Decomposing some finitely generated groups into free products with amalgamation [1]

V.V. Benyash-Krivets

Introduction

. We shall say that a group G is a non-trivial free product with amalgamation if $G = G_1 *_A G_2$, where $G_1 \neq A \neq G_2$ (see [1]). Wall [2] has posed the following question:

Which one-relator groups are non-trivial free products with amalgamation?

Let $G = \langle g_1, \dots, g_m \mid R_1 = \dots = R_n = 1 \rangle$ be a group with m generators and n relations such that $\operatorname{def} G = m - n \geq 2$. It is proved in [4] that G is a non-trivial free product with amalgamation. In particular, if G is a group with $m \geq 3$ generators and one relation, then G is a non-trivial free product with amalgamation. A case of groups with two generators and one relation is more complicated. For example, the free abelian group $G = \langle a, b \mid [a, b] = 1 \rangle$ of rank 2, where $[a, b] = aba^{-1}b^{-1}$, obviously, is not a non-trivial free product with amalgamation. Other examples are given by groups $G_n = \langle a, b \mid aba^{-1} = b^n \rangle$. For any n, the group G_n is solvable and using results from [3], it is easy to show that for $n \neq -1$ the group G_n is not decomposable into a non-trivial free product with amalgamation. The following conjecture was stated in [4].

Conjecture 1 *Let $G = \langle a, b \mid R^m(a, b) = 1 \rangle$, $m \geq 2$, be a group with two generators and one relation with torsion. Then G is a non-trivial free product with amalgamation.*

Zieschang [5] has studied the problem of decomposing into non-trivial free products with amalgamation for discontinuous groups of transformations of the plane. He has given a complete answer to the question when such a group is a non-trivial free product with amalgamation in all cases except for the groups $H_1 = \langle a, b \mid [a, b]^n = 1 \rangle$ and $H_2 = \langle a, b \mid a^2 = [a, b]^n = 1 \rangle$, $n \geq 2$. Rosenberger [6] has proved that the groups H_1 and H_2 are non-trivial free products with amalgamation if n is not a power of 2. In recent papers [7, 8] it was proved that H_1 is a non-trivial free product with amalgamation for arbitrary $n \geq 2$. The independent proof of this fact was given in [9, 10].

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In the present paper we study the more general case, namely, we consider so-called *generalized triangle groups*, that is, groups G having a presentation of the form

$$G = \langle a, b \mid a^m = b^n = R^l(a, b) = 1 \rangle,$$

where $l \geq 2$, $R(a, b)$ is a cyclically reduced word in the free group on a, b. Not all of these groups are decomposable into non-trivial free products with amalgamation. For example, Zieschang [5] has proved that *the ordinary triangle group*

$$T(m, n, l) = \langle a, b \mid a^m = b^n = (ab)^l = 1 \rangle,$$

where $m, n, l \geq 2$, is not a non-trivial free product with amalgamation. On the other hand, it was shown in [10] that the group $G = \langle a, b \mid a^{2m} = R^l(a, b) = 1 \rangle$, where $m \geq 1$, $l \geq 2$, is a free product with amalgamation. Theorems 2 and 3 of the present paper contain more general results about decomposing of the generalized triangle groups into non-trivial free products with amalgamation.

In Theorem 1 we prove that a finitely generated group Γ is a non-trivial free product with amalgamation if the dimension of some algebraic variety (so-called character variety of irreducible representations of the group Γ into $\mathrm{SL}_2(\mathbb{C})$) is more than 1. To formulate this result we recall some notations and facts from the geometric representation theory (see also [11, 12, 13, 14]).

Let $\Gamma = \langle g_1, \dots, g_m \rangle$ be a finitely generated group and let $G \subset \mathrm{GL}_n(K)$ be a connected linear algebraic group defined over an algebraically closed field K of characteristic zero. Obviously, for each homomorphism $\rho : \Gamma \to G(K)$ the set of elements

$$(\rho(g_1), \dots, \rho(g_m)) \in G(K) \times \cdots \times G(K)$$

satisfies all defining relations of Γ. So the correspondence $\rho \to (\rho(g_1), \dots, \rho(g_m))$ is a bijection between the set $\mathrm{Hom}\,(\Gamma, G(K))$ and the set of K-points in some affine K-variety $R(\Gamma, G) \subset G^m$. The variety $R(\Gamma, G)$ is usually called the representation variety of the group Γ into the algebraic group G.

The group G acts on $R(\Gamma, G)$ in a natural way (by simultaneous conjugation of components) and its orbits are in one-to-one correspondence with the equivalence classes of representations of Γ. In a general case the orbits of the group G under this action are not necessarily closed and hence the variety of orbits (the geometric quotient) is not an algebraic variety. However, if G is a reductive group, then one can consider the categorical quotient $X(\Gamma, G) = R(\Gamma, G)/G$ (see [15]). Its points parametrize closed G-orbits. In the case $G = \mathrm{GL}_n(K)$ or $G = \mathrm{SL}_n(K)$ an orbit of G is closed if and only if the corresponding representation is completely reducible. Therefore in this case points of the variety $X(\Gamma, G)$ are in one-to-one correspondence with the equivalence classes of completely reducible representations of the group Γ into G or, in other words, with characters of representations of Γ into G.

Throughout we shall consider only the case $G = \mathrm{SL}_2(K)$ and for brevity we shall denote $R(\Gamma), \mathrm{SL}_2(K)) = R(\Gamma)$, $X(\Gamma, \mathrm{SL}_2(K)) = X(\Gamma)$. One can find all used below information about varieties $R(\Gamma)$, $X(\Gamma)$ in [12, 16, 17, 18]. We set

$$R^s(\Gamma) = \{\rho \in R(\Gamma) \mid \rho \text{ is irreducible}\}, \qquad X^s(\Gamma) = \pi(R^s(\Gamma)),$$

where $\pi : R(\Gamma) \to X(\Gamma)$ is the canonical projection. It is shown in [12] that $R^s(\Gamma)$, $X^s(\Gamma)$ are open (in Zariski topology) subset of $R(\Gamma)$, $X(\Gamma)$ respectively. The aim of the present paper is to prove the following theorems.

THEOREM 1 *Let Γ be a finitely generated group such that $\dim X^s(\Gamma) \geq 2$. Then Γ is a non-trivial free product with amalgamation.*

THEOREM 2 *Let $k, m \in \mathbb{Z}$, $k, m \geq 2$, let $F_2 = \langle g, h \rangle$ be the free group of rank 2 with generators g, h, and let $R(g, h) = g^{u_1}h^{v_1} \dots g^{u_s}h^{v_s}$ be a cyclically reduced word in F_2 such that $u_i \neq 0$, $0 < v_i < k$, $s \geq 1$. Suppose that there exists $i \in \{1, \dots, s\}$ such that $|u_i| \geq 2$. Let p be a prime number such that $u_i p$ does not divide u_j for $j \neq i$ and let $0 \neq f \in \mathbb{Z}$. We set $n = u_i p f$. Then in the following cases the group*

$$\Gamma_n = \langle a, b \mid a^n = b^k = R^m(a, b) = 1 \rangle$$

is a non-trivial free product with amalgamation.

1) $m = 2$, p does not belong to some finite set of prime numbers S. The set S is completely determined by the exponent k and the word R.

2) $m = 3$ or $m = 2^l > 3$, $p \neq 2$;

3) $m > 3$ and $m \neq 2^l$.

Note that the condition $u_i p \nmid u_j$ for $j \neq i$ in Theorem 2 is carried out automatically if $u_i = \max_{1 \leq j \leq s} u_j \geq 2$ or $u_i \nmid u_j$ for each $j \neq i$.

THEOREM 3 *Let $\Gamma = \langle a, b \mid a^n = R^m(a, b) = 1 \rangle$, where $n = 0$ or $n \geq 2$, $m \geq 2$, $R(a, b) = a^{u_1}b^{v_1} \dots a^{u_s}b^{v_s}$, $s \geq 1$, $v_i \neq 0$, $0 < u_i < n$. Then Γ is a non-trivial free product with amalgamation.*

As a direct consequence of Theorem 3 we obtain the proof of Conjecture 1.

COROLLARY 1 *Let $\Gamma = \langle a, b \mid R^m(a, b) = 1 \rangle$, $m \geq 2$, be a group with two generators and one relation with torsion. Then Γ is a non-trivial free product with amalgamation.*

At the end of Section 2 we shall prove that the group Γ from Corollary 1 satisfies the assumptions of Theorem 1, that is, $\dim X_2^s(\Gamma) = 2$ so that we obtain an another proof of Conjecture 1.

COROLLARY 2 *Fuchsian groups $H_1 = \langle a, b \mid [a, b]^n = 1 \rangle$ and $H_2 = \langle a, b \mid a^2 = [a, b]^n = 1 \rangle$, $n \geq 2$, are non-trivial free products with amalgamation.*

1. The proof of Theorem 1

In what follows we shall denote the field of p-adic numbers by $\mathbb{Q}_p$, the ring of p-adic integers by $\mathbb{Z}_p$, the group of invertible elements in $\mathbb{Z}_p$ by $\mathbb{Z}_p^*$, the p-adic valuation by $|\ |_p$, the trace of a matrix A by $\operatorname{tr} A$, the identity 2×2 matrix by E.

We recall some facts about the character variety $X(\Gamma)$ of representations of a finitely generated group Γ into $\mathrm{SL}_2(\mathbb{C})$ (see [12]). For an arbitrary element $g \in \Gamma$ one can consider the regular function

$$\tau_g : R(\Gamma) \to \mathbb{C}, \qquad \tau_g(\rho) = \operatorname{tr} \rho(g).$$

Usually, τ_g is called *a Fricke character* of the element g. It is known that the $\mathbb{Z}$-algebra $T(\Gamma)$ generated by all functions τ_g, $g \in \Gamma$ is finitely generated. Moreover, if $\tau_{g_1}, \ldots, \tau_{g_s}$ are the generators of $T(\Gamma)$ then the $\mathbb{C}$-algebra of $\mathrm{SL}_2(\mathbb{C})$-invariant regular functions $\mathbb{C}[R(\Gamma)]^{\mathrm{SL}_2(\mathbb{C})}$ is equal to $\mathbb{C}[\tau_{g_1}, \ldots, \tau_{g_s}]$. Consider the morphism

$$\pi : R(\Gamma) \to \mathbb{A}^s, \qquad \pi(\rho) = (\tau_{g_1}(\rho), \ldots, \tau_{g_s}(\rho)).$$

It is shown in [12] that the image $\pi(R(\Gamma))$ is closed in $\mathbb{A}^s$. Since $X(\Gamma)$ and $\pi(R(\Gamma))$ are biregularly isomorphic, then throughout we shall identify $X(\Gamma)$ and $\pi(R(\Gamma))$.

The idea of the proof of Theorem 1 is to construct a representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{Q}_p)$ for some prime p such that the group $\rho(\Gamma)$ is dense in $\mathrm{SL}_2(\mathbb{Q}_p)$ in p-adic topology. After that Theorem 1 will follow from the following well known facts.

1) If H is a dense in p-adic topology subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ then H is a non-trivial free product with amalgamation (see [19]).

