Equivariant evaluation on free loop spaces

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January 30, 2000

Abstract

There is a natural evaluation map on the free loop space $\Lambda X \to X^k$ which sends a loop to its values at the kth roots of unity. This map is equivariant with respect to the action of the cyclic group on k elements C_k. We study the induced map in C_k-equivariant cohomology with mod two coefficients in the cases where $k = 2^m$ for $m \geq 1$.

1 Introduction

For each positive integer k there is an evaluation map on the free loop space $\Lambda X \to X^k$ which is equivariant with respect to the action of the cyclic group C_k. The evaluation map sends a loop to its values at the kth roots of unity. As k approaches infinity this evaluation distinguishes more loops. One hopes that the induced map in cohomology becomes more and more informative as k increases and hence provides new input to the ongoing problem of computing the cohomology of free loop spaces. Since evaluating at two different points of the circle gives homotopic maps, one should use C_k-equivariant cohomology in order to get interesting results.

In this paper we use cohomology with $\mathbb{Z}/2$-coefficients. We consider the cases $k = 2^m$ with $m \geq 1$. Under these assumptions we compute the map induced by evaluation in C_k-equivariant cohomology in terms of the approximation functor ℓ introduced in [BO].

The functor ℓ comes equipped with a ring homomorphism as follows

$$\psi : \ell(H^*X) \to H^*(ES^1 \times_{S^1} \Lambda X).$$

*1991 *Mathematics Subject Classification*: 55N91, 55P35. I would like to thank Université de Paris Nord and Universität Bielefeld where this work was developed, supported by the European Union TMR network ERB FMRX CT-97-0107: Algebraic K-Theory, Linear Algebraic Groups and Related Structures.

It is an isomorphism for 1-connected spaces X with polynomial cohomology. It is not hard to see that $H^*(BC_k) \otimes_{H^*(BS^1)} \ell(H^*X)$ approximates $H^*(ES^1 \times_{C_k} \Lambda X)$ similarly.

In §2 we introduce some notation and conventions which are fixed through the paper. In §3 we review the definition and some basic properties of the functor ℓ and the related twisted de Rham functor Ω_λ.

In §4 we show that ℓ and Ω_λ are functors from the category of connected unstable $\mathcal{A}$-algebras to the category of unstable $\mathcal{A}$-algebras. Here $\mathcal{A}$ denotes the mod two Steenrod algebra. We give a useful pullback description of $\ell(K)$ when K is the cohomology of a product of Eilenberg-MacLane spaces of type $K(\mathbb{F}_2, n)$ with $n \geq 1$. The description is similar to the one for the quadratic construction $C_2(K)$ as described in [HLS].

In §5 we generalize the exactness result for the approximation ψ to 0-connected spaces with polynomial cohomology and a finite 2-group as fundamental group. The sections §4 and §5 give a convenient setup for working with the functor ℓ.

In §7 we set up the computation. There is a commutative diagram

$$\begin{tikzcd}
S^1 \times \Lambda X \arrow[d, "\eta"] \arrow[dr, "f"] & \\
ES^1 \times_{C_k} \Lambda X \arrow[r, "ev"] & ES^1 \times_{C_k} X^k \\
BC_k \times X \arrow[u, "i"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

where i comes from the constant loop inclusion $X \hookrightarrow \Lambda X$ and η comes from the S^1-action on ΛX. The pair (i, η) induces an injective map in cohomology by the pullback in §5. The map induced by the diagonal Δ was computed in [O]. The induced map of f is computed in §7. From this data ev^* is determined in §8. Finally in §9 we give an extension of the evaluation map ev^* to an algebraic setting.

2 Notation and conventions

We always use $\mathbb{F}_2 = \mathbb{Z}/2$ coefficients unless otherwise is specified. We say that a space X is of finite type if H_iX is finitely generated for each i. The mod two Steenrod algebra is denoted $\mathcal{A}$, the category of unstable $\mathcal{A}$-modules is denoted $\mathcal{U}$ and the category of unstable $\mathcal{A}$-algebras is denoted $\mathcal{K}$. The category $\mathcal{K}_0$ is the full subcategory of $\mathcal{K}$ consisting of connected unstable $\mathcal{A}$-algebras. For any $\mathcal{A}$-module M we define $\lambda : M \to M$ by $\lambda(x) = Sq^{|x|-1}x$.

The cyclic group of order k is denoted C_k. We view it as a subgroup of the circle group S^1 generated by $\zeta_k = \exp(2\pi i/k)$. Assume that $k = 2^m$ with $m \geq 0$. For an S^1-space Y and a C_k-space Z we use the following short hand notation for Borel constructions:

$$E_\infty Y := ES^1 \times_{S^1} Y \quad , \quad E_m Z := ES^1 \times_{C_k} Z$$

For integers n and m with $0 \leq n < m \leq \infty$ there is a map q_m^n induced by the quotient and a transfer map τ_n^m as follows

$$q_m^n : H^* E_m Y \to H^* E_n Y \quad ; \quad \tau_n^m : H^* E_n Y \to H^* E_m Y$$

Here τ_n^∞ is the S^1-transfer from Definition 4.4 in [BO].

The free loop space is the mapping space $\Lambda X = X^{S^1}$. Precomposition with the inclusion map $j_k : C_k \hookrightarrow S^1$ defines a map $j_k^* : X^{S^1} \to X^{C_k}$ which is C_k-equivariant. We get a map of Borel constructions.

$$1 \times_{C_k} j_k^* : ES^1 \times_{C_k} X^{S^1} \to ES^1 \times_{C_k} X^{C_k}$$

When $k = 2^m$ we put $\mathcal{C}_m X = ES^1 \times_{C_k} X^k$ and we write $ev_m : E_m \Lambda X \to \mathcal{C}_m X$ for the evaluation map $1 \times_{C_k} j_k^*$.

Finally, we fix the names on the algebra generators of the following cohomology groups. $H^*(S^1) = \Lambda(v)$ where $|v| = 1$, $H^*(BS^1) = \mathbb{F}_2[u]$ where $|u| = 2$ and

$$H^*(BC_{2^m}) = \begin{cases} \Lambda(v) \otimes \mathbb{F}_2[u] & \text{when } m \geq 2 \\ \mathbb{F}_2[t] & \text{when } m = 1 \end{cases}$$

where $|t| = |v| = 1$ and $|u| = 1$.

3 The functors Ω_λ and ℓ

In this section we review the definitions of the approximation functors Ω_λ and ℓ from [BO] and provide some extra material on them.

Definition 3.1. Let R be a unital, connected $\mathbb{F}_2$-algebra. A linear map $\lambda : R \to R$ is called a derivation over Frobenius if $|\lambda x| = 2|x| - 1$ and $\lambda(xy) = x^2 \lambda y + y^2 \lambda x$. We refer to (R, λ) as an algebra with derivation over Frobenius, and use the abbreviation FGA for such gadgets. They form a category FGA where the morphisms are algebra morphisms respecting the structure given by the derivations over Frobenius.

Definition 3.2. Let (R, λ) be a FGA. The algebra $\Omega_\lambda(R)$ is the quotient of the free commutative algebra on

$$x, dx \text{ for } x \in R$$

where $|dx| = |x| - 1$, by the ideal generated by the elements

$$d(x+y) + dx + dy \tag{1}$$

$$d(xy) + xdy + ydx \tag{2}$$

$$(dx)^2 + d(\lambda x) \tag{3}$$

There is a differential on $\Omega_\lambda(R)$ defined by the formula $d(x) = dx$ as a derivation over R.

Clearly Ω_- is a functor from the category of FGA's to the category of DGA's. The following result shows that the category of FGA's is equipped with a tensor product. We omit the proof which is an easy direct computation.

Proposition 3.3. *Let (A, λ) and (B, γ) be FGA's and define an $\mathbb{F}_2$-linear map $\lambda * \gamma : A \otimes B \to A \otimes B$ by*

$$\lambda * \gamma(a \otimes b) = \lambda(a) \otimes b^2 + a^2 \otimes \gamma(b)$$

Then $(A \otimes B, \lambda * \gamma)$ is an FGA.

The functor Ω_- preserves tensor products in the following sense:

Proposition 3.4. *The canonical inclusion maps $i : A \to A \otimes B$; $a \mapsto a \otimes 1$ and $j : B \to A \otimes B$; $b \mapsto 1 \otimes b$ are morphisms of FGA's and the composite map*

$$\kappa : \Omega_\lambda(A) \otimes \Omega_\gamma(B) \xrightarrow{i_* \otimes j_*} \Omega_{\lambda*\gamma}(A \otimes B) \otimes \Omega_{\lambda*\gamma}(A \otimes B) \xrightarrow{\mu} \Omega_{\lambda*\gamma}(A \otimes B)$$

is an isomorphism of DGA's.

Proof. i and j are morphisms of FGA's since an FGA is non negatively graded. Thus i_* and j_* are morphisms of DGA's hence κ is a morphism of DGA's. We check that there is a well defined map $\Omega_{\lambda*\gamma}(A \otimes B) \to \Omega_\lambda(A) \otimes \Omega_\gamma(B)$ such that $y \mapsto y$ and $dy \mapsto d_\otimes y$ for $y \in A \otimes B$ Such a map is clearly an inverse to κ. As for the usual de Rham complex one sees that elements of the form (1) and (2) are mapped to zero. For $a \in A$ and $b \in B$ we have

$$\begin{aligned}
&(d(a \otimes b))^2 + d(\lambda * \gamma(a \otimes b)) = \\
&(d(a \otimes b))^2 + d(\lambda(a) \otimes b^2 + a^2 \otimes \gamma(b)) \mapsto \\
&(d(a) \otimes b + a \otimes d(b))^2 + d(\lambda a) \otimes b^2 + a^2 \otimes d(\gamma b) = 0
\end{aligned}$$

and since (3) is additive in x we are done. $\square$

The forgetful functor $U : \mathcal{K} \to FGA$ provide the main examples of FGA's. For an object K in $\mathcal{K}$ we let $\lambda x = Sq^{|x|-1}x$. It is a derivation over Frobenius by the Cartan formula.

Let Y be a S^1-space with action map $\eta : S^1 \times Y \to Y$. Then there is a map $d : H^*Y \to H^{*-1}Y$ defined by $\eta^*(y) = 1 \otimes y + v \otimes dy$. Proposition 3.2 in [BO] shows that (H^*Y, d) is a DGA and $Sq^i(dy) = d(Sq^i y)$ such that $(dy)^2 = d(\lambda y)$. It is not necessary to assume that Y is connected for these results even though we did so in [BO]. When Y is a free loop space we get the following.

Proposition 3.5. *For any connected space X there is a morphism of DGA's*

$$e : (\Omega_\lambda(H^*X), d) \to (H^*(\Lambda X), d) \quad ; \quad e(x) = ev_0^*(x) \quad ; \quad e(dx) = dev_0^*(x)$$

which is natural in X. For any pair of connected spaces X and Y there is a commutative diagram

$$\begin{array}{ccc} \Omega_\lambda(H^*X) \otimes \Omega_\lambda(H^*Y) & \xrightarrow{\kappa} & \Omega_\lambda(H^*X \otimes H^*Y) \\ {\scriptstyle e\otimes e}\downarrow & & {\scriptstyle e}\downarrow \\ H^*(\Lambda X) \otimes H^*(\Lambda Y) & \longrightarrow & H^*(\Lambda(X \times Y)) \end{array}$$

where the lower horizontal map is the Künneth isomorphism.

