, $(s, 2r+1) = 1$, *then* $0 \neq P_{s-1}(\gamma) \notin \mathcal{O}^*$.

5) Let $\gamma \in \mathcal{O}$. *Assume that* γ *is not equal to* $2\cos(r\pi/s)$, *where* $r, s \in \mathbb{Z}$. *Then there exists an integer* $l > 0$ *such that* $P_l(\gamma) \notin \mathcal{O}^*$.

Proof. 1) We fix s and proceed the induction on t. If $t = 0$, then $P_s(\lambda)P_0(\lambda) = P_s(\lambda)$. If $t = 1$, then $P_s(\lambda)P_1(\lambda) = P_s(\lambda)\lambda = P_{s+1}(\lambda) + P_{s-1}(\lambda)$ by definition. Furthermore, we have by induction

$$\begin{aligned} P_s(\lambda)P_t(\lambda) &= P_s(\lambda)(\lambda P_{t-1}(\lambda) - P_{t-2}(\lambda)) = \lambda \sum_{i=0}^{t-1} P_{s-t+1+2i}(\lambda) - \sum_{i=0}^{t-2} P_{s-t+2+2i}(\lambda) \\ &= \sum_{i=0}^{t-1} (P_{s-t+2+2i}(\lambda) + P_{s-t+2i}(\lambda)) - \sum_{i=0}^{t-2} P_{s-t+2+2i}(\lambda) \\ &= P_{s+t}(\lambda) + \sum_{i=0}^{t-1} P_{s-t+2i}(\lambda) = \sum_{i=0}^{t} P_{s-t+2i}(\lambda) \end{aligned}$$

as required.

2) Bearing in mind (4), we obtain

$$P_s(\lambda_r) - P_{s-1}(\lambda_r) = \frac{\sin((2r+1)(s+1)\pi/(2s+1)) - \sin((2r+1)s\pi/(2s+1))}{\sin((2r+1)\pi/(2s+1))} = 0.$$

3) Taking into account (4), we have

$$\frac{1}{P_{s+1}(\gamma) - P_s(\gamma)} = \frac{\sin(2r\pi/(2s+1))}{2\sin(r\pi/(2s+1))\cos(r\pi)} = \pm\cos(\tfrac{r\pi}{2s+1}) \notin \mathcal{O}$$

by item 1) of Lemma 7.

4) Using (4), we obtain

$$\frac{1}{P_{s-1}(\gamma)} = \frac{\sin((2r+1)\pi/(2s)}{\sin((2r+1)\pi/2)} = (-1)^r \cos\left(\tfrac{(s-2r-1)\pi}{2s}\right) \notin \mathcal{O}$$

by item 1) of Lemma 7.

5) Since by (5) the polynomial $P_l(\lambda)$ has the roots $2\cos(r\pi/(l+1))$, $r = 1, \dots, l$, then one can write $P_l(\gamma) = \prod_{r=1}^{l}(\gamma - 2\cos(r\pi/(l+1)))$. Hence it is sufficient to prove that $\gamma - (\varepsilon + \varepsilon^{-1}) \notin \mathcal{O}^*$, where $\varepsilon \neq \pm 1$ is some root of 1. Let $f(\lambda)$ be the irreducible polynomial for γ over $\mathbb{Q}$, let K_0 be the splitting field of $f(\lambda)$ and let $K_1 = K_0(x_0)$, where x_0 is a root of the equation $x + x^{-1} = \gamma$. Let Z_1 be the integer closure of $\mathbb{Z}$ in K_1 and let $p \neq 2$ be some prime number. Let $\mathfrak{p}_1$ be some prime ideal in Z_1 lying above (p). Then $k_1 = Z_1/\mathfrak{p}_1 \supset \mathbb{Z}/p\mathbb{Z} = k$ is the finite extension of fields. We have $x_0, y_0 \in Z_1$. Let $\overline{x}_0, \overline{\gamma}$ be the images of x_0, γ in the field k_1, respectively. Then the equality holds

$$\overline{x}_0 + \overline{x}_0^{-1} = \overline{\gamma}.$$

Let $l = |k_1^*|$ be the order of the multiplicative group of the field k_1. Then $\overline{x}_0^l = 1$ in k_1. Let $K_2 = K_1(\xi)$, where ξ is a primitive root of 1 of degree l in $\mathbb{C}$. Let Z_2 be the integer closure of Z_1 in K_2, let $\mathfrak{p}_2$ be some prime ideal of Z_2 lying above $\mathfrak{p}_1$, $k_2 = Z_2/\mathfrak{p}_2 \supset k_1$. Let Δ be the group of roots of 1 of the degree l in K_2 and let $\overline{\Delta}$ be its image in k_2. Prove that $\overline{\Delta} = k_1^*$. Assume the contrary, that is, $\overline{\Delta} \neq k_1^*$. Then for some integer r, $0 < r < l$, we have $\overline{\xi}^r = 1$, where $\overline{\xi}$ is the image of ξ in k_2. Then $(1+y)^l = 1$, that is, $1 + C_l^1 y + \cdots + C_l^l y^l = 1$, where C_l^i is the corresponding binomial coefficient. Hence $y(l + yy_1) = 0$, where $y_1 = C_l^2 y + \cdots + C_l^l y^{l-1}$. Since $y \neq 0$, then we obtain that $l \in \mathfrak{p}_2 \cap \mathbb{Z} = (p)$. But $l = |k_1^*| = p^t - 1$ for some t which is a contradiction. So there exists a root ε of 1 of degree l such that $\overline{\varepsilon} = \overline{x}_0$. This means that $\gamma - (\varepsilon + \varepsilon^{-1}) \in \mathfrak{p}_2$ and hence $\gamma - (\varepsilon + \varepsilon^{-1})$ is not a unit in the ring $\mathcal{O}$. Lemma 8 is proved.