2) If $f : G_1 \to G_2$ is an epimorphism of groups and G_2 is a non-trivial free product with amalgamation then G_1 is a non-trivial free product with amalgamation.

LEMMA 1 *Let H be a subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Then H is dense in $\mathrm{SL}_2(\mathbb{Q}_p)$ in p-adic topology if and only if H is absolute irreducible (that is, irreducible over algebraic closure of $\mathbb{Q}_p$), unbounded, and non-discrete.*

Proof. If H is dense in $\mathrm{SL}_2(\mathbb{Q}_p)$ then obviously H is absolute irreducible, unbounded and non-discrete. Prove the inverse assertion. The unboundedness of H means that there exists an element $h \in H$ such that $|\operatorname{tr} h|_p > 1$. Indeed, otherwise the traces of all elements from H belong to $\mathbb{Z}_p$ whence the group H is conjugated to a subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ (see [20] or [12, lemma I.4.3]), that is, H is bounded. Let us show that the eigenvalues of the matrix h belong to $\mathbb{Q}_p$. Let $\operatorname{tr} h = p^{-s}\alpha$, where $\alpha \in \mathbb{Z}_p^*$, $s > 0$. Then the characteristic polynomial of h is of the form $f(y) = y^2 - p^{-s}\alpha y + 1$ and its discriminant is equal to $D = p^{-2s}\alpha^2 - 4 = p^{-2s}(\alpha^2 - 4p^{2s})$. Thus, D is a square in

$\mathbb{Q}_p$ whence the roots of $f(y)$ belong to $\mathbb{Q}_p$. So h is conjugate in $\mathrm{SL}_2(\mathbb{Q}_p)$ to a diagonal matrix of the form

$$\mathrm{diag}(\lambda, \lambda^{-1}), \quad \lambda = p^{-s}\gamma, \quad s \geq 0, \quad \gamma \in \mathbb{Z}_p^*. \tag{1}$$

Without loss of generality we can suppose considering if necessary a group conjugated to H that $h = \mathrm{diag}(\lambda, \lambda^{-1}) \in H$. Consider the following unipotent subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$:

$$U_1 = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \middle| a \in \mathbb{Q}_p \right\}, \qquad U_2 = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{Q}_p \right\}.$$

It is well known that U_1 and U_2 generate $\mathrm{SL}_2(\mathbb{Q}_p)$. Therefore it is sufficient to show that $U_1, U_2 \subset \overline{H}$, where $\overline{H}$ is the closure of H in p-adic topology. For example, let us prove that $U_1 \subset \overline{H}$. To this end, first we shall show that $\overline{H}$ contains a non-trivial unipotent element $u = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, $a \in \mathbb{Q}_p^*$. Let

$$\Gamma_j = \left\{ \begin{pmatrix} 1 + p^j a & p^j b \\ p^j c & 1 + p^j d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p) \middle| a, b, c, d \in \mathbb{Z}_p \right\}$$

be the principal congruence subgroup of a level j in $\mathrm{SL}_2(\mathbb{Q}_p)$. The groups Γ_j, $j \geq 1$, form a base for the neighborhoods of the identity in $\mathrm{SL}_2(\mathbb{Q}_p)$. Then non-discreteness of H implies that for each $j > 0$ there exists an element $E \neq x_j \in H \cap \Gamma_j$. Let

$$x_j = \begin{pmatrix} 1 + p^j a_j & p^j b_j \\ p^j c_j & 1 + p^j d_j \end{pmatrix},$$

where $a_j, b_j, c_j, d_j \in \mathbb{Z}_p$. Then $\lim_{j\to\infty} x_j = E$. Since H is an absolutely irreducible subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$, then without loss of generality we can suppose that almost all elements b_j, c_j (with the exception of a finite number) are not equal to 0 (otherwise one can conjugate all x_j by a suitable element y from H, and again we shall have $\lim_{j\to\infty} y x_j y^{-1} = E$). Let $c_j = p^{k_j}\varepsilon_j$, where $k_j \geq 0$, $\varepsilon_j \in \mathbb{Z}_p^*$, and let $t_j = [(k_j + j)/(2s)]$ be the integer part of the number $(k_j + j)/(2s)$, where s is defined in (1). Then we have $r_j = k_j + j - 2st_j \in \{0, 1, \ldots, 2s - 1\}$. Now consider the following sequence $\{x_j'\}$ of elements of H:

$$x_j' = h^{-t_j} x_j h^{t_j} = \begin{pmatrix} 1 + p^j a_j & p^{j+2st_j}\gamma^{-2t_j} b_j \\ p^{r_j}\gamma^{2t_j}\varepsilon_j & 1 + p^j d_j \end{pmatrix}. \tag{2}$$

The infinite sequence $\{p^{r_j}\gamma^{2t_j}\varepsilon_j\}$ is contained in $\mathbb{Z}_p$ hence it is bounded. Therefore one can choose from it the convergent subsequence

$$a_{j_m} = p^{r_{j_m}}\gamma^{2t_{j_m}}\varepsilon_{j_m}.$$

Since $|a_{j_m}|_p = p^{-r_{j_m}} \geq p^{-2s+1}$, then $\lim_{j_m\to\infty} a_{j_m} = a \neq 0$. Thus, it follows from (2) that

$$\lim_{j_m\to\infty} x_{j_m}' = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = u \in \overline{H}.$$

Conjugating u by a suitable power of h, we can suppose that $a \notin \mathbb{Z}_p$. Furthermore, $u^n = \begin{pmatrix} 1 & 0 \\ an & 1 \end{pmatrix}$ for arbitrary $n \in \mathbb{Z}$. Since the closure of $a\mathbb{Z}$ in $\mathbb{Q}_p$ contains $\mathbb{Z}_p$, then $\overline{H}$ contains a group $U_1' = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \middle| x \in \mathbb{Z}_p \right\}$. Furthermore, let $u_1 = \begin{pmatrix} 1 & 0 \\ p^{-r}\beta & 1 \end{pmatrix} \in U_1$ be an element such that $r > 0$ and $\beta \in \mathbb{Z}_p^*$. Choose an integer m such that $2sm - r \geq 0$. Then $h^m u_1 h^{-m} \in U_1'$ whence $u_1 \in \overline{H}$. Thus, we have proved that $U_1 \subset \overline{H}$. Similarly, one can proved that $U_2 \subset \overline{H}$. Hence $\overline{H} = \mathrm{SL}_2(\mathbb{Q}_p)$. Lemma 1 is proved.

LEMMA 2 *Let X,Y be irreducible $\mathbb{Q}$-defined affine varieties, let* $\dim Y \geq 1$, *and let $f : X \to Y$ be a dominant $\mathbb{Q}$-defined regular morphism. Then there exists a prime number $p \neq 2$ and a point $x \in X(\mathbb{Q}_p)$ such that not all coordinates of the image $f(x) \in Y(\mathbb{Q}_p)$ belong to a ring $\mathbb{Z}_p$.*

Proof. Let K be the algebraic closure of $\mathbb{Q}$. Let D be an arbitrary irreducible curve in $Y(K)$, and let L be an arbitrary irreducible curve such that $L \subset f^{-1}(D)$ and $f(L)$ is dense in D. Let $\overline{D}$ and $\overline{L}$ be the projective closure of D and L respectively, and let $\widetilde{L}$ be the smooth projective model of $\overline{L}$. The regular morphism $f : L \to D$ determines a rational morphism $\widetilde{f} : \widetilde{L} \to \overline{D}$. Since any rational morphism from a smooth curve to a projective variety is regular and the image of a projective variety under regular map is closed (see [21]), then $\widetilde{f}$ is a regular surjective morphism. Let $v \in \overline{D} \backslash D$ be a point at infinity on $\overline{D}$ and let $w \in \widetilde{f}^{-1}(v)$. The coordinates of points v and w generate the finite extension $K_1/\mathbb{Q}$. By Chebotarev's density theorem there exist infinitely many prime numbers p such that $K_1 \subset \mathbb{Q}_p$. Choose one of such p. Then $w \in \widetilde{L}(\mathbb{Q}_p)$, $v \in \overline{D}(\mathbb{Q}_p)$. Since w is a non-singular point on $\widetilde{L}$, then w has p-adic neighborhood $W \subset \widetilde{L}(\mathbb{Q}_p)$ such that W is homeomorphic to an area in $\mathbb{Q}_p$ (see [21, chapter II]). This means that there exists an infinite sequence of elements $w_i \in W$ such that $w_i \in L(\mathbb{Q}_p)$ and $\lim_{i \to \infty} w_i = w$ in p-adic topology. Then by continuity of $\widetilde{f}$ we have $\lim_{i \to \infty} \widetilde{f}(w_i) = v$. Since $v \in \overline{D}(\mathbb{Q}_p)$ is a point at infinity, then the sequence of elements $f(w_i) = \widetilde{f}(w_i) \in D(\mathbb{Q}_p)$ is not bounded. This means that there exists i such that not all of the coordinates of the point $f(w_i)$ belong to $\mathbb{Z}_p$. Lemma 2 is proved.

The proof of Theorem 1. Let $g_1 \dots, g_s \in \Gamma$ be elements such that the functions $\tau_{g_1}, \dots, \tau_{g_s}$ generate the ring $T(\Gamma)$. Then the projection $\pi : R(\Gamma) \to X(\Gamma)$ is defined by the formula $\pi(\rho) = (\tau_{g_1}(\rho), \dots, \tau_{g_s}(\rho))$. Since by assumption of Theorem 1 we have $\dim X^s(\Gamma) \geq 2$, then there exists an irreducible component Z of the variety $X(\Gamma)$ such that $\dim Z \geq 2$ and $U = Z \cap X^s(\Gamma) \neq \varnothing$. Let $p_i : Z \to \mathbb{A}^1$ be the projection defined by the formula $p_i(z_1, \dots, z_s) = z_i$. Since $\dim Z \geq 2$, then there exists i such that the projection p_i is dominant, that is $p_i(U)$ is dense in $\mathbb{A}_1$. Therefore there exists an integer $n > 2$ such that $n \in p_i(U)$. Let $Y = p_i^{-1}(n) \subset Z$. Then by the Dimension Theorem $\dim Y \geq \dim Z - 1 \geq 1$ and $Y \cap U \neq \varnothing$. Furthermore, let X be an irreducible component of $\pi^{-1}(Y)$ such that $\pi(X))$ is dense in Y. Applying Lemma 2 to varieties X, Y and morphism π, we obtain that there exists a prime p such that $X(\mathbb{Q}_p)$ contains a representation ρ with the following property: ρ is irreducible and not all coordinates of the point $\pi(\rho)$ belong to $\mathbb{Z}_p$. The last means that there exists j such that $\tau_{g_j}(\rho) = \operatorname{tr}\rho(g_j) \notin \mathbb{Z}_p$. Hence the group $\rho(\Gamma)$ is an unbounded subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Moreover, it follows from the construction of the representation ρ that $\tau_{g_i}(\rho) = \operatorname{tr}\rho(g_i) = n > 2$. Thus, the cyclic subgroup of $\rho(\Gamma)$ generated by $\rho(h_i)$ is infinite and bounded. Hence, $\rho(\Gamma)$ is a non-discrete subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. It follows from Lemma 1 that $\rho(\Gamma)$ is dense in $\mathrm{SL}_2(\mathbb{Q}_p)$ in p-adic topology whence $\rho(\Gamma)$ (and consequently Γ) is a non-trivial free product with amalgamation. Theorem 1 is proved.