Definition 3.6. Let (R, λ) be an FGA. The algebra $\ell(R)$ is the quotient of the free commutative algebra on generators

$$\phi(x), q(x), \delta(x) \text{ for } x \in R \text{ and } u$$

of degree $|\phi(x)| = 2|x|$, $|q(x)| = 2|x| - 1$, $|\delta(x)| = |x| - 1$ and $|u| = 2$, by the ideal generated by the elements

$$\phi(a+b) + \phi(a) + \phi(b) \tag{4}$$
$$\delta(a+b) + \delta(a) + \delta(b) \tag{5}$$
$$q(a+b) + q(a) + q(b) + \delta(ab) \tag{6}$$
$$\delta(xy)\delta(z) + \delta(yz)\delta(x) + \delta(zx)\delta(y) \tag{7}$$
$$\phi(xy) + \phi(x)\phi(y) + uq(x)q(y) \tag{8}$$
$$q(xy) + q(x)\phi(y) + \phi(x)q(y) \tag{9}$$
$$\delta(x)^2 + \delta(\lambda x) \tag{10}$$
$$q(x)^2 + \phi(\lambda x) + \delta(x^2\lambda x) \tag{11}$$
$$\delta(x)\phi(y) + \delta(xy^2) \tag{12}$$
$$\delta(x)q(y) + \delta(x\lambda y) + \delta(xy)\delta(y) \tag{13}$$
$$u\delta(x) \tag{14}$$

where a, b, x, y, z are homogeneous elements in R and $|a| = |b|$.

Theorem 3.7. *Let (R,λ) be an FGA. There is a natural algebra homomorphism $DR : \ell(R) \to \Omega_\lambda(R)$ defined by*

$$\phi(x) \mapsto x^2 \quad , \quad q(x) \mapsto xdx + \lambda x \quad , \quad \delta(x) \mapsto dx \quad , \quad u \mapsto 0$$

If R is a symmetric algebra $R = S(V)$ where V is a non negatively graded $\mathbb{F}_2$-vector space with V^n finitely generated for each n, there is a short exact sequence

$$0 \to (u) \longrightarrow \ell(R) \xrightarrow{DR} \ker(d) \to 0$$

Proof. This was shown in [BO] Proposition 8.1, Theorem 8.2 and Theorem 8.5. □

We can apply the results from chapter 6 of [BO] component vise and get the following.

Theorem 3.8. *For any connected space X there is an algebra homomorphism $\psi : \ell(H^*X) \to H^*(ES^1 \times_{S^1} \Lambda X)$ defined by*

$$\begin{array}{ccccccc} \phi(x) & \mapsto & \tau_1^\infty \circ ev_1^*(t \otimes x^{\otimes 2}) & ; & q(x) & \mapsto & \tau_1^\infty \circ ev_1^*(1 \otimes x^{\otimes 2}) \\ \delta(x) & \mapsto & \tau_0^\infty \circ ev_0^*(x) & ; & u & \mapsto & \tau_1^\infty \circ ev_1^*(t^3 \otimes 1^{\otimes 2}) \end{array}$$

It is natural in X and the following diagram commutes

$$\begin{array}{ccc} \ell(H^*X) & \xrightarrow{\psi} & H^*(ES^1 \times_{S^1} \Lambda X) \\ {\scriptstyle DR}\downarrow & & {\scriptstyle q^0_\infty}\downarrow \\ \Omega_\lambda(H^*X) & \xrightarrow{e} & H^*(\Lambda X) \end{array}$$

4 Unstable algebras.

In this section we show that Ω_λ and ℓ restricts to functors $\mathcal{K}_0 \to \mathcal{K}$. We do this algebraically instead of using the result in the next chapter, since it gives a better understanding of the functors. For the functor ℓ we follow the treatment of the quadratic construction as close as possible. A certain pull back diagram appears, which will become useful later.

We use the notation $Sq_i(x) := Sq^{|x|-i}x$. For a sequence of positive integers $n_1, \dots, n_r$ we define the following object in $\mathcal{K}_0$:

$$A(n_1, \dots, n_r) = H^*(K(\mathbb{F}_2, n_1) \times \cdots \times K(\mathbb{F}_2, n_r))$$

Lemma 4.1. *The following relations are valid in any unstable $\mathcal{A}$-module*

$$Sq^{2i}(\lambda x) = \lambda(Sq^i x) \tag{15}$$

$$Sq^{2i+1}(\lambda x) = (i + |x|)Sq^{i+|x|}Sq^i x \tag{16}$$

Proof. Follows by the Adem relation. See the proof of Proposition 5.2 in [O]. ∎

Proposition 4.2. *For any $K \in ob(\mathcal{K}_0)$ we can define an $\mathcal{A}$-action on $\Omega_\lambda(K)$ by $Sq^i(dx) = d(Sq^i x)$ and the Cartan formula. With this action Ω_λ becomes a functor $\Omega_\lambda : \mathcal{K}_0 \to \mathcal{K}$. The differential $d : \Omega_\lambda(K) \to \Omega_\lambda(K)$ is $\mathcal{A}$-linear.*

Proof. Let dK be the $\mathcal{A}$-module given by $(dK)^n = K^{n+1}$. The symmetric algebra $S(K \oplus dK)$ is an $\mathcal{A}$-module by the Cartan formula. $\Omega_\lambda(K)$ is by definition the quotient of this module by the ideal generated by (1),(2) and (3). We verify that this ideal is closed under the $\mathcal{A}$-action. We have

$$Sq^i(d(x+y) + dx + dy) = d(Sq^i x + Sq^i y) + d(Sq^i x) + d(Sq^i y)$$

which is in the ideal (in fact (1) is already zero in dK). Further

$$\begin{aligned}&Sq^i(d(xy) + xdy + ydx) = \\ &\sum_{j=0}^{i} \big(d(Sq^j(x)Sq^{i-j}(y)) + (Sq^j x)d(Sq^{i-j}y) + (Sq^j y)d(Sq^{i-j}x)\big)\end{aligned}$$

which is in the ideal by (2). We use Lemma 4.1 for the last type of elements:

$$\begin{aligned}Sq^{2i}((dx)^2 + d(\lambda x)) &= (Sq^i(dx))^2 + d(Sq^{2i}(\lambda x)) = (d(Sq^i x))^2 + d(\lambda(Sq^i x)) \\ Sq^{2i+1}((dx)^2 + d(\lambda x)) &= d(Sq^{2i+1}(\lambda x)) = d((i + |x|)(Sq^i x)^2) = 0\end{aligned}$$

We conclude that $\Omega_\lambda(K)$ has a natural $\mathcal{A}$-module structure. It is a non negatively graded module since $d1 = 0$ by (2). We have $Sq^{|x|}(dx) = d(x^2) = 0$ and clearly $Sq^i(dx) = 0$ for $i > |x|$ so $\Omega_\lambda(K)$ is an unstable $\mathcal{A}$-module. The Cartan formula holds by definition and $Sq^{|x|-1}(dx) = d(\lambda x) = (dx)^2$ so $\Omega_\lambda(K)$ is an unstable $\mathcal{A}$-algebra. The differential on $\Omega_\lambda(K)$ is $\mathcal{A}$-linear by definition. ∎

Definition 4.3. For $K \in \mathcal{K}_0$ we define the modified de Rham cohomology by $H^*_\lambda(K) = H^*(\Omega_\lambda(K), d)$.

Note that this cohomology theory has the nice property of being a functor $H^*_\lambda : \mathcal{K}_0 \to \mathcal{K}$ as well as it respects tensor products: $H^*_\lambda(A \otimes B) \cong H^*_\lambda(A) \otimes H^*_\lambda(B)$.

Definition 4.4. Let K be an unstable $\mathcal{A}$-algebra. For $x \in K$ we define the elements $St_0(x) = \sum u^i \otimes Sq_{2i}x$ and $St_1(x) = \sum u^i \otimes Sq_{2i+1}x$ in $\mathbb{F}_2[u] \otimes K$ where the sums are taken over all integers $i \geq 0$. Further we let $R(K)$ denote the sub $\mathbb{F}_2$-algebra of $\mathbb{F}_2[u] \otimes K$ generated by u and $St_0(x)$, $St_1(x)$ for $x \in K$.

Proposition 4.5. *For $i \geq 0$ one has $Sq^i R(K) \subseteq R(K)$ hence we have defined a functor $R : \mathcal{K} \to \mathcal{K}$. Explicitly, the following formula holds for $\nu = 0, 1$:*

$$Sq^i St_\nu(x) = \sum_{j=0}^{\infty} \binom{|x| - j + 1 - \nu}{i - 2j} u^{[\frac{i+1-\nu}{2}]-j} St_{i+\nu}(Sq^j x) \tag{17}$$

where the lower index on the right should be taken modulo 2.

Proof. Let $St(x) = \sum t^i \otimes Sq_i(x)$ in $\mathbb{F}_2[t] \otimes K$ where $|t| = 1$. The following formula holds

$$Sq^i(St(x)) = \sum_{j=0}^{\infty} \binom{|x| - j}{i - 2j} t^{i-2j} St(Sq^j x) \tag{18}$$

It expresses that the diagonal map $\Delta : \mathcal{C}_1(K) \to \mathbb{F}_2[t] \otimes K$ which is given by $\Delta(1 \otimes (x \otimes y + y \otimes x)) = 0$ and $\Delta(1 \otimes x^{\otimes 2}) = St(x)$ is $\mathcal{A}$-linear. Here $\mathcal{C}_1$ is the quadratic functor [HLS] page 58-59. The formulas for the $\mathcal{A}$-module structure on $\mathcal{C}_1 K$ can be found in [M]. There is an $\mathcal{A}$-linear transfer map of degree -1

$$\tau : \mathbb{F}_2[t] \to \mathbb{F}_2[u] \quad ; \quad \tau(t^{2i}) = 0 \quad , \quad \tau(t^{2i+1}) = u^i$$

Note that the following formula holds for $\nu = 0, 1$:

$$St_\nu(x) = (\tau \otimes 1)(t^{1-\nu} St(x)) \tag{19}$$

It implies that $Sq^i St_\nu(x) = (\tau \otimes 1) \circ Sq^i(t^{1-\nu} St(x))$. By this and formula (18) we see that equation (17) holds. □

Proposition 4.6. *For $K \in ob(\mathcal{K}_0)$ there is an $\mathbb{F}_2$-algebra homomorphism*

$$St : \ell(K) \to \mathbb{F}_2[u] \otimes K \quad ;$$
$$\phi(x) \mapsto St_0(x) \quad , \quad q(x) \mapsto St_1(x) \quad , \quad \delta(x) \mapsto 0 \quad , \quad u \mapsto u$$

and $R(K)$ is the image of this map.

