Generic Splitting Towers and Generic Splitting Preparation of Quadratic Forms

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ABSTRACT. This manuscript describes how a generic splitting tower of a regular anisotropic quadratic form digests the form down to a form which is totally split.

Introduction

We work with quadratic forms on finite dimensional vector spaces over an arbitrary field k. We call such a form $q: V \to k$ *regular* if the radical $V^\perp$ of the associated bilinear form B_q has dimension ≤ 1 and the quasilinear part $q|V^\perp$ of q is anisotropic. If k has characteristic char $k \neq 2$ this means that $V^\perp = \{0\}$. If char $k = 2$ it means that either $V^\perp = \{0\}$ or $V^\perp = kv$ with $q(v) \neq 0$.

In the present article a "form" always means a regular quadratic form. Our first goal is to develop a generic splitting theory of forms. Such a theory has been given in [K_2] for the case of char $k \neq 2$. Without any restriction on the characteristic, a generic splitting theory for complete quotients of reductive groups was given in [KR], which is closely related to our topic.

In §1 we present a generic splitting theory of forms in a somewhat different manner than in [K_2]. We start with a key result from [KR] (cf. Theorem 1.3 below), then develop the notion of a *generic splitting tower* of a given form q over k with associated higher indices and kernel forms, and finally explain how such a tower $(K_r \mid 0 \leq r \leq h)$ together with the sequence of higher kernel forms $(q_r \mid 0 \leq r \leq h)$ of q controls the splitting of $q \otimes L$ into a sum of hyperbolic planes and an anisotropic form (called the anisotropic part or kernel form of $q \otimes L$), cf. 1.19 below. More generally we explain how the generic splitting tower $(K_r \mid 0 \leq r \leq h)$ together with $(q_r \mid 0 \leq r \leq h)$ controls the splitting of the specialization $\gamma_*(q)$ of q by a place $\gamma: k \to L \cup \infty$, if q has good reduction under q, cf. Theorem 1.18 below. Then in §2

1991 *Mathematics Subject Classification.* Primary 11E04, 11E81.

Key words and phrases. quadratic forms, generic splitting.

This research was supported by the European Union TMR Network FMRX CT-97-0107 "Algebraic K-Theory, Linear Algebraic Groups and Related Structures", and by the Sonderforschungsbereich "Diskrete Mathematik" at the University of Bielefeld.

we study how a generic splitting tower of $q \otimes L$ can be constructed from a generic splitting tower of q for any field extension L/k. All these results are an expansion of the corresponding results in [K$_2$] to fields of any characteristic.
We mention that a reasonable generic splitting theory holds more generally for a quadratic form $q: V \to k$ such that the quasilinear part $q|V^\perp$ is anisotropic, without the additional assumption $\dim V^\perp \leq 1$. This needs more work. It will be contained in the forthcoming book [K$_5$].
In §3 we prove that for any form q over k there exists a generic splitting tower $(K_r \mid 0 \leq r \leq h)$ of q which contains a subtower $(K'_r \mid 0 \leq r \leq h)$ of field extensions of k such that K'_r/K'_{r-1} is purely transcendental, and such that the anisotropic part of $q \otimes K_r$ can be defined over K'_r for every $r \in [1, h]$. {We have $K'_0 = K_0 = k$.} This result, which may be surprising at first glance, leads us in §4 to the second theme of this article, namely *generic splitting preparations* (Def. 4.3) and the closely related *generic splitting decompositions* (Def. 4.8) of a form q. We focus now on the second notion, since its meaning can slightly more easily be grasped than that of the first (more general) notion.
A generic splitting decomposition of a form q over k consists of a purely transcendental field extension K'/k and an orthogonal decomposition

$$q \otimes K' \cong \eta_0 \perp \eta_1 \perp \cdots \perp \eta_h \perp \varphi_h \qquad (*)$$

with certain properties. In particular, $\dim \varphi_h \leq 1$, all η_i have even dimension, and η_0 is the hyperbolic part of $q \otimes K'$ (which comes from the hyperbolic part of q by going up from k to K'). The generic splitting decomposition in a certain sense controls the splitting behavior of $q \otimes L$ for any field extension L of k, more generally of $\gamma_*(q)$, for any place $\gamma: k \to L \cup \infty$ such that q has good reduction under γ. This control can be made explicit in much the same way as the control by generic splitting towers, using "*quadratic places*" (or "Q-places" for short) instead of ordinary places, cf. §6.
Quadratic places have been introduced in the recent article [K$_4$] and used there for another purpose. We recapitulate here what is necessary in §5. We are sorry to say that our theory in §6 demands that the occurring fields have characteristic $\neq 2$. This is forced by the article [K$_4$], where the specialization theory of forms under quadratic places is only done in the case of characteristics $\neq 2$. It seems that major new work and probably also new concepts are needed to establish a specialization theory of forms under quadratic places in all characteristics.
An overall idea behind generic splitting decompositions is the following. If we allow for the form q over k a suitable linear change of coordinates with coefficients in a purely transcendental field extension $K' \supset k$, then the form – now called $q \otimes K'$ – decomposes orthogonally into subforms $\eta_0, \eta_1, \ldots, \eta_h, \varphi_h$ such that the forms $\eta_k \perp \cdots \perp \eta_h \perp \varphi_h$ with $1 \leq k \leq h$ give the higher kernel forms of q, when we go up further from K' to suitable field extensions of K'. Thus, after the change of coordinates, the form q is "well prepared" for an investigation of its splitting behavior. This reminds a little of the Weierstrass preparation theorem, where an analytic function germ becomes well prepared after a linear change of coordinates. In contrast to Weierstrass preparation we allow a purely transcendental field extension for the coefficients of the linear change of coordinates. But no essential information about the form q is lost by passing from q to $q \otimes K'$, since q is the specialization $\lambda_*(q \otimes K')$ of $q \otimes K'$ under any place $\lambda: K' \to k \cup \infty$ over k.

The idea behind generic splitting preparations is similar. {Generic splitting decompositions form a special class of generic splitting preparations.} Now the forms η_i are defined over fields K'_i such that $K'_0 = k$ and every K'_i is a purely transcendental extension of K'_{i-1}.
Generic splitting decompositions and, more generally, generic splitting preparations give new possibilities for manipulations with forms. For example, if $q \otimes K' \cong \eta_0 \perp \cdots \perp \eta_h \perp \varphi_h$ is a generic splitting decomposition of q, then we may look how many hyperbolic planes split off in $q \otimes E_r$ for E_r the total generic splitting field of one of the summands η_r. We do not enter these matters here, leaving all experiments to the future and to the interested reader.

§1. Generic splitting in all characteristics

1.0. NOTATIONS. For a, b elements of a field k we denote the form $a\xi^2 + \xi\eta + b\eta^2$ over k by $[a, b]$. Since we only allow regular forms, we demand $1 - 4ab \neq 0$. By $H := [0, 0]$ we denote the hyperbolic plane.
If q is a (regular quadratic) form over k then we have the Witt decomposition ([W], [A]) $q \cong r \times H \perp \varphi$ with an anisotropic form φ and $r \in \mathbb{N}_0$. We call r the *index* of q and write $r = \text{ind}\,(q)$. We further call φ the *kernel form* [1)] or *anisotropic part* of q and use both notations $\varphi = \ker(q)$, $\varphi = q_{\text{an}}$.
If $L \supset k$ is a field extension then $q \otimes L$ or q_L denotes the form over L obtained from q by extension of the base field k to L. Thus, if q lives on the k-vector space V, then $q \otimes L$ lives on the L-vector space $L \otimes_k V$. A major theme of this article is the study of ind $(q \otimes L)$ and $\ker(q \otimes L)$ for varying extensions L/k.
$\dim q$ denotes the dimension of the vector space V on which q lives, i.e., the number of variables occurring in the form q. We have $\dim q = \dim(q \otimes L)$. The zero form $q = 0$ is not excluded. Then $V = \{0\}$ and $\dim q = 0$.
We say that q *splits totally* if $\dim(q_{\text{an}}) \leq 1$. This is equivalent to ind $(q) = [\dim q/2]$.
For another form φ over k we write $\varphi < q$ if φ is isometric to a subform of q (including the case $\varphi \cong q$).

