On Hurwitz groups of low rank

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1. Introduction

Let $\Delta(2,3,7)$ denote the infinite triangle group, defined by the presentation $\langle X, Y \mid X^2 = Y^3 = (XY)^7 = 1\rangle$. A non-trivial group G is said to be $(2,3,7)$-generated (or a *Hurwitz* group, when finite) if it is an epimorphic image of $\Delta(2,3,7)$. Hurwitz groups are particularly interesting for the theory of Riemann surfaces. Namely, if H is the automorphism group of a compact Riemann surface of genus $g > 1$, then H is finite of order not exceeding $84(g-1)$, and the Hurwitz groups are exactly those for which $|H| = 84(g-1)$. As a quotient of a Hurwitz group is again Hurwitz, it is crucial to determine the simple Hurwitz groups. The values of q for which the groups $\mathrm{PSL}(2,q)$ are Hurwitz were determined by A. M. Macbeath in 1969 ([Mac]), whereas J. Cohen in 1981 ([Coh]) showed that $PSL(3,2)$ is the only Hurwitz group in the series $\mathrm{PSL}(3,q)$, and none of the groups in the series $\mathrm{PSU}(3,q^2)$ are Hurwitz. M. Conder in 1980 ([Con]) proved that the alternating groups A_n are Hurwitz, provided $n > 167$. More recently G. Malle has shown that the exceptional simple groups $G_2(q)$, $q \geq 5$; ${}^2G_2(q)$, $q > 3$; ${}^3D_4(q)$ for $q = p^m$, $p \neq 3$, $q \neq 4$, and ${}^2F_4(2^{2m+1})'$ for $m \equiv 1$ (3) are Hurwitz (cfr. [Mal1], [Mal2]). The sporadic Hurwitz groups are also known, with the exception of the Monster. (For a bibliography and related comments, see [Jon]).
Recent constructive results of Lucchini, Tamburini and Wilson ([LTW], [Lu], [Wil]) show that the family of $(2,3,7)$-generated groups, and more generally of $(2,3,k)$-generated groups, is very large.

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In fact, for every $k \geq 7$ there are $2^{\aleph_0}$ isomorphism classes of simple $(2,3,k)$-generated groups. In particular, most finite classical groups of large rank are Hurwitz ([LT]). In this paper we prove, by contrast, that most quasi-simple classical groups of small rank are not $(2,3,7)$-generated.
Note that, whenever the centre of a quasi-simple group is non-trivial, the problem of deciding whether the corresponding simple central quotient is Hurwitz requires further analysis. This is why a general treatment of projective classical groups is not envisaged in this paper. Our first result, however, deals with projective groups. Namely, in Theorem 1 *ii*) we determine the (2,3,7)-generated subgroups of $\mathrm{PSL}(3,\overline{F})$, where $\overline{F}$ is the algebraic closure of an arbitrary field F. This extends the results of [Mac] and [Coh], which deal with the case $F = F_p$, the Galois field of order p. Our approach essentially amounts to bringing the infinite triangle groups $\Delta(2,3,k)$, $k \geq 7$, into the picture. While referring to [Mag] II.5 for details and proofs, we recall here that the group $\mathrm{PSL}(2,\mathbb{R})$ over the reals acts naturally on the complex upper half-plane $\mathcal{U} = \{\alpha \in \mathbb{C} \mid \operatorname{Im}\alpha > 0\}$, preserving the hyperbolic metric which turns $\mathcal{U}$ into a model of the Poincaré plane. In this realization, the subgroup of $\mathrm{PSL}(2,\mathbb{R})$ generated by rotations of angles $\frac{2\pi}{2}$, $\frac{2\pi}{3}$, $\frac{2\pi}{k}$ around the vertices of a hyperbolic triangle having angles of size $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{k}$ is isomorphic to $\Delta(2,3,k)$.

Throughout the paper F denotes a field of characteristic $p \geq 0$ and $\overline{F}$ denotes the algebraic closure of F. Moreover, we observe explicitly that, for any prime $k \geq 7$, a $(2,3,k)$-generated group G is a perfect group. It follows that, for any representation $f: G \to \mathrm{GL}(n,F)$, $f(G)$ is a subgroup of $\mathrm{SL}(n,F)$. We start with the following:

Theorem 1 *Let $k \geq 7$ be a prime, and $\epsilon \in \overline{F}$ be a primitive k^{th} root of unity ($\epsilon = 1$ if $p = k$).*
i) Up to isomorphism, a $(2,3,k)$-generated subgroup H of $\mathrm{PSL}(2,\overline{F})$ is the projective image of the group

$$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} \epsilon & 1 \\ 0 & \epsilon^{-1} \end{pmatrix} \right\rangle.$$

If $p = 0$, H is isomorphic to $\Delta(2,3,k)$. If $p > 0$, H is a subgroup of $\mathrm{PSL}(2,p^{k-1})$. In particular H is finite and, for $k = 7$, by [Mac] $H \simeq \mathrm{PSL}(2,p)$ if $p \equiv 0, \pm 1 \pmod 7$, $H \simeq \mathrm{PSL}(2,p^3)$ if $p \not\equiv 0, \pm 1 \pmod 7$.

ii) If $p \neq 2$, the image of H under the orthogonal embedding $\varphi: \mathrm{PSL}(2,\overline{F}) \to \mathrm{SO}(3,\overline{F})$ defined by

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

is irreducible for all primes $k \geq 7$. Assuming $k = 7$, let K be an irreducible $(2,3,7)$-generated subgroup of $\mathrm{SL}(3,\overline{F})$. Then either K is isomorphic to $\mathrm{PSL}(2,7)$, or $p \neq 2$ and K is isomorphic to $\varphi(H)$. The irreducible $(2,3,7)$-generated subgroups of $\mathrm{PSL}(3,\overline{F})$ are the projective images of those of $\mathrm{SL}(3,\overline{F})$.

iii) If $p = 0$ or $p \not\equiv 0, \pm 1 \pmod k$, the group H embeds irreducibly into $\mathrm{SL}(4,\overline{F})$, for all primes $k \geq 7$. The embedding is orthogonal if $p \neq 2$.

iv) If $p \neq 3$, the group H embeds irreducibly into a symplectic subgroup of $\mathrm{PSL}(4,\overline{F})$, for all primes $k \geq 7$.

We have included items *iii*) and *iv*) in order to emphasize the occurrence of orthogonal or symplectic representations of Hurwitz groups in dimension 4 (cfr. Theorem 2). We are aware that they are special cases of much more general embeddings in tensor (or twisted tensor) product spaces.

Our subsequent results only concern linear groups, and are based on the application of a formula due to L.L. Scott (see Section 2) to appropriate representations of a group, and the possible contradictions that one may obtain assuming that the group is $(2,3,7)$-generated. Scott himself observed that his formula could be used for proving that certain linear groups are not $(2,3,7)$-generated (he considered, as examples, the groups $\mathrm{SL}(6,3)$ and $\mathrm{SL}(9,3)$). A more systematic application of Scott's formula in the context of Hurwitz groups appears in [TV], where it is shown that the groups $\mathrm{SL}(4,q)$ and $\mathrm{SU}(4,q^2)$, q odd, are not Hurwitz. Although these techniques work in principle over an arbitrary field, we mainly concentrate on classical groups over finite fields (Theorems 2,3,4), and on the field of rational numbers (Theorem 5). We prove the following "negative" results.

Theorem 2 *Let H denote an absolutely irreducible subgroup of $\mathrm{SL}(n,F)$, $n \in \{4,5,6,7,10\}$. Assume that H is not contained in an orthogonal group if $p \neq 2$, and that H is not contained in a symplectic group if $p=2$. If $n=6$ and $p=2$, assume further that F is finite and $H = \mathrm{SL}(6,F)$ or $SU(6,F)$. Then H is not $(2,3,7)$-generated. In particular, if $n \in \{4,5,6,7,10\}$, then:*

i) the groups $\mathrm{SL}(n,q)$, $\mathrm{Sp}(n,q)$, $\mathrm{SU}(n,q^2)$ are not Hurwitz, with the only possible exception of $\mathrm{Sp}(n,2^t)$;

ii) every complex irreducible character of degree n of a $(2,3,7)$-generated group, is real.

Both Theorem 1 and Theorem 2 may be rephrased as statements about representations of a (2,3,7)-generated group G. Clearly, if $p \neq 2$, G has no absolutely irreducible representation of degree 2, since the only involution of $\mathrm{SL}(2,\overline{F})$ is $-I$. In particular G has no complex irreducible representation of degree 2. It follows from Theorem 1 that, if G has a complex irreducible representation of degree 3, then it has a quotient isomorphic to $\mathrm{PSL}(2,7)$. In particular, $\mathrm{PSL}(2,7)$ does have complex irreducible representations of degree 3. The remaining irreducible representations of $\mathrm{PSL}(2,7)$ over $\mathbb{C}$ have degrees 6,7 and 8, with real characters. Similarly, the next Hurwitz group $\mathrm{SL}(2,8)$ has non-real irreducible characters of degree 9 and real characters of degree 7 and 8 (e.g. cfr. [Isa], p. 289). Theorem 2 states that, under the given assumptions on F and for the stated values of n, every irreducible representation $f: G \to \mathrm{SL}(n,\overline{F})$ is orthogonal (symplectic if $p=2$). It is worth to recall that, in the natural characteristic, the groups of type G_2 do have orthogonal representations of degree 7 if $p \neq 2$, and symplectic representations of degree 6 if $p=2$; whereas the groups of type ${}^2G_2$ and ${}^3D_4$ have orthogonal representations of degree 7 and 8, respectively. Finally, note that Holt, Plesken and Souvignier [HPS] have constructed a 7-dimensional complex representation of $\Delta(2,3,7)$, whose kernel is generated by the element $[X,Y]^{11}$.

Theorem 3 *Let H be one of the groups:*

a) $\mathrm{SL}(n,q)$, $q \neq 3^6$ *and, if* $q \equiv 1 \pmod 7$, $q \neq p, p^3$; $\mathrm{Sp}(n,q)$, q *odd;* $\mathrm{SU}(n,q^2)$, $q \neq 3^3$; $n \in \{8,9,11\}$;

b) $\mathrm{SL}(12,q)$, $q \not\equiv 1 \pmod 7$; $\mathrm{Sp}(12,q)$, q *odd;* $\mathrm{SU}(12,q^2)$, $q \not\equiv -1 \pmod 7$.

Then H is not Hurwitz .

Theorem 4 *Assume* $13 \leq n \leq 19$, *and let H be one of the groups* $\mathrm{SL}(n,q)$, $q \not\equiv 1 \pmod 3$; $\mathrm{Sp}(n,q)$, q *odd*; $\mathrm{SU}(n,q^2)$, $q \not\equiv -1 \pmod 3$. *Then H is not Hurwitz, except possibly for the following cases*:

a) $\mathrm{SL}(16,q)$, $q \equiv 8 \pmod{21}$, $\mathrm{SL}(16,3^{6h})$, $\mathrm{SU}(16,q^2)$, $q \equiv -8 \pmod{21}$, $\mathrm{SU}(16,3^{6h+3})$.

b) $\mathrm{SL}(18,q)$, q *odd* $\equiv 8 \pmod{21}$, $\mathrm{SL}(18,3^{6h})$; $\mathrm{SU}(18,q^2)$, q *odd* $\equiv -8 \pmod{21}$, $\mathrm{SU}(18,3^{6h+3})$.

c) $\mathrm{SL}(n,q)$, $n \in \{15,17,19\}$, $q \equiv 2,8,-10 \pmod{21}$ *or* $q \in \{3^{6h}, 3^{6h+2}, 3^{6h+4}\}$.

d) $\mathrm{SU}(n,q^2)$ $n\in\{15,17,19\}$, $q\equiv 10,-2,-8 \pmod{21}$ *or* $q\in\{3^{6h+1},3^{6h+3},3^{6h+5}\}$.

e) $\mathrm{SL}(n,2^t)$, $n\in\{14,18\}$, $2^t\equiv 2,8,-10 \pmod{21}$.