Proof. We must check that St is well defined. The elements (5), (7), (10), (12), (13), (14) are clearly mapped to zero since $St \circ \delta = 0$. The elements (4), (6) are mapped to zero since $St \circ \phi$ and $St \circ q$ are additive and $St \circ \delta = 0$. For the elements (8) and (9) we proceed as follows. Let again $St(x) = \sum t^i \otimes Sq_i(x)$ instead of the above map. By the Cartan formula we have $St(xy) = St(x)St(y)$ and applying (19) to this we find

$$St_1(ab) = St_1(a)St_0(b) + St_0(a)St_1(b)$$
$$St_0(ab) = St_0(a)St_0(b) + uSt_1(a)St_1(b)$$

Finally (11) is mapped to zero since Lemma 4.1 implies $St_0(\lambda x) = (St_1 x)^2$. By its definition $R(K)$ is the image of the map St. □

Lemma 4.7. *The Eilenberg-MacLane space $K(\mathbb{F}_2, n)$ with $n \geq 1$ has the following mod two cohomology algebra.*

$$\mathbb{F}_2[\lambda^k Sq^I \iota_n \,|\, I \text{ admissible}\ ,\ e(I) \leq n-2\ ,\ k \geq 0]$$

Proof. The usual set of algebra generators is $\{Sq^J \iota_n\}$ where J runs through the admissible sequences with excess $e(J) \leq n-1$. We verify that this set is in fact the same as the set of generators in the lemma.

Assume that $I = (i_1, i_2, \dots, i_s)$ is admissible with excess $e(I) \leq n-1$. We have $\lambda Sq^I \iota_n = Sq^{(n-1+|I|,I)}\iota_n$. Since $e(I) = i_1 - i_2 - \dots - i_s \leq n-1$ we have $n-1+|I| \geq 2i_1$ such that $(n-1+|I|, I)$ is admissible and clearly $e(n-1+|I|, I) = n-1$. Thus the set of generators in the lemma is a subset of the usual set of algebra generators.

Conversely, if $J = (j_1, \dots, j_r)$ is admissible and $e(J) \leq n-2$ then $Sq^J \iota_n$ is one of the generators with $k = 0$ in the lemma. Assume that $e(J) = n-1$ and put $J(t) = (j_t, j_{t+1}, \dots, j_r)$. The map $t \mapsto e(J(t))$ is weakly decreasing since $e(J(t)) - e(J(t+1)) = j_t - 2j_{t+1} \geq 0$ and eventually it becomes zero. Let $k = max\{t | e(J(t)) = n-1\}$. Then we have $Sq^J \iota_n = \lambda^k Sq^{J(k+1)} \iota_n$ which is one of the generators in the lemma. □

Definition 4.8. When R is an FGA we let $J(R)$ denote the subring of $\ell(R)$ generated by the elements $\delta(x)$ for $x \in R$. By (12), (13) and (14) we see that $J(R)$ is an ideal in $\ell(R)$.

Proposition 4.9. *When $K = A(n_1, \dots, n_r)$ where $n_i \geq 1$ for each i we have* $\ker(St) = J(K)$.

Proof. By definition $J(K) \subseteq \ker(St)$ so it suffices to show that the induced map $St : \ell(K)/J(K) \to \mathbb{F}_2[u] \otimes K$ is injective. The algebra $\ell(K)/J(K)$ has generators u, $\phi(x)$, $q(x)$ for $x \in K$. The relations are that ϕ and q are additive and further

$$\begin{aligned} \phi(xy) &= \phi(x)\phi(y) + uq(x)q(y) \\ q(xy) &= q(x)\phi(y) + \phi(x)q(y) \\ q(x)^2 &= \phi(\lambda x) \end{aligned}$$

Put $x_i = \iota_{n_i}$ for each i. We see that the following elements are algebra generators: $\phi(Sq^{I_i}x_i)$, $q(\lambda^{k_i}Sq^{I_i}x_i)$, u where I_i is admissible, $e(I_i) \leq n_i - 2$ and $0 \leq k_i$ for $1 \leq i \leq r$. It suffices to show that these are mapped to algebraic independent elements. The coefficient to u^0 in $St_0(Sq^{I_i}x_i)$ is $(Sq^{I_i}x_i)^2$ and in $St_1(\lambda^{k_i}Sq^{I_i}x_i)$ it is $\lambda^{k_i+1}Sq^{I_i}x_i$. Using the projection homomorphisms $p_i : \mathbb{F}_2[u] \otimes K \to K$ given by $p_i(u^j \otimes x) = \delta_{i,j}x$ and induction the algebraic independence follows by Lemma 4.7. □

Theorem 4.10. *For $K \in ob(\mathcal{K}_0)$ there is a commutative diagram in the category of $\mathbb{F}_2$-algebras*

$$\begin{array}{ccc} \ell(K) & \xrightarrow{St} & R(K) \\ {\scriptstyle DR}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ \ker(d) & \xrightarrow{q} & K \end{array}$$

where $p : \mathbb{F}_2[u] \otimes K \to K$ and $q : \Omega_\lambda(K) \to K$ are the projections defined by $q(x) = x$, $q(dx) = 0$ and $p(x) = x$ and $p(u) = 0$ for $x \in K$. If $K = A(n_1, \dots, n_r)$ where $n_i \geq 1$ for each i the diagram is a pull back square.

Proof. By evaluating on generators it is easy to check that the diagram is commutative. Assume K is the cohomology of a product of Eilenberg-MacLane spaces as above. The diagram is a pull back if and only if the restriction $St : \ker(DR) \to \ker(p)$ is an isomorphism. By Proposition 4.9 and Theorem 3.7 we have

$$\ker(DR) \cap \ker(St) = (u) \cap J(K) = 0$$

so the restriction of St is injective. The kernel of $p : \mathbb{F}_2[u] \otimes K \to K$ is $(u) \subseteq \mathbb{F}_2[u] \otimes K$ so $(u) \cap R(K)$ is the kernel of $p : R(K) \to K$. Since $St(\phi(x)) = St_0(x)$, $St(q(x)) = St_1(x)$ and $St(u) = u$ it is obvious that the restriction of St is surjective. □

Theorem 4.11. *The functor ℓ restricts to a functor $\ell : \mathcal{K}_0 \to \mathcal{K}$ For $K \in ob(\mathcal{K}_0)$ the $\mathcal{A}$-module structure on $\ell(K)$ is given by the following formulas:*

$$Sq^i \delta(x) = \delta(Sq^i x) \tag{20}$$

$$Sq^i \phi(x) = \sum_{j=0}^{\infty} \binom{|x| + 1 - j}{i - 2j} u^{[\frac{i+1}{2}]-j} \big((i+1)\phi(Sq^j x) + iq(Sq^j x)\big) \tag{21}$$

$$Sq^i q(x) = \sum_{j=0}^{\infty} \binom{|x| - j}{i - 2j} u^{[\frac{i}{2}]-j} \big(i\phi(Sq^j x) + (i+1)q(Sq^j x)\big) + \delta(Q^i(x)) \tag{22}$$

$$Sq^i u = \binom{2}{i} u^{[\frac{i}{2}]+1} \tag{23}$$

Here by convention a binomial coefficient is zero when its lower parameter is negative. The Q^i operation in the last formula is defined by

$$Q^i(x) = \sum_{r=0}^{[\frac{i}{2}]} Sq^r(x) Sq^{i-r}(x)$$

There is a commutative diagram in the category $\mathcal{K}$

$$\begin{array}{ccc} \ell(K) & \xrightarrow{St} & R(K) \\ {\scriptstyle DR}\downarrow & & \downarrow{\scriptstyle p} \\ \ker(d) & \xrightarrow{q} & K \end{array}$$

It is natural in K and when $K = A(n_1, \ldots, n_r)$ for a sequence of positive integers $n_1, \ldots, n_r$ it is a pull back square.

Proof. When $K = A(n_1, \ldots, n_r)$ we see that $\ell(K)$ is an unstable $\mathcal{A}$-algebra by the pull back diagram in Theorem 4.10. We verify that (20)-(23) are formulas for this $\mathcal{A}$-module structure by using the injectivity of the pair (St,DR). The verification of equation (20) and (23) is trivial. When we apply DR to the right hand side of (21) we only need to consider the $j = [\frac{i+1}{2}]$ term since $DR(u) = 0$. This term equals $\phi(Sq^{i/2}x)$ for i even and zero for i odd. hence we get $(Sq^{i/2}x)^2$ for i even and zero for i odd. This equals $Sq^iDR(\phi(x)) = Sq^i(x^2)$ as it should. When we apply DR to the right hand side of (22) only the $j = [\frac{i}{2}]$ term contribute. This term equals $q(Sq^rx) + \delta(Q^{2r}(x))$ for $i = 2r$ and $(|x| - r)\phi(Sq^rx) + \delta(Q^{2r+1}(x))$ for $i = 2r + 1$. Thus we must check that

$$\begin{aligned} &(Sq^rx)d(Sq^rx) + \lambda(Sq^rx) + dQ^{2r}(x) = Sq^{2r}(xdx + \lambda x) \\ &(|x| - r)(Sq^rx)^2 + dQ^{2r+1}(x) = Sq^{2r+1}(xdx + \lambda x) \end{aligned}$$

We have $dQ^i(x) = Sq^i(xdx) + (i + 1)Sq^{i/2}(x)Sq^{i/2}(dx)$ by the definition of $Q^i(x)$ and this together with Lemma 4.1 gives the desired result. For the map St the verification follows directly by formula (17).

For a general connected unstable $\mathcal{A}$-algebra K we proceed as follows. Let T denote the free, graded, unital, non commutative algebra on Sq^i for $i \geq 0$ where $Sq^0 = 1$. Let $F(K)$ be the free, unital, graded, commutative algebra on u, $\phi(x)$, $q(x)$, $\delta(x)$ for $x \in K$. We let T act on $F(K)$ by the formulas (20)-(23) and the Cartan formula. Let $I(K) \subseteq F(K)$ denote the ideal generated by the elements (4)-(14). First we check that $T \cdot I(K) \subseteq I(K)$ such that we have a well defined action $T \times \ell(K) \to \ell(K)$. Let $g(y_1, \ldots, y_r)$ be one of the elements (4)-(14). Put $n_j = |y_j|$, $x_j = \iota_{n_j}$ and $A = A(n_1, \ldots, n_r)$. Let $f : A(n_1, \ldots, n_r) \to K$ be the morphism in $\mathcal{K}$ defined by $f(x_j) = y_j$. By the first part of this proof we have $T \cdot F(A) \subseteq F(A)$ so naturality gives $Sq^ig(y_1, \ldots, y_r) \in I(K)$.