2. Some auxiliary results

In this section we shall prove several auxiliary results used in the proofs of Theorems 2 and 3. Throughout we shall denote the ring of algebraic integers in $\mathbb{C}$ by $\mathcal{O}$, the group of units in $\mathcal{O}$ by $\mathcal{O}^*$, the free group of a rank 2 with generators g and h by $F_2 =< g, h >$, the greatest common divisor of integers a and b by (a, b). If $K \supset L$ is a finite extension of fields and $x \in K$, then we shall denote the norm of x by $N_{K/L}(x)$. The following lemma characterizes elements of the finite order in $\mathrm{SL}_2(\mathbb{C})$.

Lemma 3 *Let $2 < m \in \mathbb{Z}$ and $\pm E \neq X \in \mathrm{SL}_2(\mathbb{C})$. Then $X^m = E$ if and only if $\operatorname{tr} X = \varepsilon + \varepsilon^{-1}$, where $\varepsilon^m = 1$, $\varepsilon \neq \pm 1$ (in other words, $\operatorname{tr} X = 2\cos(2r\pi/m)$ for some $r \in \{1, \dots, m-1\}$). In particular, if $\operatorname{tr} X = 0$, then $X^2 = -E$.*

Proof. If $X^m = E$, then the assertion is obvious. If $\operatorname{tr} X = \varepsilon + \varepsilon^{-1}$, then $\varepsilon, \varepsilon^{-1}$ are the eigenvalues of the matrix X. Hence, X is conjugated with the matrix $\operatorname{diag}(\varepsilon, \varepsilon^{-1})$, that is, $X^m = E$ as required.

Obviously, the representation variety $R(F_2)$ of the free group $F_2 = \langle g, h\rangle$ is equal to $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. It is known that the ring $T(F_2)$ is generated by the functions $\tau_g, \tau_h, \tau_{gh}$ (see [12, 16, 17]). For an element $u \in F_2$ the function τ_u is usually called a Fricke character of the element u.

Lemma 4 *For all $\alpha, \beta, \gamma \in \mathbb{C}$ there exist matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that $\tau_g(A, B) = \operatorname{tr} A = \alpha$, $\quad \tau_h(A, B) = trB = \beta$, $\quad \tau_{gh}(A, B) = \operatorname{tr} AB = \gamma$.*

This lemma can be easily proved by straightforward computations.

Lemma 4 implies that $X(F_2) = \pi(R(F_2)) = \mathbb{A}^3$. Moreover, the functions $\tau_g, \tau_h, \tau_{gh}$ are algebraically independent over $\mathbb{C}$ and for all $u \in F_2$ we have

$$\tau_u = Q_u(\tau_g, \tau_h, \tau_{gh}),$$

where $Q_u \in \mathbb{Z}[x, y, z]$ is a uniquely determined polynomial with integer coefficients. The polynomial Q_u is usually called the Fricke polynomial of the element u. The following relations for Fricke characters follow from the relations between the traces of arbitrary matrices in $\mathrm{SL}_2(\mathbb{C})$:

$$1)\tau_{u^{-1}} = \tau_u; \quad 2)\tau_{uv} = \tau_{vu}; \quad 3)\tau_{vuv^{-1}} = \tau_u; \quad 4)\tau_{uv} = \tau_u\tau_v - \tau_{uv^{-1}}. \tag{3}$$

Furthermore, we require the more detailed information on the Fricke polynomials (see [22]). Consider polynomials $P_n(\lambda)$ satisfying the initial conditions $P_{-1}(\lambda) = 0$, $P_0(\lambda) = 1$ and the recurrence relation

$$P_n(\lambda) = \lambda P_{n-1}(\lambda) - P_{n-2}(\lambda).$$

If $n < 0$, then we set $P_n(\lambda) = -P_{|n|-2}(\lambda)$. The degree of the polynomial $P_n(\lambda)$ is equal to n if $n > 0$ and to $|n| - 2$ if $n < 0$. It is easy to verify by induction on n that

$$P_n(2\cos(\varphi)) = \frac{\sin((n+1)\varphi)}{\sin(\varphi)}. \tag{4}$$

It follows from (4) that the polynomial $P_n(\lambda)$, $n \geq 1$, has n zeros described by the formula

$$\lambda_{n,k} = 2\cos(\tfrac{k\pi}{n+1}), \qquad k = 1, 2, \ldots, n. \tag{5}$$

Moreover, it is easy to verify by induction that for $n \geq 0$ we have

$$\begin{aligned} P_{2n}(\lambda) &= \lambda^{2n} + \cdots + (-1)^n \\ P_{2n-1}(\lambda) &= \lambda(\lambda^{2n-2} + \cdots + (-1)^{n-1}n). \end{aligned} \tag{6}$$

Furthermore, let $w = g^{\alpha_1}h^{\beta_1} \ldots g^{\alpha_s}h^{\beta_s}$ be a cyclically reduced word in F_2 and let $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$. Let us treat the Fricke polynomial $Q_w(x, y, z)$ as a polynomial in z. Let

$$Q_w(x, y, z) = M_n(x, y)z^n + M_{n-1}(x, y)z^{n-1} + \ldots + M_0(x, y)$$

Lemma 5 ([22]) *The degree of the Fricke polynomial $Q_w(x, y, z)$ with respect to z is equal to s, that is, the number of blocks of the form $g^{\alpha_i}h^{\beta_i}$ in w. The leading coefficient $M_s(x, y)$ of the polynomial $Q_w(x, y, z)$ has the following form*

$$M_s(x, y) = \prod_{i=1}^{s} P_{\alpha_i - 1}(x))P_{\beta_i - 1}(y). \tag{7}$$

The following lemma plays an important role in the proof of Theorems 2 and 3.

Lemma 6 *Let* $\Gamma = \langle a, b \mid a^n = R^m(a,b) = 1\rangle$*, where* $n = 0$ *or* $n \geq 2$*,* $m \geq 2$*,* $R(a,b)$ *is a cyclically reduced containing* b *word in the free group on* a *and* b*. Assume that there exists matrices* $A, B \in \mathrm{SL}_2(\mathbb{C})$ *such that* $\operatorname{tr} A = \alpha = 2\cos(t\pi/n)$ *for some* $t \in \{1, \ldots, n-1\}$ *and* $\operatorname{tr} R(A,B) = Q_R(\alpha, y, z) = c$*, where* Q_R *is the Fricke polynomial of an element* $R(g,h) \in F_2$*,* $c = 2\cos(r\pi/m)$ *for some* $r \in \{1, \ldots, m-1\}$*,* $y = \operatorname{tr} B$*,* $z = \operatorname{tr} AB$*. Let* $H =< A, B >$ *be the group generated by matrices* A *and* B*. Assume that the following two conditions hold:*

1) *there exists a unipotent (or finite-order) element* $W \in H$ *of the form* $W = A^{\alpha_1}B^{\beta_1}\ldots A^{\alpha_s}B^{\beta_s}$*, where* $\alpha_i, \beta_i \neq 0$ *for* $i = 1, \ldots, s$*, such that* $l = \sum_{i=1}^{s} \beta_i \neq 0$*.*

2) *There exists an element* $h \in H$ *such that* $\operatorname{tr} h \notin \mathcal{O}$*.*

Then the group Γ *is a non-trivial free product with amalgamation.*

If instead of the condition 1) the following condition holds

1′) B *has finite order, that is,* $\operatorname{tr} B = 2\cos(k_1\pi/k)$ *for some* $k \geq 2$*,* $(k_1, k) = 1$*,*

then the group $\Gamma_1 = \langle a, b \mid a^n = b^{kv} = R^m(a,b) = 1\rangle$ *is a non-trivial free product with amalgamation for any integer* v*.*

The proof of this lemma is based on the Bass' classification of finitely generated subgroups in $\mathrm{SL}_2(\mathbb{C})$ [23].

Proposition 1 ([23]) *Let* $H \subset \mathrm{GL}_2(\mathbb{C})$ *be a finitely generated subgroup. Then one of the following cases must occur:*

1) *there exists an epimorphism* $f : H \to \mathbb{Z}$ *such that* $f(u) = 0$ *for all unipotent elements* $u \in H$*;*

2) $\operatorname{tr} h \in \mathcal{O}$ *for any element* $h \in H$*;*

3) H *is a non-trivial free product with amalgamation.*

Proof of lemma 6. It is easy to see that the group H does not satisfy the conditions 1), 2) of Proposition 1. Indeed, assume that $f : H \to \mathbb{Z}$ is an epimorphism such that $f(z) = 0$ for all unipotent elements $z \in H$. Then $f(A) = 0$ because $A^{2n} = E$ by Lemma 3. Furthermore, $f(u) = lf(B) = 0$ whence $f(B) = 0$ because by assumption u is either unipotent or has finite order and $l \neq 0$. Thus, $f(H) = \{0\}$ which is a contradiction. By assumption H does not satisfy condition 2) of Proposition 1. Hence H is a non-trivial free product with amalgamation, that is, $H = H_1 *_F H_2$, where $H_1 \neq F \neq H_2$. Let $\overline{A}$, $\overline{B}$, $\overline{H}$, $\overline{H}_1$, $\overline{H}_2$, $\overline{F}$ be the images of A, B, H, H_1, H_2, F in $\mathrm{PSL}_2(\mathbb{C})$, respectively. If $-E \notin H$, then groups H and $\overline{H}$ are isomorphic. If $-E \in H$, then $-E$ belongs to the centre of H hence $-E \in F$. In all of these cases, $\overline{H}_1 \neq \overline{F} \neq \overline{H}_2$ and hence $\overline{H} = \overline{H}_1 *_{\overline{F}} \overline{H}_2$ is a non-trivial free product with amalgamation. The conditions $\operatorname{tr} A = \alpha$ and $Q_R(\alpha, y, z) = c$ imply by Lemma 3 that $A^{2n} = R^{2m}(A,B) = E$. Hence $\overline{A}^n = R^m(\overline{A}, \overline{B}) = 1$ in $\mathrm{PSL}_2(\mathbb{C})$. Thus, $\overline{H}$ is an epimorphic image of Γ therefore Γ is a non-trivial free product with amalgamation too.

Furthermore, if we replace the condition 1) to 1′), then again the group $\overline{H}$ is a non-trivial free product with amalgamation. Moreover, we have $\overline{A}^n = \overline{B}^k = R^m(\overline{A}, \overline{B}) = 1$ in $\mathrm{PSL}_2(\mathbb{C})$. Hence $\overline{H}$ is an epimorphic image of Γ_1. Thus, Γ_1 is a non-trivial free product with amalgamation. Lemma 6 is proved.