1.1. DEFINITION/FURTHER NOTATIONS. If $q \neq 0$ and $\dim q$ is even, let

$$\delta q := \begin{cases} (\text{discriminant of } q) \in k^*/k^{*2} & \text{if char}\, k \neq 2 \\ (\text{Arf invariant of } q) \in k^+/\wp k & \text{if char}\, k = 2, \end{cases}$$

$$p_{\delta q}(X) := \begin{cases} X^2 - \delta q & \text{if char}\, k \neq 2 \\ X^2 + X + \delta q & \text{if char}\, k = 2. \end{cases}$$

The separable polynomial $p_{\delta q}(X)$ splits over k if and only if δq is trivial. If $\dim q$ is odd or if δq is trivial we say that q is of *inner type*, otherwise we say that q is of *outer type*.
These notions are adjusted to the corresponding notions in the theory of reductive groups. q is inner (resp. outer) if and only if $SO(q)$ is inner (resp. outer). {N.B. $SO(q)$ is almost simple for $\dim q \geq 3$ since the form q is regular.}
We define

$$k_{\delta q} := \begin{cases} k & \text{if } q \text{ is of inner type} \\ k[X]/p_{\delta q}(X) & \text{if } q \text{ is of outer type.} \end{cases}$$

For $i = 1, \ldots, [\dim q/2]$, we denote by $V_i(q)$ the projective variety of totally isotropic subspaces of dimension i in the underlying space of q, and by $k_i(q)$ we denote the

[1)] = “Kernform” in [W]

function field of $V_i(q)$, unless $\dim q = 2$. In the latter case V_1 consists of two irreducible components defined over $k_{\delta q}$. We then set $k_1(q) = k_{\delta q}$.
In general, we will also write $k(q) = k_1(q)$, which is, with the above interpretation, the function field of the quadric $V_1(q)$ associated to q by the equation $q = 0$. □

1.2. LEMMA. *Let q be a (regular quadratic) form over k.*

i) *If K/k is any field extension such that q_K is of inner type, then K contains a subfield isomorphic to $k_{\delta q}$.*
ii) *Let $i \in \{1, \ldots, [\dim q/2]\}$. If q is of inner type or if $i \leq \dim q/2 - 1$, then $V_i(q)$ is defined over k. If q is of outer type, then $V_{\dim q/2}(q)$ is defined over $k_{\delta q}$.*
iii) *$V_i(q)$ is geometrically irreducible unless q is of outer type and $i = \dim q/2-1$, in which case it decomposes, over $k_{\delta q}$, into two geometrically irreducible components isomorphic to $V_{\dim q/2}(q)$.*

PROOF. i): Let $\dim q$ be even. Clearly q_K is inner if and only if the polynomial $p_{\delta q}$ has a zero in K, hence i) follows.
ii), iii): Since the stabilizer of an i-dimensional totally isotropic subspace of the underlying space of q is a parabolic subgroup of $SO(q)$, the statements follow from [KR, 3.7, p. 44f]. □

A key observation for the generic splitting theory of quadratic forms is the following theorem, which has a generalization for arbitrary homogeneous projective varieties [KR, 3.16, p. 47]:

1.3. THEOREM. *Let q be a form over k, let F_i denote the function field of $V_i(q)$ as a regular extension of k resp. $k_{\delta q}$ according to 1.2.ii, and let L/k be a field extension. The following statements are equivalent.*

i) $\operatorname{ind}(q \otimes L) \geq i$.
ii) *The projective variety $V_i(q)$ has an L-rational point.*
iii) *There is a k-place $F_i \to L \cup \infty$.*
iv) *L contains a subfield isomorphic to the algebraic closure F_i^0 of k in F_i, and the free composite LF_i over F_i^0 is a purely transcendental extension of L.*

REMARK. Only in the case of an outer q and $i = \dim q/2$, we have $F_i^0 = k_{\delta q} \neq k$; in all other cases, i.e., if q is inner or $i \leq \dim q/2 - 1$, we have $F_i^0 = k$ in iv).

PROOF of 1.3. The equivalence of i) and ii) is obvious. The other equivalences follow from [KR, 3.16, p.47], again after observing that the stabilizer of an i-dimensional totally isotropic subspace is a parabolic subgroup of $SO(q)$. □

1.4. DEFINITION. We call two field extensions $K \supset k$ and $L \supset k$ *specialization equivalent* over k, and we write $K \sim_k L$, if there exists a place from K to L over k and also a place from L to K over k. □

1.5. COROLLARY. *If $q' = l \times H \perp q$, and if F_i' is the function field of $V_i(q')$, then F_{l+i}' and F_i are specialization equivalent over k.* [2)].

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2) *We will denote the hyperbolic plane $[0,0]$ over* **any** *field by H*

PROOF. This is obvious by the equivalence of i) and iii) in 1.3. $\square$

In the following q is a (regular quadratic) form over k. We want to associate to q partial generic splitting fields and partial generic splitting towers as has been done in [K$_2$] for char $k \neq 2$. We will proceed in a different way than in [K$_2$], starting with a formal consequence of Theorem 1.3.

1.6. COROLLARY. *Let L and L' be field extensions of k. Assume there exists a place $\lambda\colon L \to L' \cup \infty$ over k. Then* $\operatorname{ind}(q \otimes L') \geq \operatorname{ind}(q \otimes L)$.

PROOF. Let $i := \operatorname{ind}(q \otimes L)$. By the theorem there exists a place $\rho\colon F_i \to L \cup \infty$ over k. Then $\lambda \circ \rho$ is a place from F_i to $L' \cup \infty$. Again by the theorem, $\operatorname{ind}(q \otimes L') \geq i$. $\square$

1.7. DEFINITION. (Cf. [HR$_1$]). The *splitting pattern* $\mathrm{SP}(q)$ is the (naturally ordered) sequence of Witt indices $\operatorname{ind}(q \otimes L)$ with L running through all field extensions of k. $\square$

Notice that the sequence $\mathrm{SP}(q)$ is finite, consisting of at most $[\dim q/2]+1$ elements $j_0 < j_1 < \cdots < j_h$. Of course, $j_0 = \operatorname{ind}(q)$ and $j_h = [\dim q/2]$. We call h the *height* of q, and we write $h = h(q)$. Notice also that $\mathrm{SP}(q)$ is the sequence of all numbers $i \leq [\dim q/2]$ with $\operatorname{ind}(q \otimes F_i) = i$.

1.8. DEFINITION. Let $r \in \{0, 1, \ldots, h\} = [0, h]$. A *generic splitting field of* q of level r is a field extension F/k with the following properties:

a) $\operatorname{ind}(q \otimes F) = j_r$.
b) For every field $L \supset k$ with $\operatorname{ind}(q \otimes L) \geq j_r$ there exists a place $\lambda\colon F \to L \cup \infty$ over k.

Such a field extension F/k, for any level r, is also called a *partial generic splitting field of* q, and, in the case $r = h$, a *total generic splitting* field of k. $\square$

It is evident from the definitions and from Corollary 1.6 that, if K is a generic splitting field of q of some level r and L is a field extension of k, then L is a generic splitting field of q of level r if and only if K and L are specialization equivalent over k.

1.9. PROPOSITION. *Let $r \in [0, h]$. All the fields F_i from Theorem* 1.3 *with $j_{r-1} < i \leq j_r$ are generic splitting fields of q of level r. {Read $j_{-1} = -1$.} In particular, the fields $k(q_{\mathrm{an}}), F_{j_0+1}, \ldots, F_{j_1}$ are generic splitting fields of q of level* 1.