Theorem 5 *Let* $\mathbb{Q}$ *be the field of rational numbers, and let* H *denote an absolutely irreducible subgroup of* $\mathrm{SL}(n,\mathbb{Q})$, $n\le 19$ *or* $n=22$. *If* H *is not contained in an orthogonal group, then* H *is not* $(2,3,7)$ *-generated. In particular, for these values of* n, *the group* $\mathrm{SL}(n,\mathbb{Z})$ *is not* $(2,3,7)$ *-generated.*

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2. Scott' s formula

We denote shortly by M the algebra $\mathrm{Mat}(n,F)$ of $n\times n$ matrices over F, and view M as acting on the left on the vector space $V=F^n$ of column vectors. Given a representation $f:H\to\mathrm{GL}(n,F)$ of a group H over F, for any subset X of H we define d_V^X as the dimension of the subspace of V fixed pointwise by $f(X)$. In a similar way we define $\hat{d}_V^X$, with respect to the dual representation $\hat{f}:H\to\mathrm{GL}(n,F)$ given by $\hat{f}(h):=(f(h^{-1}))^t$. The following formula, which is a special case of a result of L.L.Scott ([Sc], Theorem 1), will be crucial.

Theorem 2.1 *Let* $f:H\to\mathrm{GL}(n,F)$ *be a representation of* $H=\langle x,y\rangle$. *Then :*

$$d_V^x+d_V^y+d_V^{xy}\le \dim V+d_V^H+\hat{d}_V^H.$$

The above formula was conceived by Scott as a generalization of the well-known *genus formula* $\left[\frac{n}{2}\right]+2\left[\frac{n}{3}\right]+6\left[\frac{n}{7}\right]\ge 2n-2$ for the index n of a subgroup S of a $(2,3,7)$-generated group G. Indeed, the genus formula is a consequence of Theorem 2.1. For, let $G=\langle x,y\rangle$ with $x^2=y^3=(xy)^7=1$, and consider the transitive permutation representation $f:G\to\mathrm{GL}(n,\mathbb{C})$ given by the action on the cosets of S. Since the multiplicity in f of the trivial representation is 1, $d_V^G=\hat{d}_V^G=1$. Scott' s formula then gives: $d_V^x+d_V^y+d_V^{xy}\le n+2$. The genus formula follows easily. Indeed, for every $g\in G$ of prime order k, whose cyclic structure admits a cycles of length k, $n-d_V^g=(k-1)a\le(k-1)\left[\frac{n}{k}\right]$.

Scott's formula will be applied essentially to the following representations of a subgroup H of $\mathrm{SL}(n,F)$, with generating pair $\{x,y\}$.

1) The representation of H by conjugation on M (also called the 'adjoint' representation of H) given by $f(h):=m\mapsto hmh^{-1}$ ($h\in H$, $m\in M$). In this case, the fixed-points subspace of M is the centralizer of H in M, and therefore, if H is supposed to be absolutely irreducible, by Schur's Lemma it consists of scalar matrices. It follows $d_M^H=\hat{d}_M^H=1$, hence Scott's formula reads

$$(*)\qquad d_M^x+d_M^y+d_M^{xy}\le n^2+2.$$

Failing of the above inequality means that H is reducible over $\overline{F}$ (and to some extent our problem is reduced to linear groups of smaller degree). Dealing with $(*)$ requires detailed information on minimal values affordable by d_M^x, d_M^y and d_M^{xy}. These values will be worked out in Section 3.

2) The representation of H on the symmetric square S of $\overline{F}^n$. If H is absolutely irreducible, then Scott's formula takes the shape

$$(**) \qquad d_S^x + d_S^y + d_S^{xy} \leq \dim S \quad , \quad \text{or} \qquad d_M^x + d_M^y + d_M^{xy} \leq \dim S + 2$$

(the latter occurring only when H is orthogonal if $p \neq 2$, symplectic if $p = 2$). This will be shown in Section 4.

3. The defect $\mathrm{def}_{\overline{M}}$

We first recall a well-known formula, due to Frobenius (e.g. cfr. [Ja], p. 207). Namely, let $B \in M := \mathrm{Mat}(n, F)$ and let $n_1 \leq \ldots \leq n_s$ be the degrees of the non-trivial invariant factors of $\lambda I - B$ (also called the similarity invariants of B). The dimension d_M^B over F of the centralizer of B in M is given by the formula

$$d_M^B \;=\; \sum_{j=1}^{s}(2s-2j+1)\, n_j \;=\; (2s+1)n - 2\sum_{j=1}^{s} j n_j .$$

In particular $d_M^B \;\geq\; n + s^2 - s$, since

$$\sum_{j=1}^{s} j n_j \;=\; \sum_{j=1}^{s} n_j + \sum_{j=2}^{s} n_j + \ldots + (n_{s-1} + n_s) + n_s \;\leq$$

$$\leq\; n + (n-1) + \ldots + (n-s+2) + (n-s+1) \leq\; sn - \frac{s(s-1)}{2}.$$

Moreover, considering the algebraic closure $\overline{F}$ and denoting by $d_{\overline{M}}^B$ the dimension over $\overline{F}$ of the centralizer of B in $\overline{M} = \mathrm{Mat}(n, \overline{F})$, clearly $d_M^B = d_{\overline{M}}^B$, since the similarity invariants of B remain unchanged in $\overline{M}$. Now, for a fixed prime k, define $d_{\overline{M}}^k$ as the minimal dimension of the centralizer of an element of order k in $\mathrm{SL}(n, \overline{F})$, i.e.

$$d_{\overline{M}}^k \;:=\; \min\,\left\{ d_{\overline{M}}^A \;\mid\; A \in \mathrm{SL}(n, \overline{F})\;,\; A^k = I.\right\}$$

Lemma 3.1 *Write* $n = k\ell + r,\ 0 \leq r < k$.

i) If $k = 2$ *and* $n \equiv 2 \pmod 4$, *assume* $p = 2$. *Let* $B \in \mathrm{SL}(n, \overline{F})$ *have order* k, *and let* $n_1 \leq \ldots \leq n_s$ *be the degrees the non-trivial invariant factors of* $\lambda I - B$. *Then* $d_{\overline{M}}^B = d_{\overline{M}}^k$ *iff* $n_i = k$ *for every* $i > 1$, *and* $d_{\overline{M}}^k \;=\; (2\ell+1)r + k\ell^2$.

ii) If $k = 2$, $n \equiv 2 \pmod 4$ *and* $p \neq 2$, *then* $d_{\overline{M}}^2 \;=\; 2(\ell^2+1)$.

Proof

Let $A \in \mathrm{GL}(n, \overline{F})$ be a matrix of order k, such that $d_{\overline{M}}^A \leq d_{\overline{M}}^C$ for all C of order k in $\mathrm{GL}(n, \overline{F})$, and denote by $n_1 \leq \ldots \leq n_s$ the degrees of the non-trivial invariant factors of $\lambda I - A$. Suppose that $n_i < k$ for some $i > 1$ and let m be the maximum of such i's. If $n_1 > 1$, there exists $C \in \mathrm{GL}(n, \overline{F})$ of order k such that $\lambda I - C$ has s invariant factors $\neq 1$ of degrees $\overline{n}_1 = n_1 - 1$, $\overline{n}_m = n_m + 1$, $\overline{n}_i = n_i$, $i \neq 1, m$. In this case we get :

$$d_{\overline{M}}^C \;=\; \sum_{j=1}^{s}(2s-2j+1)\overline{n}_j \;=\; 2(1-m) + \sum_{j=1}^{s}(2s-2j+1)n_j \;<\; d_{\overline{M}}^A,$$

a contradiction. On the other hand, if $n_1 = 1$, there exists $C \in \mathrm{GL}(n, \overline{F})$ of order k such that $\lambda I - C$ has $s-1$ invariant factors $\neq 1$ with degrees $\overline{n}_i = n_{i+1}$ for $i \neq m-1$, $\overline{n}_{m-1} = n_m + 1$. A similar calculation gives again the contradiction $d^C_{\overline{M}} < d^A_{\overline{M}}$. It follows $n_i = k$ for all $i > 1$ and $n_1 = r$ if $r > 0$, $n_1 = k$ if $r = 0$. Hence $d^A_{\overline{M}} = (2\ell+1)r + k\ell^2 \leq d^k_{\overline{M}}$.

i) We need to show that there exists $B \in \mathrm{SL}(n, \overline{F})$ such that $B^k = I$ and $d^B_{\overline{M}} = d^A_{\overline{M}}$. If $k = p$ or $r = 0$ we may choose $B = A$. Otherwise, we write $r = 2h + h_0$ $(h_0 \leq 1)$ and choose B such that $\lambda I - B$ has non-trivial invariant factors

$$d_1(\lambda) = (\lambda \mp 1)^{h_0} \prod_{j=1}^{h} (\lambda - \alpha^j)(\lambda - \alpha^{-j}) \quad , \quad d_i(\lambda) = \lambda^k - 1 \ , \ 2 \leq i \leq \ell + 1,$$

where α is a primitive k-th root of 1 in $\overline{F}$. We conclude $(2\ell+1)r + k\ell^2 = d^k_{\overline{M}}$. Finally, assume $d^B_{\overline{M}} = d^k_{\overline{M}}$. If $n_i < k$ for some $i > 1$, by the above argument there exists $C \in \mathrm{GL}(n, \overline{F})$ such that $d^C_{\overline{M}} < d^B_{\overline{M}} \leq d^C_{\overline{M}}$, a contradiction.

ii) Let $B \in \mathrm{SL}(n, F)$ be an involution such that $d^B_{\overline{M}} = d^2_{\overline{M}}$. It is easily seen that the non-trivial invariant factors of $\lambda I - B$ must have degrees $n_1 = n_2 = 1$, $n_3 = \ldots = n_s = 2$, and the formula can be computed directly. □

From the formulae given in Lemma 3.1, we obtain Table 1. The last two columns list the values of the *defect* of $\overline{M}$, defined by

$$\mathrm{def}_{\overline{M}} := n^2 + 2 - (d^2_{\overline{M}} + d^3_{\overline{M}} + d^7_{\overline{M}}).$$

4. The symmetric square action

Let $\overline{V} = \overline{F}^n$, with basis $\{v_i \mid 1 \leq i \leq n\}$. We denote by S the symmetric square of $\overline{V}$, i.e. the subspace of $\overline{V} \otimes \overline{V}$ generated by the set:

$$\{v_i \otimes v_i \mid 1 \leq i \leq n\} \ \cup \ \{v_i \otimes v_j + v_j \otimes v_i \mid 1 \leq i < j \leq n\},$$

and denote by E the exterior square, i.e. the subspace generated by the set:

$$\{v_i \otimes v_j - v_j \otimes v_i \mid 1 \leq i < j \leq n\}.$$

Clearly $\dim(S) = n(n+1)/2$ and $\dim(E) = n(n-1)/2$. If $p \neq 2$, $\overline{V} \otimes \overline{V} = S \dot{+} E$ where each of the summands is invariant under the action $g(u \otimes v) := g(u) \otimes g(v)$ for every $g \in \mathrm{GL}(n, \overline{F})$ (diagonal action).

Lemma 4.1 *Let H be an irreducible subgroup of $\mathrm{GL}(n, \overline{F})$ which fixes a non-zero vector of S. Then:*

i) if $p \neq 2$, H is contained in an orthogonal group;

ii) if $p = 2$, H is contained in a symplectic group;

iii) the identical representation of H is equivalent to the dual one. In particular $d^H_S = \hat{d}^H_S = 1$.