LEMMA 9 *Let $F_2 = \langle g, h\rangle$ be the free group with generators g and h. Set $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$, $t = \tau_{ghg^{-1}h^{-1}}$. Then the following assertions hold.*

1) $t = x^2 + y^2 + z^2 - xyz - 2$.

2) *Let $R = gh(ghg^{-1}h^{-1})^s$. Then*

$$\tau_R = (P_s(t) - P_{s-1}(t))z.$$

3) *Let $T = (gh)^{-1}(ghg^{-1}h^{-1})^s(gh)^2(ghg^{-1}h^{-1})^s$. Then*

$$\tau_T = (t-2)P_{s-1}(t)^2 z^3 + (2 - P_{2s-1}(t) + P_{2s-2}(t))z.$$

Proof. 1) One can prove the equality by straightforward computations using relations (3) (see [16]).

2) Let u, v be arbitrary elements of F_2. Then using induction and relations (3), it is easy to show that for arbitrary integers p and q the following equality is valid

$$\tau_{u^p v^q} = P_{p-1}(\tau_u)P_{q-1}(\tau_v)\tau_{uv} - P_{p-2}(\tau_u)P_q(\tau_v) - P_p(\tau_u)P_{q-2}(\tau_v). \qquad (10)$$

Set now $u = gh$, $v = ghg^{-1}h^{-1}$. Then $\tau_u = z$, $\tau_v = t$, $\tau_{uv} = \tau_{gh(ghg^{-1}h^{-1})} = zt - \tau_{g^{-1}h^{-1}} = z(t-1)$. Hence,

$$\begin{aligned}
\tau_{uv^s} &= P_{s-1}(\tau_v)\tau_{uv} - P_{s-2}(\tau_v)\tau_u = P_{s-1}(t)(t-1)z - P_{s-2}(t)z \\
&= z(tP_{s-1}(t) - P_{s-1}(t) - P_{s-2}(t)) = z(P_s(t) + P_{s-2}(t) - P_{s-1}(t) - P_{s-2}(t)) \\
&= z(P_s(t) - P_{s-1}(t)).
\end{aligned}$$

3) Let u and v be such as above. Then using relations (3), (10), we have

$$\begin{aligned}
\tau_{u^{-1}v^s} &= \tau_{u^{-1}}\tau_{v^s} - \tau_{uv^s} = z(P_s(t) - P_{s-2}(t)) - z(P_s(t) - P_{s-1}(t)) \\
&= z(P_{s-1}(t) - P_{s-2}(t)); \\
\tau_{u^2v^s} &= \tau_u\tau_{uv^s} - \tau_{v^s} = z^2(P_s(t) - P_{s-1}(t)) - P_s(t) + P_{s-2}(t); \\
\tau_{u^3} &= z^3 - 3z.
\end{aligned}$$

Hence,

$$\begin{aligned}
\tau_{u^{-1}v^su^2v^s} &= \tau_{u^{-1}v^s}\tau_{u^2v^s} - \tau_{u^3} = z^3((P_s(t) - P_{s-1}(t))(P_{s-1}(t) - P_{s-2}(t)) - 1) \\
&\quad + z(3 - (P_{s-1}(t) - P_{s-2}(t))(P_s(t) - P_{s-2}(t))).
\end{aligned}$$

Using (9), we simplify the last expression. First consider the coefficient by z^3.

$$\begin{aligned}
&(P_s(t) - P_{s-1}(t))(P_{s-1}(t) - P_{s-2}(t)) - 1 = P_s(t)P_{s-1}(t) + P_{s-1}(t)P_{s-2}(t) \\
&- P_s(t)P_{s-2}(t) - P_{s-1}(t)^2 - 1 = P_{s-1}(t)(P_s(t) + P_{s-2}(t)) - \sum_{i=1}^{s-1} P_{2i}(t) - P_0(t) \\
&- P_{s-1}(t)^2 = tP_{s-1}(t)^2 - 2P_{s-1}(t)^2 = (t-2)P_{s-1}(t)^2.
\end{aligned}$$

Now consider the coefficient by z.

$$\begin{aligned}
&3 - (P_{s-1}(t) - P_{s-2}(t))(P_s(t) - P_{s-2}(t)) = 3 - P_s(t)P_{s-1}(t) + P_{s-1}(t)P_{s-2}(t) \\
&+ P_s(t)P_{s-2}(t) - P_{s-2}(t)^2 = 3 - \sum_{i=1}^{s} P_{2i-1}(t) + \sum_{i=1}^{s-1} P_{2i-1}(t) + \sum_{i=1}^{s-1} P_{2i}(t) - \sum_{i=0}^{s-2} P_{2i}(t) \\
&= 2 - P_{2s-1}(t) + P_{2s-2}(t).
\end{aligned}$$

Lemma 9 is proved.

At the end of Section 2 we shall show how one can deduce Corollary 1 from Theorem 1. This gives us another proof of Conjecture 1. Let $\Gamma = \langle a, b \mid R^m(a,b) = 1\rangle$, where $m \geq 2$, $R(a,b) = a^{u_1}b^{v_1} \dots a^{u_s}b^{v_s}$, $u_i, v_i \neq 0$, $s \geq 1$ and $R(a,b)$ is not a proper power.