PROOF. By Theorem 1.3, we certainly have $\operatorname{ind}(q \otimes F_i) \geq i$, hence $\operatorname{ind}(q \otimes F_i) \geq j_r$. If L/k is any field extension with $\operatorname{ind}(q \otimes L) \geq j_r$, then, again by Theorem 1.3, there exists a place $\lambda\colon F_i \to L \cup \infty$ over k. Thus condition b) in Definition 1.8 is fulfilled. We can choose L as an extension of k with $\operatorname{ind}(q \otimes L) = j_r$. By Corollary 1.6 we have $\operatorname{ind}(q \otimes F_i) \leq \operatorname{ind}(q \otimes L)$. Thus $\operatorname{ind}(q \otimes F_i) = j_r$.
Moreover, since $k(q_{\mathrm{an}})$ is the function field of $V_1(q_{\mathrm{an}})$, it follows from 1.5 that this field is specialization equivalent over k to F_{j_0+1}. $\square$

1.10. COROLLARY. *If F is a generic splitting field of q (of some level r), then the algebraic closure of k in F is always k, except if q is outer and* $\operatorname{ind}(q_F) = \dim q/2$, *in which case it is $k_{\delta q}$.*

PROOF. Clearly $F \sim_k F_i$ for $i = \operatorname{ind} q_F$, hence we have k-places from F to F_i and vice versa, which are of course injective on the algebraic closure of k in F resp. F_i. Our claim now follows from 1.3 and the remark after 1.3. $\square$

1.11. SCHOLIUM. *Let $(K_r \mid 0 \le r \le h)$ be a sequence of field extensions of k such that for each $r \in [0,h]$ the field K_r is a generic splitting field of q of level r. Let L/k be a field extension of k. We choose $s \in [0,h]$ maximal such that there exists a place from K_s to L over k. Then* $\operatorname{ind}(q \otimes L) = j_s$.

PROOF. By Corollary 1.6 we have $\operatorname{ind}(q \otimes L) \ge j_s$. Suppose that $\operatorname{ind}(q \otimes L) > j_s$. Then $\operatorname{ind}(q \otimes L) = j_r$ for some $r \in [0,h]$ with $r > s$. Thus there exists a place from K_r to L over k. This contradicts the maximality of s. We conclude that $\operatorname{ind}(q \otimes L) = j_s$. □

If q is anisotropic and $\dim q \ge 2$, then a generic splitting field of q of level 1 is called a *generic zero field* of q. Proposition 1.9 tells us that, in general, F_{j_0+1} and $k(q_{\mathrm{an}})$ are generic zero fields of q_{an}. {N.B. The notion of generic zero field has also been established if q is isotropic, cf. [K_2, p. 69]. Then it still is true that $k(q)$ is a generic zero field of q.}

1.12. DEFINITION. A *generic splitting tower* of q is a sequence of field extensions $K_0 \subset \cdots \subset K_h$ of k such that K_0 is specialization equivalent over k with k, and such that K_{r+1} is specialization equivalent over K_r with $K_r(q_{K_r,\mathrm{an}})$. [3)] In particular, the inductively defined sequence $K_0 = k, K_{r+1} = K_r(q_{K_r,\mathrm{an}})$ is the *standard generic splitting tower of* q (cf. [K_2, p. 78]). We call $q_r := (q_{K_r})_{\mathrm{an}}$ the *r-th higher kernel form* of q (with respect to the tower). We define $i_0 := \operatorname{ind} q$ and $i_r := \operatorname{ind} q_{r-1} \otimes K_r$ for $1 \le r \le h$, and we call i_r the *r-th higher index* of q $\quad (0 \le r \le h)$.

1.13. THEOREM. *If $K_0 \subset \cdots \subset K_h$ is a generic splitting tower of q, then, for every $r \in [0,h]$, the field K_r is a generic splitting field of q of level r.*

PROOF. We denote the function fields of the varieties $V_i(q)$ by F_i, as in 1.3. By 1.9, it suffices to show $K_r \sim_k F_{j_r}$, for every $r \ge 0$. For $r = 0$ this is obvious, since F_{j_0} is a purely transcendental extension of k. For $r = 1$ we have, by 1.5, applied to $q_{K_0} = j_0 \times H \perp q_{K_0,\mathrm{an}}$,

$$K_1 \sim_{K_0} K_0(q_{K_0,\mathrm{an}}) \sim_{K_0} F_{j_0+1}K_0 \sim_k F_{j_0+1},$$

hence $K_1 \sim_k F_{j_0+1} \sim_k F_{j_1}$, by 1.9.
We proceed by induction on $\dim q_{\mathrm{an}}$. By induction assumption, our claim is true for $q_1 := q_{K_1,\mathrm{an}}$ over K_1, and hence for $q_{K_1} = (j_0 + j_1) \times H \perp q_1$ by 1.5. That is, for $r \ge 1$, the field K_r is a generic splitting field of q_{K_1} of level $r-1$, and, as such, specialization equivalent over K_1 with the function field of $V_{j_r}(q_{K_1}) \cong V_{j_r}(q) \times_k K_1$ resp. $\cong V_{j_r}(q) \times_{k_{\delta q}} K_1 k_{\delta q}$, which is $F_{j_r} \cdot K_1$ (free product over k resp. $k_{\delta q}$). Hence it remains to show that $F_{j_r} \cdot K_1$ is specialization equivalent to F_{j_r} over k. We have a trivial k-place $F_{j_r} \to F_{j_r} \cdot K_1 \cup \infty$. On the other hand, since $r \ge 1$, we also have a k-place $K_1 \to F_{j_r} \cup \infty$, which gives us a k-place $F_{j_r} \cdot K_1 \to F_{j_r} \cup \infty$. □

The rest of this paragraph will be used in paragraphs 5 and 6 only. For the next statements we need the notion of "good reduction" of a quadratic form.

[3)] This definition of generic splitting towers is slightly broader than the definition in [K_2, p.78]. There it is demanded that $K_0 = k$.

1.14. DEFINITION/REMARK. Let $q\colon K^n \to K$ be a quadratic form over a field K and $\lambda\colon K \to L \cup \infty$ be a place to a second field L. Let $\mathfrak{o} = \mathfrak{o}_\lambda$ denote the valuation ring of λ.

a) We say that q has *good reduction* (abbreviated: GR) under the place λ, if there exists a linear change of coordinates $T \in \mathrm{GL}(n, K)$ such that ($x := (x_1, \ldots, x_n)$)
$$q(Tx) = \sum_{i \le j} a_{ij} x_i x_j$$
with coefficients $a_{ij} \in \mathfrak{o}$, and such that the form $\sum_{i \le j} \lambda(a_{ij}) x_i x_j$ over L is regular.

b) In this situation it can be proved that, up to isometry, the form
$$\sum_{i \le j} \lambda(a_{ij}) x_i x_j$$
does not depend on the choice of T (cf. [K$_1$, Lemma 2.8], [K$_5$, §8]). Abusively we denote this form by $\lambda_*(q)$, and we call $\lambda_*(q)$ *"the" specialization of q under λ.*

c) Let q be a regular form over k, let K, L be field extensions of k and let $\lambda : K \to L \cup \infty$ be a k-place with valuation ring $\mathfrak{o}$. Then q_K has GR under λ and $\lambda_*(q_K) = q_L$. By Lemma 1.14.b below it follows that also $q_{K,\mathrm{an}}$ has GR under λ. □

If q has GR under λ then certainly q itself is regular. Moreover it can be proved that

$$(*) \qquad q \cong [a_1, b_1] \perp \cdots \perp [a_m, b_m] \quad (\perp [\varepsilon])$$

with elements $a_i, b_i \in \mathfrak{o}$ and $\varepsilon \in \mathfrak{o}^*$ (cf. [K$_1$], [K$_5$, §6]). Here the last summand $[\varepsilon]$ denotes the form εX^2 in one variable X. It appears if and only if $n = \dim q$ is odd. Of course, $(*)$ implies

$$\lambda_*(q) \cong [\lambda(a_1), \lambda(b_1)] \perp \cdots \perp [\lambda(a_m), \lambda(b_m)] \quad (\perp [\lambda(\varepsilon)]).$$

1.15. LEMMA. *Let q and q' be forms over K, and assume that* $\dim q$ *is even.*

a) *If q and q' have GR under λ then $q \perp q'$ has GR under λ, and*
$$\lambda_*(q \perp q') \cong \lambda_*(q) \perp \lambda_*(q').$$

b) *If q and $q \perp q'$ have GR under λ, then q' has GR under λ.*

PROOF. Part a) of this lemma is trivial, but b) needs a proof. A proof can be found in [K$_1$, §2] in the case that also q' has even dimension, and in [K$_5$, §8] in general. □

Part b) will be crucial for the arguments to follow.

1.16. PROPOSITION. *Let $\lambda\colon K \to L \cup \infty$ be a place and φ a form over K with GR under λ. Then $\varphi_0 := \ker(\varphi)$ has again GR under λ and $\lambda_*(\varphi) \sim \lambda_*(\varphi_0)$,* $\mathrm{ind}\,(\lambda_*(\varphi)) \ge \mathrm{ind}\,(\varphi)$. *If* $\mathrm{ind}\,(\lambda_*(\varphi)) = \mathrm{ind}\,(\varphi)$, *then* $\ker \lambda_*(\varphi) = \lambda_*(\varphi_0)$.