Table 1: the defect of $\overline{M}$

n	$d^2_{\overline{M}}$	$d^2_{\overline{M}}$	$d^3_{\overline{M}}$	$d^7_{\overline{M}}$	$\mathrm{def}_{\overline{M}}$	$\mathrm{def}_{\overline{M}}$
	$p \neq 2$	$p = 2$			$p \neq 2$	$p = 2$
4	8	8	6	4	0	0
5	13	13	9	5	0	0
6	20	18	12	6	0	2
7	25	25	17	7	2	2
8	32	32	22	10	2	2
9	41	41	27	13	2	2
10	52	50	34	16	0	2
11	61	61	41	19	2	2
12	72	72	48	22	4	4
13	85	85	57	25	4	4
14	100	98	66	28	4	6
15	113	113	75	33	6	6
16	128	128	86	38	6	6
17	145	145	97	43	6	6
18	164	162	108	48	6	8
19	181	181	121	53	8	8
20	200	200	134	58	10	10
21	221	221	147	63	12	12
22	244	242	162	70	10	12

Proof (*of Lemma* 4.1)

i) and *ii*) Let $\{v_1,\ldots,v_n\}$ be the standard basis of $\overline{V}=\overline{F}^n$. The map $v_i\otimes v_j\mapsto v_i\,v_j^t$ extends to an isomorphism $\overline{V}\otimes\overline{V}\to M(n,\overline{F})$ of $\overline{F}$-vector spaces. Via this isomorphism, the elements of S are identified with the symmetric matrices. Clearly, for every $g\in\mathrm{GL}(n,\overline{F})$ and every $v_i,v_j\in V$,

$$gv_i\otimes gv_j\ \mapsto\ g\,v_iv_j^t\,g^t.$$

Thereby, the diagonal action of $\mathrm{GL}(n,\overline{F})$ on $\overline{V}\otimes\overline{V}$ is identified with the action of $\mathrm{GL}(n,\overline{F})$ on $M(n,\overline{F})$ given by $g\mapsto\gamma$, where γ is the linear transformation $B\mapsto gBg^t$, for every $B\in M(n,\overline{F})$. As H fixes a non-zero vector of S, there exists a non-zero symmetric matrix B such that $hBh^t=B$, for every $h\in H$. Note that $B\overline{V}$ is a non-zero H-invariant subspace of $\overline{V}$. Hence, as H is irreducible, B is non-degenerate. Since $(h^t)^tBh^t=B$ implies $(h^{-1})^tB^{-1}h^{-1}=B^{-1}$ for every $h\in H$, it follows that, if $p\neq 2$, H is orthogonal. Thus *i*) holds. If $p=2$, *ii*) will be proved if we show that $v^tBv=0$ for every $v\in\overline{V}$. To this purpose let $\overline{V}_0:=\{v\in\overline{V}\mid v^tBv=0\}$. From

$$(w+\alpha u)^tB(w+\alpha u)=w^tBw+2\alpha w^tBu+u^tBu=w^tBw+u^tBu\quad\text{for }\alpha\in\overline{F},$$

it follows that $\overline{V}_0$ is a subspace of $\overline{V}$. As it is H-invariant and $\neq\{0\}$, we obtain $\overline{V}_0=\overline{V}$.

iii) As $B^{-1}hB=(h^{-1})^t$ for every $h\in H$, the identical representation of H is equivalent to the dual one. It follows immediately that $d_S^H=\hat{d}_S^H=1$, since absolutely irreducible groups can fix only one form, up to a scalar multiple (e.g. see [KL], Lemma 2.10.3).

For a matrix $A\in M(n,\overline{F})$, define d_S^A as the dimension of the subspace of the symmetric square of $\overline{V}\otimes\overline{V}$ fixed pointwise by $A\otimes A$. Define d_E^A in a similar way. The following result is straightforward.

Lemma 4.2 *Let $g\in\mathrm{GL}(n,F)$ be an element of finite order coprime to p. If ν is an eigenvalue of g, let m_ν denote the multiplicity of ν. Then:*

$$d_S^g=\frac{m_1(m_1+1)+m_{-1}(m_{-1}+1)}{2}+\sum m_\nu m_{\nu^{-1}}$$

where the summation runs over all pairs ν,ν^{-1} of eigenvalues of g in $\overline{F}$, with $\nu\neq\nu^{-1}$. □

In the sequel, for every $r\in\mathbb{N}$ we denote by π_r the companion matrix of λ^r-1 and by J_r the Jordan unipotent block of size r. In particular, if $r=p$, π_p and J_p are conjugate, and in this case we will usually write π_p.

Lemma 4.3

i) *Assume $p=2$ and let $g\sim\left(\underbrace{1,\ldots,1}_{k_1},\ \underbrace{\pi_2,\ldots,\pi_2}_{k_2}\right)\in\mathrm{GL}(n,F)$. Then*

$$d_S^g=\frac{k_1^2+k_1}{2}+k_2^2+k_2+k_1k_2\ .$$

ii) *Assume $p=3$ and let $g\sim\left(\underbrace{1,\ldots,1}_{k_1},\ \underbrace{J_2,\ldots,J_2}_{k_2},\ \underbrace{\pi_3,\ldots,\pi_3}_{k_3}\right)\in\mathrm{GL}(n,F)$. Then*

$$d_S^g=\frac{k_1^2+k_1}{2}+k_2^2+\frac{3k_3^2+k_3}{2}+k_1k_2+k_1k_3+2k_2k_3\ .$$

ii) Assume $p=7$ *and let* $g \in \mathrm{GL}(n,F)$,

$$g \sim \Big(\underbrace{1,\ldots,1}_{k_1}\ ,\ \underbrace{J_2,\ldots,J_2}_{k_2}\ ,\ \underbrace{J_3,\ldots,J_3}_{k_3}\ ,\ \underbrace{J_4,\ldots,J_4}_{k_4}\ ,\ \underbrace{J_5,\ldots,J_5}_{k_5}\ ,\ \underbrace{J_6,\ldots,J_6}_{k_6}\ ,\ \underbrace{\pi_7,\ldots,\pi_7}_{k_7} \Big).$$

Then $d_S^g =$

$$\frac{k_1^2+k_1}{2}+k_2^2+\frac{3k_3^2+k_3}{2}+2k_4^2+\frac{5k_5^2+k_5}{2}+3k_6^2+\frac{7k_7^2+k_7}{2}+$$
$$k_1\sum_{j=2}^{7}k_j+2k_2\sum_{j=3}^{7}k_j+3k_3\sum_{j=4}^{7}k_j+4k_4\sum_{j=5}^{7}k_j+5k_5\sum_{j=6}^{7}k_j+6k_6k_7.$$

Proof

All these formulae can be computed directly. For p odd, they can also be obtained from Lemma 1.2.3 in [GM], where a corresponding formula is given for d_S^g, g a unipotent in $\mathrm{GL}(n,F)$. $\square$

5. Constraints on (2,3,7)-generating triples

Our aim, in this section, is to provide an explanation for the data collected in the Tables 2, 3, 4, 4′ and 5, which will play a crucial role in the proof of our results. For this purpose we assume that $x,y,z=xy$ are elements of order 2, 3 and 7 respectively, which generate an absolutely irreducible subgroup of $\mathrm{SL}(n,F)$. It follows that these generators satisfy condition $(*)$ of Section 2. In particular Tables 2, 3, 4 and 4′ describe the conjugacy classes to which x,y and z are constrained to belong in order to satisfy the bounds:

$$d_M^x \le d_{\overline{M}}^2 + def_{\overline{M}} \quad , \quad d_M^y \le d_{\overline{M}}^3 + def_{\overline{M}} \quad , \quad d_M^z \le d_{\overline{M}}^7 + def_{\overline{M}}$$

which follow from $(*)$. Note that explicit values for the right-hand sides are deducible from Table 1.

A) Constraints on similarity invariants

In this subsection, as a consequence of the above mentioned bounds, we obtain constraints on the number and the degrees of the similarity invariants of x,y,z.

Lemma 5.1 *Let* $g \in \mathrm{GL}(n,F)$ *be an involution with* s *similarity invariants, and set* $\delta := d_{\overline{M}}^2 + def_{\overline{M}}$. *Then* $d_M^g \le \delta$ *iff*

$$\frac{n}{2} \le s \le \frac{n+\sqrt{2\delta-n^2}}{2}.$$

Proof

Clearly g has has $2s-n$ invariants of degree 1 and $n-s$ of degree 2. Hence, by the Frobenius formula, $d_M^g = 2s^2-2ns+n^2$. Our claim follows immediately. $\square$

From the previous Lemma we obtain Table 2, which lists representatives of all the conjugacy classes to which x may belong, and the corresponding values of d_S^x, for every $n \le 22$.

Lemma 5.2 *Let* $g \in \mathrm{GL}(n,F)$ *have order* 3, s *be the number of its similarity invariants, and denote by* k_1 *be the multiplicity of the invariant of degree* 1*. Setting* $\delta := d_{\overline{M}}^{3} + def_{\overline{M}}$, $d_M^g \leq \delta$ *iff the following conditions are satisfied:*

$$k_1 \ \leq \ \min\left\{\frac{3s-n}{2}\,,\,\sqrt{\frac{6\delta-2n^2}{3}}\right\} \quad ; \quad \frac{n}{3} \leq s \ \leq \ \frac{3k_1+2n+\sqrt{6\delta-2n^2-3{k_1}^2}}{6}.$$

Proof

It is easy to see that g has $k_2 = 3s - 2k_1 - n$ invariants of degree 2 and $k_3 = -2s + k_1 + n$ of degree 3. Hence, by the Frobenius formula, $d_M^g = 6s^2 - 6sk_1 - 4sn + 2k_1n + 2k_1^2 + n^2$. Our claim follows, noting that $k_3 \leq \frac{n-k_1}{3}$ implies $k_1 \ \leq \ \frac{3s-n}{2}$. □

From Lemma 5.2 we obtain Table 3, which lists representatives of all the conjugacy classes to which y may belong, and the corresponding values of d_S^y, for every $n \leq 22$. In this table we set $d_2 := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. If $p \neq 3$, d_2 is conjugate to $(\,\omega, \omega^{-1}\,)$, where ω is a primitive cubic root of 1. If $p = 3$, d_2 is conjugate to J_2. If $k_\ell > 0$ set $d^\ell(\lambda)$ for the similarity invariant of degree ℓ of y and note that, by the condition $\det y = 1$:

$$\begin{cases} k_1 \not\equiv 0 \pmod 3 \\ k_2 \equiv 0 \pmod 3 \end{cases} \Rightarrow\ d^1(\lambda) = \lambda - 1 \quad ; \qquad k_2 \not\equiv 0 \pmod 3 \ \Rightarrow\ d^1(\lambda) = \lambda - \omega^{\pm 1};$$

$$k_1 \not\equiv 0 \pmod 3 \ \Rightarrow\ d^2(\lambda) = (\lambda-1)(\lambda-\omega^{\pm1})\,; \begin{cases} k_1 \equiv 0 \pmod 3 \\ k_2 \not\equiv 0 \pmod 3 \end{cases} \Rightarrow\ d^2(\lambda) = (\lambda-\omega)(\lambda-\omega^{-1}).$$

Lemma 5.3 *Let* $g \in \mathrm{GL}(n,F)$ *be an element of order* 7*, with* s *similarity invariants of degrees* $n_1 \leq \ldots \leq n_s \leq 7$*. Set* $t_1 := n_1 - 1$, $t_i := n_i - n_{i-1}$ $(2 \leq i \leq s-1)$, $\delta := \frac{1}{2}\,(d_{\overline{M}}^{7} + \mathrm{def}_{\overline{M}} - n)$. *For the stated values of* s, $d_M^g \leq d_{\overline{M}}^{7} + def_{\overline{M}}$ *iff:*

$$s=3 \quad \begin{cases} 3t_1 + t_2 \ \leq \ \delta - 3 \\[2ex] 2t_1 + t_2 \geq n - 9 \\[2ex] 3t_1 + 2t_2 \leq n - 3 \end{cases} \qquad\qquad s=4 \quad \begin{cases} 6t_1 + 3t_2 + t_3 \ \leq \ \delta - 6 \\[2ex] 3t_1 + 2t_2 + t_3 \geq n - 10 \\[2ex] 4t_1 + 3t_2 + 2t_3 \leq n - 4 \end{cases}$$

$$s=5 \quad \begin{cases} 10t_1 + 6t_2 + 3t_3 + t_4 \ \leq \ \delta - 10 \\[2ex] 4t_1 + 3t_2 + 2t_3 + t_4 \geq n - 11 \\[2ex] 5t_1 + 4t_2 + 3t_3 + 2t_4 \leq n - 5 \end{cases} \qquad s=6 \quad \begin{cases} 15t_1 + 10t_2 + 6t_3 + 3t_4 + t_5 \ \leq \ \delta - 15 \\[2ex] 5t_1 + 4t_2 + 3t_3 + 2t_4 + t_5 \geq n - 12 \\[2ex] 6t_1 + 5t_2 + 4t_3 + 3t_4 + 2t_5 \leq n - 6. \end{cases}$$

Proof

Note that $n_s = n - \sum_{i<s} n_i$, $n \leq 7s$, $n + s^2 - s \leq d_M^g$. For each of the stated values of s, the first inequality follows from the Frobenius formula, whereas the second and the third one follow from the conditions $7 \geq n_s \geq n_{s-1}$. □

Using the above Lemma (and ruling out greater values of s via similar mutually incompatible inequalities) we obtain Table 4, which lists representatives of all the conjugacy classes to which z may belong in the modular case, and the corresponding values of d_S^z, for every $n \leq 22$.