LEMMA 7 1) *Let $r, s \in \mathbb{Z}$, where $s \geq 3$ and $(r, s) = 1$. Then $\cos(r\pi/s) \notin \mathcal{O}$.*

2) *Let $s \in \mathbb{Z}$, $s \geq 1$, and $r \not\equiv 0 \,(\mathrm{mod}\ 2s+1)$. Then $2\cos(r\pi/(2s+1)) \in \mathcal{O}^*$.*

3) *Let $u \in \mathbb{Z}$, $|u| \geq 1$, and let p be a prime number. Set*

$$x_r = 2\cos(\tfrac{r\pi}{2pu}), \quad y_r = 2\sin(\tfrac{r\pi}{2pu}), \quad K_r = \mathbb{Q}(x_r), \quad L_r = \mathbb{Q}(y_r).$$

Then there exist $r, r_1 \not\equiv 0 \,(\mathrm{mod}\ p)$ such that p divides $N_{K_r/\mathbb{Q}}(x_r)$ and $N_{L_{r_1}/\mathbb{Q}}(y_{r_1})$. In particular, $x_r, y_{r_1} \notin \mathcal{O}^*$.

4) *Let $u, c \in \mathbb{Z}$, $|u| \geq 2$, $c \neq 0$, and let p be a prime number not dividing c. Set $x_0 = -2\cos(\pi/u)$, $x_r = 2\cos(r\pi/(pu))$. Then there exists $r \not\equiv 0 \,(\mathrm{mod}\ p)$ such that $c/(x_r - x_0) \notin \mathcal{O}$.*

5) *Let $p > 2$ be a prime number. Then for each $r \not\equiv 0 \,(\mathrm{mod}\ p)$ and $s \geq 1$ we have $\sin(r\pi/p^s) \notin \mathcal{O}^*$.*

6) *Let $t \geq 1$. Then for each odd r we have $2sin(r\pi/2^t) \notin \mathcal{O}^*$.*

Proof. 1) Assume that $\cos(r\pi/s) \in \mathcal{O}$. Then for each $d \in \mathbb{Z}$ we have $\cos(dr\pi/s) \in \mathcal{O}$. Since by assumption $(r, s) = 1$, then for each integer l there exists d such that $dr \equiv l$ $(\mathrm{mod}\ s)$. Hence for each integer l we have $\cos(l\pi/s) \in \mathcal{O}$. By (5) the polynomial $P_{s-1}(\lambda)$ has the roots $2\cos(l\pi/s)$, $l = 1, \ldots, s-1$. Then the polynomial $P_{s-1}(2\lambda)$ has the roots $\cos(l\pi/s)$, $l = 1, \ldots, s-1$. If $s = 2s_1 + 1$ is odd, then by (6) we have $P_{2s_1}(2\lambda) = 2^{2s_1}\lambda^{2s_1} + \cdots + (-1)^{s_1}$. Since $1/2^{2s_1} \notin \mathbb{Z}$, then $P_{2s_1}(2\lambda)$ has a root not belonging to $\mathcal{O}$, that is, there exists l such that $\cos(l\pi/s) \notin \mathcal{O}$. If $s = 2s_1$ is even, then (6) implies that $P_{2s_1-1}(2\lambda) = 2\lambda(2^{2s_1-2}\lambda^{2s_1-2} + \cdots + (-1)^{s_1-1}s_1)$. By assumption $s \geq 3$ hence $s_1 \geq 2$. Then $s_1/2^{2s_1-2} \notin \mathbb{Z}$ and $P_{2s_1-1}(2\lambda)$ has a root not belonging to $\mathcal{O}$.

2) By (5), (6) the number $2\cos(r\pi/(2s+1))$ is a root of the polynomial $P_{2s}(\lambda) = \lambda^{2s} + \cdots + (-1)^s$ and therefore belongs to $\mathcal{O}^*$.

3) Since $y_r = 2\cos((pu - r)\pi/(2pu)) = x_{pu-r}$, then it is sufficient to prove the assertion for x_r. Let $u = p^f u'$, where $f \geq 0$, $p \nmid u'$, and let $r = r_1 u'$, where $p \nmid r_1$. Then $x_r = 2\cos(r_1\pi/(2p^{f+1}))$. By (5), (6) the polynomial

$$P_{2p^{f+1}-1}(\lambda) = \lambda(\lambda^{2p^{f+1}-2} + \cdots + (-1)^{p^{f+1}-1}p^{f+1})$$

has the roots $2\cos(r'\pi/(2p^{f+1}))$, $r' = 1, \ldots, 2p^{f+1} - 1$, and the polynomial

$$P_{2p^f-1}(\lambda) = \lambda(\lambda^{2p^f-2} + \cdots + (-1)^{p^f-1}p^f)$$

has the roots $2\cos(r'\pi/(2p^f))$, $r' = 1, \ldots, 2p^f - 1$. Hence the polynomial $P_{2p^f-1}(\lambda)$ divides the polynomial $P_{2p^{f+1}-1}(\lambda)$, that is,

$$P_{2p^{f+1}-1}(\lambda) = P_{2p^f-1}(\lambda)F(\lambda), \tag{8}$$

where, as it is easy to see, $F(\lambda)$ is the polynomial of degree $2(p^{f+1} - p^f)$ with the leading coefficient 1 and the constant term p. The roots of $F(\lambda)$ are the numbers $2\cos(r'\pi/(2p^{f+1}))$, $r' \not\equiv 0 \,(\mathrm{mod}\ p)$. Let K be the splitting field of $F(\lambda)$. Obviously, there exists $r' \not\equiv 0 \,(\mathrm{mod}\ p)$ such that $N_{K/\mathbb{Q}}(2\cos(r'\pi/(2p^{f+1}))) = \pm p$ as required.

4) Note that

$$x_r - x_0 = 2\cos(\tfrac{r\pi}{pu}) + 2\cos(\tfrac{\pi}{u}) = \left(2\cos(\tfrac{(r+p)\pi}{2pu})\right)\left(2\cos(\tfrac{(r-p)\pi}{2pu})\right).$$

Therefore it is sufficient to show that for some $r \not\equiv 0 \,(\mathrm{mod}\ p)$ we have $c/\alpha_r \notin \mathcal{O}$, where $\alpha_r = 2\cos((r+p)\pi/(2pu))$. Let $K_r = \mathbb{Q}(\alpha_r)$. By item 3) proven above, there exists $r \not\equiv 0$ $(\mathrm{mod}\ p)$ such that p divides $N_{K_r/\mathbb{Q}}(\alpha_r)$. Then

$$N_{K_r/\mathbb{Q}}\left(\frac{c}{\alpha_r}\right) = \frac{c}{N_{K_r/\mathbb{Q}}(\alpha_r)} \notin \mathbb{Z}$$

because by assumption $p \nmid c$. Hence $c/\alpha_r \notin \mathcal{O}$, as required.

5) Note that $1/\sin(r\pi/p^s) = 2/(2\cos((p^s - 2r)\pi/(2p^s)))$. It follows from the proof of item 4) that there exists $r_0 \not\equiv 0 \,(\mathrm{mod}\ p)$ such that

$$\frac{2}{2\cos((p^s - 2r_0)\pi/(2p^s))} \notin \mathcal{O}.$$

Now we shall show that for every $r \not\equiv 0 \,(\mathrm{mod}\ p)$ we have $\sin(r\pi/p^s) \notin \mathcal{O}$. Assume the contrary. Let for some r, $(r,p) = 1$, we have $1/\sin(r\pi/p^s) \in \mathcal{O}$. Since $p \nmid (p^s - 2r_0)$, then there exists d such that $r \equiv d(p^s - 2r_0) \,(\mathrm{mod}\ p^s)$. Then by (4) we have

$$P_d\left(2\cos(\tfrac{(p^s-2r_0)\pi}{p^s})\right) = \frac{\sin(d(p^s - 2r_0)\pi/p^s)}{\sin((p^s - 2r_0)\pi/p^s)} = \pm\frac{\sin(r\pi/p^s)}{\sin((p^s - 2r_0)\pi/p^s)}.$$

So we immediately obtain that

$$\frac{1}{\sin((p^s - 2r_0)\pi/p^s)} = \pm\frac{1}{\sin(r\pi/p^s)}P_d\left(2\cos(\tfrac{(p^s-2r_0)\pi}{p^s})\right) \in \mathcal{O}$$

which is a contradiction.

6) For $t = 1$ the assertion is obvious. Suppose that $t > 1$. By item 3) there exists an odd r_0 such that $2\sin(r_0\pi/2^t) \notin \mathcal{O}^*$. Prove that for each odd r we have $2\sin(r\pi/2^t) \notin \mathcal{O}^*$. Assume the contrary. Let for some odd r we have $2\sin(r\pi/2^t) \in \mathcal{O}^*$. Obviously, there exists an integer d such that $r \equiv dr_0 \,(\mathrm{mod}\ 2^t)$. Then by (4) we have

$$P_d\left(2\cos(\tfrac{r_0\pi}{2^t})\right) = \pm\frac{2\sin(r\pi/2^t)}{2\sin(r_0\pi/2^t)}.$$

The last equality implies that $2\sin(r_0\pi/2^t) \in \mathcal{O}^*$ which is a contradiction. Lemma 7 is proved.

LEMMA 8 1) *Let $s, t \geq 0$. Then*

$$P_s(\lambda)P_t(\lambda) = \sum_{i=0}^{t} P_{s-t+2i}(\lambda). \tag{9}$$

2) *The polynomial $P_s(\lambda) - P_{s-1}(\lambda)$ has the roots $\lambda_r = 2\cos((2r+1)\pi/(2s+1))$, $r \in \{0, 1, \dots, s-1\}$.*

3) *If $\gamma = 2\cos(2r\pi/(2s+1))$, where $s \geq 1$, $r \in \{1 \dots, s\}$, $(r, 2s+1) = 1$, then $P_s(\gamma) - P_{s-1}(\gamma) \notin \mathcal{O}^*$.*

4) *If $\gamma = 2\cos((2r+1)\pi/(2s))$, where $s \geq 2$, $(s, 2r+1) = 1$, then $0 \neq P_{s-1}(\gamma) \notin \mathcal{O}^*$.*

5) Let $\gamma \in \mathcal{O}$. Assume that γ is not equal to $2\cos(r\pi/s)$, where $r, s \in \mathbb{Z}$. Then there exists an integer $l > 0$ such that $P_l(\gamma) \notin \mathcal{O}^*$.

Proof. 1) We fix s and proceed the induction on t. If $t = 0$, then $P_s(\lambda)P_0(\lambda) = P_s(\lambda)$. If $t = 1$, then $P_s(\lambda)P_1(\lambda) = P_s(\lambda)\lambda = P_{s+1}(\lambda) + P_{s-1}(\lambda)$ by definition. Furthermore, we have by induction

$$\begin{aligned} P_s(\lambda)P_t(\lambda) &= P_s(\lambda)(\lambda P_{t-1}(\lambda) - P_{t-2}(\lambda)) = \lambda \sum_{i=0}^{t-1} P_{s-t+1+2i}(\lambda) - \sum_{i=0}^{t-2} P_{s-t+2+2i}(\lambda) \\ &= \sum_{i=0}^{t-1}(P_{s-t+2+2i}(\lambda) + P_{s-t+2i}(\lambda)) - \sum_{i=0}^{t-2} P_{s-t+2+2i}(\lambda) \\ &= P_{s+t}(\lambda) + \sum_{i=0}^{t-1} P_{s-t+2i}(\lambda) = \sum_{i=0}^{t} P_{s-t+2i}(\lambda) \end{aligned}$$

as required.