Proof. Let $\varphi_0 := \ker \varphi$. We have $\varphi \cong j \times H \perp \varphi_0$ with $j = \text{ind}\,(\varphi)$. The form $j \times H$ has GR under λ. By Lemma 1.15 it follows that φ_0 has GR under λ and $\lambda_*(\varphi) \cong j \times H \perp \lambda_*(\varphi_0)$. Now all the claims are evident. $\square$

1.17. Proposition. *Let φ be an anisotropic form over K of dimension ≥ 2 which has GR under a place $\lambda: K \to L \cup \infty$. Let $K_1 \supset K$ be a generic zero field of φ. Then $\lambda_*(\varphi)$ is isotropic if and only if λ extends to a place $\mu: K_1 \to L \cup \infty$.*

Sketch of Proof. a) If λ extends to a place $\mu: K_1 \to L \cup \infty$ then it is obvious that $\varphi \otimes K_1$ has GR under μ and $\mu_*(\varphi \otimes K_1) = \lambda_*(\varphi)$. Since $\varphi \otimes K_1$ is isotropic we conclude by Proposition 1.16 that $\lambda_*(\varphi)$ is isotropic.
b) Assume now that $\lambda_*(\varphi)$ is isotropic. We denote this form by $\overline{\varphi}$ for short. By use of elementary valuation theory it is rather easy to extend λ to a place $\tilde{\lambda}: K(\varphi) \to L(\overline{\varphi}) \cup \infty$ (cf. [K$_5$, §9]; here we do not need that $\overline{\varphi}$ is isotropic). Since $\overline{\varphi}$ is isotropic the field extension $L(\overline{\varphi})/L$ is purely transcendental (cf. Th. 1.3). Thus there exists a place $\rho: L(\overline{\varphi}) \to L \cup \infty$ over L. Now $\rho \circ \tilde{\lambda}: K(\varphi) \to L \cup \infty$ is a place extending λ. Since $K(\varphi) \sim_K K_1$, there exists also a place $\mu: K_1 \to L \cup \infty$ extending λ. $\square$

We return to our form q over k.

1.18. Theorem. *Let $(K_r \mid 0 \leq r \leq h)$ be a generic splitting tower of q with higher indices i_r and higher kernel forms q_r $(0 \leq r \leq h)$. Let $\gamma: k \to L \cup \infty$ be a place such that q has GR under γ. Moreover let $m \in [0,h]$ and $\lambda: K_m \to L \cup \infty$ be a place extending γ. Assume in the case $m < h$ that λ does not extend to a place from K_{m+1} to L. Then* $\text{ind}\,(\gamma_*(q)) = i_0 + \cdots + i_m = j_m$*. The form q_m has GR under λ and* $\ker(\gamma_*(q)) \cong \lambda_*(q_m)$.

Proof. We have $i_0 + \cdots + i_m = j_m$ and $q \otimes K_m \cong j_m \times H \perp q_m$. This implies, that q_m has GR under λ and

$$\gamma_*(q) = \lambda_*(q \otimes K_m) \cong j_m \times H \perp \lambda_*(q_m)$$

(cf. Proof of Prop. 1.16.) It remains to prove that $\lambda_*(q_m)$ is anisotropic. This is trivial if $m = h$. Assume now that $m < h$. If $\lambda_*(q_m)$ would be isotropic then Proposition 1.17 would imply that λ extends to a place from K_{m+1} to L, contradicting our assumptions in the theorem. Thus $\lambda_*(q_m)$ is anisotropic. $\square$

Applying the theorem to the special case that L is a field extension of k and γ is the trivial place $k \hookrightarrow L$, we obtain a result on the Witt decomposition of $q \otimes L$ which is much stronger than 1.11.

1.19. Corollary. *Let $(K_r \mid 0 \leq r \leq h)$ be a generic splitting tower of q. If L/k is a field extension and* $\text{ind}\,(q \otimes L) = j_m$*, and if $\rho: K_r \to L \cup \infty$ is a place over k for some $r \in [0,h]$, then $r \leq m$ and ρ extends to a place $\lambda: K_m \to L \cup \infty$. For every such place λ the kernel form q_m of $q \otimes K_m$ has GR under λ and $\lambda_*(q_m) \cong$* $\ker(q \otimes L)$.$\square$

An easy consequence is the following statement.

1.20. Scholium. *("Uniqueness" of generic splitting towers and higher kernel forms). Let $(K_r \mid 0 \leq r \leq h)$ and $(K'_r \mid 0 \leq r \leq h)$ be generic splitting towers of q with associated sequences of higher indices $(i_r \mid 0 \leq r \leq h)$, $(i'_r \mid 0 \leq r \leq h)$ and sequences of kernel forms $(q_r \mid 0 \leq r \leq h)$, $(q'_r \mid 0 \leq r \leq h)$. Then $i_r = i'_r$ for every $r \in [0,h]$. There exists a place $\lambda: K_h \to K'_h \cup \infty$ over k which restricts to a* *place* $\lambda_r\colon K_r \to K'_r \cup \infty$ *for every* $r \in [0,h]$. *If there is given a place* $\mu\colon K_r \to K'_r \cup \infty$ *over* k *for some* $r \in [0,h]$ *then* q_r *has good reduction under* μ *and* $\mu_*(q_r) \cong q'_r$. □

§2. Behavior of generic splitting fields and generic splitting towers under base field extension

2.1. Definition/Remark. If K/k is a partial generic splitting field of q of some level r, then we denote the algebraic closure of k in K by K°. The extension K°/k is k or $k_{\delta q}$ (cf. 1.10).

For systematic reasons we retain the notation K° for later use, although most often $K^\circ = k$.

2.2. Definition. We call a generic splitting field K of q of some level $r \in [0,h]$ *regular*, if K is regular over the algebraic closure K° of k in K. We then denote by $L \cdot K$, or more precisely by $L \cdot_k K$, the free composite of $L \cdot K^\circ$ and K over K°.
Explanation. Here we have to read $K^\circ = k$, $L \cdot K^\circ = L$ if $r < h$ or if $r = h$ and q is inner. If $r = h$ and q is outer we have two cases. Either L splits the discriminant of q. In this case $K^\circ = k_{\delta q}$ embeds into L and we read $L \cdot K^\circ = L$. Or L does not split δq. In this case $L \cdot K^\circ = L \otimes_k K^\circ = L_{\delta(q \otimes L)}$. □

2.3. Definition. We call a generic splitting tower $(K_r \mid 0 \le r \le h)$ of q *regular* if K_r/K_{r-1} is a regular field extension for every r with $1 \le r < h$, and also for $r = h$, if the form q is inner. If $r = h$ and q is outer, we demand that K_h is regular over the composite $K_{h-1} \cdot K_h^\circ = K_{h-1} \cdot k_{\delta q} = K_{h-1} \otimes_k k_{\delta q}$ over k. □

Let L/k be any field extension. We want to construct partial generic splitting fields and generic splitting towers for $q \otimes L$ from corresponding data for q.
Assume that $(K_r \mid 0 \le r \le h)$ is a *regular* generic splitting tower of q. For every $r \in [0,h]$ we have the free composite $L \cdot K_r = L \cdot_k K_r$ as explained in 2.2. (The existence of the free products $L \cdot K_r$ is the only assumption needed for the following theorem. This is generally true if either L or K_r is regular over k resp. K^0. Thus, instead of the regularity of the generic splitting tower, we could also assume that the field L is separable over k.)

2.4. Theorem. *Let* $J = (r_0, \dots r_e)$ *denote the sequence of increasing numbers* $r \in \{0, \dots, h\}$ *such that* $\operatorname{ind}(q \otimes K_r) = \operatorname{ind}(q \otimes L \cdot K_r)$.
a) *Then the sequence*

$$L \cdot K_{r_0} \subset L \cdot K_{r_1} \subset \cdots \subset L \cdot K_{r_e}$$

is a regular generic splitting tower of $q \otimes L \cdot K_0$, *and hence of* $q \otimes L$.
b) *For every* $r \in [0,h] \setminus J$ *we have an* $L \cdot K_r$*-place* $L \cdot K_{r+1} \to L \cdot K_r \cup \infty$.