First consider the case $m \geq 3$. Prove that $\dim X_2^s(\Gamma) \geq 2$. Then Theorem 1 immediately implies that Γ is a non-trivial free product with amalgamation. Consider in the character variety $X_2(F_2) = \mathbb{A}^3$ of the free group $F_2 = \langle g, h \rangle$ the hypersurface V defined by the equation

$$\tau_{R(g,h)}(x, y, z) = 2\cos(\tfrac{2\pi}{m}), \tag{11}$$

where $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$. By Lemma 5 one can write the equation (11) in the form

$$f(x, y, z) = M_s(x, y)z^s + \cdots + M_0(x, y) - 2\cos(\tfrac{2\pi}{m}) = 0. \tag{12}$$

We claim that $V \subset X_2(\Gamma)$. Indeed, let $v = (x_0, y_0, z_0) \in V$ and let $A, B \in \mathrm{SL}_2(\mathbb{C})$ be matrices such that $\operatorname{tr} A = x_0$, $\operatorname{tr} B = y_0$, $\operatorname{tr} AB = z_0$. Then by Lemma 3 we have $R^m(A, B) = E$. Hence the pair of matrices (A, B) defines a representation ρ of the group Γ into $\mathrm{SL}_2(\mathbb{C})$. Moreover, the image of ρ in $X_2(\Gamma)$ coincides with v. Hence, $v \in X_2(\Gamma)$. Furthermore, let $V_1, \dots, V_r$ be the irreducible components of V. It is easy to see (see [21]) that $\dim V_i = 2$ for each i. It remains to show that $V \cap X_2^s(\Gamma) \neq \varnothing$. Let us assume the contrary. Then all representations corresponding to points from V are reducible. This means that the regular function $\tau_{ghg^{-1}h^{-1}} - 2$ is identically equal to 0 on V. Hence by item 1 of Lemma 9 we have

$$g(x, y, z) = x^2 + y^2 + z^2 - xyz - 4 \equiv 0$$

on V. Thus,

$$f(x, y, z) = g(x, y, z)^d \tag{13}$$

for some $d \geq 1$. If for some i we have $|u_i| \geq 2$ or $|v_i| \geq 2$, then by Lemma 5 the leading coefficient $M_s(x, y)$ in (12) is not a constant and the equality (13) is impossible.

Let now $|u_i| = |v_i| = 1$ for $i = 1, \dots, s$. First, if for some i we have $u_i = u_{i+1}$ or $v_i = v_{i+1}$ ($u_1 = u_s$ or $v_1 = v_s$ for $i = s$), then we can consider new generators of the group Γ. Assume for definiteness that $u_1 = u_2$. Set $a_1 = a^{u_1}b^{v_1}$, $b_1 = b$. Then $\Gamma = \langle a_1, b_1 \mid R_1^m(a_1, b_1) = 1 \rangle$, where $R_1^m(a_1, b_1) = a_1^{u_1'}b_1^{v_1'} \dots a_1^{u_r'}b_1^{v_r'}$, $u_i', v_i' \neq 0$, $r \geq 1$ and $u_1' \geq 2$. This case has been considered above.

So we can assume that $u_{i+1} = -u_i$, $v_{i+1} = -v_i$. Since by assumption $R(a, b)$ is not a proper power, then we have for $R(a, b)$ the unique possibility up to cyclic rearrangement $R(a, b) = aba^{-1}b^{-1}$. Hence

$$f(x, y, z) = x^2 + y^2 + z^2 - xyz - 2 - 2\cos(\tfrac{2\pi}{m}) = g(x, y, z) + 2 - 2\cos(\tfrac{2\pi}{m}).$$

Since $2 - 2\cos(\frac{2\pi}{m}) \neq 0$, then it is obvious that in this case $g(x, y, z)$ has no zeros on V. So that for $m \geq 3$ we have proved that the group Γ is a non-trivial free product with amalgamation.

Let now $m = 2$. In this case one can consider a group $\Gamma_1 = \langle a, b \mid R^4(a, b) = 1 \rangle$. We have proved above that $\dim X_2^s(\Gamma_1) \geq 2$. Then the proof of Theorem 1 implies that there exists a representation $\rho : \Gamma_1 \to \mathrm{SL}_2(\mathbb{Q}_p)$ for some prime p such that $\rho(\Gamma_1)$

is dense in $\mathrm{SL}_2(\mathbb{Q}_p)$ in p-adic topology. Hence $\rho(\Gamma_1)$ is a non-trivial free product with amalgamation. Let G be the image of $\rho(\Gamma_1)$ in $\mathrm{PSL}_2(\mathbb{Q}_p)$. Then it is easy to see that G is a non-trivial free product with amalgamation too. But G is an epimorphic image of Γ whence Γ is a non-trivial free product with amalgamation.

3. The proof of Theorem 2

1) Let $\Gamma_n = \langle a, b \mid a^n = b^k = R^2(a,b) = 1\rangle$ and let $F_2 = \langle g, h\rangle$ be the free group with generators g and h. Set $x = \tau_g$, $\beta = \tau_h = 2\cos(\pi/k)$, $z = \tau_{gh}$. Consider the equation

$$Q_{R(g,h)}(x, \beta, z) = 0, \tag{14}$$

where $Q_{R(g,h)}$ is the Fricke polynomial of the element $R(g,h) \in F_2$. By Lemma 5 we can write (14) in the form

$$A_0(x)z^s + \cdots + A_s(x) = 0, \tag{15}$$

where $A_0(x) = \prod_{i=1}^s P_{u_i-1}(x)P_{v_i-1}(\beta)$. Since by assumption there exists i such that $|u_i| \geq 2$, then $\deg P_{u_i-1}(x) \geq 1$. Let $x_0 = -2\cos(\pi/u_i)$ be one of the roots of $P_{u_i-1}(x)$. Then $x - x_0$ divides $A_0(x)$. Let $A_0(x) = (x - x_0)B_0(x)$, where $B_0(x) \in \mathcal{O}[x]$. Write (15) in the form

$$(x - x_0)B_0(x)z^s + \cdots + A_s(x) = 0. \tag{16}$$

First we assume that all polynomials $A_1(x), \ldots, A_s(x)$ are divided on $x - x_0$. Then one can write (16) in the form

$$(x - x_0)f(x, z) = 0, \tag{17}$$

where $f(x,z)$ is some polynomial of x, z. Let z_0 be an arbitrary non-integer element of $\mathbb{C}$, that is, $z_0 \notin \mathcal{O}$, and let $A, B \in \mathrm{SL}_2(\mathbb{C})$ be matrices such that $\operatorname{tr} A = x_0$, $\operatorname{tr} B = \beta$, $\operatorname{tr} AB = z_0$. By construction the pair of matrices (A, B) defines a representation of the group Γ_n into $\mathrm{PSL}_2(\mathbb{C})$. Applying Lemma 6, we obtain that Γ_n is a non-trivial free product with amalgamation.