2) Bearing in mind (4), we obtain

$$P_s(\lambda_r) - P_{s-1}(\lambda_r) = \frac{\sin((2r+1)(s+1)\pi/(2s+1)) - \sin((2r+1)s\pi/(2s+1))}{\sin((2r+1)\pi/(2s+1))} = 0.$$

3) Taking into account (4), we have

$$\frac{1}{P_{s+1}(\gamma) - P_s(\gamma)} = \frac{\sin(2r\pi/(2s+1))}{2\sin(r\pi/(2s+1))\cos(r\pi)} = \pm\cos(\tfrac{r\pi}{2s+1}) \notin \mathcal{O}$$

by item 1) of Lemma 7.

4) Using (4), we obtain

$$\frac{1}{P_{s-1}(\gamma)} = \frac{\sin((2r+1)\pi/(2s)}{\sin((2r+1)\pi/2)} = (-1)^r \cos\left(\tfrac{(s-2r-1)\pi}{2s}\right) \notin \mathcal{O}$$

by item 1) of Lemma 7.

5) Since by (5) the polynomial $P_l(\lambda)$ has the roots $2\cos(r\pi/(l+1))$, $r = 1, \dots, l$, then one can write $P_l(\gamma) = \prod_{r=1}^{l}(\gamma - 2\cos(r\pi/(l+1)))$. Hence it is sufficient to prove that $\gamma - (\varepsilon + \varepsilon^{-1}) \notin \mathcal{O}^*$, where $\varepsilon \neq \pm 1$ is some root of 1. Let $f(\lambda)$ be the irreducible polynomial for γ over $\mathbb{Q}$, let K_0 be the splitting field of $f(\lambda)$ and let $K_1 = K_0(x_0)$, where x_0 is a root of the equation $x + x^{-1} = \gamma$. Let Z_1 be the integer closure of $\mathbb{Z}$ in K_1 and let $p \neq 2$ be some prime number. Let $\mathfrak{p}_1$ be some prime ideal in Z_1 lying above (p). Then $k_1 = Z_1/\mathfrak{p}_1 \supset \mathbb{Z}/p\mathbb{Z} = k$ is the finite extension of fields. We have $x_0, y_0 \in Z_1$. Let $\overline{x}_0, \overline{\gamma}$ be the images of x_0, γ in the field k_1, respectively. Then the equality holds

$$\overline{x}_0 + \overline{x}_0^{-1} = \overline{\gamma}.$$

Let $l = |k_1^*|$ be the order of the multiplicative group of the field k_1. Then $\overline{x}_0^l = 1$ in k_1. Let $K_2 = K_1(\xi)$, where ξ is a primitive root of 1 of degree l in $\mathbb{C}$. Let Z_2 be the integer closure of Z_1 in K_2, let $\mathfrak{p}_2$ be some prime ideal of Z_2 lying above $\mathfrak{p}_1$, $k_2 = Z_2/\mathfrak{p}_2 \supset k_1$. Let Δ be the group of roots of 1 of the degree l in K_2 and let $\overline{\Delta}$ be its image in k_2. Prove that $\overline{\Delta} = k_1^*$. Assume the contrary, that is, $\overline{\Delta} \neq k_1^*$. Then for some integer r, $0 < r < l$, we have $\overline{\xi}^r = 1$, where $\overline{\xi}$ is the image of ξ in k_2. Then $(1+y)^l = 1$, that is, $1 + C_l^1 y + \dots + C_l^l y^l = 1$, where C_l^i is the corresponding binomial coefficient. Hence $y(l + yy_1) = 0$, where $y_1 = C_l^2 y + \dots + C_l^l y^{l-1}$. Since $y \neq 0$, then we obtain that $l \in \mathfrak{p}_2 \cap \mathbb{Z} = (p)$. But $l = |k_1^*| = p^t - 1$ for some t which is a contradiction. So there exists a root ε of 1 of degree l such that $\overline{\varepsilon} = \overline{x}_0$. This means that $\gamma - (\varepsilon + \varepsilon^{-1}) \in \mathfrak{p}_2$ and hence $\gamma - (\varepsilon + \varepsilon^{-1})$ is not a unit in the ring $\mathcal{O}$. Lemma 8 is proved.

LEMMA 9 *Let $F_2 = \langle g, h \rangle$ be the free group with generators g and h. Set $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$, $t = \tau_{ghg^{-1}h^{-1}}$. Then the following assertions hold.*

1) *$t = x^2 + y^2 + z^2 - xyz - 2$.*

2) *Let $R = gh(ghg^{-1}h^{-1})^s$. Then*

$$\tau_R = (P_s(t) - P_{s-1}(t))z.$$

3) *Let $T = (gh)^{-1}(ghg^{-1}h^{-1})^s(gh)^2(ghg^{-1}h^{-1})^s$. Then*

$$\tau_T = (t-2)P_{s-1}(t)^2 z^3 + (2 - P_{2s-1}(t) + P_{2s-2}(t))z.$$

Proof. 1) One can prove the equality by straightforward computations using relations (3) (see [16]).

2) Let u, v be arbitrary elements of F_2. Then using induction and relations (3), it is easy to show that for arbitrary integers p and q the following equality is valid

$$\tau_{u^p v^q} = P_{p-1}(\tau_u)P_{q-1}(\tau_v)\tau_{uv} - P_{p-2}(\tau_u)P_q(\tau_v) - P_p(\tau_u)P_{q-2}(\tau_v). \quad (10)$$

Set now $u = gh$, $v = ghg^{-1}h^{-1}$. Then $\tau_u = z$, $\tau_v = t$, $\tau_{uv} = \tau_{gh(ghg^{-1}h^{-1})} = zt - \tau_{g^{-1}h^{-1}} = z(t-1)$. Hence,

$$\begin{aligned}
\tau_{uv^s} &= P_{s-1}(\tau_v)\tau_{uv} - P_{s-2}(\tau_v)\tau_u = P_{s-1}(t)(t-1)z - P_{s-2}(t)z \\
&= z(tP_{s-1}(t) - P_{s-1}(t) - P_{s-2}(t)) = z(P_s(t) + P_{s-2}(t) - P_{s-1}(t) - P_{s-2}(t)) \\
&= z(P_s(t) - P_{s-1}(t)).
\end{aligned}$$

3) Let u and v be such as above. Then using relations (3), (10), we have

$$\begin{aligned}
\tau_{u^{-1}v^s} &= \tau_{u^{-1}}\tau_{v^s} - \tau_{uv^s} = z(P_s(t) - P_{s-2}(t)) - z(P_s(t) - P_{s-1}(t)) \\
&= z(P_{s-1}(t) - P_{s-2}(t)); \\
\tau_{u^2v^s} &= \tau_u\tau_{uv^s} - \tau_{v^s} = z^2(P_s(t) - P_{s-1}(t)) - P_s(t) + P_{s-2}(t); \\
\tau_{u^3} &= z^3 - 3z.
\end{aligned}$$

Hence,

$$\begin{aligned}
\tau_{u^{-1}v^su^2v^s} &= \tau_{u^{-1}v^s}\tau_{u^2v^s} - \tau_{u^3} = z^3((P_s(t) - P_{s-1}(t))(P_{s-1}(t) - P_{s-2}(t)) - 1) \\
&\quad + z(3 - (P_{s-1}(t) - P_{s-2}(t))(P_s(t) - P_{s-2}(t))).
\end{aligned}$$

Using (9), we simplify the last expression. First consider the coefficient by z^3.

$$\begin{aligned}
&(P_s(t) - P_{s-1}(t))(P_{s-1}(t) - P_{s-2}(t)) - 1 = P_s(t)P_{s-1}(t) + P_{s-1}(t)P_{s-2}(t) \\
&- P_s(t)P_{s-2}(t) - P_{s-1}(t)^2 - 1 = P_{s-1}(t)(P_s(t) + P_{s-2}(t)) - \sum_{i=1}^{s-1} P_{2i}(t) - P_0(t) \\
&- P_{s-1}(t)^2 = tP_{s-1}(t)^2 - 2P_{s-1}(t)^2 = (t-2)P_{s-1}(t)^2.
\end{aligned}$$

Now consider the coefficient by z.

$$\begin{aligned}
&3 - (P_{s-1}(t) - P_{s-2}(t))(P_s(t) - P_{s-2}(t)) = 3 - P_s(t)P_{s-1}(t) + P_{s-1}(t)P_{s-2}(t) \\
&+ P_s(t)P_{s-2}(t) - P_{s-2}(t)^2 = 3 - \sum_{i=1}^{s} P_{2i-1}(t) + \sum_{i=1}^{s-1} P_{2i-1}(t) + \sum_{i=1}^{s-1} P_{2i}(t) - \sum_{i=0}^{s-2} P_{2i}(t) \\
&= 2 - P_{2s-1}(t) + P_{2s-2}(t).
\end{aligned}$$

Lemma 9 is proved.

At the end of Section 2 we shall show how one can deduce Corollary 1 from Theorem 1. This gives us another proof of Conjecture 1. Let $\Gamma = \langle a, b \mid R^m(a, b) = 1 \rangle$, where $m \geq 2$, $R(a, b) = a^{u_1}b^{v_1} \dots a^{u_s}b^{v_s}$, $u_i, v_i \neq 0$, $s \geq 1$ and $R(a, b)$ is not a proper power.

First consider the case $m \geq 3$. Prove that $\dim X_2^s(\Gamma) \geq 2$. Then Theorem 1 immediately implies that Γ is a non-trivial free product with amalgamation. Consider in the character variety $X_2(F_2) = \mathbb{A}^3$ of the free group $F_2 = \langle g, h\rangle$ the hypersurface V defined by the equation

$$\tau_{R(g,h)}(x, y, z) = 2\cos(\tfrac{2\pi}{m}), \tag{11}$$

where $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$. By Lemma 5 one can write the equation (11) in the form

$$f(x, y, z) = M_s(x, y)z^s + \cdots + M_0(x, y) - 2\cos(\tfrac{2\pi}{m}) = 0. \tag{12}$$

We claim that $V \subset X_2(\Gamma)$. Indeed, let $v = (x_0, y_0, z_0) \in V$ and let $A, B \in \mathrm{SL}_2(\mathbb{C})$ be matrices such that $\operatorname{tr} A = x_0$, $\operatorname{tr} B = y_0$, $\operatorname{tr} AB = z_0$. Then by Lemma 3 we have $R^m(A, B) = E$. Hence the pair of matrices (A, B) defines a representation ρ of the group Γ into $\mathrm{SL}_2(\mathbb{C})$. Moreover, the image of ρ in $X_2(\Gamma)$ coincides with v. Hence, $v \in X_2(\Gamma)$. Furthermore, let $V_1, \dots, V_r$ be the irreducible components of V. It is easy to see (see [21]) that $\dim V_i = 2$ for each i. It remains to show that $V \cap X_2^s(\Gamma) \neq \varnothing$. Let us assume the contrary. Then all representations corresponding to points from V are reducible. This means that the regular function $\tau_{ghg^{-1}h^{-1}} - 2$ is identically equal to 0 on V. Hence by item 1 of Lemma 9 we have

$$g(x, y, z) = x^2 + y^2 + z^2 - xyz - 4 \equiv 0$$

on V. Thus,

$$f(x, y, z) = g(x, y, z)^d \tag{13}$$

for some $d \geq 1$. If for some i we have $|u_i| \geq 2$ or $|v_i| \geq 2$, then by Lemma 5 the leading coefficient $M_s(x, y)$ in (12) is not a constant and the equality (13) is impossible.