Proof. The claim is obvious if $\dim q_{\mathrm{an}} \le 1$. We proceed by induction on $\dim q_{\mathrm{an}}$. Let $r' := \min J \setminus \{0\}$. The induction hypothesis, applied to q_{K_1}, gives a regular generic splitting tower $L \cdot K_{r'} \subset \cdots \subset L \cdot K_{r_e}$ for $q_{L \cdot K_1}$, as well as b) for $r \ge 1$.
In particular, the latter implies that $L \cdot K_{r'} \sim_{L \cdot K_1} L \cdot K_1 \sim_{L \cdot K_0} L \cdot K_0(q_{L \cdot K_0, \mathrm{an}})$, and this proves a).
It remains to show b) for $r = 0$. But $0 \notin J$ means $\operatorname{ind} q_L > \operatorname{ind} q$, hence $\operatorname{ind} q_{L \cdot K_0} > \operatorname{ind} q_{K_0}$. Therefore there is a K_0-place $K_1 \to L \cdot K_0 \cup \infty$, which yields an $L \cdot K_0$-place $L \cdot K_1 \to L \cdot K_0 \cup \infty$. □

In [K_2, p. 85] another proof of Theorem 2.4 and its corollary has been given, which clearly remains valid if char $k = 2$. We believe that the present proof albeit shorter gives more insight than the proof in [K_2].

The sequence $\mathrm{SP}(q \otimes L)$ is a subsequence of $\mathrm{SP}(q) = (j_0, \dots, j_h)$, say $\mathrm{SP}(q \otimes L) = (j_{t(0)}, \dots, j_{t(e)})$, with

$$0 \leq t(0) < t(1) < \cdots < t(e) = h.$$

{ It is evident, that $j_{t(e)} = j_h = [\dim q/2] \in \mathrm{SP}(q \otimes L.$}

It follows from Theorem 2.4 that the $t(i)$ coincide with the numbers r_i there. Thus we have the following corollary.

2.5. COROLLARY. *a) For every $s \in [0, e]$ the anisotropic part of $q \otimes L \cdot K_{t(s)}$ is $q_{t(s)} \otimes L \cdot K_{t(s)}$.*
b) $\mathrm{SP}(q \otimes L)$ is the sequence of all $r \in [0, h]$ such that the form $q_r \otimes L \cdot K_r$ is anisotropic. □

2.6. PROPOSITION. *Let K be a regular generic splitting field of q of some level $r \in [0, h]$. Then $L \cdot K$ is a generic splitting field of $q \otimes L$ of level s, where $s \in [0, e]$ is the number with $t(s-1) < r \leq t(s)$. {Read $t(-1) = -1$.}*

PROOF. We return to the fields F_i in Theorem 1.3. Let $i := j_r$. We have $K \sim_k F_i$ by Proposition 1.9. This implies $K \cdot L \sim_L F_i \cdot L$. Thus it suffices to prove the claim for F_i instead of K. Now $L \cdot F_i$ is the function field of the variety $V_i(q \otimes L)$. Proposition 1.9 gives the claim. □

For later use, it is convenient to insert a digression about "inessential" field extensions.

2.7. DEFINITION. We call a field extension E/k *inessential*, if there exists a place $\alpha: E \to k \cup \infty$ over k, i.e., $E \sim_k k$. □

The idea behind this definition is that, if E/k is inessential, then $q \otimes E$ has essentially the same splitting behavior as q. This will now be verified.
We know already from Corollary 1.6 that ind $(q \otimes E) =$ ind (q), hence $\ker(q \otimes E) = \ker(q) \otimes E$.

2.8. COROLLARY. *Assume again that E/k is an inessential field extension.*
   i) *If $(E_r \mid 0 \leq r \leq h')$ is a generic splitting tower of $q \otimes E$, then it is also a generic splitting tower of q. In particular $h' = h$, i.e., $h(q \otimes E) = h(q)$. Moreover $\mathrm{SP}(q \otimes E) = \mathrm{SP}(q)$.*
   ii) *If K/E is a generic splitting field of $q \otimes E$ of some level $r \in [0, h]$, then K/k is a generic splitting field of q of the same level r.*

PROOF. i): This follows from the definition 1.12 of the notion of a generic splitting tower, together with 2.4.
ii): We have $K \sim_E E_r$. This implies $K \sim_k E_r$, and we are done. □

2.9. REMARK. Theorem 2.4 tells us that the splitting pattern of a quadratic form becomes coarser under base field extension. This may even happen with anisotropic k-forms, which stay anisotropic over the extension field L. The classical example is a quadratic form ψ of dimension 4 and with a non trivial discriminant (or Arf invariant) $\delta\psi$. The form ψ remains anisotropic over the quadratic discriminant extension $L = k_{\delta\psi}$. Of course the height of ψ_L is one, which means, that, over L, the form ψ is 'simpler' than over k. Such a transition $\psi \mapsto \psi_L$ is called an *anisotropic splitting*, since it reduces the complexity of the quadratic form ψ without disturbing its anisotropy.

This phenomenon can be more subtle than in the example just given. For the rest of this remark we assume that char $k \neq 2$.

i) For an an anisotropic r-Pfister form φ with pure part φ', and ψ as above, we study the form $q := \varphi' \otimes \psi$. As mentioned above, ψ_L is anisotropic for $L = k(\sqrt{\delta q})$. We also assume that φ and hence φ' as well as q stay anisotropic over L: For example, we can start with some ground field k_0, and let $k = k_0(X_1, X_2, Y_1, \dots Y_r)$ be the function field of $r + 2$ indeterminates over k_0, and then take

$$\psi = \langle 1, X_1, X_2, \delta X_1 X_2\rangle \text{ with } \delta \in k_0^* \setminus k_0^{*2},\ \varphi = \langle\langle Y_1, \dots, Y_r\rangle\rangle.$$

Then $(\varphi \otimes \psi)_L$ is an anisotropic $r + 2$-Pfister form, and q_L is a Pfister neighbor of that form with complement ψ_L.

Hence q_L is an excellent form of height 2 with splitting pattern

$$\mathrm{SP}(q_L) = (2^{r+1} - 4,\ 2^{r+1} - 2).$$

On the other hand, if $E = k(\sqrt{-X_1})$, then $\psi_E = H \perp \psi_{E,\mathrm{an}}$, and hence $q_E = (\varphi' \otimes \psi)_E = \varphi'_E \otimes H \perp \varphi'_E \otimes \psi_{E,\mathrm{an}}$. Using, e.g., [HR$_2$, 1.2, p. 165] one sees easily that $\varphi'_E \otimes \psi_{E,\mathrm{an}}$ is anisotropic. It is similar to a Pfister neighbor with complement $\psi_{E,\mathrm{an}}$, hence of height two with splitting pattern $(2^r - 1, 2^{r+1} - 3)$.

The splitting pattern of q therefore contains the numbers

$$2^r - 1,\ 2^{r+1} - 4,\ 2^{r+1} - 3,\ 2^{r+1} - 2,$$

and possibly more, but only two of them survive for q_L.

ii) The following example may be even more instructive. We refer to [HR$_2$, 2.5–2.10, p. 167ff.]. Assume $n \geq r > 0$, and let $k = k_0(X_1, \dots, X_n, Y_1, \dots, Y_r, Z$ be the function field over some field k_0 in $n + r + 1$ indeterminates. We let k' denote the subfield $k_0(X_1, \dots, X_n, Y_1, \dots, Y_r)$ of k, hence $k = k'(Z)$ is an inessential extension of k'.