Assume now that not all polynomials $A_1(x), \ldots, A_s(x)$ are multiples of $x - x_0$. Let, for example, $A_1(x)$ is not a multiple of $x - x_0$ and let $0 \neq \delta = A_1(x_0) \in \mathcal{O}$ be the residue of $A_1(x)$ modulo $x - x_0$. Let $c = N_{\mathbb{Q}(\delta)/\mathbb{Q}}(\delta)$. For a finite set of prime numbers S from assertion of the theorem we take $S = \{p \in \mathbb{Z} \mid p \text{ divides } c\}$. Assume that $n = u_i p f$ for some integer f and prime $p \notin S$ such that $u_i p \nmid u_j$ for $j \neq i$. Let $x_r = 2\cos(r\pi/(pu_i))$ for some $r \not\equiv 0 \pmod p$, $K_r = \mathbb{Q}(\delta, x_r - x_0)$. By item 3) of Lemma 7 one can choose r such that p divides $N_{K_r/\mathbb{Q}}(x_r - x_0)$. Moreover, by construction p does not divide c. Then $N_{K_r/\mathbb{Q}}(\delta/(x_r - x_0)) \notin \mathbb{Z}$ hence $\delta/(x_r - x_0) \notin \mathcal{O}$. Thus, we have

$$\frac{A_1(x_r)}{x_r - x_0} \notin \mathcal{O}.$$

Furthermore, since $p \nmid r$ and $pu_i \nmid u_j$ for each $j \neq i$, then $B_0(x_r) \neq 0$. Set now $x = x_r$ and write the equation (16) in the form

$$z^s + \tfrac{A_1(x_r)}{(x_r - x_0)B_0(x_r)} z^{s-1} + \cdots + \tfrac{A_s(x_r)}{(x_r - x_0)B_0(x_r)} = 0 \tag{18}$$

Clearly, we have $A_1(x_r)/((x_r - x_0)B_0(x_r)) \notin \mathcal{O}$ because $B_0(x_r) \in \mathcal{O}$. Hence the equation (18) has a root $z_0 \notin \mathcal{O}$. Consider matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr} A = x_r, \qquad \operatorname{tr} B = \beta, \qquad \operatorname{tr} AB = z_0.$$

By construction the pair of matrices (A, B) defines a representation of the group Γ_n into $\mathrm{PSL}_2(\mathbb{C})$. Applying Lemma 6, we obtain that the group Γ_n is a non-trivial free product with amalgamation.

2) We keep notations of item 1). Consider the equation

$$Q_{R(g,h)}(x, \beta, z) = \gamma_t, \tag{19}$$

where $\gamma_t = 2\cos(t\pi/m)$, $m \nmid t$. By Lemma 5 one can write (19) in the form

$$(x - x_0)B_0(x)z^s + \cdots + A_s(x) - \gamma_t = 0. \tag{20}$$

Let $x_r = 2\cos(r\pi/(pu_i))$, where $r \not\equiv 0 \,(\mathrm{mod}\ p)$. Prove that there exist t and r such that $(A_s(x_r) - \gamma_t)/(x_r - x_0) \notin \mathcal{O}$. Indeed, let us assume the contrary.

First, consider the case $m = 3$. Then $\gamma_1 = 1$, $\gamma_2 = -1$. Since both of the numbers $(A_s(x_r) - 1)/(x_r - x_0)$ and $(A_s(x_r) + 1)/(x_r - x_0)$ belong to $\mathcal{O}$, then their difference $2/(x_r - x_0) \in \mathcal{O}$ for each $r \not\equiv 0 \,(\mathrm{mod}\ p)$. Since by assumption $p \neq 2$, then we obtain a contradiction to item 4) of Lemma 7.

Now let $m = 2^l$. Then $\gamma_{2^{l-1}} = 0$, $\gamma_{2^{l-2}} = \sqrt{2}$. Since both of the numbers $A_s(x_r)/(x_r - x_0)$ and $(A_s(x_r) - \sqrt{2})/(x_r - x_0)$ belong to $\mathcal{O}$, then their difference $\sqrt{2}/(x_r - x_0) \in \mathcal{O}$ and therefore $2/(x_r - x_0) \in \mathcal{O}$. Again we obtain a contradiction to item 4) of Lemma 7.

So let us choose t and r such that $(A_s(x_r) - \gamma_t)/(x_r - x_0) \notin \mathcal{O}$. Since $p \nmid r$ and $pu_i \nmid u_j$ for each $j \neq i$, then $B_0(x_r) \neq 0$. Set $x = x_r$ and write (20) in the form:

$$z^s + \cdots + \tfrac{A_s(x_r) - \gamma_t}{(x_r - x_0)B_0(x_r)} = 0. \tag{21}$$

By construction $(A_s(x_r) - \gamma_t))/((x_r - x_0)B_0(x_r)) \notin \mathcal{O}$ hence (21) has a root $z_0 \notin \mathcal{O}$. Consider matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr} A = x_r, \qquad \operatorname{tr} B = \beta, \qquad \operatorname{tr} AB = z_0.$$

By construction the pair of matrices (A, B) defines a representation of the group Γ_n into $\mathrm{PSL}_2(\mathbb{C})$. Applying Lemma 6, we obtain that the group Γ_n is a non-trivial free product with amalgamation.

3) Let $m > 3$ and $m \neq 2^l$. We keep notations of item 2). Show that there exist $t \not\equiv 0 \pmod m$ and $r \not\equiv 0 \pmod p$ such that $(A_s(x_r) - \gamma_t)/(x_r - x_0) \notin \mathcal{O}$. Indeed, let us assume the contrary. Let for each $t \not\equiv 0 \pmod m$ and each $r \not\equiv 0 \pmod p$ we have $(A_s(x_r) - \gamma_t)/(x_r - x_0) \in \mathcal{O}$.