B) Constraints on eigenvalue multiplicities

An element $g \in \mathrm{GL}(n,F)$ of finite order $o(g)$ is called *modular* if $o(g)$ is a power of p, *semisimple* if $o(g)$ is coprime to p. In the semisimple case, g is called *real* if it is conjugate to g^{-1}. Now, assume that g is a semisimple element of order 7. Denote by ϵ a primitive 7-th root of 1 in $\overline{F}$ and let $m_{\pm i}$ be the multiplicity of $\epsilon^{\pm i}$, $0 \leq i \leq 3$, in the Jordan canonical form of g over $\overline{F}$. We note that g is real whenever $m_1 = m_{-1}$, $m_2 = m_{-2}$, $m_3 = m_{-3}$. Furthermore, we say that g is *rational* when it is conjugate to all its non-trivial powers, i.e. when $m_1 = m_2 = m_3 = m_{-1} = m_{-2} = m_{-3}$; and we say that g is *semirational* when it is conjugate to g^2, i.e when $m_1 = m_2 = m_{-3}$, $m_{-1} = m_{-2} = m_3$. Let us remark that, if g has s similarity invariants, then obviously $m_{\pm i} \leq s$ for every i, and w.l.o.g. we may assume $m_1 \geq m_{\pm i}$, for every $i > 0$. With the above notations, and assuming $\det g = 1$:

$$d_M^g = m_0^2 + \sum_{i>0}(m_i^2 + m_{-i}^2) \quad ; \quad m_1 + 2m_2 + 3m_3 \equiv m_{-1} + 2m_{-2} + 3m_{-3} \pmod 7.$$

Using these conditions one may easily obtain representatives of the conjugacy classes to which the generator z may belong, listed in Table $4'$.

Lemma 5.4 *Let $g \in \mathrm{GL}(n,F)$ have order 7. If g is real, then $2d_S^g \geq d_{\overline{M}}^7$. If g is semirational, then:*

$$\begin{cases} m_0 + 3m_1 + 3m_{-1} = n \\ m_0^2 + 3{m_1}^2 + 3m_{-1}^2 = d_M^g \\ m_0^2 + m_0 + 6m_1 m_{-1} = 2d_S^g \end{cases}.$$

Proof

Assume that g is real. Then $d_M^g = m_0^2 + 2m_1^2 + 2m_2^2 + 2m_3^2$ and, by Lemma 4.2 , $2d_S^g = m_0^2 + m_0 + 2m_1^2 + 2m_2^2 + 2m_3^2$. It follows $2d_S^g - d_M^g = m_0 \geq 0$, hence $2d_S^g \geq d_M^g \geq d_{\overline{M}}^7$. The semirationality conditions are obvious. □

The next three lemmas are devoted to obtain Table 5

Lemma 5.5 *Let $h, g \in \mathrm{SL}(n,q)$ be elements of order 3 and 7, respectively.*

i) If $q \equiv 2 \pmod 3$, then h is real;

ii) if $q \equiv 3,5 \pmod 7$, then g is rational;

iii) if $q \equiv 2,4 \pmod 7$, then g is semirational;

iv) if $q \equiv 6 \pmod 7$, then g is real.

Proof

i) As $|F_q(\omega) : F_q| = 2$, the minimum polynomial of ω over F_q has degree 2. It follows that the similarity invariants of h can only be

$$\lambda - 1 \quad , \quad \prod_{k=1}^{2}(\lambda - \omega^k) \quad \text{or} \quad \prod_{k=0}^{2}(\lambda - \omega^k).$$

ii) As 3 and 5 have order 6 (mod 7), $[F_q(\epsilon) : F_q] = 6$. Thus the minimum polynomial of ϵ over F_q has degree 6. It follows that the similarity invariants of g can only be

$$\lambda - 1 \quad , \quad \prod_{k=1}^{6}(\lambda - \epsilon^k) \quad \text{or} \quad \prod_{k=0}^{6}(\lambda - \epsilon^k).$$

iii) and *iv*) are proved with similar arguments. □

Remark We recall here that any companion matrix is conjugate to its transpose: in fact they have the same minimum and characteristic polynomial, which coincide. It follows immediately that any matrix in $\mathrm{GL}(n, F)$ is conjugate to its transpose.

Lemma 5.6 *Let g be a semisimple element of a symplectic or orthogonal group over F.*

i) g is real;

ii) if $F = F_q$ with $q \equiv 2, 3, 4, 5 \pmod 7$ and g has order 7, then g is rational.

Proof

i) Let J be the matrix of the non-degenerate symplectic or orthogonal form defining the relevant group. From $g^t Jg = J$ it follows that g is conjugate to $(g^{-1})^t$, hence to g^{-1} by the remark above.

ii) $q \equiv 3, 5 \pmod 7$ implies g rational, $q \equiv 2, 4 \pmod 7$ implies g conjugate to g^2. On the other hand, by *i*), g is conjugate to g^{-1}. Then it is readily seen that g is conjugate to all its non-trivial powers, hence is rational.

Lemma 5.7 *Let $h, g \in \mathrm{SU}(n, q^2)$ be elements of order 3 and 7, respectively.*

i) If $q \equiv 1 \pmod 3$, then h is real;

ii) if $q \equiv 2, 4 \pmod 7$, then g is rational;

iii) if $q \equiv 1 \pmod 7$, then g is real;

iv) if $q \equiv 3, 5 \pmod 7$, then g is semirational.

Proof

Let $A \in \mathrm{SU}(n, q^2)$ be semisimple. W.l.o.g. we may assume that $A^t\overline{A} = I$, where $\overline{A}$ is obtained from A applying the field automorphism $\alpha \mapsto \alpha^q$ to all its entries. By the above remark, A is conjugate to $(\overline{A}^{-1})$. Moreover, considering the Jordan canonical form of A over $\overline{F}$, it is clear that $\overline{A}$ is conjugate to A^q. Hence A is conjugate to A^{-q}.
i) h is conjugate to $h^{-q} = h^{-1}$ by the previous observation. *ii*) g is semirational and conjugate to g^{-2} or to g^{-4}. It follows g rational. *iii*) g is conjugate to g^{-1}. *iv*) In this case $q^2 \equiv 2, 4 \pmod 7$. Therefore g is semirational. □

Table 2: admissible conjugacy classes for x

n	s	Representatives	d_S^x
4	2	$(\,(\pi_2)^2\,)$	6
5	3	$(\,1, (\pi_2)^2\,)$	9
6	4	$(\,1^2, (\pi_2)^2\,)$	13
	3 , p= 2	$(\,(\pi_2)^3\,)$ p= 2	12
7	4	$(\,-1, (\pi_2)^3\,)$	16
8	4	$(\,(\pi_2)^4\,)$	20
	5 , p= 2	$(\,1^2, (\pi_2)^3\,)$ p= 2	21
9	5	$(\,1, (\pi_2)^4\,)$	25
10	6	$(\,1^2, (\pi_2)^4\,)$	31
	5 , p= 2	$(\,(\pi_2)^5\,)$ p= 2	30
11	6	$(\,-1, (\pi_2)^5\,)$	36
12	6	$(\,(\pi_2)^6\,)$	42
	7 , p= 2	$(\,1^2, (\pi_2)^5\,)$ p= 2	43
13	7,8	$(\,1, (\pi_2)^6\,)$, $(\,(-1)^3, (\pi_2)^5\,)$	49, 51
14	8	$(\,1^2, (\pi_2)^6\,)$	57
	7 , p= 2	$(\,(\pi_2)^7\,)$ p= 2	56
15	8,9	$(\,-1, (\pi_2)^7\,)$, $(\,1^3, (\pi_2)^6\,)$	64, 66
16	8	$(\,(\pi_2)^8\,)$	72
	9 , p= 2	$(\,1^2, (\pi_2)^7\,)$ p= 2	73
17	9,10	$(\,1, (\pi_2)^8\,)$, $(\,(-1)^3, (\pi_2)^7\,)$	81, 83
18	10	$(\,1^2, (\pi_2)^8\,)$	91
	9,11 , p= 2	$(\,(\pi_2)^9\,)$, $(\,1^4, (\pi_2)^7\,)$ p=2	90,94
19	10,11	$(\,-1, (\pi_2)^9\,)$, $(\,1^3, (\pi_2)^8\,)$	100,102
20	10,12	$(\,(\pi_2)^{10}\,)$, $(\,1^4, (\pi_2)^8\,)$	110,114
	11 , p= 2	$(\,1^2, (\pi_2)^9\,)$ p=2	111
21	11,12,13	$(\,1, (\pi_2)^{10}\,)$, $(\,(-1)^3, (\pi_2)^9\,)$, $(\,1^5, (\pi_2)^8\,)$	121,123,127
22	12	$(\,1^2, (\pi_2)^{10}\,)$	133
	11,13, p= 2	$(\,(\pi_2)^{11}\,)$, $(\,1^4, (\pi_2)^9\,)$ p=2	132,136

Table 3: admissible conjugacy classes for y

n	s	Representatives	d_S^y
4	2	$(\,1,\pi_3\,)$	4
5	2	$(\,d_2,\pi_3\,)$	5
6	2	$(\,(\pi_3)^2\,)$	7
7	3	$(\,1,(\pi_3)^2\,)\ ,\ (\,(d_2)^2,\pi_3\,)$	10
8	3, 4	$(\,d_2,(\pi_3)^2\,)\ ,\ (\,(1)^2,(\pi_3)^2\,)$	12,14
9	3,4	$(\,(\pi_3)^3\,)$	15
		$(\,J_1,J_2,(\pi_3)^2\,)\ p=3$	16
10	4	$(\,1,(\pi_3)^3\,)$	19
		$(\,(d_2)^2,(\pi_3)^2\,)\ p=2$	19
11	4,5	$(\,d_2,(\pi_3)^3\,)\ ,\ (\,(1)^2,(\pi_3)^3\,)$	22,24
12	4,5	$(\,(\pi_3)^4\,)$	26
		$(\,J_1,J_2,(\pi_3)^3\,)\ p=3$	27
13	5,6	$(\,1,(\pi_3)^4\,)\ ,\ (\,(d_2)^2,(\pi_3)^3\,)$	31
		$(\,(\omega)^2,(1,\omega),(\pi_3)^3\,)\ p\neq 3$	28
		$(\,(J_1)^2,J_2,(\pi_3)^3\,)\ p=3$	33
14	5,6	$(\,d_2,(\pi_3)^4\,)\ ,\ (\,(1)^2,(\pi_3)^4\,)$	35,37
		$(\,\omega,(1,\omega)^2,(\pi_3)^3\,)\ p\neq 3$	33
		$(\,J_1,(J_2)^2,(\pi_3)^3\,)\ p=3$	37
15	5,6,7	$(\,(\pi_3)^5\,)\ ,\ (\,d_2^3,(\pi_3)^3\,)\ ,\ (\,(1)^3,(\pi_3)^4\,)$	40, 42, 44
		$(\,(\omega)^3,(\pi_3)^4\,)\ ,\ (\,(1,\omega)^3,(\pi_3)^3\,)\ p\neq 3$	38, 39
		$(\,J_1,J_2,(\pi_3)^4\,)\ p=3$	41
16	6,7	$(\,1,(\pi_3)^5\,)\ ,\ (\,(d_2)^2,(\pi_3)^4\,)$	46
		$(\,(\omega)^2,(1,\omega),(\pi_3)^4\,)\ p\neq 3$	43
		$(\,(J_1)^2,J_2,(\pi_3)^4\,)\ p=3$	48
17	6,7	$(\,d_2,(\pi_3)^5\,)\ ,\ (\,(1)^2,(\pi_3)^5\,)$	51, 53
		$(\,\omega,(1,\omega)^2,(\pi_3)^4\,)\ p\neq 3$	49
		$(\,J_1,(J_2)^2,(\pi_3)^4\,)\ p=3$	53
18	6,7,8	$(\,(\pi_3)^6\,)\ ,\ (\,(d_2)^3,(\pi_3)^4\,)\ ,\ (\,(1)^3,(\pi_3)^5\,)$	57,59,61
		$(\,(\omega)^3,(\pi_3)^5\,)\ ,\ (\,(1,\omega)^3,(\pi_3)^4\,)\ p\neq 3$	55,56
		$(\,(J_1,J_2,(\pi_3)^5\,)\ p=3$	58
19	7,8	$(\,1,(\pi_3)^6\,)\ ,\ (\,d_2^2,(\pi_3)^5\,)$	64
		$(\,1,(1,\omega)^3,(\pi_3)^4\,)\ ,\ (\,(\omega)^2,(1,\omega),(\pi_3)^5\,)\ p\neq 3$	64,61
		$(\,J_1,(J_2)^3,(\pi_3)^4\,)\ ,\ (\,(J_1)^2,J_2,(\pi_3)^5\,)\ p=3$	67,66