We consider the anisotropic forms

$$q := \langle\langle X_1, \dots, X_n\rangle\rangle \perp Z\langle\langle Y_1, \dots, Y_r\rangle\rangle \qquad \text{over } k$$

and

$$\psi := \langle\langle X_1, \dots, X_n\rangle\rangle \perp \langle\langle Y_1, \dots, Y_r\rangle\rangle \qquad \text{over } k'.$$

According to [l.c., 2.6], their splitting pattern is given by

$$(*) \qquad \begin{array}{ll} (0,\ 2^0,\ 2^1, \dots, 2^r,\ 2^{n-1},\ 2^{n-1} + 2^{r-1}) & \text{if } 1 \leq r \leq n-2 \\ (0,\ 2^0,\ 2^1, \dots, 2^{n-1},\ 2^{n-1} + 2^{n-2}) & \text{if } r = n-1 \\ (0,\ 2^0,\ 2^1, \dots, 2^n) & \text{if } r = n \end{array}$$

(The proof is given in [l.c.] for q, but works, mutatis mutandis, for ψ as well.) We consider the standard generic splitting tower $K_0 = k, K_1, \dots, K_h$ of ψ_k, which is a generic splitting tower of ψ as well, since k is inessential over k', and note that $h = r + 3, r + 2, r + 1$ respectively in the three cases distinguished above.

The forms $\sigma := \langle\langle X_1, \dots, X_n\rangle\rangle_{K_i}$ and $\tau := \langle\langle Y_1, \dots, Y_r\rangle\rangle_{K_i}$ are anisotropic for $r = 0, \dots, r$. Hence, using [l.c., Thm. 1.2, p. 165], we conclude that q_{K_i} is anisotropic for $i = 0, \dots, r$.
In [l.c., 2.5, p. 167] the following well known linkage result is stated: If a linear combination of two anisotropic Pfister forms σ, τ over a given field is isotropic, then its index is the dimension of a Pfister form of maximal dimension dividing both σ and τ.
Since ψ_{K_i} is isotropic, it follows that its index is a power of two, since it is the dimension of the common maximal Pfister divisor of σ and τ. Hence, by the same result, the first higher index of q_{K_i} is exactly the dimension of this Pfister divisor. Therefore, for $i \leq r$, the splitting pattern of q_{K_i} consists of 0, followed by the suffix starting with 2^i of the appropriate sequence $(*)$.
This shows that the gaps occurring in a splitting pattern by base field extension can be arbitrarily large, even for a form which stays anisotropic over the extension.

§3. Defining higher kernel forms over purely transcendental field extensions

3.1. Definition. Let L/K be a field extension and φ a form over L. If we have $\varphi \cong \varphi'_L = \varphi' \otimes L$ with some form φ' over K, then we say that φ is *definable over K (by φ')*. We say that φ is *defined over K (by φ')* if φ' is unique up to isometry over K.

It is known for a field k of characteristic $\neq 2$, that all higher kernels of a form q over k are defined over k if and only the form q is excellent [K_3, 7.14,p. 6], which is a very strong condition on that form: q is excellent if either $\dim q \leq 3$, or if q is a Pfister neighbor with excellent complement. E.g., $\sum_{i=1}^{n} X_i^2$ is excellent over any field of characteristic $\neq 2$.
In this section, as before, q is a (regular quadratic) form over a field k. We want to prove the surprising fact, that, for a suitable generic splitting tower of q, every higher kernel form of q is definable over some finitely generated purely transcendental extension of k.
The following lemma is well known, but we will need its precise statement as given here later on.

3.2. Lemma. *Assume that q is anisotropic. Let L be a separable quadratic field extension of k, such that q_L is isotropic. Then*

$$q \cong \alpha \perp \beta$$

for some regular quadratic forms α, β over k, such that α_L is hyperbolic and β_L is anisotropic. Let $L = k[X]/(aX^2 + X + b)$ with $a \neq 0$, $b \neq 0$ (which can always be achieved). Then α is divisible by $[a, b]$. More precisely, if $i = \operatorname{ind}(q_L)$, then there are pairwise orthogonal vectors $y_1, \dots, y_i$ over k such that

$$\alpha = [a, b] \otimes \langle q(y_1), \dots, q(y_i)\rangle.$$

In case char $k \neq 2$, we may assume that $L = k(\sqrt{\delta})$. Then, α is divisible by $\langle 1, -\delta\rangle$.

Proof. Let $e \neq 0$ be an isotropic vector for q_L. We denote the image of X in L by θ. Then, for $e = x + y\theta$, where x, y have coordinates in k, we obtain [4] $0 = aq_L(e) =$

[4] We briefly write $(x, y) := B_q(x, y)$ with B_q the bilinear form associated to q

$aq(x) + aq(y)\theta^2 + a(x,y)\theta = aq(x) - bq(y) + (a(x,y) - q(y))\theta$, hence $aq(x) = bq(y)$ and $a(x,y) = q(y)$. Therefore $(x,y) \neq 0$ and $q(\xi y + \eta x) = (a\xi^2 + \xi\eta + b\eta^2)(x,y)$ for arbitrary $\xi, \eta \in k$, which gives a binary orthogonal summand of q of the requested type. If its complement is anisotropic over L we are done. Otherwise, an induction on $\dim q$ gives the general result, and the special result for char $k \neq 2$ is obtained as usual by the substitution $X = X' - 1/(2a)$, $\delta = b/a - 1/(4a^2)$. □

3.3. COROLLARY. *Let $k(q)$ denote the function field of the quadric given by $q = 0$, let $k' \subset k(q)$ be a subfield containing k such that $k(q)/k'$ is separable quadratic and k'/k is purely transcendental. Then there is a decomposition*

$$q_{k'} = \alpha \perp q',$$

over k', such that $\alpha \neq 0$, $\alpha_{k(q)}$ is hyperbolic and $q'_{k(q)}$ is anisotropic. Hence the first higher kernel form of q is definable by q' over the purely transcendental extension k' of k.

PROOF. This follows immediately from 3.2. □

As before $h = h(q)$ denotes the height of q.

3.4. THEOREM. *There exists a regular (cf. 2.3) generic splitting tower $(K_r \mid 0 \leq r \leq h)$ of q, a tower of fields $(K'_r \mid 0 \leq r \leq h)$ with $k \subset K'_r \subset K_r$ for every r, $k = K'_0 = K_0$, and a sequence $(\varphi_r \mid 0 \leq r \leq h)$ of forms φ_r over K'_r, such that the following holds. {N.B. All the fields K_r, K'_r are subfields of K_h.}*

(1) $\varphi_r \otimes K_r = \ker(q \otimes K_r)$ *for every* $r \in [0,h]$.
(2) $\varphi_{r+1} < \varphi_r \otimes K'_{r+1} \quad (0 \leq r < h)$. [5]
(3) *K'_{r+1}/K'_r is purely transcendental of finite transcendence degree* $(0 \leq r < h)$.
(4) *K_r/K'_r is a finite multiquadratic extension* $(0 \leq r \leq h)$.

PROOF. We proceed by induction on $\dim q$. We may assume that q is anisotropic. If $\dim q \leq 1$ nothing has to be done. Assume now that $\dim q > 1$. Let $K_1 = k(q)$, the function field of the projective quadric $q = 0$. We choose for K'_1 a subfield of K_1 containing k such that K'_1/k is purely transcendental and K_1/K'_1 is quadratic, which is possible. By 3.3 we have a (not unique) decomposition $q \otimes K'_1 = \eta_1 \perp \varphi_1$ with $\dim \varphi_1 < \dim q$, $\varphi_1 \otimes K_1$ anisotropic, $\eta_1 \otimes K_1$ hyperbolic. If the height $h = 1$, we have finished with K_1, K'_1, φ_1.
Assume now that $h > 1$. We apply the induction hypothesis to φ_1. Let $h(\varphi_1) = e$. There exists a regular generic splitting tower $(L_j \mid 0 \leq j \leq e)$ of φ_1, a tower of fields $(L'_j \mid 0 \leq j \leq e)$, and forms ψ_j over $L'_j \quad (0 \leq j \leq e)$, such that $K'_1 \subset L'_j \subset L_j$, $\psi_{j+1} < \psi_j \otimes L'_{j+1}$, $\psi_j \otimes L_j = \ker(\varphi_1 \otimes L_j)$, L'_{j+1}/L'_j is purely transcendental of finite degree, and L_j/L'_j is finite multiquadratic. Certainly $e \geq h - 1 \geq 1$ since $h(\varphi_1 \otimes K_1) = h - 1$.
We form the field composites $K_1 \cdot L_j = K_1 \cdot_{K'_1} L_j$ as explained in 2.2. Let J denote the set of indices $j \in [0,e]$ with $\psi_j \otimes K_1 \cdot L_j$ anisotropic, and let $\mu(0) < \mu(1) < \cdots < \mu(t)$ be a list of these indices. (N.B. $\mu(0) = 0, \mu(t) = e$.) By 2.4 and 2.9 the sequence of fields $(K_1 \cdot L_{\mu(r)} \mid 0 \leq r \leq t)$ is a regular generic splitting tower of $\varphi_1 \otimes K_1$, hence of $\varphi \otimes K_1$, and $\varphi \otimes (K_1 \cdot L_{\mu(r)})$ has the kernel form $\psi_{\mu(r)} \otimes (K_1 \cdot L_{\mu(r)})$. Clearly the tower $(K_1 \cdot L_{\mu(r)} \mid 0 \leq r \leq t)$ is regular, and $t = h - 1$.