We have $\mathcal{A} = T/J$ where $J \subseteq T$ is the two sided ideal generated by the Adem elements. We must check that $J \cdot \ell(K) = 0$ such that we have a well defined action $\mathcal{A} \times K \to K$. But this follows by a similar naturality argument as above. Hence we have shown that the formulas (20)-(23) define a natural $\mathcal{A}$-module structure on $\ell(K)$, which satisfy the Cartan formula. It is immediate from the formulas that the instability condition holds for each generator of $\ell(K)$ so the action is unstable. Using the relations (4)-(14) one verifies that $Sq^{|y|}y = y^2$ when y is either of the generators $\phi(x)$, $q(x)$, $\delta(x)$, u and so this relation holds in general. Thus the action makes $\ell(K)$ an unstable $\mathcal{A}$-algebra. It is obvious that p and q are $\mathcal{A}$-linear. $\square$

5 A more general condition for exactness of the approximation functors

In this section we prove a stronger version of the main theorem in [BO], Theorem 10.1, which states that e and ψ are isomorphisms for 1-connected spaces with polynomial cohomology. We start by an examination of the case $X = B\mathbb{Z}/2 = \mathbb{R}\mathrm{P}^\infty$.

For a connected space X one has $\pi_0\Lambda X \cong [S^1_+, X] \cong < \pi_1 X >$ where $< \cdot >$ denotes the set of conjugacy classes. Especially $\pi_0\Lambda BG \cong < G >$ for any group G. By Lemma 7.11 of [BHM] we have the following result.

Lemma 5.1. *For a discrete group G*

$$\Lambda BG = \coprod_{[g]\in<G>} \Lambda_{[g]}BG \quad \text{and} \quad \Lambda_{[g]}BG \cong BC_G[g]$$

where $C_G[g]$ denotes the centralizer of g.

As a special case $\Lambda B\mathbb{Z}/2 \cong \Lambda_0 B\mathbb{Z}/2 \sqcup \Lambda_1 B\mathbb{Z}/2$ where $\Lambda_\nu B\mathbb{Z}/2 \simeq B\mathbb{Z}/2$ for $\nu = 0, 1$. Define a group action $a_\nu : \mathbb{Z} \times \mathbb{Z}/2 \to \mathbb{Z}/2$ by $(r, s) \mapsto \nu r + s$. Then $Ba_\nu : B\mathbb{Z} \times B\mathbb{Z}/2 \to B\mathbb{Z}/2$ is an S^1-action which is the trivial one for $\nu = 0$. Let $B_\nu\mathbb{Z}/2$ denote the space $B\mathbb{Z}/2$ with S^1-action Ba_ν.

Lemma 5.2. $(Ba_\nu)^*(t) = 1 \otimes t + \nu t \otimes 1$ *for* $\nu = 0, 1$.

Proof. Since $\pi_1(Ba_\nu) = a_\nu$ we have $H_1(Ba_\nu; \mathbb{Z}) = a_\nu$. The result follows by tensoring with $\mathbb{F}_2$ and dualizing. $\square$

Proposition 5.3. *For $\nu = 0, 1$ the following map is S^1-equivariant and also a homotopy equivalence*

$$j_\nu : B_\nu\mathbb{Z}/2 \to \Lambda_\nu B\mathbb{Z}/2 \quad ; \quad j_\nu(x)(z) = Ba_\nu(z, x)$$

Proof. The conjugacy class of the neutral element corresponds to the component containing constant loops so the index of the target of j_0 is 0 as stated.

Let i be the index of the target of j_1. Since Ba_ν is an action map we see that j_μ is equivariant and that $ev_0 \circ j_\nu = id$. Thus $\pi_q(j_\nu)$ is injective. But the homotopy groups of its domain and target space are the same finite groups so it is injective as well. By the Whitehead theorem j_ν is a homotopy equivalence. $H^*(B_\nu\mathbb{Z}/2)$ is a DGA and $dt = \nu$ by Lemma 5.2. Since j_ν induces an isomorphism of DGA's we have $i = 1$. $\square$

Proposition 5.4. *The map $e : (\Omega_\lambda(H^*B\mathbb{Z}/2), d) \to (H^*(\Lambda B\mathbb{Z}/2), d)$ is an isomorphism.*

Proof. Let $R = H^*B\mathbb{Z}/2 = \mathbb{F}_2[t]$. It suffices to show that $(j_0 \sqcup j_1)^* \circ e : \Omega_\lambda(R) \to R \oplus R$ is an isomorphism. Since $ev_0 \circ j_\nu = id$ it follows that $t \mapsto (t,t)$ and by Lemma 5.2 we see that $dt \mapsto (0,1)$. In $\Omega_\lambda(R)$ we have $(dt)^2 = d\lambda t = dt$ such that $(1+dt)^2 = 1 + dt$, $(dt)(1+dt) = 0$ and clearly $dt + (1+dt) = 1$. This gives an algebra splitting: $\Omega_\lambda(R) = (dt)\Omega_\lambda(R) \oplus (1+dt)\Omega_\lambda(R)$. The components are simply $(dt)\Omega_\lambda(R) = (dt)R$ and $(1+dt)\Omega_\lambda(R) = (1+dt)R$ thus the above map is an isomorphism. $\square$

Recall that an action of a group π on an Abelian group M is said to be *nilpotent* if there is a finite π-filtration of M such that π acts trivially on the filtration quotients.

Theorem 5.5. *Let X be a connected space of finite type. Then the two maps*

$$e : \Omega_\lambda(H^*X) \to H^*(\Lambda X) \quad \text{and} \quad \psi : \ell(H^*X) \to H^*(ES^1 \times_{S^1} \Lambda X)$$

are morphisms in $\mathcal{K}$ which are natural in X. Assume further that the group $\pi_1(X) \times \pi_1(X)$ acts nilpotently on $H_i(\Omega X)$ for each i and that the cohomology of X is a symmetric algebra $H^*X = S(V)$ where V is a graded $\mathbb{F}_2$-vector space with V^n finitely generated for each n. Then both e and ψ are isomorphisms of unstable $\mathcal{A}$-algebras.

Remark 5.6. Any action of a finite 2-group on a mod 2 vector space is nilpotent. Hence the nilpotent action condition in the theorem always holds when $\pi_1 X$ is a finite 2-group.

Proof. We must check that the $\mathbb{F}_2$-algebra maps e and ψ are $\mathcal{A}$-linear. The $\mathcal{A}$-action on $\Omega_\lambda(H^*X)$ was described in Proposition 4.2. Since the differential d on $H^*\Lambda X$ is $\mathcal{A}$-linear we see that e is $\mathcal{A}$-linear. The $\mathcal{A}$-action on $\ell(H^*X)$ is given in Theorem 4.11. By Proposition 6.7 of [BO] we see that the $\mathbb{F}_2$-algebra map ψ is also $\mathcal{A}$-linear.

Making the further assumptions of nilpotent action and polynomial cohomology we first show that e is an isomorphism. According to [S] there is a fiber square with common fiber ΩX as follows.

$$\begin{array}{ccc} \Lambda X & \xrightarrow{ev} & X \\ {\scriptstyle ev}\big\downarrow & & \big\downarrow{\scriptstyle \Delta} \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

The associated cohomology Eilenberg-Moore spectral sequence lies in the second quadrant and has the following E_2-page

$$E_2^{-p,q} = \mathrm{Tor}_p^{K \otimes K}(K, K)^q$$

where $K = H^*X = S(V)$. It converges strongly to $H^*(\Lambda X)$ by [B] 4.1. The E_2-page can be interpreted by Hochschild Homology of K and then via the Hochschild Konstant Rosenberg theorem by differential forms

$$E_2^{-p,*} = HH_p(K) \cong \Omega^p(K)$$

Here the bidegree of $x_0 dx_1 \dots dx_p$ is $(-p, |x_0| + \cdots + |x_p|)$. Since all algebra generators for E_2 sits in $E_2^{0,*}$ or $E_2^{-1,*}$ we have $E_2 = E_\infty$. The assumption that V^n is finitely generated for each n combined with the formula for the bidegree shows that for m fixed $E_\infty^{-p,m+p} = 0$ for p sufficiently large. Hence the filtration of $H^m := H^m(\Lambda X)$ is finite for any m. As in the proof of Theorem 10.1 in [BO] one easily sees that $e(x) \in H^*(\Lambda X)$ represents $x \in E_2^{0,*}$ for $x \in K$. We now show that $e(dx) \in H^*(\Lambda X)$ represents $dx \in E_2^{-1,*}$. By the naturality argument given in the proof of Theorem 10.1 in [BO] it suffices to verify this when $X = K(\mathbb{F}_2, n)$ where $n \geq 1$ and $x = \iota_n$ is the fundamental class. For $n \geq 2$ this is done in [BO] and for $n = 1$ we proceed as follows. Since $E_\infty = E_2 = \Omega(\mathbb{F}_2[\iota_1])$ we have $E_\infty^{-p,*} = 0$ for $p \geq 2$. Thus the filtration has length two:

$$H^n = F^{-1}H^n \supseteq F^0 H^n \supseteq 0 \quad , \quad E_\infty^{-p,n+p} \cong F^{-p}H^n / F^{-p+1}H^n$$

From the E_2 page we read that $F^{-1}H^0 \cong \mathbb{F}_2 \oplus \mathbb{F}_2$ and $F^0 H^0 \cong \mathbb{F}_2$ so it suffices to show that $e(d\iota_1) \neq 0$ and $e(d\iota_1) \neq 1 = (1,1) \in H^*(\Lambda K(\mathbb{F}_2, 1))$. But this is a consequence of Proposition 5.4.

Defining a compatible filtration of $\Omega_\lambda(K)$ one sees that e is an isomorphism as in [BO]. The same kind of Serre spectral sequence argument as given in [BO] shows that ψ is an isomorphism. $\square$

6 Detection maps

Let X be a connected space of finite type. In this section we shall see how it is possible to describe the cohomology of the evaluation map $ev_m : E_m\Lambda X \to \mathcal{C}_m X$ via the functor ℓ. The cohomology of its target space has a classical interpretation. For a positive integer k and a positively graded $\mathbb{F}_2$-vector space V we let $C_k(V)$ denote the group cohomology $C_k(V) := H^*(C_k; V^{\otimes k})$ where the action on the coefficient module is by cyclic permutation. We put $\mathcal{C}_m(V) := C_{2^m}(V)$ for $m \geq 0$. Our reason for choosing the same notation as for cyclic constructions is the following. By classical work of Dold and Steenrod there is an isomorphism of $\mathbb{F}_2$-algebras

$$\Phi : \mathcal{C}_m(H^*X) \to H^*(\mathcal{C}_m X) \tag{24}$$

In [O] we found formulas for an $\mathcal{A}$-action on $\mathcal{C}_m(M)$ when M is an unstable $\mathcal{A}$-module. This gave a functor $\mathcal{C}_m : \mathcal{U} \to \mathcal{U}$ such that (24) is an isomorphism of unstable $\mathcal{A}$-modules.