First, consider the case, where m is odd. Then $1 + \sum_{i=1}^{2g} \gamma_{2i} = 0$ as the sum of all roots of 1 of degree m. Note that $-\gamma_i = \gamma_{m-i}$. Set $C_i = (A_s(x_r) - (-1)^i \gamma_{2i})/(x_r - x_0)$. Then we have

$$\sum_{i=1}^{m-1} (-1)^i C_i = -\sum_{i=1}^{m-1} \frac{\gamma_{2i}}{x_r - x_0} = \frac{1}{x_r - x_0} \in \mathcal{O}$$

for each $r \not\equiv 0 \pmod p$ which contradicts to item 4) of Lemma 7.

Now let $m = m_1 2^g$, where $g \geq 1$, $m_1 > 1$ is odd. Consider the numbers $\gamma_{i2^g} = 2\cos(i\pi/m_1)$. Arguing just as in the case of odd m above, we obtain a contradiction to item 4) of Lemma 7.

So let us choose t and r such that $(A_s(x_r) - \gamma_t)/(x_r - x_0) \notin \mathcal{O}$. Then the constant term in the equation (21) does not belong to $\mathcal{O}$ and (21) has a root $z_0 \notin \mathcal{O}$. Let $A, B \in \mathrm{SL}_2(\mathbb{C})$ be matrices such that

$$\operatorname{tr} A = x_r, \qquad \operatorname{tr} B = \beta, \qquad \operatorname{tr} AB = z_0.$$

By construction the pair of matrices (A, B) defines a representation of the group Γ_n into $\mathrm{PSL}_2(\mathbb{C})$. Applying Lemma 6, we complete the proof of Theorem 2.

Remark. In several cases one can obtain the more precise information when a generalized triangle group Γ is a non-trivial free product with amalgamation. For example, consider the following group $\Gamma_k = \langle a, b \mid a^2 = b^k = (ab^2)^3 = 1 \rangle$. Then Theorem 2 implies that if $k = 2k_1$, where $1 < k_1 \neq 2^l$, then Γ_k is a non-trivial free product with amalgamation. However it is easy to see that for $k_1 = 2^l$, $l \geq 1$, the group Γ_k is a non-trivial free product with amalgamation too. Indeed, let $F_2 = \langle g, h \rangle$ be the free group, let $x = \tau_g = 0$, $y = \tau_h$, $z = \tau_{gh}$. Consider the equation

$$Q_{gh^2}(0, y, z) = yz = 2\cos(\pi/3) = 1.$$

Let $y = y_r = 2\cos(r\pi/2^{l+1})$. Then by item 6) of Lemma 7 for each odd r, we have that $z_r = 1/y_r \notin \mathcal{O}$. Now Lemma 6 implies that Γ_k is a non-trivial free product with amalgamation.

4. Proof of Theorem 3

First, assume that for a word $R(a, b) = a^{u_1} b^{v_1} \ldots A^{u_s} b^{v_s}$, we have $v = \max_{1 \leq i \leq s} |v_i| \geq 2$. Then by Theorem 2 there exists a prime p such that the group $\Gamma_1 = \langle a, b \mid a^n = b^{pv} =$

$R^m(a,b) = 1\rangle$ is a non-trivial free product with amalgamation. Since Γ_1 is an epimorphic image of Γ, then Γ is a non-trivial free product with amalgamation too.

Thus one can assume that

$$R(a,b) = a^{u_1}b^{v_1}\dots a^{u_s}b^{v_s},$$

where $v_i \in \{-1,1\}$, $i = 1,\dots,s$. Suppose for a moment that there exists i, $1 \le i \le s$, such that either $v_i = v_{i+1}$ or $v_1 = v_s$. For definiteness, let $v_1 = v_2$. In this case one can consider the new generators of the group Γ: $a_1 = a$, $b_1 = a^{u_2}b^{v_1}$. Then it is easy to check that $\Gamma = \langle a_1, b_1 \mid a_1^n = R_1^m(a_1,b_1) = 1\rangle$, where $R_1(a_1,b_1) = a_1^{u_1'}b_1^{v_1'}\dots A_1^{u_l'}b_1^{v_l'}$, $0 < u_i' < n$, $v_i' \neq 0$ for $i = 1,\dots,l$. In addition, we have $v' = \max_{1\le i\le l}|v_i'| \ge 2$. But we have proved just above that in this case the group Γ is a non-trivial free product with amalgamation.

Thus without loss of generality we can assume that

$$R(a,b) = a^{u_1}ba^{u_2}b^{-1}\dots a^{u_{2k-1}}ba^{u_{2k}}b^{-1},$$

where $k \ge 1$, $0 < u_i < n$ for $i = 1,\dots,2k$. Set $c = ba^{-1}b^{-1}$. Then

$$R(a,b) = a^{u_1}c^{-u_2}\dots a^{u_{2k-1}}c^{-u_{2k}} = R_1(a,c).$$

Let $F_2 = \langle g,h\rangle$ be the free group of rank 2, $f = hg^{-1}h^{-1}$. Set $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$, $t = \tau_{gf}$. Then $\tau_f = \tau_g = x$, $t = \tau_{gf} = \tau_{ghg^{-1}h^{-1}} = x^2+y^2+z^2-xyz-2$ by item 1 of Lemma 9. Consider the element $R_1(g,f) \in F_2$ as a word of g and f. Let $q(x,t)$ be the Fricke polynomial of $R_1(g,f)$, that is,