[5] *cf. Notations 1.0*

For $2 \leq i \leq h$ we put $K_i' := L'_{\mu(i-1)}$, $K_i = K_1 \cdot L_{\mu(i-1)}$, $\varphi_i := \psi_{\mu(i-1)}$. Adding to these fields and forms the fields K_1', K_1, $K_0 = K_0' = k$, and the forms $\varphi_1, \varphi_0 := q$, we have towers $(K_r \mid 0 \leq r \leq h)$, $(K_r' \mid 0 \leq r \leq h)$ and a sequence $(\varphi_r \mid 0 \leq r \leq h)$ of anisotropic forms with all the properties listed in the theorem. $\square$

We add to this theorem a further observation.

3.5. PROPOSITION. *We stay in the situation of Theorem 3.4. Assume that $h \geq 1$, i.e., q is not split. By property (2) we have a sequence $(\eta_r \mid 1 \leq r \leq h)$ of forms η_r over K_r' such that*

$$\varphi_{r-1} \otimes K_r' \cong \eta_r \perp \varphi_r \qquad (1 \leq r \leq h).$$

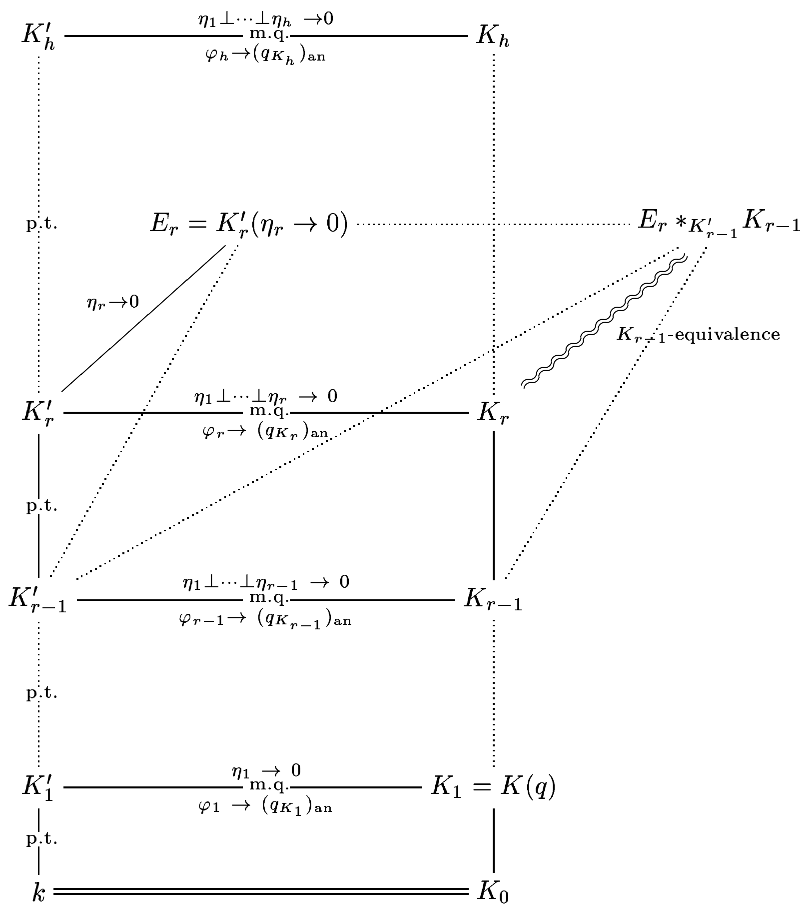
We choose for each $r \in [1, h]$ a total generic splitting field E_r of η_r. Let q_r denote the kernel form of $q \otimes K_r$, $0 \leq r \leq h$.

CLAIM. *The field composite $K_{r-1} \cdot_{K'_{r-1}} E_r =: L_r$ is specialization equivalent to K_r over K_{r-1}. Thus L_r is a generic zero field of q_{r-1} and a generic splitting field of q of level r.*

PROOF. $q_{r-1} \otimes L_r$ is isotropic. Thus there exists a place $\lambda: K_r \to L_r \cup \infty$ over K_{r-1}. On the other hand $\eta_r \otimes K_r \sim 0$. Thus there exists a place $\rho: E_r \to K_r \cup \infty$ over K_r'. The field extension K_{r-1}/K'_{r-1} is finite multiquadratic. By standard valuation theoretic arguments ρ extends to a place μ: $K_{r-1} \cdot_{K'_{r-1}} E_r \to K_r \cup \infty$ over $K_{r-1} \cdot K_r'$, hence over K_{r-1}. $\square$

EXPLANATIONS ON THE DIAGRAM. The tower on the left consists of purely transcendental extensions (labeled by "p.t."), the tower on the right is a generic splitting tower of q. The "horizontal" extensions labeled by "m.q." are multiquadratic, splitting the direct sums $\varepsilon_r = \mathop{\perp}\limits_{i=1}^{r-1} \eta_{i,K_i'} \perp \eta_r$ (for which we simply have written $\eta_1 \perp \cdots \perp \eta_r$) totally and leaving φ_r anisotropic, making it isometric to the r-th higher kernel form q_r of q. The field $E_r = K_r'(\eta_r \to 0)$ is a generic total splitting field of the form η_r over K_r'.

A Generic Splitting Tower According To 3.4 and 3.5



§4. Generic splitting preparation

In 3.4 and 3.5 we have associated to the form q over k – among other things – a tower $(K'_r \mid 0 \le r \le h)$ of purely transcendental field extensions of k together with a sequence $(\eta_r \mid 1 \le r \le h)$ of anisotropic subforms η_r of $q \otimes K'_r$. We want to understand in which way these data control the splitting behavior of q under field extensions, forgetting the generic splitting tower $(K_r \mid 0 \le r \le h)$ in 3.4. (We will have only partial success, cf. §6 below.) In addition we strive for an abstraction of the situation established in 3.4 and 3.5.

As before q is any regular quadratic form over a field k and h denotes the height $h(q)$.

4.1. Definition. Let $r \in [0,h]$ and let K/k be an inessential field extension (cf. 2.7). A form η over K is called a *generic splitting form of q of* level r, if $\dim \eta$ is even and there exists an orthogonal decomposition $q \otimes K \cong \eta \perp \psi$ such that the following holds: If E/K is a total generic splitting field of η, then E/k is a partial generic splitting field of q of level r, while $\psi \otimes E$ is anisotropic. We further call ψ the *complement* (or complementary form) *of* η. □

N.B. This property does not depend on the choice of the total generic splitting field of E.
Generic splitting forms occur whenever a higher kernel form of q is definable over an inessential field extension of k.