The cohomology of the domain space of ev_m is of course unknown in general but the ℓ functor gives an approximation for it. It is easy to see that Theorem 4.2 in [BO] also holds for non connected spaces Y. Hence we have an isomorphism

$$\theta : H^*(BC_{2^m}) \otimes_{\mathbb{F}_2[u]} H^*(E_\infty \Lambda X) \to H^*(E_m \Lambda X)$$

for $m \geq 1$. Via this we get a morphism of unstable $\mathcal{A}$-algebras

$$\Psi = \theta \circ (1 \otimes \psi) : H^*(BC_{2^m}) \otimes_{\mathbb{F}_2[u]} \ell(H^*X) \to H^*(E_m \Lambda X) \tag{25}$$

We know that this is an isomorphism when $X = K(\mathbb{F}_2, n)$ with $n \geq 1$ so we can attempt to compute the map ev_m^* in these cases. By naturality this would give information on the general case.

We shall use two detection maps, i_m and η_m, which we now describe. In the following a lower index 0 on a space means the space equipped with the trivial S^1-action. Firstly, the constant loop inclusion $i : X_0 \to \Lambda X$ gives maps

$$i_m := E_m i : BC_{2^m} \times X \to E_m \Lambda X$$

Secondly, the action map $\eta : S^1 \times (\Lambda X)_0 \to \Lambda X$ gives maps

$$E_m \eta : E_m(S^1 \times (\Lambda X)_0) \to E_m \Lambda X$$

The domain space can be rewritten $E_m(S^1 \times (\Lambda X)_0) \cong E_m(S^1) \times \Lambda X$ and the projection on the second factor $pr_2 : E_m(S^1) \to S^1/C_{2^m}$ is a homotopy equivalence. For our purpose it is better with a map in the opposite direction of pr_2. We choose a point $e \in ES^1$ and define $s_0 : S^1 \to ES^1 \times S^1$ by $s_0(z) = (ze, z)$. It is an S^1-map so we can define

$$\begin{aligned} s_m = s_0/C_{2^m} &: S^1/C_2^m \to E_m(S^1) \\ s_0 = s_0/S^1 &: * \to E_\infty(S^1) \end{aligned}$$

Since $pr_2 \circ s_m = id$ we see that s_m induces an isomorphism in cohomology. Our second detection map is the composite

$$\begin{aligned} \eta_m : S^1/C_{2^m} \times \Lambda X &\xrightarrow{s_m \times 1} E_m(S^1) \times \Lambda X \xrightarrow{\cong} E_m(S^1 \times (\Lambda X)_0) \\ &\xrightarrow{E_m \eta} \quad E_m \Lambda X \end{aligned} \tag{26}$$

The following result explains why we call i_m and η_m detection maps.

Theorem 6.1. *Let X be a connected space of finite type and let $m \geq 2$ be an integer. There is a commutative digram*

$$\begin{array}{ccc} \Lambda(v) \otimes \ell(H^*X) & \xrightarrow{1 \otimes St} & \Lambda(v) \otimes \mathbb{F}_2[u] \otimes H^*X \\ {\scriptstyle 1\otimes\psi}\downarrow & & \downarrow{\scriptstyle \cong} \\ \Lambda(v) \otimes H^*(E_\infty \Lambda X) & \xrightarrow{1 \otimes i_\infty^*} & \Lambda(v) \otimes H^*(BS^1 \times X) \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ H^*(E_m \Lambda X) & \xrightarrow{i_m^*} & H^*(BC_{2^m} \times X) \end{array} \tag{27}$$

and a commutative diagram

$$\begin{array}{ccc} \Lambda(v) \otimes \ell(H^*X) & \xrightarrow{1 \otimes DR} & \Lambda(v) \otimes \Omega_\lambda(H^*X) \\ {\scriptstyle 1\otimes\psi}\downarrow & & \downarrow{\scriptstyle 1\otimes e} \\ \Lambda(v) \otimes H^*(E_\infty \Lambda X) & \xrightarrow{1 \otimes q_\infty^0} & \Lambda(v) \otimes H^*(\Lambda X) \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ H^*(E_m \Lambda X) & \xrightarrow{\eta_m^*} & H^*(S^1/C_{2^m} \times \Lambda X) \end{array} \tag{28}$$

Thus the map

$$(i_m^*, \eta_m^*) : H^*(E_m \Lambda X) \to H^*(BC_{2^m} \times X) \oplus H^*(S^1/C_{2^m} \times \Lambda X)$$

is injective when $X = K(\mathbb{F}_2, n)$ with $n \geq 1$.

Proof. By the argument given in the proof of Proposition 7.1 of [BO] we see that the lower square in (27) commutes. By Proposition 7.2 of [BO] the upper square in this diagram also commutes (in [BO] we assumed that ΛX was connected, but it is easy to see that this assumption is not necessary).

The composite (26) defining η_m sits over the following composite via quotient maps:

$$* \times \Lambda X \xrightarrow{s_\infty \times 1} E_\infty(S^1) \times \Lambda X \xrightarrow{\cong} E_\infty(S^1 \times (\Lambda X)_0) \xrightarrow{E_\infty \eta} E_\infty \Lambda X$$

which is given by $(*, \omega) \mapsto [e, \omega]$. So the top square in the following diagram commutes. Mapping down by projections pr_1 we get the whole commutative cube:

$$\begin{tikzcd}
S^1/C_{2^m} \times \Lambda X \arrow[rr, "\eta_m"] \arrow[dr] \arrow[dd] & & E_m \Lambda X \arrow[dr, "Q_m^\infty"] \arrow[dd] & \\
& * \times \Lambda X \arrow[rr, "Q_0^\infty"] \arrow[dd] & & E_\infty \Lambda X \arrow[dd] \\
S^1/C_{2^m} \arrow[rr, "j"] \arrow[dr] & & BC_{2^m} \arrow[dr] & \\
& * \arrow[rr] & & BS^1
\end{tikzcd}$$

Here Q_m^∞ and Q_0^∞ are quotient maps and j is the fiber inclusion of the fibration following it in the cube. From this cube and the definition of θ in Theorem 4.2 of [BO] we see that the lower square of (28) commutes. The top square commutes by Theorem 3.8.

When $X = K(\mathbb{F}_2, n)$ for $n \geq 1$ the vertical maps in (27) and (28) are all isomorphisms so (i_m^*, η_m^*) is injective by Theorem 4.11. $\square$

7 The twisted diagonal map

Let Y be an S^1-space with action map $\eta : S^1 \times Y \to Y$ and associated differential d on H^*Y.

Definition 7.1. For $m \geq 0$ we define the map

$$\begin{aligned} f'_{Y,m} : S^1 \times Y &\to ES^1 \times Y^{2^m} \\ (z, y) &\mapsto (ze, zy, z\xi_m y, z\xi_m^2 y, \ldots, z\xi_m^{2^m-1} y) \end{aligned}$$

where $\xi_m = exp(2\pi i/2^m)$ and e is a point in ES^1. We let C_{2^m} act on the domain space by $\xi_m \cdot (z, y) = (\xi_m z, y)$ and on the target space by

$$\xi_m \cdot (e, y_1, \ldots, y_{2^m}) = (\xi_m e, y_2, \ldots, y_{2^m}, y_1)$$

Then the above map is C_{2^m}-equivariant. Passing to the quotients we get a map

$$f_{Y,m} : S^1/C_{2^m} \times Y \to \mathcal{C}_m Y$$

which we call the twisted diagonal of order 2^m.

Remark 7.2. $f_{Y,m}$ is natural in Y with respect to C_{2^m}-equivariant maps.

Remark 7.3. Our reason for introducing the twisted diagonal map is that it can be used to compute $\eta_m^* \circ ev_m^*$ since there is a commutative diagram

$$\begin{array}{ccc} S^1/C_{2^m} \times \Lambda X & \xrightarrow{\eta_m} & E_m \Lambda X \\ {\scriptstyle f_{\Lambda X,m}} \downarrow & & {\scriptstyle ev_m} \downarrow \\ E_m \Lambda X & \xrightarrow{E_m(ev_0)} & \mathcal{C}_m X \end{array}$$

Lemma 7.4. *Let X be a space with trivial S^1-action and $m \geq 1$ an integer. The map $f_{X,m}^* : H^*(\mathcal{C}_m X) \to H^*(S^1 \times X)$ satisfies*

$$f_{X,m}(1 \otimes x^{\otimes 2^m}) = 1 \otimes x^{2^m} + v \otimes x^{2^m - 2} \lambda x$$

Proof. There is a factorization

$$f_{X,m} : S^1/C_{2^m} \times X \xrightarrow{i \times 1} ES^1/C_{2^m} \times X \xrightarrow{1 \times \Delta_m} \mathcal{C}_m X$$

where $i : S^1 \to ES^1$ is the inclusion associated to e and Δ_m is the diagonal $\Delta_m(y) = (y, \dots, y)$. When $m = 1$ the result follows by Steenrod's formula for the diagonal. For $m \geq 2$ we use Theorem 6.7 of [O] where the map $(1 \times \Delta_m)^*$ was determined. We find

$$f_{X,m}^*(1 \otimes x^{\otimes 2^m}) = 1 \otimes x^{2^m} + v \otimes Q_{m-1}^{2^{m-1}|x|-1}(x)$$

The operation Q_n^i is defined in Definition 5.5 of [O]. We have $Q_1^{2|x|-1}(x) = x^2 \lambda x$ giving $Q_n^{2^n|x|-1}(x) = x^{2^{n+1}-2} \lambda x$ in general. $\square$

Proposition 7.5. *There is a commutative diagram*

$$\begin{array}{ccc} H^*(\mathcal{C}_m Y) & \xrightarrow{f_{Y,m}^*} & H^*(S^1/C_{2^m} \times Y) \\ {\scriptstyle \tau_{m-1}^m} \uparrow & & {\scriptstyle \tau_{m-1}^m \otimes 1} \uparrow \\ H^*(\mathcal{C}_{m-1}(Y^2)) & \xrightarrow{(f'_{Y,m}/C_{2^{m-1}})^*} & H^*(S^1/C_{2^{m-1}} \times Y) \\ {\scriptstyle \mathcal{C}_{m-1}(\Delta_1)^*} \downarrow & & \parallel \\ H^*(\mathcal{C}_{m-1} Y) & \xrightarrow{f_{Y,m-1}^*} & H^*(S^1/C_{2^{m-1}} \times Y) \end{array}$$

where $\Delta_1 : Y \to Y \times Y$ denotes the diagonal $\Delta_1(y) = (y, y)$. Especially we have $f_{Y,m}^*(1 \otimes Na_1 \otimes \cdots \otimes a_{2^m}) = v \otimes d(a_1 \dots a_{2^m})$.

Proof. It is obvious that the upper square commutes. The lower square is induced by a homotopy commutative diagram of spaces. The norm class equals $\tau_0^m(1 \otimes a_1 \otimes \cdots \otimes a_{2^m})$ so we can find its image under $f^*_{Y,m}$ by use of the diagram. From the factorization

$$f_{Y,0} : S^1 \times Y \xrightarrow{\Delta_1 \otimes 1} S^1 \times S^1 \times Y \xrightarrow{i \times \eta} ES^1 \times Y$$

we see that $f^*_{Y,0}(1 \otimes y) = 1 \otimes y + v \otimes dy$ and the result follows. $\square$

Lemma 7.6. *For $m \geq 1$ the map $f^*_{S^1,m} : H^*(\mathcal{C}_m S^1) \to H^*(S^1/C_{2^m} \times S^1)$ is zero on all classes of positive degrees except for $1 \otimes Nv \otimes 1 \otimes \cdots \otimes 1$ which is mapped to $v \otimes 1$.*

Proof. When $m = 1$ this is a special case of Lemma 4.11 and Theorem 4.12 of [BO]. The differential on $H^*(S^1)$ is given by $dv = 1$ since $\eta^*(v) = 1 \otimes v + v \otimes 1$.