$$Q(x,t) = Q_{R_1(g,f)}(\tau_g,\tau_f,\tau_{gf}) = Q_{R_1(g,f)}(x,x,t).$$

Since $R_1(g,f)$ contains k blocks of the form $g^{u_j}f^{-u_{j+1}}$, then by Lemma 5 the degree of the polynomial $q(x,t)$ with respect to t is equal to k and the leading coefficient of $q(x,t)$ is equal to $(-1)^k\prod_{i=1}^{2k}P_{u_i-1}(x)$. Since by construction $R(g,h) = R_1(g,f)$, then

$$Q_{R(g,h)}(x,y,z) = q(x,t) = q(x, x^2+y^2+z^2-xyz-2). \tag{22}$$

Set now $x = \tau_g = \alpha_r = 2\cos(r\pi/n)$, $\gamma_l = 2\cos(l\pi/m)$, where $r \not\equiv 0 \pmod n$, $l \not\equiv 0 \pmod m$, and consider the equation

$$Q_{R(g,h)}(\alpha_r,y,z) = \gamma_l. \tag{23}$$

By (22) one can write the equation (23) in the form

$$q(\alpha_r,t) = \gamma_l. \tag{24}$$

Lemma 10 *There exist $r, l \in \mathbb{Z}$, where $r \not\equiv 0 \pmod n$ and $l \not\equiv 0 \pmod m$, such that $P_{u_i-1}(\alpha_r) \neq 0$ for $i = 1, \dots, 2k$ and the equation (24) has a root $t = t_0 \neq 2$.*

Proof. First let $m \geq 3$. In this case $\gamma_1 \neq \gamma_2$. Set $r = 1$. Then the degree of the polynomial $q(\alpha_1, t)$ is equal to k. It is evident that at least one of the equations $q(\alpha_1, t) = \gamma_1$, $q(\alpha_1, t) = \gamma_2$ has a root $t_0 \neq 2$.

Now we assume that $m = 2$. Suppose that the equation $q(\alpha_r, t) = 0$ has a unique root $t = 2$. This means that for arbitrary matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that $\operatorname{tr} A = \operatorname{tr} B = \alpha_r$, the condition $\operatorname{tr} R_1(A, B) = \operatorname{tr} A^{u_1} B^{-u_2} \dots A^{u_{2k-1}} B^{-u_{2k}} = 0$ implies that $\operatorname{tr} AB = 2$. Prove that this is not the case. To obtain a contradiction, it is sufficient to find matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ satisfying to the conditions:

1) $\operatorname{tr} A = \operatorname{tr} B = \alpha_r$;
2) $\operatorname{tr} AB \neq 2$;
3) $\operatorname{tr} R_1(A, B) = \operatorname{tr} A^{u_1} B^{-u_2} \dots A^{u_{2k-1}} B^{-u_{2k}} = 0$.

We shall find A, B in the form

$$A = \begin{pmatrix} \varepsilon_r & w \\ 0 & \varepsilon_r^{-1} \end{pmatrix}, \qquad B = \begin{pmatrix} \varepsilon_r & 0 \\ w & \varepsilon_r^{-1} \end{pmatrix},$$

where $\varepsilon_r + \varepsilon_r^{-1} = \alpha_r = 2\cos(r\pi/n)$ and w is a variable. It is easy to see that $\operatorname{tr} AB = w^2 + \varepsilon_r^2 + \varepsilon_r^{-2}$. Hence the condition $\operatorname{tr} AB \neq 2$ is equivalent to $w^2 + \varepsilon_r^2 + \varepsilon_r^{-2} \neq 2$, that is,

$$w^2 \neq 2 - (\varepsilon_r^2 + \varepsilon_r^{-2}) = 2 - 2\cos(\tfrac{2r\pi}{n}) = 4\sin^2(\tfrac{r\pi}{n}).$$

It is easy to verify by induction that

$$A^i = \begin{pmatrix} \varepsilon_r^i & P_{i-1}(\alpha_r)w \\ 0 & \varepsilon_r^{-i} \end{pmatrix}, \qquad B^i = \begin{pmatrix} \varepsilon_r^i & 0 \\ P_{i-1}(\alpha_r)w & \varepsilon_r^{-i} \end{pmatrix}.$$

Furthermore, it is easy to show that

$$R_1(A, B) = \begin{pmatrix} \varepsilon_r^d + C_1(\alpha_r)w^2 + \dots + C_k(\alpha_r)w^{2k} & wf_1(w) \\ wf_2(w) & \varepsilon_r^{-d} + D_1(\alpha_r)w^2 + \dots + D_{k-1}(\alpha_r)w^{2k-2} \end{pmatrix}$$

where $d = \sum_{i=1}^{2k} u_i$, $C_k(\alpha_r) = (-1)^k \prod_{i=1}^{2k} P_{u_i-1}(\alpha_r)$, $f_1(w)$, $f_2(w)$ are some polynomials of w. Hence,

$$\operatorname{tr} R_1(A, B) = C_k(\alpha_r)w^{2k} + \dots + (C_1(\alpha_r) + D_1(\alpha_r))w^2 + (\varepsilon_r^d + \varepsilon_r^{-d}) = g(w^2).$$

Prove that there exists r, $1 \leq r < n$, such that $C_k(\alpha_r) \neq 0$ and the polynomial $g(w^2)$ has a root w_0 with $w_0^2 \neq 4\sin^2(r\pi/n)$. Let's assume the contrary. Suppose that for each r such that $C_k(\alpha_r) \neq 0$ we have

$$g(w) = C_k(\alpha_r)(w - 4\sin^2(\tfrac{r\pi}{n}))^k. \tag{25}$$

Comparing the constant terms in the left and right part of (25) and taking into account the expression for $C_k(\alpha_r)$, we obtain

$$\left(\prod_{i=1}^{2k} P_{u_i-1}(2\cos(\tfrac{r\pi}{n}))\right) 4^k \left(\sin(\tfrac{r\pi}{n})\right)^{2k} = 2\cos(\tfrac{dr\pi}{n}). \tag{26}$$

By (4) we have $P_{u_i-1}(2\cos(r\pi/n)) = \sin(u_i r\pi/n)/\sin(r\pi/n)$. Write u_i/n in the form u_i'/n_i, where $(u_i', n_i) = 1$. Then (26) has the form

$$\prod_{i=1}^{2k} \left(2\sin\left(\tfrac{u_i' r\pi}{n_i}\right)\right) = 2\cos(\tfrac{dr\pi}{n}). \tag{27}$$

Thus, to complete the proof of the lemma it is sufficient to prove that one gets a contradiction assuming that (27) holds for each r such that the left part in (27) is non-zero.