Table 3 (continuation)

n	s	y	d_S^y
20	7,8,9	$(d_2,(\pi_3)^6)$, $((d_2)^4,(\pi_3)^4)$, $((1)^2,(\pi_3)^6)$	70,74,72
		$(\omega,(\omega,1)^2,(\pi_3)^5)$, $((\omega)^3,d_2,(\pi_3)^5)$ $p\neq 3$	68,69
		$(J_1,(J_2)^2,(\pi_3)^5)$, $((J_1)^3,J_2,(\pi_3)^5)$ $p=3$	72,75
21	7,8,9	$((\pi_3)^7)$, $((d_2)^3,(\pi_3)^5)$, $((1)^3,(\pi_3)^6)$	77,79,81
		$((\omega)^3,(\pi_3)^6)$, $((\omega,1)^3,(\pi_3)^5)$ $p\neq 3$	75,76
		$(J_1,J_2,(\pi_3)^6)$, $((J_1)^2,(J_2)^2,(\pi_3)^5)$ $p=3$	78,81
22	8,9,10	$(1,(\pi_3)^7)$, $((d_2)^2,(\pi_3)^6)$, $((1)^4,(\pi_3)^6)$	85,91
		$((\omega)^2,(1,\omega),(\pi_3)^6)$, $(1,(1,\omega)^3,(\pi_3)^5)$ $p\neq 3$	82,85
		$((J_1)^2,J_2,(\pi_3)^6)$, $(J_1,(J_2)^3,(\pi_3)^5)$ $p=3$	87,88
		$((\omega)^3,(d_2)^2,(\pi_3)^5)$ $p=2$	85

Table 4 : admissible conjugacy classes for z modular

n	s	Representatives	d_S^z
4	1	(J_4)	2
5	1	(J_5)	3
6	1	(J_6)	3
7	1,2	(π_7) , (J_1,J_6)	4,5
8	2	(J_1,J_7) , (J_2,J_6)	6
9	2,3	(J_2,J_7) , (J_3,J_6) , $((J_1)^2,J_7)$	7,8,9
10	2	(J_3,J_7)	9
11	2,3	(J_4,J_7) , (J_5,J_6) , (J_1,J_3,J_7)	10,11,12
12	2,3	(J_5,J_7) , $((J_6)^2)$	12
		(J_1,J_4,J_7) , (J_1,J_5,J_6) , (J_2,J_3,J_7)	13,14
13	2,3	(J_6,J_7) , (J_1,J_5,J_7)	13,15
		$(J_1,(J_6)^2)$, (J_2,J_4,J_7)	15
14	2,3	$((J_7)^2)$, (J_1,J_6,J_7) , (J_2,J_5,J_7)	15,16,17
15	3,4	$(J_1,(J_7)^2)$, (J_2,J_6,J_7) , (J_3,J_5,J_7) , $(J_3,(J_6)^2)$	18, 20
		$((J_4)^2,J_7)$, $((J_1)^2,J_6,J_7)$, (J_1,J_2,J_5,J_7)	20, 21
16	3,4	$(J_2,(J_7)^2)$, (J_3,J_6,J_7) , (J_4,J_5,J_7) , $(J_4,(J_6)^2)$	20, 21, 22
		$((J_1)^2,(J_7)^2)$, (J_1,J_2,J_6,J_7) , (J_1,J_3,J_5,J_7)	22,24

Table 4 (continuation)

n	s	z	d_S^z
17	3,4,5	$(\,J_3,(J_7)^2\,)\ ,\ (\,J_4,J_6,J_7\,)\ ,\ (\,(J_5)^2,J_7\,)\ (\,J_5,(J_6)^2\,)$	23, 25
		$(\,J_1,J_2,(J_7)^2\,)\ ,\ (\,J_1,J_3,J_6,J_7\,)\ ,\ (\,J_1,J_4,J_5,J_7\,)$	24,25
		$(\,(J_2)^2,J_6,J_7\,)\ ,\ (\,(J_1)^3,(J_7)^2\,)$	26,25,27
18	3,4,5	$(\,J_4,(J_7)^2\,)\ ,\ (\,J_5,J_6,J_7\,)\ ,\ (\,(J_6)^3\,)\ ,\ (\,J_1,J_3,(J_7)^2\,)$	25,26,27
		$(\,J_1,(J_5)^2,J_7\,)\ ,\ (\,(J_2)^2,(J_7)^2\,)\ ,\ (\,J_2,J_3,J_6,J_7\,)$	29,27,28
		$(\,J_1,J_4,J_6,J_7\,)\ ,\ (\,(J_1)^2,J_2,(J_7)^2\,)$	27,29
19	3,4,5	$(\,J_5,(J_7)^2\,)\ ,\ (\,(J_6)^2,J_7\,)\ ,\ (\,J_1,J_4,(J_7)^2\,)\ ,\ (\,J_1,J_5,J_6,J_7\,)$	28,29,30
		$(\,J_1,(J_6)^3\,)\ ,\ (\,J_2,J_3,(J_7)^2\,)\ ,\ (\,J_2,J_4,J_6,J_7\,)$	31,30
		$(\,J_2,(J_5)^2,J_7\,)\ ,\ (\,(J_3)^2,J_6,J_7\,)\ ,\ (\,(J_1)^2,J_3,(J_7)^2\,)$	32
		$(\,(J_1)^2,J_4,J_6,J_7\,)\ ,\ (\,J_1,(J_2)^2,(J_7)^2\,)$	32
20	3,4,5	$(\,J_6,(J_7)^2\,)\ ,\ (\,J_1,(J_6)^2,J_7\,)\ ,\ (\,J_1,J_5,(J_7)^2\,)\ ,\ (\,J_2,J_4,(J_7)^2\,)$	30,32
		$(\,J_2,(J_6)^3\,)\ ,\ (\,(J_3)^2,(J_7)^2\,)\ ,\ (\,J_3,J_4,J_6,J_7\,)$	34
		$(\,J_2,J_5,J_6,J_7\,)\ ,\ (\,J_3,(J_5)^2,J_7\,)\ ,\ (\,(J_1)^2,J_4,(J_7)^2\,)$	33,36,34
		$(\,(J_1)^2,J_5,J_6,J_7\,)\ ,\ (\,J_1,J_2,J_3,(J_7)^2\,)\ ,\ (\,J_1,J_2,J_4,J_6,J_7\,)$	35
21	3,4,5,6	$(\,(J_7)^3\,)\ ,\ (\,J_1,J_6,(J_7)^2\,)\ ,\ (\,J_2,J_5,(J_7)^2\,)\ ,\ (\,J_2,(J_6)^2,J_7\,)$	33,34,35
		$(\,J_3,J_4,(J_7)^2\,)\ ,\ (\,J_3,J_5,J_6,J_7\,)\ ,\ (\,(J_4)^2,J_6,J_7\,)\ ,\ (\,J_3,(J_6)^3\,)$	36,37,38
		$(\,J_4,(J_5)^2,J_7\,)\ ,\ (\,(J_1)^2,J_5,(J_7)^2\,)\ ,\ (\,(J_1)^2,(J_6)^2,J_7\,)$	39,37
		$(\,J_1,J_2,J_4,(J_7)^2\,)\ ,\ (\,J_1,J_2,J_5,J_6,J_7\,)\ ,\ (\,J_1,(J_3)^2,(J_7)^2\,)$	37,38,39
		$(\,J_1,J_3,J_4,J_6,J_7\,)\ ,\ (\,(J_2)^2,J_3,(J_7)^2\,)\ ,\ (\,(J_1)^3,J_4,(J_7)^2\,)$	39,40
22	4,5,6	$(\,J_1,(J_7)^3\,)\ ,\ (\,J_2,J_6,(J_7)^2\,)\ ,\ (\,J_3,J_5,(J_7)^2\,)\ ,\ (\,J_3,(J_6)^2,J_7\,)$	37,39
		$(\,(J_4)^2,(J_7)^2\,)\ ,\ (\,(J_1)^2,J_6,(J_7)^2\,)\ ,\ (\,J_4,J_5,J_6,J_7\,)$	39,40
		$(\,J_1,J_2,J_5,(J_7)^2\,)\ ,\ (\,J_1,J_2,(J_6)^2,J_7\,)\ ,\ (\,J_1,J_3,J_4,(J_7)^2\,)\ ,$	40,41
		$(\,J_1,J_3,J_5,J_6,J_7\,)\ ,\ (\,(J_2)^2,J_4,(J_7)^2\,)\ ,\ (\,(J_1)^3,J_5,(J_7)^2\,)$	42,41,43

In Table $4'$ below we set $d_6 := (\epsilon, \epsilon^2, \epsilon^3, \epsilon^{-3}, \epsilon^{-2}, \epsilon^{-1})$.

Table $4'$: admissible conjugacy classes for z semisimple
($p \neq 2$ if $n = 14$)