Assume that $m \geq 2$. Since the target space is $\Lambda(v) \otimes \Lambda(v)$ all classes of higher degree than two is mapped to zero. The degree two class

$$1 \otimes (1 + T + \cdots + T^{2^{m-1}-1})(v \otimes 1^{\otimes 2^{m-1}-1})^{\otimes 2} = \tau_1^m(1 \otimes (v \otimes 1^{\otimes 2^{m-1}-1})^{\otimes 2})$$

is mapped to zero by Proposition 7.5 and the $m = 1$ case. All other elements of degree at most two are norm elements and Proposition 7.5 takes care of these. $\square$

We now introduce an expansion formula which will be used in the proof of the next theorem and again later on. Let M be an $\mathbb{F}_2$-vector space and $k \geq 1$ an integer. The cyclic group C_k act on $M^{\otimes k}$ by cyclic permutation of the factors. For $x, y \in M$ we let $T_k(x, y)$ denote the subset of $M^{\otimes k}$ consisting of all strings $a_1 \otimes \cdots \otimes a_k$ with $a_i = x$ or $a_i = y$ for each i. Note that $T_k(x, y)$ is stable under the C_k action and that $(x + y)^{\otimes k}$ is the sum of the elements in $T_k(x, y)$. For an orbit $\beta \in T_k(x, y)/C_k$ we let $|\beta|$ denote the length of β and $s\beta$ the sum of the elements in β. Thus we have the following expansion formula among C_k-invariants in $M^{\otimes k}$:

$$(x + y)^{\otimes k} = \sum s\beta \tag{29}$$

where the summation is taken over $\beta \in T_k(x, y)/C_k$.

Theorem 7.7. *For $m \geq 1$ we have*

$$f^*_{Y,m}(1 \otimes y^{\otimes 2^m}) = 1 \otimes y^{2^m} + v \otimes y^{2^m - 2}(ydy + \lambda y)$$

Proof. Let Y_0 denote the space Y with trivial S^1-action. The action map $\eta : S^1 \times Y_0 \to Y$ is S^1-equivariant so by naturality there is a commutative diagram

$$\begin{array}{ccc} S^1/C_{2^m} \times S^1 \times Y_0 & \xrightarrow{f_{S^1\times Y_0,m}} & \mathcal{C}_m(S^1 \times Y_0) \\ {\scriptstyle 1\times\eta}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{C}_m(\eta)} \\ S^1/C_{2^m} \times Y & \xrightarrow{f_{Y,m}} & \mathcal{C}_m(Y) \end{array}$$

The map $\gamma : Y \to S^1 \times Y$ defined by $\gamma(y) = (1, y)$ has the property $\eta \circ \gamma = id$ thus

$$f_{Y,m} = \mathcal{C}_m(\eta) \circ f_{S^1\times Y_0,m} \circ (1 \times \gamma)$$

We pull back step by step according to this factorization.

$$\mathcal{C}_m(\eta)^*(1 \otimes y^{\otimes 2^m}) = 1 \otimes (1 \otimes y + v \otimes dy)^{\otimes 2^m} = \sum 1 \otimes s\beta$$

where the summation is taken over $\beta \in T_{2^m}(1 \otimes y, v \otimes dy)$. Consider an orbit β with $|\beta| = 2^i$ where $1 \leq i \leq m$. A representative for this has the form $\overline{\beta} = (a_1 \otimes \cdots \otimes a_{2^i})^{\otimes 2^{m-i}}$. By Proposition 7.5 we have

$$f^*_{S^1\times Y_0,m}(1 \otimes s\beta) = (\tau^m_{m-i} \otimes 1) \circ f^*_{S^1\times Y_0,m-i}(1 \otimes (a_1 \dots a_{2^i})^{\otimes 2^{m-i}}) \tag{30}$$

Since v is an exterior class we see that (30) is zero when two or more of the elements $a_1, \dots, a_{2^i}$ equals $v \otimes dy$. If there is exactly one of these we have $a_1 \dots a_{2^i} = v \otimes y^{2^i-1}dy$ and (30) equals zero unless $i = m$ by Lemma 7.6. When $i = m$ we have a norm element so Proposition 7.5 gives

$$\begin{aligned} f^*_{S^1\times Y_0,m}(1 \otimes s\beta) &= v \otimes d_{S^1\times Y_0}(v \otimes y^{2^m-1}dy) \\ &= v \otimes d_{S^1}(v) \otimes y^{2^m-1}dy + v \otimes v \otimes d_{Y_0}(y^{2^m-1}dy) \\ &= v \otimes 1 \otimes y^{2^m-1}dy \end{aligned}$$

We have now taken care of all the terms except of $1 \otimes (1 \otimes y)^{\otimes 2^m}$ and $1 \otimes (v \otimes dy)^{\otimes 2^m}$ which corresponds to orbits of length one. By Lemma 7.4 the first is mapped as follows

$$f^*_{S^1\times Y_0,m}(1 \otimes (1 \otimes y)^{\otimes 2^m}) = 1 \otimes 1 \otimes y^{\otimes 2^m} + v \otimes 1 \otimes y^{2^m-2}\lambda y$$

and Lemma 7.6 gives that the second is mapped to zero. We conclude that

$$f^*_{S^1\times Y_0,m} \circ \mathcal{C}_m(\eta)^*(1 \otimes y^{\otimes 2^m}) = 1 \otimes 1 \otimes y^{\otimes 2^m} + v \otimes 1 \otimes y^{2^m-2}(ydy + \lambda y)$$

The result follows by applying $1 \otimes \gamma^*$. ∎

Corollary 7.8. *For $m \geq 2$ one has*

$$\eta^*_m \circ ev^*_m(1 \otimes x^{\otimes 2^m}) = 1 \otimes e(x^{2^m}) + v \otimes e(x^{2^m-2}(xdx + \lambda x))$$

Proof. Follows directly from Theorem 7.7 and Remark 7.3. ∎

8 Formulas for the evaluation maps

In this section we find a formula for $ev_m^*(1 \otimes x^{\otimes 2^m})$ in terms of the map (25). This is done by the two detection maps η_m and i_m. Note that $i_m \circ ev_m = \Delta_m$ where $\Delta_m = E_m(D_m)$ with $D_m : X \to X^{2^m}$ the diagonal map. We computed Δ_m^* in [O]. The result involved some notation which we now introduce. In this section K always denotes an unstable $\mathcal{A}$-algebra.

Definition 8.1. For integers $k \geq 1$ and $n \geq 0$ we define $\lambda_k : K \to K$ and $Q_k^n : K \to K$ by

$$\lambda_k(x) = x^{2^k-2}\lambda x \tag{31}$$

$$Q_1^n(x) = \sum_{i=0}^{[\frac{n}{2}]} Sq^i(x)Sq^{n-i}(x) \tag{32}$$

$$Q_k^n(x) = \sum_{i=0}^{n} Sq^i(x^{2^k-2})Q_1^{n-i}(x) \quad , \quad k \geq 2 \tag{33}$$

We put $Q_k^n = 0$ when $n \leq -1$.

Definition 8.2. For integers $k \geq 1$ we define three maps $K \to \mathbb{F}_2[u] \otimes K$ as follows

$$F_{k+1}(x) = \sum_{i=0}^{\infty} u^{2^{k-1}(|x|-i)} \otimes (Sq^i x)^{2^k} \tag{34}$$

$$G'_{k+1}(x) = \sum_{j=0}^{\infty} u^{2^{k-1}|x|-j} \otimes Sq^{2j}(\lambda_k x) \tag{35}$$

$$G''_{k+1}(x) = \sum_{j=0}^{\infty} u^{2^{k-1}|x|-j} \otimes Q_k^{2j-1}(x) \tag{36}$$

Note that the sums are finite. F_m from Definition 6.5 of [O] is the same as above and G_m from Definition 6.5 of [O] equals the sum $G'_m + G''_m$. Thus by Theorem 6.7 of [O] we have

$$\Delta_m^*(1 \otimes x^{\otimes 2^m}) = F_m(x) + v(G'_m(x) + G''_m(x))$$

for $m \geq 2$.

Proposition 8.3. *For $x \in K$ and $k \geq 1$ the following equations hold.*

$$St(\phi(x^{2^k})) = F_{k+1}(x) \tag{37}$$

$$St(uq(\lambda_k x)) = G'_{k+1}(x) \tag{38}$$

$$St(\phi(x^{2^k-1})q(x)) = G''_{k+1}(x) \tag{39}$$

Proof. Equation (37) and (38) follows easily from the definition of St in Proposition 4.6. We first prove (39) for $k = 1$. Since St is a ring homomorphism we have that

$$St(\phi(x)q(x)) = \sum_{j=0}^{\infty} u^j \otimes \sum_{r+s=j} Sq_{2r}(x)Sq_{2s+1}(x)$$

So we must show that

$$Q_1^{2|x|-2j-1}(x) = \sum_{r+s=j} Sq_{2r}(x)Sq_{2s+1}(x) \tag{40}$$

We can rewrite the left hand side using the definition followed by the substitution $t = |x| - i$.

$$\begin{aligned} Q_1^{2|x|-2j-1}(x) &= \sum_{t=0}^{|x|-j-1} Sq^t(x)Sq^{2|x|-2j-1-t}(x) \\ &= \sum_{i=j+1}^{|x|} Sq_i(x)Sq_{2j+1-i}(x) \\ &= \sum_{i=j+1}^{2j+1} Sq_i(x)Sq_{2j+1-i}(x) \end{aligned} \tag{41}$$

The map, $f_j : \{0, 1, \dots, j\} \to \{j+1, j+2, \dots, 2j+1\}$ defined by $f_j(r) = 2(j-r)+1$ for $r \leq [j/2]$ and $f_j(r) = 2r$ otherwise, is a bijection from the set of summation indexes of (40) to the set of summation indexes of (41). Since the r'th term of the right hand side of (40) equals the $i = f_j(r)$'th term of (41) the result follows.