Let now $|u_i| = |v_i| = 1$ for $i = 1, \dots, s$. First, if for some i we have $u_i = u_{i+1}$ or $v_i = v_{i+1}$ ($u_1 = u_s$ or $v_1 = v_s$ for $i = s$), then we can consider new generators of the group Γ. Assume for definiteness that $u_1 = u_2$. Set $a_1 = a^{u_1}b^{v_1}$, $b_1 = b$. Then $\Gamma = \langle a_1, b_1 \mid R_1^m(a_1, b_1) = 1\rangle$, where $R_1^m(a_1, b_1) = a_1^{u_1'}b_1^{v_1'} \dots a_1^{u_r'}b_1^{v_r'}$, $u_i', v_i' \neq 0$, $r \geq 1$ and $u_1' \geq 2$. This case has been considered above.

So we can assume that $u_{i+1} = -u_i$, $v_{i+1} = -v_i$. Since by assumption $R(a, b)$ is not a proper power, then we have for $R(a, b)$ the unique possibility up to cyclic rearrangement $R(a, b) = aba^{-1}b^{-1}$. Hence

$$f(x, y, z) = x^2 + y^2 + z^2 - xyz - 2 - 2\cos(\tfrac{2\pi}{m}) = g(x, y, z) + 2 - 2\cos(\tfrac{2\pi}{m}).$$

Since $2 - 2\cos(\frac{2\pi}{m}) \neq 0$, then it is obvious that in this case $g(x, y, z)$ has no zeros on V. So that for $m \geq 3$ we have proved that the group Γ is a non-trivial free product with amalgamation.

Let now $m = 2$. In this case one can consider a group $\Gamma_1 = \langle a, b \mid R^4(a, b) = 1\rangle$. We have proved above that $\dim X_2^s(\Gamma_1) \geq 2$. Then the proof of Theorem 1 implies that there exists a representation $\rho : \Gamma_1 \to \mathrm{SL}_2(\mathbb{Q}_p)$ for some prime p such that $\rho(\Gamma_1)$

is dense in $\mathrm{SL}_2(\mathbb{Q}_p)$ in p-adic topology. Hence $\rho(\Gamma_1)$ is a non-trivial free product with amalgamation. Let G be the image of $\rho(\Gamma_1)$ in $\mathrm{PSL}_2(\mathbb{Q}_p)$. Then it is easy to see that G is a non-trivial free product with amalgamation too. But G is an epimorphic image of Γ whence Γ is a non-trivial free product with amalgamation.

3. The proof of Theorem 2

1) Let $\Gamma_n = \langle a, b \mid a^n = b^k = R^2(a, b) = 1\rangle$ and let $F_2 = \langle g, h\rangle$ be the free group with generators g and h. Set $x = \tau_g$, $\beta = \tau_h = 2\cos(\pi/k)$, $z = \tau_{gh}$. Consider the equation

$$Q_{R(g,h)}(x, \beta, z) = 0, \tag{14}$$

where $Q_{R(g,h)}$ is the Fricke polynomial of the element $R(g, h) \in F_2$. By Lemma 5 we can write (14) in the form

$$A_0(x)z^s + \cdots + A_s(x) = 0, \tag{15}$$

where $A_0(x) = \prod_{i=1}^{s} P_{u_i-1}(x)P_{v_i-1}(\beta)$. Since by assumption there exists i such that $|u_i| \geq 2$, then $\deg P_{u_i-1}(x) \geq 1$. Let $x_0 = -2\cos(\pi/u_i)$ be one of the roots of $P_{u_i-1}(x)$. Then $x - x_0$ divides $A_0(x)$. Let $A_0(x) = (x - x_0)B_0(x)$, where $B_0(x) \in \mathcal{O}[x]$. Write (15) in the form

$$(x - x_0)B_0(x)z^s + \cdots + A_s(x) = 0. \tag{16}$$

First we assume that all polynomials $A_1(x), \ldots, A_s(x)$ are divided on $x - x_0$. Then one can write (16) in the form

$$(x - x_0)f(x, z) = 0, \tag{17}$$

where $f(x, z)$ is some polynomial of x, z. Let z_0 be an arbitrary non-integer element of $\mathbb{C}$, that is, $z_0 \notin \mathcal{O}$, and let $A, B \in \mathrm{SL}_2(\mathbb{C})$ be matrices such that $\mathrm{tr}\, A = x_0$, $\mathrm{tr}\, B = \beta$, $\mathrm{tr}\, AB = z_0$. By construction the pair of matrices (A, B) defines a representation of the group Γ_n into $\mathrm{PSL}_2(\mathbb{C})$. Applying Lemma 6, we obtain that Γ_n is a non-trivial free product with amalgamation.

Assume now that not all polynomials $A_1(x), \ldots, A_s(x)$ are multiples of $x - x_0$. Let, for example, $A_1(x)$ is not a multiple of $x - x_0$ and let $0 \neq \delta = A_1(x_0) \in \mathcal{O}$ be the residue of $A_1(x)$ modulo $x - x_0$. Let $c = N_{\mathbb{Q}(\delta)/\mathbb{Q}}(\delta)$. For a finite set of prime numbers S from assertion of the theorem we take $S = \{p \in \mathbb{Z} \mid p \text{ divides } c\}$. Assume that $n = u_i p f$ for some integer f and prime $p \notin S$ such that $u_i p \nmid u_j$ for $j \neq i$. Let $x_r = 2\cos(r\pi/(pu_i))$ for some $r \not\equiv 0 \pmod p$, $K_r = \mathbb{Q}(\delta, x_r - x_0)$. By item 3) of Lemma 7 one can choose r such that p divides $N_{K_r/\mathbb{Q}}(x_r - x_0)$. Moreover, by construction p does not divide c. Then $N_{K_r/\mathbb{Q}}(\delta/(x_r - x_0)) \notin \mathbb{Z}$ hence $\delta/(x_r - x_0) \notin \mathcal{O}$. Thus, we have

$$\frac{A_1(x_r)}{x_r - x_0} \notin \mathcal{O}.$$

Furthermore, since $p \nmid r$ and $pu_i \nmid u_j$ for each $j \neq i$, then $B_0(x_r) \neq 0$. Set now $x = x_r$ and write the equation (16) in the form

$$z^s + \tfrac{A_1(x_r)}{(x_r - x_0)B_0(x_r)} z^{s-1} + \cdots + \tfrac{A_s(x_r)}{(x_r - x_0)B_0(x_r)} = 0 \tag{18}$$

Clearly, we have $A_1(x_r)/((x_r - x_0)B_0(x_r)) \notin \mathcal{O}$ because $B_0(x_r) \in \mathcal{O}$. Hence the equation (18) has a root $z_0 \notin \mathcal{O}$. Consider matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr} A = x_r, \qquad \operatorname{tr} B = \beta, \qquad \operatorname{tr} AB = z_0.$$

By construction the pair of matrices (A, B) defines a representation of the group Γ_n into $\mathrm{PSL}_2(\mathbb{C})$. Applying Lemma 6, we obtain that the group Γ_n is a non-trivial free product with amalgamation.

2) We keep notations of item 1). Consider the equation

$$Q_{R(g,h)}(x, \beta, z) = \gamma_t, \tag{19}$$

where $\gamma_t = 2\cos(t\pi/m)$, $m \nmid t$. By Lemma 5 one can write (19) in the form

$$(x - x_0)B_0(x)z^s + \cdots + A_s(x) - \gamma_t = 0. \tag{20}$$

Let $x_r = 2\cos(r\pi/(pu_i))$, where $r \not\equiv 0 \,(\mathrm{mod}\ p)$. Prove that there exist t and r such that $(A_s(x_r) - \gamma_t)/(x_r - x_0) \notin \mathcal{O}$. Indeed, let us assume the contrary.

First, consider the case $m = 3$. Then $\gamma_1 = 1$, $\gamma_2 = -1$. Since both of the numbers $(A_s(x_r) - 1)/(x_r - x_0)$ and $(A_s(x_r) + 1)/(x_r - x_0)$ belong to $\mathcal{O}$, then their difference $2/(x_r - x_0) \in \mathcal{O}$ for each $r \not\equiv 0 \,(\mathrm{mod}\ p)$. Since by assumption $p \neq 2$, then we obtain a contradiction to item 4) of Lemma 7.

Now let $m = 2^l$. Then $\gamma_{2^{l-1}} = 0$, $\gamma_{2^{l-2}} = \sqrt{2}$. Since both of the numbers $A_s(x_r)/(x_r - x_0)$ and $(A_s(x_r) - \sqrt{2})/(x_r - x_0)$ belong to $\mathcal{O}$, then their difference $\sqrt{2}/(x_r - x_0) \in \mathcal{O}$ and therefore $2/(x_r - x_0) \in \mathcal{O}$. Again we obtain a contradiction to item 4) of Lemma 7.

So let us choose t and r such that $(A_s(x_r) - \gamma_t)/(x_r - x_0) \notin \mathcal{O}$. Since $p \nmid r$ and $pu_i \nmid u_j$ for each $j \neq i$, then $B_0(x_r) \neq 0$. Set $x = x_r$ and write (20) in the form:

$$z^s + \cdots + \tfrac{A_s(x_r) - \gamma_t}{(x_r - x_0)B_0(x_r)} = 0. \tag{21}$$

By construction $(A_s(x_r) - \gamma_t))/((x_r - x_0)B_0(x_r)) \notin \mathcal{O}$ hence (21) has a root $z_0 \notin \mathcal{O}$. Consider matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr} A = x_r, \qquad \operatorname{tr} B = \beta, \qquad \operatorname{tr} AB = z_0.$$

By construction the pair of matrices (A, B) defines a representation of the group Γ_n into $\mathrm{PSL}_2(\mathbb{C})$. Applying Lemma 6, we obtain that the group Γ_n is a non-trivial free product with amalgamation.

3) Let $m > 3$ and $m \neq 2^l$. We keep notations of item 2). Show that there exist $t \not\equiv 0 \pmod m$ and $r \not\equiv 0 \pmod p$ such that $(A_s(x_r) - \gamma_t)/(x_r - x_0) \notin \mathcal{O}$. Indeed, let us assume the contrary. Let for each $t \not\equiv 0 \pmod m$ and each $r \not\equiv 0 \pmod p$ we have $(A_s(x_r) - \gamma_t)/(x_r - x_0) \in \mathcal{O}$.

First, consider the case, where m is odd. Then $1 + \sum_{i=1}^{2g} \gamma_{2i} = 0$ as the sum of all roots of 1 of degree m. Note that $-\gamma_i = \gamma_{m-i}$. Set $C_i = (A_s(x_r) - (-1)^i \gamma_{2i})/(x_r - x_0)$. Then we have

$$\sum_{i=1}^{m-1} (-1)^i C_i = -\sum_{i=1}^{m-1} \frac{\gamma_{2i}}{x_r - x_0} = \frac{1}{x_r - x_0} \in \mathcal{O}$$

for each $r \not\equiv 0 \pmod p$ which contradicts to item 4) of Lemma 7.