First consider the case, where n is odd. Let $n_0 = \min_j n_j$ and let for definiteness $n_0 = n_1$. Then n_1 is odd and let $p > 2$ be an arbitrary prime divisor of n_1. Set $r = n_1/p$. Then $2\sin(u_1' r\pi/n_1) = 2\sin(u'1\pi/p)$. If $j > 1$, then $2\sin(u_j' r\pi/n_j) = 2\sin(u_j' n_1\pi/(pn_j)) \neq 0$ because by construction pn_j does not divide $u_j' n_1$. It follows from (27) that

$$\prod_{i=2}^{2k} \left(2\sin\left(\tfrac{u_i' n_1\pi}{pn_i}\right)\right) = \frac{2\cos(dn_1\pi/(pn))}{\sin(u_1'\pi/p)} \in \mathcal{O}. \tag{28}$$

If pn divides dn_1, then the equality (28) implies that $1/(\sin(u_1'\pi/p)) \in \mathcal{O}$ which contradicts to item 5 of Lemma 7. Further if pn does not divide dn_1, then $2\cos(dn_1\pi/(pn)) \in \mathcal{O}^*$ by item 2 of Lemma 7. Thus it follows from (28) that $1/(2\sin(u_1'\pi/p)) \in \mathcal{O}$ which again contradicts to item 5 of Lemma 7.

Let now $n = 2^l n'$, where $l \geq 1$ and n' is odd. Let $n_i = 2^{l_i} n_i'$, where $l_i \geq 0$, n_i' is odd, and let $n_0' = \min_j n_j'$.

If $n_0' > 1$, then we set $r = 2^l r'$, where $r' \not\equiv 0 \pmod{n'}$. Then (27) has the form

$$\prod_{i=1}^{2k} \left(2\sin\left(\tfrac{u_i' 2^{l-l_i} r'\pi}{n_i'}\right)\right) = 2\cos(\tfrac{dr'\pi}{n'}), \tag{29}$$

where n' is odd. We have proved above that there exists an r' such that the left part of (29) is non-zero and the equality (29) is not valid.

Let now $n_0' = 1$. Set

$$I = \{i \mid n_i' = 1\}, \qquad l_0 = \min_{i\in I} l_i, \qquad I_0 = \{i \in I \mid l_i = l_0\}.$$

Furthermore, we set $r = 2^{l_0-1} r'$, where r' is odd. Then for $i \in I_0$ we have

$$2\sin\left(\tfrac{u_i' r\pi}{n_i}\right) = 2\sin\left(\tfrac{u_i' 2^{l_0-1} r'\pi}{2^{l_0}}\right) = 2\sin\left(\tfrac{u_i' r'\pi}{2}\right) = \pm 2.$$

Now we can write the equality (27) in the form:

$$\prod_{i\notin I_0}\left(2\sin\left(\frac{u_i'r'\pi}{2^{l_i-l_0+1}n_i'}\right)\right)=\pm\frac{1}{2^{|I_0|-1}}\cos\left(\frac{dr'\pi}{2^{l-l_0+1}n'}\right). \tag{30}$$

Choose r' such that the left part of (30) is non-zero. Then the right part of (30) should be non-zero too. If $|I_0|>1$ or $|I_0|=1$ and $\cos(dr'\pi/(2^{l-l_0+1}n'))\neq\pm1$, then the left part of (30) belongs to $\mathcal{O}$. But by item 1) of Lemma 7 the right part of (30) does not belong to $\mathcal{O}$ which is a contradiction.

Thus, it remains to consider the case $|I_0|=1$ and $\cos(dr'\pi/(2^{l-l_0+1}n'))=\pm1$. In this case (30) has the form

$$\prod_{i\neq i_0}\left(2\sin\left(\frac{u_i'r'\pi}{2^{l_i-l_0+1}n_i'}\right)\right)=\pm1. \tag{31}$$

If $|I|>1$ and $i_0\neq i\in I$, then $l_i>l_0$, $n_i=1$. Hence for each odd r' the left part of (31) is non-zero and (31) implies that $1/(2\sin(u_i'r'\pi/(2^{l_i-l_0+1})))\in\mathcal{O}$. We obtain a contradiction to item 5 of Lemma 7.

Let now $I=I_0=\{i_0\}$. Set

$$n_{j_0}=\min_{j\neq i_0}n_j\geq3,\qquad J=\{j\mid n_j=n_{j_0}\},\qquad l_{j_0}=\min_{j\in J}l_j.$$

If $l_{j_0}-l_0+1>0$, then we set $r'=n_{j_0}$. It is easy to verify that in this case the left part of (31) is not equal to 0 and (31) implies that $1/(2\sin(u_{j_0}'\pi/2^{l_{j_0}-l_0+1}))\in\mathcal{O}$. We obtain a contradiction to item 6 of Lemma 7.

At last, if $l_{j_0}-l_0+1\leq0$, then we take an arbitrary prime divisor $p\geq3$ of n_{j_0} and set $r'=n_{j_0}/p$. Then, as above, the left part of (31) is not equal to 0 and we obtain from (31) that

$$2\sin\left(\frac{u_{j_0}'r'\pi}{2^{l_{j_0}-l_0+1}n_{j_0}'}\right)=2\sin\left(\frac{u_{j_0}'2^{-l_{j_0}+l_0-1}\pi}{p}\right)\in\mathcal{O}^*.$$

We have a contradiction to item 5 of Lemma 7. Lemma 10 is proved.