Finally we prove (39) for $k \geq 2$. By (9) we have $q(x^{2^k-2}) = 0$ such that $\phi(x^{2^k-1})q(x) = \phi(x^{2^k-2})\phi(x)q(x)$ by (8). Thus we can apply the $k = 1$ case as follows:

$$\begin{aligned} St(\phi(x^{2^k-1})q(x)) &= St(\phi(x^{2^k-2}))St(\phi(x)q(x)) \\ &= \sum_{j=0}^{\infty} u^j \otimes \sum_{r+s=j} Sq_{2r}(x^{2^k-2})Q_1^{2|x|-2s-1}(x) \end{aligned}$$

and we must prove the following equation for each j:

$$Q_k^{2^k|x|-2j-1}(x) = \sum_{r+s=j} Sq_{2r}(x^{2^k-2})Q_1^{2|x|-2s-1}(x) \tag{42}$$

By its definition and the fact that $Sq^i(y^2) = 0$ we can rewrite the left hand side as

$$Q_k^{2^k|x|-2j-1}(x) = \sum_{t=0}^{\infty} Sq^{2t}(x^{2^k-2})Q_1^{2^k|x|-2(j+t)-1}(x)$$

We substitute $t = (2^{k-1}-1)|x| - r$ and get that

$$Q_k^{2^k|x|-2j-1}(x) = \sum_{r=-\infty}^{(2^{k-1}-1)|x|} Sq_{2r}(x^{2^k-2})Q_1^{2|x|-2(j-r)-1}(x)$$

If $r < 0$ the first factor in each summand is zero so we can start the summation at $r = 0$. If $r > j$ the second factor in each summand is zero and if $r > (2^{k-1}-1)|x|$ the first factor in each summand is zero. So we can end the summation at j obtaining the formula (42). $\square$

Theorem 8.4. *Let X be a connected space of finite type and let $x, y \in H^*X$. For integers $m \geq 2$ the following equation holds*

$$ev_m^*(1 \otimes x^{\otimes 2^m}) = \Psi\big(\phi(x^{2^{m-1}}) + v\phi(x^{2^{m-1}-1})q(x) + vuq(x^{2^{m-1}-2}\lambda x)\big)$$

For $m = 1$ we have

$$ev_1^*(1 \otimes x^{\otimes 2}) = \Psi(\phi(x) + tq(x)) \quad \textit{and} \quad ev_1^*(1 \otimes (1+T)x \otimes y) = \Psi(t\delta(xy))$$

Further the square with transfer maps, the square with quotient maps and the triangle in the following diagram commutes.

$$\begin{tikzcd}
H^*(\mathcal{C}_m X) \arrow[r, "ev_m^*"] \arrow[d, shift left, "q_m^{m-1}"] & H^*(E_m \Lambda X) \arrow[d, shift left, "q_m^{m-1}"] \\
H^*(\mathcal{C}_{m-1}(X^2)) \arrow[r] \arrow[u, shift left, "\tau_{m-1}^m"] \arrow[d, "\Delta_1^*"'] & H^*(E_{m-1}\Lambda X) \arrow[u, shift left, "\tau_{m-1}^m"] \\
H^*(\mathcal{C}_{m-1} X) \arrow[ur, "ev_{m-1}^*"'] &
\end{tikzcd}$$

where $\Delta_1 : X \to X \times X$ denotes the diagonal.

Proof. By naturality it is enough to prove the formula for ev_m^* when $X = K(\mathbb{F}_2, n)$ with $n \geq 1$ and $x = \iota_n$ is the fundamental class. By Theorem 6.1 it suffices to check that the right hand side maps correctly by η_m^* and i_m^*. By Theorem 6.7 of [O] and Proposition 8.3 it does by i_m^*. For η_m^* we find

$$\begin{aligned}&1 \otimes DR(\phi(x^{2^{m-1}}) + v\phi(x^{2^{m-1}-1})q(x) + vuq(x^{2^{m-1}-2}\lambda x)) = \\ &x^{2^m} + vx^{2^m-2}(xdx + \lambda x)\end{aligned}$$

which is the correct result by Corollary 7.8. The formulas for $m = 1$ follows from Lemma 6.4 and Proposition 6.5 of [BO].

The squares in the diagram commute since there is a corresponding commutative diagram of spaces with evaluation and quotient maps. The triangle comes from a homotopy commutative diagram of spaces. $\square$

Remark 8.5. The map ev_m^* is $H^*(BC_{2^m})$-linear and elements in its domain which are not of highest symmetry are hit by the transfer. So we have given a complete description of ev_m^* in terms of the approximation Ψ.

9 Algebraic evaluation maps

In this section we show that the formulas for the map ev_m^* also define a ring map when we replace H^*X by a general FGA. This allows us to state our main theorem in a nice form.

Recall that the deviation from linearity of a set map $F : A \to B$ between $\mathbb{F}_2$-algebras is defined as $\Delta F(x, y) = F(x + y) + F(x) + F(y)$. If $L : A \to B$ is linear and we let $G(x) = L(x)F(x)$ we have

$$\Delta G(x, y) = L(x)F(y) + L(y)F(x) + (L(x) + L(y))\Delta F(x, y) \qquad (43)$$

Definition 9.1. Let K be an FGA. The maps $f_i, g_i, h_i : K \to \ell(K)$ for integer $i \geq 0$ are defined by $f_0(x) = f_1(x) = 0$, $g_0(x) = \delta(x)$ and

$$\begin{aligned} f_i(x) = &\quad uq(x^{2^{i-1}-2}\lambda x) &&\text{for } i \geq 2 \\ g_i(x) = &\quad \phi(x^{2^{i-1}-1})q(x) &&\text{for } i \geq 1 \\ h_i(x) = &\quad f_i(x) + g_i(x) &&\text{for } i \geq 0 \end{aligned}$$

Lemma 9.2. *For $w = f, g, h$ and $m \geq 1$ the following formulas are valid.*

$$w_m(ab) = \phi(a^{2^{m-1}})w_m(b) + \phi(b^{2^{m-1}})w_m(a) \qquad (44)$$

$$\Delta w_m(x, y) = \sum_{i=0}^{m-1} \sum_{r+s=2^{m-i}} w_i(x^r y^s) \qquad (45)$$

Proof. It is of course enough to prove this for $w = f$ and $w = g$. We first prove (44). For $m = 1$ and $w = f$ it is trivial. For $m = 1$ and $w = g$ it holds by (9). The $m = 2$ case requires a little computation. For $w = f$ we have

$$\begin{aligned} f_2(ab) &= uq(\lambda(ab)) = uq(a^2\lambda b + b^2\lambda a) \\ &= u(\phi(a^2)q(\lambda b) + \phi(b^2)q(\lambda a)) = \phi(a^2)f_2(b) + \phi(b^2)f_2(a) \end{aligned}$$

and for $w = g$ we find

$$\begin{aligned}
g_2(xy) &= (\phi(x)\phi(y) + uq(x)q(y))(\phi(x)q(y) + q(x)\phi(y)) \\
&= \phi(x)^2 g_2(y) + \phi(y)^2 g_2(x) + uq(y)^2 g_2(x) + uq(x)^2 g_2(y) \\
&= (\phi(x)^2 + uq(x)^2) g_2(y) + (\phi(y)^2 + uq(y)^2) g_2(x) \\
&= \phi(x^2) g_2(y) + \phi(y^2) g_2(x)
\end{aligned}$$

Note that when $m \geq 3$ the following formula holds for both $w = f$ and $w = g$.

$$w_m(x) = \phi(x^{2^{m-1}-2}) w_2(x) \tag{46}$$

So for these m we have

$$\begin{aligned}
w_m(ab) &= \phi((ab)^{2^{m-1}-2}) w_2(ab) \\
&= \phi(a^{2^{m-1}-2})\phi(b^{2^{m-1}-2})(\phi(a^2)w_2(b) + \phi(b^2)w_2(a)) \\
&= \phi(a^{2^{m-1}}) w_m(b) + \phi(b^{2^{m-1}}) w_m(a)
\end{aligned}$$

Next we prove (45). From (44) and $\delta(a^2) = 0$ we see that $w_i(a^2) = 0$ for any $i \geq 0$. Thus the terms with both r and s even are zero. For $m = 1$ we have $f_1 = 0$ and $g_1 = q$ such that $\Delta g_1(x, y) = \delta(xy) = g_0(xy)$. So here (45) holds. For $m = 2$ we have $f_2(x) = uq(\lambda x)$ which is linear. $g_2(x) = \phi(x)q(x)$ so by (43) and (44) we have

$$\Delta g_2(x, y) = g_1(xy) + (\phi(x) + \phi(y))\delta(xy) = g_1(xy) + \delta(xy^3) + \delta(x^3 y)$$

Thus (45) also holds for $m = 2$.

For $m \geq 3$ we have $w_m(x) = \phi(x^{2^{m-2}}) w_{m-1}(x)$ by (46) so (43) and (44) gives

$$\Delta w_m(x, y) = w_{m-1}(xy) + (\phi(x^{2^{m-2}}) + \phi(y^{2^{m-2}}))\Delta w_{m-1}(x, y)$$

Assume that $\Delta w_{m-1}(x, y)$ is given by (45). By (44) we have $\phi(a^{2^{m-2}})w_i(b) = w_i(a^{2^{m-1-i}} b)$ for $1 \leq i \leq m-2$. This formula also holds for $i = 0$ trivially for $w = f$ and by (12) for $w = g$. So we have

$$\Delta w_m(x, y) = w_{m-1}(xy) + \sum_{i=0}^{m-2} \sum_{r+s=2^{m-1-i}} \left(w_i(x^{2^{m-1-i}+r} y^s) + w_i(x^r y^{2^{m-1-i}+s}) \right)$$

Since $w_i(a^2) = 0$ for any $i \geq 0$ this sum equals the right hand side of (45). $\square$

The cyclic functor $C_{2^m} : \mathcal{U} \to \mathcal{U}$ was introduced in Theorem 4.8 of [O]. We denote it $\mathcal{C}_m = C_{2^m}$. Recall that there are transfer and restriction maps as follows

$$\tau^m_{m-1} : \mathcal{C}_{m-1}(M \otimes M) \to \mathcal{C}_m(M) \quad \text{and} \quad q^{m-1}_m : \mathcal{C}_m(M) \to \mathcal{C}_{m-1}(M \otimes M)$$

where $M \in ob(\mathcal{U})$. Formulas for these were given in section 3 of [O].

Proposition 9.3. *For each $m \geq 1$ the functor $\mathcal{C}_m : \mathcal{U} \to \mathcal{U}$ restricts to a functor $\mathcal{C}_m : \mathcal{K} \to \mathcal{K}$.*

Proof. Let K be an unstable $\mathcal{A}$-algebra. We must verify that the Cartan formula and the relation $Sq^{|x|}x = x^2$ hold in $\mathcal{C}_m(K)$. This is done by induction on m. For $m = 1$ the result is well known. Assume that it holds for $m-1$. Let $\tau = \tau^m_{m-1}$ and $q = q^{m-1}_m$. We know that these are $\mathcal{A}$-linear. Using Frobenius reciprocity we can then check the Cartan formula when one of the factors in the product is hit by the transfer:

$$\begin{aligned} Sq^n(a\tau(b)) &= Sq^n(\tau(q(a)b)) = \tau(Sq^n(q(a)b)) \\ &= \tau(\sum_{i+j=n} Sq^i(q(a))Sq^j(b)) = \sum_{i+j=n} \tau(q(Sq^i(a))Sq^j(b)) \\ &= \sum_{i+j=n} Sq^i(a)\tau(Sq^j(b)) = \sum_{i+j=n} Sq^i(a)Sq^j(\tau(b)) \end{aligned}$$

Hence it suffices to verify the Cartan formula for a product of two highest symmetry classes. For $x, y \in K$ with $|x| = r$ and $|y| = s$ there is a morphism of unstable $\mathcal{A}$-algebras $f : H^*(K(\mathbb{F}_2, r) \times K(\mathbb{F}_2, s)) \to K$ given by $f(\iota_r \otimes 1) = x$ and $f(1 \otimes \iota_s) = y$. If we apply C_{2^m} to the cohomology of a space we know that this is isomorphic to the cohomology of the cyclic construction of order 2^m on the space. So here the Cartan formula holds. So by the map f we see that it also holds for the product of $v^i u^j \otimes x^{\otimes 2^m}$ and $v^k u^n \otimes y^{\otimes 2^m}$. The relation $Sq^{|x|}x = x^2$ is easy to verify directly from the formulas for the $\mathcal{A}$-action. □

We can now give the final description of the maps induced by the equivariant evaluations in mod two cohomology.