n	s	Representatives	d_S^z
4	1	$(\epsilon, \epsilon^3, \epsilon^{-3}, \epsilon^{-1})$, $(\epsilon, \epsilon^2, \epsilon^{-2}, \epsilon^{-1})$, $(1, \epsilon, \epsilon^2, \epsilon^{-3})$	2,1
5	1	$(1, \epsilon, \epsilon^3, \epsilon^{-3}, \epsilon^{-1})$, $(1, \epsilon, \epsilon^2, \epsilon^{-2}, \epsilon^{-1})$	3
6	1	d_6	3
	2	$(\epsilon, 1, \epsilon, \epsilon^3, \epsilon^{-3}, \epsilon^{-2})$, $(\epsilon, 1, \epsilon, \epsilon^2, \epsilon^{-3}, \epsilon^{-1})$ p=2	2,3
	2	$(1, 1, \epsilon, \epsilon^3, \epsilon^{-3}, \epsilon^{-1})$, $(1, 1, \epsilon, \epsilon^2, \epsilon^{-2}, \epsilon^{-1})$ p=2	5
7	1	(π_7)	4
10	2	$(\epsilon, \epsilon^2, \epsilon^{-3}, \pi_7)$, $(1, \epsilon, \epsilon^{-1}, \pi_7)$	7,9
	2	$(\epsilon, \epsilon^2, \epsilon^{-2}, \epsilon^{-1}, d_6)$, $(\epsilon, \epsilon^3, \epsilon^{-3}, \epsilon^{-1}, d_6)$	9
	2	$(\epsilon, \epsilon^{-3}, \epsilon^{-2}, \epsilon^{-1}, 1, \epsilon, \epsilon^3, \epsilon^{-3}, \epsilon^{-2}, \epsilon^{-1})$, $(\epsilon, \epsilon^3, \epsilon^{-3}, \epsilon^{-2}, 1, \epsilon, \epsilon^2, \epsilon^3, \epsilon^{-3}, \epsilon^{-2})$	7
	2	$(\epsilon, \epsilon^2, \epsilon^3, \epsilon^{-2}, 1, \epsilon, \epsilon^2, \epsilon^3, \epsilon^{-2}, \epsilon^{-1})$, $(\epsilon, \epsilon^2, \epsilon^3, \epsilon^{-1}, 1, \epsilon, \epsilon^2, \epsilon^3, \epsilon^{-3}, \epsilon^{-1})$	7
	3,2	$(\epsilon, \epsilon, \epsilon^{-2}, \pi_7)$, $(1, \epsilon, \epsilon^3, \epsilon^{-2}, 1, \epsilon, \epsilon^3, \epsilon^{-3}, \epsilon^{-2}, \epsilon^{-1})$	7
	2	$(1, \epsilon, \epsilon^{-3}, \epsilon^{-2}, 1, \epsilon, \epsilon^2, \epsilon^{-3}, \epsilon^{-2}, \epsilon^{-1})$, $(1, \epsilon, \epsilon^2, \epsilon^3, 1, \epsilon, \epsilon^2, \epsilon^3, \epsilon^{-3}, \epsilon^{-2})$	7
13	2	(d_6, d_7)	13
	3	$(\epsilon, \epsilon^{-3}, \epsilon, \epsilon^{-3}, \epsilon^{-2}, \epsilon^{-1}, \pi_7)$, $(\epsilon, \epsilon^{-2}, \epsilon, \epsilon^{-2}, \epsilon^3, \epsilon^{-1}, \pi_7)$	12
	3	$(\epsilon, \epsilon^{-1}, \epsilon, \epsilon^{-1}, \epsilon^3, \epsilon^{-3}, \pi_7)$, $(\epsilon, \epsilon^{-1}, \epsilon, \epsilon^{-1}, \epsilon^2, \epsilon^{-2}, \pi_7)$	15
	3	$(\epsilon, \epsilon^3, \epsilon, \epsilon^3, \epsilon^2, \epsilon^{-3}, \pi_7)$, $(\epsilon, \epsilon^2, \epsilon, \epsilon^2, \epsilon^3, \epsilon^{-2}, \pi_7)$	12
	3	$(\epsilon, 1, \epsilon, \epsilon^3, \epsilon^{-3}, \epsilon^{-2}, \pi_7)$, $(\epsilon, \epsilon^3, 1, \epsilon, \epsilon^3, \epsilon^{-1}, \pi_7)$	12,13
	3	$(\epsilon, \epsilon^{-2}, 1, \epsilon, \epsilon^2, \epsilon^{-2}, \pi_7)$, $(\epsilon, 1, \epsilon, \epsilon^2, \epsilon^{-3}, \epsilon^{-1}, \pi_7)$	13
	3	$(\epsilon, 1, \epsilon, \epsilon^2, \epsilon^3, \epsilon^{-2}, \epsilon^{-1}, 1, \epsilon, \epsilon^2, \epsilon^3, \epsilon^{-2}, \epsilon^{-1})$, $(1, \epsilon, 1, \epsilon, \epsilon^2, \epsilon^3, \pi_7)$	13
	3	$(1, 1, \epsilon, \epsilon^3, \epsilon^{-3}, \epsilon^{-1}, \pi_7)$, $(1, 1, \epsilon, \epsilon^2, \epsilon^{-2}, \epsilon^{-1}, \pi_7)$	15
14	3	$(\epsilon, \epsilon^{-2}, \epsilon, \epsilon^2, \epsilon^3, \epsilon^{-3}, \epsilon^{-2}, \pi_7)$, $(\epsilon, \epsilon^3, \epsilon, \epsilon^2, \epsilon^3, \epsilon^{-2}, \epsilon^{-1}, \pi_7)$	14
	3	$(\epsilon, \epsilon^{-3}, \epsilon, \epsilon^2, \epsilon^3, \epsilon^{-3}, \epsilon^{-1}, \pi_7)$, $(\epsilon, \epsilon^2, \epsilon, \epsilon^2, \epsilon^{-3}, \epsilon^{-2}, \epsilon^{-1}, \pi_7)$	15
	3	$((\pi_7)^2)$, $(\epsilon, \epsilon^{-3}, 1, \epsilon, \epsilon^{-3}, \epsilon^{-2}, \epsilon^{-1}, \pi_7)$	15,14
	3	$(\epsilon, \epsilon^{-2}, 1, \epsilon, \epsilon^3, \epsilon^{-2}, \epsilon^{-1}, \pi_7)$	14
	3	$(\epsilon, \epsilon^{-1}, 1, \epsilon, \epsilon^3, \epsilon^{-3}, \epsilon^{-1}, \pi_7)$, $(\epsilon, \epsilon^{-1}, 1, \epsilon, \epsilon^2, \epsilon^{-2}, \epsilon^{-1}, \pi_7)$	17
	3	$(\epsilon, \epsilon^3, 1, \epsilon, \epsilon^2, \epsilon^3, \epsilon^{-3}, \pi_7)$, $(\epsilon, \epsilon^2, 1, \epsilon, \epsilon^2, \epsilon^3, \epsilon^{-2}, \pi_7)$	14
	3	$(1, \epsilon, 1, \epsilon, \epsilon^3, \epsilon^{-3}, \epsilon^{-2}, \pi_7)$, $(1, \epsilon, 1, \epsilon, \epsilon^2, \epsilon^{-3}, \epsilon^{-1}, \pi_7)$	15,16

Table 5 (Lemmas 5.5, 5.6, 5.7)

q	$\mathrm{SL}(n,q)$	$\mathrm{Sp}(n,q)$	$\mathrm{SU}(n,q^2)$
$\equiv 1\ (21)$		y real, z real	y real, z real
$\equiv 2,-10\ (21)$	y real, z semirat.	y real, z rational	z rational
$\equiv 4,-5\ (21)$	z semirat.	y real, z rational	y real, z rational
$\equiv 5,-4\ (21)$	y real, z rational	y real, z rational	z semirat.
$\equiv 8\ (21)$	y real	y real, z real	z real
$\equiv 10,-2\ (21)$	z rational	y real, z rational	y real, z semirat.
$\equiv -8\ (21)$	z real	y real, z real	y real
$\equiv -1\ (21)$	y real, z real	y real, z real	
3^{6h}	y modular	y modular, z real	y modular, z real
3^{6h+1}	y modular, z rational	y modular, z rational	y modular, z semirat.
3^{6h+2}, 3^{6h+4}	y modular, z semirat.	y modular, z rational	y modular, z rational
3^{6h+3}	y modular, z real	y modular, z real	y modular
3^{6h+5}	y modular, z rational	y modular, z rational	y modular, z semirat.
7^t	z modular	y real, z modular	y real, z modular

6. Proof of Theorem 1

We start with an easy but useful remark. Namely, let $k \geq 7$ be a prime, G be a $(2,3,k)$-generated group, and $f : G \to \mathrm{SL}(n,F)$ be a representation of degree $n \leq 2$. If $p \neq 2$, the only involution of $\mathrm{SL}(2,F)$ is $-I$, and therefore $f(G)$ is abelian. Hence $f(G) = \{1\}$. This readily implies that a $(2,3,k)$-generated subgroup G of $\mathrm{SL}(3,F)$ or $\mathrm{SL}(4,F)$ which does not fix any non-zero vector is absolutely irreducible. For otherwise, by the above remark, G necessarily admits the eigenvalue 1.

i) Let $\overline{H} \leq \mathrm{SL}(2,\overline{F})$ be the preimage of a $(2,3,k)$-generated subgroup of $\mathrm{PSL}(2,\overline{F})$. Clearly we may assume $\overline{H} = \langle \overline{x}, z\rangle$ where $\overline{x}^2 = -I$, $z^k = I$. Moreover $\overline{H}$ is irreducible, since it is not soluble. Therefore, if u is an eigenvector of z, u and $\overline{x}u$ are linearly independent. It follows that, with respect to the basis $\{u, \overline{x}u\}$,

$$\overline{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad z = z_i = \begin{pmatrix} \epsilon^i & a \\ 0 & \epsilon^{-i} \end{pmatrix} \quad 1 \leq i \leq k-1$$

for a suitable $a \in \overline{F}$. Thus $y_i := \overline{x} z_i$ has trace a. On the other hand, y_i must be conjugate to $\pm\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, which gives $a = \pm 1$. Both choices for z_i give rise to conjugate subgroups. This is clear if $p = k$, $\epsilon = 1$. On the other hand, if $p \neq k$, let $t := \begin{pmatrix} 1 & \epsilon^{-i} - \epsilon^{i} \\ -\epsilon^{-i} + \epsilon^{i} & 1 \end{pmatrix}$. t is non-singular, centralizes $\overline{x}$ and

$$t^{-1}\begin{pmatrix} \epsilon^i & -1 \\ 0 & \epsilon^{-i} \end{pmatrix} t = \begin{pmatrix} \epsilon^i & 1 \\ 0 & \epsilon^{-i} \end{pmatrix}^{-1}.$$

Hence we may assume $a = 1$. Let $\overline{H}_i := \langle \overline{x}, z_i \rangle$. Note that, for every i, $(\overline{x} z_i)^3 = -I$ and $\overline{H}_i \leq \mathrm{SL}(2, R)$, where R is isomorphic to $\mathbb{Q}(\epsilon)$ if $p = 0$, $R = F_{p^{k-1}}$ if $p > 0$. If $p = 0$, we may consider the automorphism γ_i of $\mathrm{GL}(n, R)$ induced by the automorphism of R sending ϵ to ϵ^i, $1 \leq i \leq k-1$. Since $\overline{H}_i = \gamma_i(\overline{H}_1)$, the projective image of $\overline{H}_1$ is, up to isomorphism, the only $(2,3,k)$-generated subgroup of $\mathrm{PSL}(2, \overline{F})$, and therefore (cfr. [Mag]) it is isomorphic to $\Delta(2,3,k)$.
Now assume $p > 0$ and $k = 7$, hence $\overline{H}_i \leq \mathrm{SL}(2, F_{p^6})$. As $\overline{H}_i$ is perfect, it follows from the knowledge of the subgroups of $\mathrm{PSL}(2, p^6)$ that the projective image H_i of $\overline{H}_i$ is isomorphic to $\mathrm{PSL}(2, p^{m_i})$ for some $m_i \in \{1, 2, 3, 6\}$. The results of [Mac] give $H_i \simeq \mathrm{PSL}(2, p)$ if $p \equiv 0, \pm 1 \pmod 7$ and $H_i \simeq \mathrm{PSL}(2, p^3)$ if $p \not\equiv 0, \pm 1 \pmod 7$.
ii) A direct computation shows that the group

$$\varphi(H) = \langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \epsilon^2 & 2\epsilon & 1 \\ 0 & 1 & \epsilon^{-1} \\ 0 & 0 & \epsilon^{-2} \end{pmatrix} \rangle$$

preserves the bilinear symmetric form defined by the matrix $\mathrm{antidiag}(1, -2, 1)$; moreover, if $p \neq 2$, it does not fix any non-zero vector. Hence $\varphi(H)$ is an irreducible subgroup of $\mathrm{SO}(3, \overline{F})$, for all $k \geq 7$. From now on assume $k = 7$, and let $K = \langle x, z \rangle$ be an irreducible subgroup of $\mathrm{SL}(3, \overline{F})$, with $x^2 = z^7 = (xz)^3 = 1$. Clearly x has a 2-dimensional eigenspace. Therefore, by the irreducibility of K, z cannot have a 2-dimensional eigenspace. In particular, if $p \neq 7$, z is either conjugate to $z_1 = \mathrm{diag}(\eta, \eta^2, \eta^4)$ or to $z_2 = \mathrm{diag}(\eta, 1, \eta^{-1})$, where $\eta = \epsilon^i$ for some i with $1 \leq i \leq 6$. Notice that z leaves invariant a 2-dimensional subspace $\langle u, v \rangle$. If $p \neq 7$, we may assume $zu = \eta u, zv = \eta^2 v$ if $z \sim z_1$ and $zu = \eta u, zv = v$ if $z \sim z_2$. On the other hand, if $p = 7$, we may assume $zu = u$. Since K is irreducible, u, v, xu are linearly independent. Moreover, as $\det x = 1$, $xv = au - v + axu$ for some $a \in \overline{F}$. It follows that, with respect to the basis $\{u, v - au, xu\}$

$$x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ and } z = z_1 = \begin{pmatrix} \eta & b & c \\ 0 & \eta^2 & d \\ 0 & 0 & \eta^4 \end{pmatrix} \text{ or } z = z_2 = \begin{pmatrix} \eta & b & c \\ 0 & 1 & d \\ 0 & 0 & \eta^{-1} \end{pmatrix}$$

where b, c, d are suitable elements of $\overline{F}$ (and $\eta = 1$ if $p = 7$). Note that $b \neq 0$, again by the irreducibility of K. Thus, conjugating x and z_i by $\mathrm{diag}(1, b^{-1}, 1)$, we may also assume $b = 1$.