Now let $m = m_1 2^g$, where $g \geq 1$, $m_1 > 1$ is odd. Consider the numbers $\gamma_{i2^g} = 2\cos(i\pi/m_1)$. Arguing just as in the case of odd m above, we obtain a contradiction to item 4) of Lemma 7.

So let us choose t and r such that $(A_s(x_r) - \gamma_t)/(x_r - x_0) \notin \mathcal{O}$. Then the constant term in the equation (21) does not belong to $\mathcal{O}$ and (21) has a root $z_0 \notin \mathcal{O}$. Let $A, B \in \mathrm{SL}_2(\mathbb{C})$ be matrices such that

$$\operatorname{tr} A = x_r, \qquad \operatorname{tr} B = \beta, \qquad \operatorname{tr} AB = z_0.$$

By construction the pair of matrices (A, B) defines a representation of the group Γ_n into $\mathrm{PSL}_2(\mathbb{C})$. Applying Lemma 6, we complete the proof of Theorem 2.

Remark. In several cases one can obtain the more precise information when a generalized triangle group Γ is a non-trivial free product with amalgamation. For example, consider the following group $\Gamma_k = \langle a, b \mid a^2 = b^k = (ab^2)^3 = 1\rangle$. Then Theorem 2 implies that if $k = 2k_1$, where $1 < k_1 \neq 2^l$, then Γ_k is a non-trivial free product with amalgamation. However it is easy to see that for $k_1 = 2^l$, $l \geq 1$, the group Γ_k is a non-trivial free product with amalgamation too. Indeed, let $F_2 = \langle g, h\rangle$ be the free group, let $x = \tau_g = 0$, $y = \tau_h$, $z = \tau_{gh}$. Consider the equation

$$Q_{gh^2}(0, y, z) = yz = 2\cos(\pi/3) = 1.$$

Let $y = y_r = 2\cos(r\pi/2^{l+1})$. Then by item 6) of Lemma 7 for each odd r, we have that $z_r = 1/y_r \notin \mathcal{O}$. Now Lemma 6 implies that Γ_k is a non-trivial free product with amalgamation.

4. Proof of Theorem 3

First, assume that for a word $R(a, b) = a^{u_1} b^{v_1} \ldots A^{u_s} b^{v_s}$, we have $v = \max_{1 \leq i \leq s} |v_i| \geq 2$. Then by Theorem 2 there exists a prime p such that the group $\Gamma_1 = \langle a, b \mid a^n = b^{pv} =$

$R^m(a,b)=1\rangle$ is a non-trivial free product with amalgamation. Since Γ_1 is an epimorphic image of Γ, then Γ is a non-trivial free product with amalgamation too.

Thus one can assume that

$$R(a,b)=a^{u_1}b^{v_1}\dots a^{u_s}b^{v_s},$$

where $v_i\in\{-1,1\}$, $i=1,\dots,s$. Suppose for a moment that there exists i, $1\le i\le s$, such that either $v_i=v_{i+1}$ or $v_1=v_s$. For definiteness, let $v_1=v_2$. In this case one can consider the new generators of the group Γ: $a_1=a$, $b_1=a^{u_2}b^{v_1}$. Then it is easy to check that $\Gamma=\langle a_1,b_1\mid a_1^n=R_1^m(a_1,b_1)=1\rangle$, where $R_1(a_1,b_1)=a_1^{u_1'}b_1^{v_1'}\dots A_1^{u_l'}b_1^{v_l'}$, $0<u_i'<n$, $v_i'\ne 0$ for $i=1,\dots,l$. In addition, we have $v'=\max_{1\le i\le l}|v_i'|\ge 2$. But we have proved just above that in this case the group Γ is a non-trivial free product with amalgamation.

Thus without loss of generality we can assume that

$$R(a,b)=a^{u_1}ba^{u_2}b^{-1}\dots a^{u_{2k-1}}ba^{u_{2k}}b^{-1},$$

where $k\ge 1$, $0<u_i<n$ for $i=1,\dots,2k$. Set $c=ba^{-1}b^{-1}$. Then

$$R(a,b)=a^{u_1}c^{-u_2}\dots a^{u_{2k-1}}c^{-u_{2k}}=R_1(a,c).$$

Let $F_2=\langle g,h\rangle$ be the free group of rank 2, $f=hg^{-1}h^{-1}$. Set $x=\tau_g$, $y=\tau_h$, $z=\tau_{gh}$, $t=\tau_{gf}$. Then $\tau_f=\tau_g=x$, $t=\tau_{gf}=\tau_{ghg^{-1}h^{-1}}=x^2+y^2+z^2-xyz-2$ by item 1 of Lemma 9. Consider the element $R_1(g,f)\in F_2$ as a word of g and f. Let $q(x,t)$ be the Fricke polynomial of $R_1(g,f)$, that is,

$$Q(x,t)=Q_{R_1(g,f)}(\tau_g,\tau_f,\tau_{gf})=Q_{R_1(g,f)}(x,x,t).$$

Since $R_1(g,f)$ contains k blocks of the form $g^{u_j}f^{-u_{j+1}}$, then by Lemma 5 the degree of the polynomial $q(x,t)$ with respect to t is equal to k and the leading coefficient of $q(x,t)$ is equal to $(-1)^k\prod_{i=1}^{2k}P_{u_i-1}(x)$. Since by construction $R(g,h)=R_1(g,f)$, then

$$Q_{R(g,h)}(x,y,z)=q(x,t)=q(x,x^2+y^2+z^2-xyz-2). \tag{22}$$

Set now $x=\tau_g=\alpha_r=2\cos(r\pi/n)$, $\gamma_l=2\cos(l\pi/m)$, where $r\not\equiv 0 \pmod{n}$, $l\not\equiv 0 \pmod{m}$, and consider the equation

$$Q_{R(g,h)}(\alpha_r,y,z)=\gamma_l. \tag{23}$$

By (22) one can write the equation (23) in the form

$$q(\alpha_r,t)=\gamma_l. \tag{24}$$

Lemma 10 *There exist $r, l \in \mathbb{Z}$, where $r \not\equiv 0 (\operatorname{mod} n)$ and $l \not\equiv 0 (\operatorname{mod} m)$, such that $P_{u_i-1}(\alpha_r) \neq 0$ for $i = 1, \dots, 2k$ and the equation (24) has a root $t = t_0 \neq 2$.*

Proof. First let $m \geq 3$. In this case $\gamma_1 \neq \gamma_2$. Set $r = 1$. Then the degree of the polynomial $q(\alpha_1, t)$ is equal to k. It is evident that at least one of the equations $q(\alpha_1, t) = \gamma_1$, $q(\alpha_1, t) = \gamma_2$ has a root $t_0 \neq 2$.

Now we assume that $m = 2$. Suppose that the equation $q(\alpha_r, t) = 0$ has a unique root $t = 2$. This means that for arbitrary matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that $\operatorname{tr} A = \operatorname{tr} B = \alpha_r$, the condition $\operatorname{tr} R_1(A, B) = \operatorname{tr} A^{u_1} B^{-u_2} \dots A^{u_{2k-1}} B^{-u_{2k}} = 0$ implies that $\operatorname{tr} AB = 2$. Prove that this is not the case. To obtain a contradiction, it is sufficient to find matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ satisfying to the conditions:

1) $\operatorname{tr} A = \operatorname{tr} B = \alpha_r$;
2) $\operatorname{tr} AB \neq 2$;
3) $\operatorname{tr} R_1(A, B) = \operatorname{tr} A^{u_1} B^{-u_2} \dots A^{u_{2k-1}} B^{-u_{2k}} = 0$.

We shall find A, B in the form

$$A = \begin{pmatrix} \varepsilon_r & w \\ 0 & \varepsilon_r^{-1} \end{pmatrix}, \qquad B = \begin{pmatrix} \varepsilon_r & 0 \\ w & \varepsilon_r^{-1} \end{pmatrix},$$

where $\varepsilon_r + \varepsilon_r^{-1} = \alpha_r = 2\cos(r\pi/n)$ and w is a variable. It is easy to see that $\operatorname{tr} AB = w^2 + \varepsilon_r^2 + \varepsilon_r^{-2}$. Hence the condition $\operatorname{tr} AB \neq 2$ is equivalent to $w^2 + \varepsilon_r^2 + \varepsilon_r^{-2} \neq 2$, that is,

$$w^2 \neq 2 - (\varepsilon_r^2 + \varepsilon_r^{-2}) = 2 - 2\cos(\tfrac{2r\pi}{n}) = 4\sin^2(\tfrac{r\pi}{n}).$$

It is easy to verify by induction that

$$A^i = \begin{pmatrix} \varepsilon_r^i & P_{i-1}(\alpha_r) w \\ 0 & \varepsilon_r^{-i} \end{pmatrix}, \qquad B^i = \begin{pmatrix} \varepsilon_r^i & 0 \\ P_{i-1}(\alpha_r) w & \varepsilon_r^{-i} \end{pmatrix}.$$

Furthermore, it is easy to show that

$$R_1(A, B) = \begin{pmatrix} \varepsilon_r^d + C_1(\alpha_r) w^2 + \dots + C_k(\alpha_r) w^{2k} & w f_1(w) \\ w f_2(w) & \varepsilon_r^{-d} + D_1(\alpha_r) w^2 + \dots + D_{k-1}(\alpha_r) w^{2k-2} \end{pmatrix}$$

where $d = \sum_{i=1}^{2k} u_i$, $C_k(\alpha_r) = (-1)^k \prod_{i=1}^{2k} P_{u_i-1}(\alpha_r)$, $f_1(w)$, $f_2(w)$ are some polynomials of w. Hence,

$$\operatorname{tr} R_1(A, B) = C_k(\alpha_r) w^{2k} + \dots + (C_1(\alpha_r) + D_1(\alpha_r)) w^2 + (\varepsilon_r^d + \varepsilon_r^{-d}) = g(w^2).$$

Prove that there exists r, $1 \leq r < n$, such that $C_k(\alpha_r) \neq 0$ and the polynomial $g(w^2)$ has a root w_0 with $w_0^2 \neq 4\sin^2(r\pi/n)$. Let's assume the contrary. Suppose that for each r such that $C_k(\alpha_r) \neq 0$ we have

$$g(w) = C_k(\alpha_r)(w - 4\sin^2(\tfrac{r\pi}{n}))^k. \tag{25}$$

Comparing the constant terms in the left and right part of (25) and taking into account the expression for $C_k(\alpha_r)$, we obtain

$$\left(\prod_{i=1}^{2k} P_{u_i-1}(2\cos(\tfrac{r\pi}{n}))\right) 4^k \left(\sin(\tfrac{r\pi}{n})\right)^{2k} = 2\cos(\tfrac{dr\pi}{n}). \tag{26}$$

By (4) we have $P_{u_i-1}(2\cos(r\pi/n)) = \sin(u_i r\pi/n)/\sin(r\pi/n)$. Write u_i/n in the form u_i'/n_i, where $(u_i', n_i) = 1$. Then (26) has the form

$$\prod_{i=1}^{2k} \left(2\sin\left(\tfrac{u_i' r\pi}{n_i}\right)\right) = 2\cos(\tfrac{dr\pi}{n}). \tag{27}$$

Thus, to complete the proof of the lemma it is sufficient to prove that one gets a contradiction assuming that (27) holds for each r such that the left part in (27) is non-zero.