Now we can complete the proof of Theorem 3. By Lemma 10 one can choose r,l such that the equation (24) has a root $t_0\neq2$. Since by construction $t=x^2+y^2+z^2-xyz-2$, $x=\alpha_r$, then we have that y, z satisfy to the equation

$$y^2+z^2-\alpha_r yz+\alpha_r^2-2-t_0=0. \tag{32}$$

Let (y_0,z_0) be some solution of (32) and let $A,B\in\mathrm{SL}_2(\mathbb{C})$ be matrices such that $\operatorname{tr}A=\alpha_r$, $\operatorname{tr}B=y_0$, $\operatorname{tr}AB=z_0$. Then by construction $\operatorname{tr}ABA^{-1}B^{-1}=t_0$, $\operatorname{tr}R(A,B)=\gamma_l$, and a pare of matrices (A,B) defines a representation of the group Γ into $\mathrm{PSL}_2(\mathbb{C})$. Note that this representation is irreducible because $t_0\neq2$. Our goal is to apply Lemma 6. To this end, it is sufficient to find a solution (y_0,z_0) of the equation (32) such that:

1) there exists an element of finite order $W_1(A,B) = A^{\alpha_1}B^{\beta_1}\dots A^{\alpha_g}B^{\beta_g}$ such that $\alpha_i, \beta_i \neq 0$ for $i = 1,\dots,g$ and $\sum_{i=1}^{g}\beta_g \neq 0$;

2) $z_0 = \operatorname{tr} AB \notin \mathcal{O}$.

The rest of the proof depends on the form of t_0. We consider the following cases:

1) $t_0 \notin \mathcal{O}$;

2) $t_0 = 2\cos((2k+1)\pi/(2s+1))$, where $s \geq 1$, $(2k+1, 2s+1) = 1$;

3) $t_0 = 2\cos(2k\pi/(2s+1))$, where $s \geq 1$, $(k, 2s+1) = 1$;

4) $t_0 = 2\cos((2k+1)\pi/(2s))$, where $s \geq 1$, $(2k+1, s) = 1$;

5) $t_0 \in \mathcal{O}$, $t_0 \neq 2\cos(k\pi/s)$ for arbitrary integers k and s.

1) Set $y_0 = 0$ and $W_1(A,B) = B$. Then $W_1(A,B) = B$ has order 4. Since $t_0 \notin \mathcal{O}$, then the equation (32) has a solution $(0, z_0)$ such that $z_0 \notin \mathcal{O}$. Applying Lemma 6, we obtain the assertion of Theorem 3.

2) Set $W_1(A,B) = AB(ABA^{-1}B^{-1})^s$. Combining Lemmas 8 and 9, we obtain

$$\operatorname{tr} W_1(A,B) = (P_{s+1}(t_0) - P_s(t_0))z_0 = 0 \cdot z_0 = 0.$$

Hence $W_1(A,B)$ has order 4. Take an arbitrary solution (y_0, z_0) of the equation (32) such that $z_0 \notin \mathcal{O}$. Again we can apply Lemma 6.

3) Set $W_1(A,B) = AB(ABA^{-1}B^{-1})^s$ and assume that

$$\operatorname{tr} W_1(A,B) = 2\cos(\pi/3) = 1.$$

Then $W_1(A,B)$ has order 6 and item 2 of Lemma 9 implies that

$$\operatorname{tr} W_1(A,B) = (P_{s+1}(t_0) - P_s(t_0))z_0 = 1.$$

Hence by item 3 of Lemma 8 we have $z_0 = 1/(P_{s+1}(t_0) - P_s(t_0)) \notin \mathcal{O}$. Let (y_0, z_0) be a solution of (32). Now we can apply Lemma 6.

4) Set $W_1(A,B) = (AB)^{-1}(ABA^{-1}B^{-1})^s(AB)^2(ABA^{-1}B^{-1})^s$ and assume that

$$\operatorname{tr} W_1(A,B) = 2\cos(\pi/3) = 1. \tag{33}$$

Then $W_1(A,B)$ has the order 6 and by item 3 of Lemma 9 we can write (33) in the form

$$(t_0 - 2)P_{s-1}(t_0)^2 z_0^3 + (2 - P_{2s-1}(t_0) + P_{2s-2}(t_0))z_0 - 1 = 0. \tag{34}$$

By item 4 of Lemma 8 we have $0 \neq P_{s-1}(t_0) \notin \mathcal{O}^*$ whence $1/((t_0-2)P_{s-1}(t_0)^2) \notin \mathcal{O}$. Thus, (34) has a root $z_0 \notin \mathcal{O}$. Let (y_0, z_0) be a solution of (32). Applying Lemma 6, we obtain the assertion of Theorem 3.

5) Since $t_0 \in \mathcal{O}$, $t_0 \neq 2\cos(k\pi/s)$ for arbitrary integers k and s, then by item 5 of Lemma 8 there exists an integer $l > 0$ such that $0 \neq P_l(t_0) \notin \mathcal{O}^*$. Set $W_1(A,B) =$

$(AB)^{-1}(ABA^{-1}B^{-1})^{l+1}(AB)^2(ABA^{-1}B^{-1})^{l+1}$ and assume that the condition (33) holds. Then $W_1(A, B)$ has the order 6 and by item 3 of Lemma 9 we can write (33) in the form

$$(t_0 - 2)P_l(t_0)^2 z_0^3 + (2 - P_{2l+1}(t_0) + P_{2l}(t_0))z_0 - 1 = 0. \tag{35}$$

Since by construction $1/((t_0 - 2)P_l(t_0)^2) \notin \mathcal{O}$, then (35) has a root $z_0 \notin \mathcal{O}$. Let (y_0, z_0) be a solution of (32). Applying Lemma 6, we complete the proof of Theorem 3 in the last case.

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