1) Let $K_1 = \langle x, z_1 \rangle$. As $y = xz_1$ must have trace 0, it follows $c = \eta^2$. Moreover, equating the elements in position (1,1) of y^2 and y^{-1}, one gets $d = \eta^4 + \eta^5$. This gives

$$z_1 = \begin{pmatrix} \eta & 1 & \eta^2 \\ 0 & \eta^2 & \eta^4 + \eta^5 \\ 0 & 0 & \eta^4 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 & \eta^4 \\ 0 & -\eta^2 & -\eta^4 - \eta^5 \\ \eta & 1 & \eta^2 \end{pmatrix}$$

and a direct calculation shows that $z_1^7 = y^3 = [x,y]^4 = 1$. But the relations $x^2 = y^3 = (xy)^7 = [x,y]^4 = 1$ correspond to a well-known presentation for $\mathrm{PSL}(2,7)$ (see [CM], 7.5, p.96). We conclude $K_1 \simeq \mathrm{PSL}(2,7)$. Note that K_1 is irreducible, as it does not fix any non-zero vector.

2) Now let $K_2 := \langle x, z_2\rangle$, Again $y = xz_2$ has trace 0, hence $c = 1$. Moreover, equating the elements in position (1,1) of y^2 and y^{-1}, one gets $d = 2$. This gives

$$x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad z_2 = \begin{pmatrix} \eta & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \eta^{-1} \end{pmatrix}.$$

If $p = 2$, K_2 is reducible, whereas if $p \neq 2$ K_2 is irreducible, as it does not fix any non-zero vector. In the latter case, conjugating x and z_2 by $\mathrm{diag}\,(1, 2\eta^4, 1)$, one gets the matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \varphi\Big(\pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\Big), \quad \begin{pmatrix} \eta & 2\eta^4 & 1 \\ 0 & 1 & \eta^3 \\ 0 & 0 & \eta^6 \end{pmatrix} = \varphi\Big(\pm\begin{pmatrix} \eta^4 & 1 \\ 0 & \eta^{-4} \end{pmatrix}\Big)$$

where φ is the embedding given in the statement. It follows from *i)* that the projective image of

$$\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \eta^4 & 1 \\ 0 & \eta^{-4} \end{pmatrix} \rangle$$

is isomorphic to H, in other words $K_2 \simeq \varphi(H)$.

Finally, let $\langle x, z\rangle \leq \mathrm{SL}(3, \overline{F})$ be the preimage of a $(2,3,7)$-generated subgroup of $\mathrm{PSL}(3,\overline{F})$. We may assume $p \neq 3$, so that the centre of $\mathrm{SL}(3,\overline{F})$ has order 3 and is generated by the scalar matrix $\beta^3 I$, where β is a primitive 9^{th} root of 1. We may also assume that x and z have respective orders 2 and 7. In particular, x is conjugate to $\mathrm{diag}(-1,-1,1)$ and has an eigenspace of dimension 2. Assume, by way of contradiction, that $y = x\,z$ has order 9, hence $y^3 = \beta^3 I$. Since y must have pairwise distinct eigenvalues by the irreducibility of $\langle x, z\rangle$, y should be conjugate to $\mathrm{diag}(\beta, \beta^4, \beta^7)$, which has determinant $\neq 1$, a contradiction. It follows that y has order 3, hence $\langle x, z\rangle$ is conjugate either to K_1 or to K_2.

iii) Under our assumptions we may consider the automorphism γ_i of $\mathrm{GL}(n, R)$ induced by $\epsilon \mapsto \epsilon^i$, for some $i \not\equiv 0, \pm 1 \pmod k$. The image of H under the map $g \mapsto g \otimes \gamma_i(g)$ is the group

$$\langle \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \epsilon^{i+1} & \epsilon^i & \epsilon & 1 \\ 0 & \epsilon^{i-1} & 0 & \epsilon^{-1} \\ 0 & 0 & \epsilon^{-i+1} & \epsilon^{-i} \\ 0 & 0 & 0 & \epsilon^{-i-1} \end{pmatrix} \rangle.$$

As 1 is not an eigenvalue of the second generator, it follows, from the remark at the beginning of the proof, that this group is irreducible unless possibly when $p = 2$ and it leaves invariant a 2-dimensional subspace. But this possibility is ruled out, using a direct argument (omitted here, but entirely similar to the one given below, see *iv*). Note that the above group preserves the symmetric bilinear form defined by the matrix $J \otimes J$, where $J = \mathrm{antidiag}(-1, 1)$, hence it is orthogonal if $p \neq 2$.

iv) The group $\overline{H} = \langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} \epsilon & 1 \\ 0 & \epsilon^{-1} \end{pmatrix} \rangle$ considered in *i)* acts on the third symmetric power of $V := \overline{F}^2$, that is on the space of homogeneous polynomials of degree 3 in two variables t, u. With respect to the basis $\{t^3, t^2u, tu^2, u^3\}$, $\overline{H}$ acts faithfully as

$$\langle \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} \epsilon^3 & \epsilon^2 & \epsilon & 1 \\ 0 & \epsilon & 2 & 3\epsilon^{-1} \\ 0 & 0 & \epsilon^{-1} & 3\epsilon^{-2} \\ 0 & 0 & 0 & \epsilon^{-3} \end{pmatrix} \rangle.$$

$\overline{H}$ preserves the symplectic form defined by the matrix $\mathrm{antidiag}(-3, 1, -1, 3)$, and the projective image of $\overline{H}$ is isomorphic to H. If $p = 3$, this form is degenerate, thus $\overline{H}$ is reducible. Let $p \neq 3$. As $\overline{H}$ is perfect and 1 is not a common eigenvalue of $\overline{x}$ and z, it follows easily that $\overline{H}$ cannot leave invariant a subspace of dimension 1 or 3. Assume that $\overline{H}$ leaves invariant a 2-dimensional subspace $U = \langle u_1, u_2 \rangle$. If $p \neq k$, $z_{|U}$ is diagonalizable and must have determinant 1. Thus we may choose as u_1 an eigenvector of z with corresponding eigenvalue ϵ^3 or ϵ, and set $u_2 = \overline{x}\, u_1$. A direct calculation of the eigenvectors of z gives the two possibilities:

$$U = \langle e_1,\ e_4 \rangle \qquad \text{or} \qquad U = \Big\langle \ \frac{\epsilon}{1-\epsilon^2}\, e_1 + e_2 \ , \ -e_3 + \frac{\epsilon}{1-\epsilon^2}\, e_4 \ \Big\rangle ,$$

where $\{e_1, e_2, e_3, e_4\}$ is the standard basis of $\overline{F}^4$. In both cases U is not z-invariant. Finally let $p = k$, hence $\epsilon = 1$. Clearly U contains the unique eigenspace of z, generated by e_1. Since $\overline{x}\, e_1 = e_4$, we obtain $U = \langle e_1, e_4 \rangle$. But, as noted above, this subspace is not z-invariant. We conclude that $\overline{H}$ is irreducible. □

7. Case by case analysis

Our aim in this section is to prove Theorems 2,3,4 and 5. In the statements of all these theorems, H is an absolutely irreducible subgroup of $\mathrm{SL}(n, F)$. Furthermore, if $p \neq 2$, H is not contained in an orthogonal group and, if $p = 2$, H is not contained in a symplectic group. Hence, by Lemma 4.1, $d_S^H = 0$. Now, suppose that H is generated by x, y, $z = xy$ of order 2, 3 and 7 respectively. By applying Scott' s formula to the adjoint representation of H on $M = \mathrm{Mat}(n, F)$ and to the representation of H on the symmetric square S, we obtain the necessary conditions:

$$(*) \quad d_M^x + d_M^y + d_M^z \leq n^2 + 2 \qquad \text{and} \qquad (**) \quad d_S^z \leq \dim S - (d_S^x + d_S^y).$$

We conclude that H is not $(2, 3, 7)$-generated whenever the assumption that x, y, z satisfy both $(*)$ (or a refined version of it, cfr. Lemma 7.1 below) and $(**)$ leads to a contradiction.

We note that x, y, z satisfy $(*)$ precisely when x, y, z belong to the 'admissible ' conjugacy classes obtained in Section 5. These classes are listed in Tables 2 and 3 for x and y, and in Table 4 for z modular. For certain values of n they are also listed in Table $4'$ for z semisimple. The tables also give the corresponding values of d_S^x, d_S^y, d_S^z.

Proof of Theorem 2

Let H be as stated in the theorem, and assume that it is (2,3,7)-generated. The assumption that x, y, z satisfy $(*)$ and $(**)$ leads immediately to a contradiction, except in the case $n = 6$, $p = 2$, which requires a little more work.

$\underline{n=4}$. $d_S^x = 6$, $d_S^y = 4$ and $\dim S = 10$ imply $d_S^z \leq 0$. A contradiction, as $d_S^z \geq 1$.

$\underline{n=5}$. $d_S^x = 9$, $d_S^y = 5$ and $\dim S = 15$ imply $d_S^z \leq 1$. A contradiction, as $d_S^z = 3$.

$\underline{n=6}$. Assume first $d_S^x = 13$. Then $d_S^y = 7$ and $\dim S = 21$ imply $d_S^z \leq 1$. A contradiction, as $d_S^z \geq 2$. We are left with the case $d_S^x = 12$, x conjugate to $(\,(\pi_2)^3)$, $p = 2$. Now $d_S^y = 7$ implies $d_S^z \leq 2$, z conjugate to $(\,\epsilon, \epsilon, 1, \epsilon^3, \epsilon^{-3}, \epsilon^{-2}\,)$. The faithful action of H on the exterior square E, of dimension 15, gives rise to an embedding $i : H \to \mathrm{GL}(15, \overline{F})$. Recall that in the present case we are assuming $H = \mathrm{SL}(6, q)$ or $\mathrm{SU}(6, q^2)$. It is well-known (e.g. cfr. [Lie]) that $i(H)$ is irreducible. It is also easy to see that $i(x)$, $i(y)$ and $i(z)$ are respectively conjugate to the matrices

$$(\,1^3, (\pi_2)^6\,)\ ,\ (\,(\pi_3)^5\,)\ ,\ (\,(1), (\epsilon)^3, (\epsilon^2)^2, (\epsilon^3), (\epsilon^4)^3, (\epsilon^5)^3, (\epsilon^6)^2\,)\,.$$

Hence, by applying Scott' s formula to the action of $i(H)$, by conjugation, on the space $M_E = \mathrm{Mat}(15, \overline{F})$, we get the contradiction:

$$d_{M_E}^{i(x)} + d_{M_E}^{i(y)} + d_{M_E}^{i(z)} = 117 + 75 + 37 = 229 > 227 = \dim(M_E) + 2.$$

$\underline{n=7}$. $d_S^x = 16$, $d_S^y = 10$ and $\dim S = 28$ imply $d_S^z \leq 2$. A contradiction, as $d_S^z \geq 4$.

$\underline{n=10}$. $d_S^x \geq 30$, $d_S^y = 19$ and $\dim S = 55$ imply $d_S^z \leq 6$. A contradiction, as $d_S^z \geq 7$. $\square$

The following lemma gives a refined version of $(*)$, which will be needed in order to prove Theorem 3.