First consider the case, where n is odd. Let $n_0 = \min_j n_j$ and let for definiteness $n_0 = n_1$. Then n_1 is odd and let $p > 2$ be an arbitrary prime divisor of n_1. Set $r = n_1/p$. Then $2\sin(u_1' r\pi/n_1) = 2\sin(u'1\pi/p)$. If $j > 1$, then $2\sin(u_j' r\pi/n_j) = 2\sin(u_j' n_1\pi/(pn_j)) \neq 0$ because by construction pn_j does not divide $u_j' n_1$. It follows from (27) that

$$\prod_{i=2}^{2k} \left(2\sin\left(\tfrac{u_i' n_1\pi}{pn_i}\right)\right) = \frac{2\cos(dn_1\pi/(pn))}{\sin(u_1'\pi/p)} \in \mathcal{O}. \tag{28}$$

If pn divides dn_1, then the equality (28) implies that $1/(\sin(u_1'\pi/p)) \in \mathcal{O}$ which contradicts to item 5 of Lemma 7. Further if pn does not divide dn_1, then $2\cos(dn_1\pi/(pn)) \in \mathcal{O}^*$ by item 2 of Lemma 7. Thus it follows from (28) that $1/(2\sin(u_1'\pi/p)) \in \mathcal{O}$ which again contradicts to item 5 of Lemma 7.

Let now $n = 2^l n'$, where $l \geq 1$ and n' is odd. Let $n_i = 2^{l_i} n_i'$, where $l_i \geq 0$, n_i' is odd, and let $n_0' = \min_j n_j'$.

If $n_0' > 1$, then we set $r = 2^l r'$, where $r' \not\equiv 0 \pmod{n'}$. Then (27) has the form

$$\prod_{i=1}^{2k} \left(2\sin\left(\tfrac{u_i' 2^{l-l_i} r'\pi}{n_i'}\right)\right) = 2\cos(\tfrac{dr'\pi}{n'}), \tag{29}$$

where n' is odd. We have proved above that there exists an r' such that the left part of (29) is non-zero and the equality (29) is not valid.

Let now $n_0' = 1$. Set

$$I = \{i \mid n_i' = 1\}, \qquad l_0 = \min_{i\in I} l_i, \qquad I_0 = \{i \in I \mid l_i = l_0\}.$$

Furthermore, we set $r = 2^{l_0-1} r'$, where r' is odd. Then for $i \in I_0$ we have

$$2\sin\left(\tfrac{u_i' r\pi}{n_i}\right) = 2\sin\left(\tfrac{u_i' 2^{l_0-1} r'\pi}{2^{l_0}}\right) = 2\sin\left(\tfrac{u_i' r'\pi}{2}\right) = \pm 2.$$

Now we can write the equality (27) in the form:

$$\prod_{i\notin I_0}\left(2\sin\left(\tfrac{u_i'r'\pi}{2^{l_i-l_0+1}n_i'}\right)\right)=\pm\tfrac{1}{2^{|I_0|-1}}\cos(\tfrac{dr'\pi}{2^{l-l_0+1}n'}). \tag{30}$$

Choose r' such that the left part of (30) is non-zero. Then the right part of (30) should be non-zero too. If $|I_0|>1$ or $|I_0|=1$ and $\cos(dr'\pi/(2^{l-l_0+1}n'))\neq\pm1$, then the left part of (30) belongs to $\mathcal{O}$. But by item 1) of Lemma 7 the right part of (30) does not belong to $\mathcal{O}$ which is a contradiction.

Thus, it remains to consider the case $|I_0|=1$ and $\cos(dr'\pi/(2^{l-l_0+1}n'))=\pm1$. In this case (30) has the form

$$\prod_{i\neq i_0}\left(2\sin\left(\tfrac{u_i'r'\pi}{2^{l_i-l_0+1}n_i'}\right)\right)=\pm1. \tag{31}$$

If $|I|>1$ and $i_0\neq i\in I$, then $l_i>l_0$, $n_i=1$. Hence for each odd r' the left part of (31) is non-zero and (31) implies that $1/(2\sin(u_i'r'\pi/(2^{l_i-l_0+1})))\in\mathcal{O}$. We obtain a contradiction to item 5 of Lemma 7.

Let now $I=I_0=\{i_0\}$. Set

$$n_{j_0}=\min_{j\neq i_0}n_j\geq3,\qquad J=\{j\mid n_j=n_{j_0}\},\qquad l_{j_0}=\min_{j\in J}l_j.$$

If $l_{j_0}-l_0+1>0$, then we set $r'=n_{j_0}$. It is easy to verify that in this case the left part of (31) is not equal to 0 and (31) implies that $1/(2\sin(u_{j_0}'\pi/2^{l_{j_0}-l_0+1}))\in\mathcal{O}$. We obtain a contradiction to item 6 of Lemma 7.

At last, if $l_{j_0}-l_0+1\leq0$, then we take an arbitrary prime divisor $p\geq3$ of n_{j_0} and set $r'=n_{j_0}/p$. Then, as above, the left part of (31) is not equal to 0 and we obtain from (31) that

$$2\sin\left(\frac{u_{j_0}'r'\pi}{2^{l_{j_0}-l_0+1}n_{j_0}'}\right)=2\sin\left(\frac{u_{j_0}'2^{-l_{j_0}+l_0-1}\pi}{p}\right)\in\mathcal{O}^*.$$

We have a contradiction to item 5 of Lemma 7. Lemma 10 is proved.

Now we can complete the proof of Theorem 3. By Lemma 10 one can choose r,l such that the equation (24) has a root $t_0\neq2$. Since by construction $t=x^2+y^2+z^2-xyz-2$, $x=\alpha_r$, then we have that y, z satisfy to the equation

$$y^2+z^2-\alpha_r yz+\alpha_r^2-2-t_0=0. \tag{32}$$

Let (y_0,z_0) be some solution of (32) and let $A,B\in\mathrm{SL}_2(\mathbb{C})$ be matrices such that $\operatorname{tr}A=\alpha_r$, $\operatorname{tr}B=y_0$, $\operatorname{tr}AB=z_0$. Then by construction $\operatorname{tr}ABA^{-1}B^{-1}=t_0$, $\operatorname{tr}R(A,B)=\gamma_l$, and a pare of matrices (A,B) defines a representation of the group Γ into $\mathrm{PSL}_2(\mathbb{C})$. Note that this representation is irreducible because $t_0\neq2$. Our goal is to apply Lemma 6. To this end, it is sufficient to find a solution (y_0,z_0) of the equation (32) such that:

1) there exists an element of finite order $W_1(A, B) = A^{\alpha_1}B^{\beta_1}\dots A^{\alpha_g}B^{\beta_g}$ such that $\alpha_i, \beta_i \neq 0$ for $i = 1, \dots, g$ and $\sum_{i=1}^{g} \beta_g \neq 0$;

2) $z_0 = \operatorname{tr} AB \notin \mathcal{O}$.

The rest of the proof depends on the form of t_0. We consider the following cases:

1) $t_0 \notin \mathcal{O}$;

2) $t_0 = 2\cos((2k+1)\pi/(2s+1))$, where $s \geq 1$, $(2k+1, 2s+1) = 1$;

3) $t_0 = 2\cos(2k\pi/(2s+1))$, where $s \geq 1$, $(k, 2s+1) = 1$;

4) $t_0 = 2\cos((2k+1)\pi/(2s))$, where $s \geq 1$, $(2k+1, s) = 1$;

5) $t_0 \in \mathcal{O}$, $t_0 \neq 2\cos(k\pi/s)$ for arbitrary integers k and s.

1) Set $y_0 = 0$ and $W_1(A, B) = B$. Then $W_1(A, B) = B$ has order 4. Since $t_0 \notin \mathcal{O}$, then the equation (32) has a solution $(0, z_0)$ such that $z_0 \notin \mathcal{O}$. Applying Lemma 6, we obtain the assertion of Theorem 3.

2) Set $W_1(A, B) = AB(ABA^{-1}B^{-1})^s$. Combining Lemmas 8 and 9, we obtain

$$\operatorname{tr} W_1(A, B) = (P_{s+1}(t_0) - P_s(t_0))z_0 = 0 \cdot z_0 = 0.$$

Hence $W_1(A, B)$ has order 4. Take an arbitrary solution (y_0, z_0) of the equation (32) such that $z_0 \notin \mathcal{O}$. Again we can apply Lemma 6.

3) Set $W_1(A, B) = AB(ABA^{-1}B^{-1})^s$ and assume that

$$\operatorname{tr} W_1(A, B) = 2\cos(\pi/3) = 1.$$

Then $W_1(A, B)$ has order 6 and item 2 of Lemma 9 implies that

$$\operatorname{tr} W_1(A, B) = (P_{s+1}(t_0) - P_s(t_0))z_0 = 1.$$

Hence by item 3 of Lemma 8 we have $z_0 = 1/(P_{s+1}(t_0) - P_s(t_0)) \notin \mathcal{O}$. Let (y_0, z_0) be a solution of (32). Now we can apply Lemma 6.

4) Set $W_1(A, B) = (AB)^{-1}(ABA^{-1}B^{-1})^s(AB)^2(ABA^{-1}B^{-1})^s$ and assume that

$$\operatorname{tr} W_1(A, B) = 2\cos(\pi/3) = 1. \tag{33}$$

Then $W_1(A, B)$ has the order 6 and by item 3 of Lemma 9 we can write (33) in the form

$$(t_0 - 2)P_{s-1}(t_0)^2 z_0^3 + (2 - P_{2s-1}(t_0) + P_{2s-2}(t_0))z_0 - 1 = 0. \tag{34}$$

By item 4 of Lemma 8 we have $0 \neq P_{s-1}(t_0) \notin \mathcal{O}^*$ whence $1/((t_0 - 2)P_{s-1}(t_0)^2) \notin \mathcal{O}$. Thus, (34) has a root $z_0 \notin \mathcal{O}$. Let (y_0, z_0) be a solution of (32). Applying Lemma 6, we obtain the assertion of Theorem 3.

5) Since $t_0 \in \mathcal{O}$, $t_0 \neq 2\cos(k\pi/s)$ for arbitrary integers k and s, then by item 5 of Lemma 8 there exists an integer $l > 0$ such that $0 \neq P_l(t_0) \notin \mathcal{O}^*$. Set $W_1(A, B) =$

$(AB)^{-1}(ABA^{-1}B^{-1})^{l+1}(AB)^2(ABA^{-1}B^{-1})^{l+1}$ and assume that the condition (33) holds. Then $W_1(A,B)$ has the order 6 and by item 3 of Lemma 9 we can write (33) in the form

$$(t_0-2)P_l(t_0)^2z_0^3+(2-P_{2l+1}(t_0)+P_{2l}(t_0))z_0-1=0. \tag{35}$$

Since by construction $1/((t_0-2)P_l(t_0)^2)\notin\mathcal{O}$, then (35) has a root $z_0\notin\mathcal{O}$. Let (y_0,z_0) be a solution of (32). Applying Lemma 6, we complete the proof of Theorem 3 in the last case.

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