Theorem 9.4. *Let K be an FGA with multiplication $\mu : K \otimes K \to K$. There is a natural $H^*(C_{2^m})$-linear ring homomorphism*

$$Ev_m : \mathcal{C}_m(K) \to H^*(C_{2^m}) \otimes_{\mathbb{F}_2[u]} \ell(K)$$

for each $m \geq 1$. When $m = 1$ it is defined by $1 \otimes x^{\otimes 2} \mapsto \phi(x) + tq(x)$ and $1 \otimes (1 + T)a \otimes b \mapsto t\delta(ab)$ where $t^2 = u$. For $m \geq 2$ it is defined by the *following two conditions*

$$Ev_m(1 \otimes x^{\otimes 2^m}) = \phi(x^{2^{m-1}}) + v\phi(x^{2^{m-1}-1})q(x) + vuq(x^{2^{m-1}-2}\lambda x) \tag{47}$$
$$Ev_m \circ \tau^m_{m-1} = (\tau^m_{m-1} \otimes 1) \circ Ev_{m-1} \circ \mu_* \tag{48}$$

where the second condition relates maps in the following diagram.

$$\begin{tikzcd}
\mathcal{C}_m(K) \arrow[r, "Ev_m"] \arrow[d, shift left, "q_m^{m-1}"] & H^*(C_{2^m}) \otimes_{\mathbb{F}_2[u]} \ell(K) \arrow[d, shift left, "q_m^{m-1}\otimes 1"] \\
\mathcal{C}_{m-1}(K^{\otimes 2}) \arrow[r] \arrow[u, shift left, "\tau^m_{m-1}"] \arrow[d, "\mu_*"'] & H^*(C_{2^{m-1}}) \otimes_{\mathbb{F}_2[u]} \ell(K) \arrow[u, shift left, "\tau^m_{m-1}\otimes 1"] \\
\mathcal{C}_{m-1}(K) \arrow[ur, "Ev_{m-1}"'] &
\end{tikzcd}$$

The following equation holds for $m \geq 2$:

$$(q_m^{m-1} \otimes 1) \circ Ev_m = Ev_{m-1} \circ \mu_* \circ q_m^{m-1} \tag{49}$$

If K is a connected unstable $\mathcal{A}$-algebra then Ev_m is a morphisms in $\mathcal{K}$ for each $m \geq 1$. Finally if $K = H^*X$ for a connected space X of finite type then we have a commutative diagram in $\mathcal{K}$ as follows.

$$\begin{array}{ccc}
H^*(\mathcal{C}_m X) & \xrightarrow{ev_m^*} & H^*(E_m \Lambda X) \\
\cong \uparrow & & \Psi \uparrow \\
\mathcal{C}_m(H^*X) & \xrightarrow{Ev_m} & H^*(C_{2^m}) \otimes_{\mathbb{F}_2[u]} \ell(H^*X)
\end{array}$$

Proof. We first check that Ev_1 is well defined. $\phi(x)$ and $uq(x)$ are linear in x and $\delta(ab)$ is bilinear in (a, b). From this it follows directly that most relations in $H^*(C_2; K^{\otimes 2})$ are respected. In fact it suffices to verify that the following type is respected.

$$1 \otimes (x+y)^{\otimes 2} = 1 \otimes x^{\otimes 2} + 1 \otimes y^{\otimes 2} + 1 \otimes (1+T)x \otimes y$$

But since ϕ is linear this means that $tq(x+y) = tq(x) + tq(y) + t\delta(xy)$ which is true by (6).

Next we check that Ev_1 is a ring homomorphism. The following relations must be respected:

$$\begin{aligned}
(1 \otimes x^{\otimes 2})(1 \otimes y^{\otimes 2}) &= 1 \otimes (xy)^{\otimes 2} \\
(1 \otimes x^{\otimes 2})(1 \otimes (1+T)a \otimes b) &= 1 \otimes (1+T)xa \otimes xb \\
(1 \otimes (1+T)a \otimes b)(1 \otimes (1+T)c \otimes d) &= 1 \otimes (1+T)ac \otimes bd \\
&\quad + 1 \otimes (1+T)ad \otimes bc
\end{aligned}$$

The third is respected since both sides maps to zero. For the first and the second we must check that

$$(\phi(x)+tq(x))(\phi(y)+tq(y))=\phi(xy)+tq(xy)$$
$$(\phi(x)+tq(x))t\delta(ab)=t\delta(x^2ab)$$

which follows by (8), (9) and (12), (14).

For the higher evaluation maps we use induction on m. Assume that the maps $Ev_1,\dots,Ev_{m-1}$ have the stated properties. We first verify that Ev_m is well defined. All classes in its domain are hit by the transfer except for those of highest symmetry. So by induction it suffices to check that the defining conditions for Ev_m respect the relation coming from the expansion (29)

$$1\otimes(x+y)^{\otimes 2^m}=\sum 1\otimes s\beta$$

where the summation is taken over $\beta\in T_{2^m}(x,y)/C_{2^m}$. We rewrite this formula as

$$1\otimes(x+y)^{\otimes 2^m}+1\otimes x^{\otimes 2^m}+1\otimes y^{\otimes 2^m}=\sum_{k=1}^{m}\sigma_k \text{ where } \sigma_k=\sum_{|\beta|=2^k}1\otimes s\beta \tag{50}$$

When we apply Ev_m to the left hand side we get $v\Delta h_m(x,y)$ by (47). We must check that this equals Ev_m applied to the right hand side where Ev_m was defined by (48). Thus we must find $Ev_m(\sigma_k)$ where $1\le k\le m$. Let β be an orbit of length 2^k. It has a representative of the form $(a_1\otimes\cdots\otimes a_{2^k})^{\otimes 2^{m-k}}$. Let $t(\beta)$ denote the order of the set $\{a_i\,|\,1\le i\le 2^k \text{ and } a_i=x\}$. By (48) we have

$$Ev_m(1\otimes s\beta)=vh_{m-k}(a_1\dots a_{2^k})=vh_{m-k}(x^{t(\beta)}y^{2^k-t(\beta)}) \tag{51}$$

If $t(\beta)$ is even $Ev_m(1\otimes s\beta)=0$ since $h_i(a^2)=0$ for all $i\ge 0$. Let $N(2^k,n)$ denote order of the set $\{\beta\,|\,|\beta|=2^k \text{ and } t(\beta)=n\}$. A string $a_1\otimes\cdots\otimes a_{2^k}$ with an odd number of a_i's equal to x cannot be broken into two equal halfs so we have

$$N(2^k,2j+1)=\frac{1}{2^k}\binom{2^k}{2j+1}=\frac{(2^k-1)(2^k-2)\dots(2^k-2j)}{(2j+1)!}$$

The 2-adic valuation of this number is

$$\begin{aligned}v_2(N(2^k,2j+1))=&v_2(2^k-2)+v_2(2^k-4)+\cdots+v_2(2^k-2j)\\&-v_2(2)-v_2(4)-\cdots-v_2(2j)=0\end{aligned}$$

so $N(2^k, 2j+1)$ is odd and we conclude that

$$Ev_m(\sigma_k) = \sum_{j=0}^{2^{k-1}-1} vh_{m-k}(x^{2j+1}y^{2^k-2j-1}) = \sum_{r+s=2^k} vh_{m-k}(x^r y^s)$$

Taking the sum over $1 \leq k \leq m$ and substituting $i = m-k$ we obtain

$$Ev_m(\sum_{k=1}^{m} \sigma_k) = \sum_{i=0}^{m-1} \sum_{r+s=2^{m-i}} vh_i(x^r y^s)$$

By (45) this equals $v\Delta h_m(x,y)$ so the relation (50) is respected and Ev_m is well defined.

Next we verify (49). For highest symmetry element we have

$$\begin{aligned} Ev_{m-1} \circ \mu_* \circ q_m^{m-1}(1 \otimes x^{\otimes 2^m}) &= Ev_{m-1} \circ \mu_*(1 \otimes (x \otimes x)^{\otimes 2^{m-1}}) \\ &= Ev_{m-1}(1 \otimes (x^2)^{\otimes 2^{m-1}}) = \phi(x^{2^{m-1}}) \\ &= (q_m^{m-1} \otimes 1) \circ Ev_m(1 \otimes x^{\otimes 2^m}) \end{aligned}$$

For elements hit by the transfer the result follows since $\mu_* \circ q_m^{m-1} \circ \tau_{m-1}^m = 0$ and $(q_m^{m-1} \otimes 1) \circ (\tau_m^{m-1} \otimes 1) = 0$.

We can now use Frobenius reciprocity and induction to show that Ev_m is a ring map. Assume that Ev_{m-1} is a ring map. We have

$$\begin{aligned} Ev_m(a \cdot \tau_{m-1}^m(b)) &= Ev_m(\tau_{m-1}^m(q_m^{m-1}(a) \cdot b)) \\ &= (\tau_{m-1}^m \otimes 1) \circ Ev_{m-1} \circ \mu_*(q_m^{m-1}(a) \cdot b) \\ &= (\tau_{m-1}^m \otimes 1)(Ev_{m-1} \circ \mu_* \circ q_m^{m-1}(a) \cdot Ev_{m-1} \circ \mu_*(b)) \\ &= (\tau_{m-1}^m \otimes 1)((q_m^{m-1} \otimes 1) \circ Ev_m(a) \cdot Ev_{m-1} \circ \mu_*(b)) \\ &= Ev_m(a) \cdot (\tau_{m-1}^m \otimes 1) \circ Ev_{m-1} \circ \mu_*(b) \\ &= Ev_m(a) \cdot Ev_m(\tau_{m-1}^m(b)) \end{aligned}$$

Hence it suffices to check the following relation

$$Ev_m(1 \otimes x^{\otimes 2^m}) \cdot Ev_m(1 \otimes y^{\otimes 2^m}) = Ev_m(1 \otimes (xy)^{\otimes 2^m})$$

which is valid by (45).

The statement for $K = H^*X$ follows from Theorem 8.4 and Proposition 2.3 of [O]. $\square$

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