Lemma 7.1 *Let H be one of the groups $\mathrm{SL}(n,q)$, $\mathrm{Sp}(n,q)$ or $\mathrm{SU}(n,q)$. Assume either*
a) $q \equiv 1 \pmod 7$ and $q \notin \{p, p^3, 3^6\}$; or
b) $q \equiv 3, 5 \pmod 7$ and $q \neq p$.
If (x, y, z) is a $(2,3,7)$-generating triple for H and y is real or modular, then:

$$d_M^x \leq d_{\overline{M}}^2 + \mathrm{def}_{\overline{M}} - 2 \quad , \quad d_M^y \leq d_{\overline{M}}^3 + \mathrm{def}_{\overline{M}} - 2 \quad , \quad d_M^z \leq d_{\overline{M}}^7 + \mathrm{def}_{\overline{M}} - 2.$$

Proof

a) Let d be the multiplicative order of $p \pmod 7$. Then $q = p^{dk}$, $k \in \mathbb{N}$. Assume first $k > 1$ and let σ be the automorphism of F_q defined by $\alpha \mapsto \alpha^{p^d}$. Then σ is non-trivial, as the subfield of the elements fixed by σ has order $p^d < q$. Consider the representation $\gamma_1 : \mathrm{GL}(n,q) \to GL(n^2, q)$ such that, for every $g \in \mathrm{GL}(n,q)$, $\gamma_1(g)$ is the automorphism of $\mathrm{Mat}(n,q)$ defined by $A \mapsto g^\sigma A g^{-1}$. By a well-known theorem of Steinberg, $\gamma_1(H)$ and its dual are irreducible (cfr. [St]). Note that γ_1 is the representation afforded by the tensor product module $V \otimes V^{*\sigma}$, where V is the natural module for $\mathrm{GL}(n,q)$ and $V^{*\sigma}$ denotes the dual of V 'twisted' by σ). Scott' s formula then gives:

$$d_{M_1}^x + d_{M_1}^y + d_{M_1}^z \leq n^2,$$

where, as usual, $d_{M_1}^g$ denotes the dimension of the space of fixed points of $\gamma_1(g)$. Under our assumptions, y is conjugate to a matrix over F_p, while z is conjugate to a matrix over F_{p^d}. As x is clearly conjugate to a matrix over F_p, it follows that $d_{M_1}^x = d_M^x$, $d_{M_1}^y = d_M^y$, $d_{M_1}^z = d_M^z$ and we are done. Now assume $k = 1$, i.e. $q = p^d$. From $q \notin \{p,\ p^3\}$ it follows $d \in \{2, 6\}$. In particular $p \neq 2$, as 2 has order $3 \pmod 7$, and $p \neq 3$ as we exclude the case $q = 3^6$. Let σ be the automorphism of order 2 of F_q, and consider the representation $\gamma_2 :\ \mathrm{GL}(n,q) \to GL(n^2,q)$ such that, for every $g \in \mathrm{GL}(n,q)$, $\gamma_2(g)$ is the automorphism of $\mathrm{Mat}(n,q)$ defined by $A \mapsto g^\sigma A g^t$. Again $\gamma_2(H)$ and its dual are irreducible (cfr. [St]; γ_2 is the representation afforded by the module $V \otimes V^\sigma$). Hence, with obvious notation, $d^x_{M_2} + d^y_{M_2} + d^z_{M_2} \le n^2$. As $\sigma(\epsilon) = \epsilon^{-1}$, $\sigma(\omega) = \omega^{\pm 1}$ and x, y, z are diagonalizable, it follows that x^σ, y^σ, z^σ are respectively conjugate to $x, y^{\pm 1}, z^{-1}$. Our claim follows immediately.

b) In this case $q = p^t$, $t > 1$. Note that, as $q \equiv 3, 5 \pmod 7$, by Lemma 5.5 z is rational, hence is conjugate to a matrix over F_p . Now let σ be the Frobenius automorphism $\alpha \mapsto \alpha^p$, and apply the same argument used above for the case $k > 1$. □

Proof of Theorem 3

Let H be as stated in the theorem, and suppose that it is (2,3,7)-generated. Our strategy is as follows. Using Tables 2,3 and 4 and Lemma 5.4 we show that z cannot be modular, real or semirational. It follows from Lemmas 5.5, 5.6 and 5.7 that H can only be either *i)* $\mathrm{SL}(n,q)$ with $q \equiv 1 \pmod 7$, or *ii)* $\mathrm{SU}(n,q)$ with $\sqrt{q} \equiv -1 \pmod 7$. This proves our statement when H is a symplectic group and when H has degree $n = 12$. Thus we are left with the cases $H = \mathrm{SL}(n,q)$ or $\mathrm{SU}(n,q)$, $q \equiv 1 \pmod 7$, $n \in \{8, 9, 11\}$. We assume, by way of contradiction, $q \not\in \{p, p^3, 3^6\}$. Since y is real or modular (cfr. Table 3) and $\mathrm{def}_{\overline{M}} = 2$, it follows from Lemma 7.1 that z satisfies the condition $d^z_M = d^7_{\overline{M}}$. Hence, by Lemma 3.1, z has two similarity invariants of respective degrees $n-7$ and 7. We reach a contradiction by showing that z does not satisfy the condition $(**)$, i.e. $d^z_S \le \dim\, S - (d^x_S + d^y_S)$.

<u>$n=8$</u>. $d^x_S \ge 20$, $d^y_S \ge 12$ and $\dim S = 36$ imply $d^z_S \le 4$. z cannot be modular since, for $p = 7$, $d^z_S = 6$. By Lemma 5.4 z cannot be real, since $2d^z_S < d^7_{\overline{M}} = 10$. Assume z semirational. Then $d^z_M \le 12$ implies, by the same Lemma, $m_0 = 2$, $m_1 = m_{-1} = 1$. But this in turn implies z real, a contradiction. Finally, in cases *i)* and *ii)*, $d^z_M = d^7_{\overline{M}}$ implies $z \sim (\, 1, \pi_7\,)$, hence $d^z_S = 6$, a contradiction.

<u>$n=9$</u>. $d^x_S = 25$, $d^y_S \ge 15$ and $\dim S = 45$ imply $d^z_S \le 5$. z cannot be modular since, for $p = 7$, $d^z_S \ge 7$. z cannot be real, as $2d^z_S < d^7_{\overline{M}} = 13$. Assume z semirational, and non-real. $d^z_M \le d^7_{\overline{M}} + \mathrm{def}_{\overline{M}} = 15$ implies $m_0 = 0$, $m_1 = 2$, $m_{-1} = 1$. But, in this case, $d^z_S = 6$, against the bound $2d^z_S \le 10$. Finally, in cases *i)* and *ii)*, $d^z_M = d^7_{\overline{M}}$ implies $z \sim (\, \epsilon, \epsilon^{-1}, \pi_7\,)$, hence $d^z_S = 7$, a contradiction.

<u>$n=11$</u>. $d^x_S = 36$, $d^y_S \ge 22$ and $\dim S = 66$ imply $d^z_S \le 8$. z cannot be modular since, for $p = 7$, $d^z_S \ge 10$. z cannot be real, as $2d^z_S < d^7_{\overline{M}} = 19$. Assume z semirational. $d^z_M \le d^7_{\overline{M}} + \mathrm{def}_{\overline{M}} = 21$ implies $m_0 = 2$, $m_1 = 2$, $m_{-1} = 1$. But, in this case, $d^z_S = 9$, against the bound $2d^z_S \le 16$. Finally, in cases *i)* and *ii)*, $d^z_M = d^z_{\overline{M}}$ implies that z is conjugate either to $(\, \epsilon, \epsilon^2, \epsilon^{-1}, \epsilon^{-2}, \pi_7\,)$ or to $(\, 1, \epsilon, \epsilon^2, \epsilon^{-3}, \pi_7\,)$. In both cases $d^z_S \ge 9$, a contradiction.

<u>$n=12$</u>. $d^x_S \ge 42$, $d^y_S \ge 26$ and $\dim S = 78$ imply $d^z_S \le 10$. z cannot be modular since, for $p = 7$, $d^z_S \ge 12$. z cannot be real, as $2d^z_S < d^7_{\overline{M}} = 22$. Assume z semirational, and non-real. $d^z_M \le d^7_{\overline{M}} + \mathrm{def}_{\overline{M}} = 26$ implies $m_0 = 3$, $m_1 = 2$, $m_{-1} = 1$. But, in this case, $d^z_S = 12$, against the bound $2d^z_S \le 20$. □

Proof of Theorem 4

Let H be as stated in the theorem, and assume that it is (2,3,7)-generated. Note that, under our assumptions, y is real or modular by Lemmas 5.5, 5.6 and 5.7. This assumption is necessary in order to obtain contradictions by playing condition $(*)$ against $(**)$. Our claims follow from the arguments below and Table 5.

A) x, y, z cannot satisfy $(*)$ and $(**)$.

$\underline{n = 13}$. $d_S^x \geq 49$, $d_S^y \geq 31$ and $\dim S = 91$ imply $d_S^z \leq 11$, contradicting Tables 4 and $4'$.

$\underline{n = 14}$, $p \neq 2$. $d_S^x = 57$, $d_S^y \geq 35$ and $\dim S = 105$ imply $d_S^z \leq 13$, contradicting Tables 4 and $4'$. For the case $p = 2$, see below.

B) If x, y, z satisfy $(*)$ and $(**)$, then z is neither modular, nor real, nor semirational.

$\underline{n = 16}$. $d_S^x \geq 72$, $d_S^y \geq 46$ and $\dim S = 136$ imply $d_S^z \leq 18$. Thus z cannot be modular by Table 4. On the other hand, if $p \neq 7$, $2d_S^z \leq 36 < 38 = d_{\overline{M}}^7$, hence z is not real. Finally, it is easy to see, using the condition $d_M^z \leq 44$, that z can be semirational (and not real) only if $m_0 = 1$, $m_1 = 3$, $m_{-1} = 2$. But in this case $d_S^z = 19$, a contradiction.

$\underline{n = 18}$, $p \neq 2$. $d_S^x \geq 91$, $d_S^y \geq 57$ and $\dim S = 171$ imply $d_S^z \leq 23$. Using this condition and $d_M^z \leq 54$, it is easy to check that z cannot be semirational. The cases z real or modular are dealt with below.

C) If x, y, z satisfy $(*)$ and $(**)$, then z is neither modular nor real.

$\underline{n = 14}$, $p = 2$. $d_S^x \geq 56$ and $d_S^y \geq 35$ imply $d_S^z \leq 14$. Assume z real. Using Lemma 5.4, we get $28 \geq 2d_S^z \geq d_M^z \geq d_{\overline{M}}^7 = 28$, whence $d_M^z = d_{\overline{M}}^7$. By Lemma 3.1 z is conjugate to $(\,(\pi_7)^2\,)$ and $d_S^z = 15$, a contradiction.

$\underline{n = 18}$, $p \geq 2$. $d_S^x \geq 90$ implies $d_S^z \leq 24$. z cannot be modular by Table 4. Assume z real. $48 \geq 2d_S^z \geq d_M^z \geq d_{\overline{M}}^7 = 48$ implies $d_M^z = d_{\overline{M}}^7$. By Lemma 3.1 z is conjugate to $(\,(\epsilon, \epsilon^{-1}, (\epsilon^i, \epsilon^{-i}), (\pi_7)^2\,)$ $(i = 2$ or $3)$, and we get $d_S^z = 25$, a contradiction.

$\underline{n = 15}$. $d_S^x \geq 64$, $d_S^y \geq 40$ and $\dim S = 120$ imply $d_S^z \leq 16$. Thus z cannot be modular by Table 4. On the other hand, if $p \neq 7$, $2d_S^z \leq 32 < 33 = d_{\overline{M}}^7$, hence z is not real.

$\underline{n = 17}$. $d_S^x \geq 81$, $d_S^y \geq 51$ and $\dim S = 153$ imply $d_S^z \leq 21$. Thus z cannot be modular. On the other hand, if $p \neq 7$, $2d_S^z \leq 42 < 43 = d_{\overline{M}}^7$, hence z is not real.

$\underline{n = 19}$. $d_S^x \geq 100$, $d_S^y \geq 64$ and $\dim S = 190$ imply $d_S^z \leq 26$. z cannot be modular. If $p \neq 7$, $2d_S^z \leq 52 < 53 = d_{\overline{M}}^7$, hence z is not real. □

Proof of Theorem 5

Let H be as stated in the theorem, and assume that it is (2,3,7)-generated. As $|\mathbb{Q}(\omega) : \mathbb{Q}| = 2$ and $|\mathbb{Q}(\epsilon) : \mathbb{Q}| = 6$, y is real and z is rational. Using Lemmas 5.3 and 4.2 it is easy to determine the admissible conjugacy classes of z and, when they exist, to compute d_S^z. (Alternatively this information can be deduced from Table 4.) Our claim follows since, for the values of n considered in the statement, conditions $(*)$ and $(**)$ are incompatible. □

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