4.2. PROPOSITION. *Let K/k be a partial generic splitting tower of q of level r. Assume there is given an inessential subextension K'/k of K/k and a subform ψ of $q \otimes K'$ such that $\psi \otimes K = \ker(q \otimes K)$. Let η denote the complement of ψ in $q \otimes K'$, i.e., $q \otimes K' \cong \eta \perp \psi$. (N.B. η is uniquely determined up to isometry.) Then η is a generic splitting form of q of level r.*

PROOF. $\dim q \equiv \dim \psi \mod 2$. Thus $\dim \eta$ is even. Let E/K' be a total generic splitting field of η. Then $q \otimes E \sim \psi \otimes E$. Thus there exists a place $\lambda: K \to E \cup \infty$ over k. On the other hand, $\eta \otimes K \sim 0$. Thus there exists a place $\mu: E \to K \cup \infty$ over K'. Since μ is also a place over k, the fields E and K are specialization equivalent over k. We conclude that also E is a generic splitting field of q of level r. Now it is evident that $\dim \ker(q \otimes E) = \dim \psi$. Thus $\psi \otimes E$ is anisotropic. □

4.3. DEFINITION. A *generic splitting preparation* of q is a tower of fields $(K'_r \mid 0 \leq r \leq h)$ together with a sequence $(\eta_r \mid 0 \leq r \leq h)$ of forms η_r over K'_r such that the following holds:

(1) $K'_0 = k$, and η_0 is the hyperbolic part of φ.
(2) K'_{r+1}/K'_r is purely transcendental for every r, $0 \leq r < h$. [6]
(3) There exist orthogonal decompositions
$$q \cong \eta_0 \perp \varphi_0, \quad \varphi_r \otimes K'_{r+1} \cong \eta_{r+1} \perp \varphi_{r+1}, \quad (0 \leq r < h).$$
(4) For every $r \in [0, h]$ the form $\underset{j=0}{\overset{r}{\perp}} \eta_j \otimes K'_r$ is a generic splitting form of q of level r. □

The forms $\varphi_r \quad (0 \leq r \leq h)$ are uniquely determined (up to isometry, as always) by condition (3). We call φ_r *the r-th residual form* and η_r the r-th *splitting form* of the given generic splitting preparation. Clearly $i_r := \dim \eta_r/2$ is the r-th higher index (cf. 1.1) of q, and $\dim \varphi_h \leq 1$. If q is anisotropic then we sometimes denote the generic splitting preparation by $(K'_r \mid 0 \leq r \leq h)$, $(\eta_r \mid 1 \leq r \leq h)$, omitting the trivial form $\eta_0 = 0$. □

4.4. SCHOLIUM. *Generic splitting preparations of q exist in abundance. Indeed, in the situation described in 3.4 and 3.5, the tower of fields $(K'_r \mid 0 \leq r \leq h)$ together with the sequence $(\eta_r \mid 0 \leq r \leq h)$ of forms η_r over K'_r, where η_r for $r \geq 1$ has been introduced in 3.5 and η_0 denotes the hyperbolic part of q, is a generic splitting preparation of q.*

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[6] Things below would not change much if we merely demanded that the extensions K'_{r+1}/K'_r are inessential.

PROOF. Let $r \in [1,h]$ be fixed and $\varepsilon_r := \perp_{j=0}^{r} \eta_j \otimes K'_r$. Then $q \otimes K'_r \cong \varepsilon_r \perp \varphi_r$ and $\ker(\varphi \otimes K_r) = \varphi_r \otimes K_r$. Proposition 4.2 tells us that ε_r is a generic splitting form of q of level r. $\square$

Notice that in this argument property (4) of Theorem 3.4 has not been used.

In the following, we study a fixed generic splitting preparation $(K'_r \mid 0 \leq r \leq h)$, $(0 \leq \eta_r \leq h)$ *of* q *with associated residual forms* φ_r $(0 \leq r \leq h)$. *For every* $r \in \{1,\dots,h\}$ *let* ε_r *denote the form* $\perp_{j=0}^{r} \eta_j \otimes K'_r$. We do *not* assume that the preparation arises in the way described in the scholium.

Our next goal is to derive a generic splitting preparation of $q \otimes L$ from these data for a given field extension L/k.

4.5. LEMMA. *Assume that* K/k *is an inessential field extension. Let* η *be a form over* K *which is a generic splitting form of* q, *and let* ψ *be the complementary form,* $\varphi \otimes K \cong \eta \perp \psi$. *Let* E/K *be a regular total generic splitting field of* η *(cf. 2.2). Finally, let* $K \cdot L = K \cdot_k L$ *denote the free field composite of* K *and* L *over* k, *and* $E \cdot L = E \cdot_k L$ *the composite of* E *and* L *as explained in 2.2. Assume that* $\psi \otimes E \cdot L$ *remains anisotropic. Then* $K \cdot L/L$ *is again inessential and* $\eta \otimes K \cdot L$ *is a generic splitting form of* $q \otimes L$ *with complementary form* $\psi \otimes K \cdot L$.

PROOF. Any place $\alpha\colon K \to L \cup \infty$ over k extends to a place from $L \cdot K$ to L over L. Thus $K \cdot L/L$ is inessential. By Proposition 2.6 the field $E \cdot_k L = E \cdot_K (K \cdot_k L)$ is a total generic splitting field of $\eta \otimes K \cdot L$ and also a partial generic splitting field of $\varphi \otimes L$. Now the claim is obvious from Definition 4.1. $\square$

As in §1 and §2 we enumerate the splitting pattern $\mathrm{SP}(q) = \{j_r \mid 0 \leq r \leq h\}$ by

$$0 \leq j_0 < j_1 < \cdots < j_h = [\frac{\dim q}{2}]$$

and write $\mathrm{SP}(q \otimes L) = \{j_r \mid r \in J\}$ with $J = \{t(s) \mid 0 \leq s \leq e\}$,

$$0 \leq t(0) < t(1) < \cdots < t(e) = h.$$

We have $e = h(q \otimes L)$.

4.6. PROPOSITION. *For* $0 < s \leq e$ *we define* $L'_s = L \cdot K'_{t(s)}$ *as the free composite of the fields* L *and* $K'_{t(s)}$ *over* k, *and we put*

$$\zeta_s := \eta_{t(s-1)+1} \otimes L'_s \perp \cdots \perp \eta_{t(s)-1} \otimes L'_s \perp \eta_{t(s)}.$$

We further define $L'_0 = L$ *and* ζ_0 *as the hyperbolic part of* $\varphi \otimes L$. *Then* $(L'_s \mid 0 \leq s \leq e)$, $(\zeta_s \mid 0 \leq s \leq e)$ *is a generic splitting preparation of* $q \otimes L$.

PROOF. For every r with $0 < r \leq h$ we choose a regular total generic splitting field F_r/K'_r of the form ε_r.*) [7] Then F_r/k is a partial generic splitting field of q of level r. Let $L \cdot F_r = L \cdot_k F_r$ denote the composite of L with F_r over k as explained in

[7] The fields F_i from 1.3 will not be used in the following.

2.2. If $0 < s \le e$, then the field $L \cdot F_{t(s)}$ is a generic splitting field of $\varphi \otimes L$ and $\ker(\varphi \otimes L \cdot F_{t(s)}) = \varphi_{t(s)} \otimes L \cdot F_{t(s)}$ by Proposition 2.6. We have

$$\varphi \otimes L'_s \cong \varepsilon_{t(s)} \otimes L'_s \perp \varphi_{t(s)} \otimes L'_s.$$

Again by 2.6, the field

$$L \cdot_k F_{t(s)} = (L \cdot_k K'_{t(s)}) \cdot_{K'_{t(s)}} F_{t(s)} = L'_s \cdot_{K'_{t(s)}} F_{t(s)}$$

is a total generic splitting field of $\varepsilon_{t(s)} \otimes L'_s$. The form $\varphi_{t(s)} \otimes L'_s$ remains anisotropic over $L \cdot_k F_{t(s)}$. Now Proposition 4.2 tells us that $\varepsilon_{t(s)} \otimes L'_s$ is a generic splitting form of $\varphi \otimes L'_s$, and we are done. □

4.7. Corollary. *Let $m \in [1, h]$ be fixed, and let F be a regular total generic splitting field of ε_m. For every r with $m < r \le h$ let $F \cdot K'_r$ denote the free composite of F and K'_r over k. Then the tower $F \subset F \cdot K'_{m+1} \subset \cdots \subset F \cdot K'_h$ together with the sequence $(\eta_r \otimes F \cdot K'_r \mid m < r \le h)$ is a generic splitting preparation of the anisotropic form $\varphi_m \otimes F$.*

Proof. We apply Proposition 4.6 with $L = F$. Now $J = \{m, m+1, \dots h\}$. Thus $e = h - m$ and $t(s) = s + m \quad (0 \le s \le h - m)$. The form $q \otimes F$ has the kernel form $\varphi_m \otimes F$. □

We now look for generic splitting preparations with $K'_1 = \cdots = K'_h$. These are the "generic decompositions" according to the following definition.

4.8. Definition. Let $K' \supset k$ be a purely transcendental field extension. A *generic splitting decomposition* of q over K' is a sequence $(\alpha_0, \alpha_1, \dots, \alpha_h)$ of forms over K' of even dimensions such that the following holds.

i) $q \otimes K' \cong \alpha_0 \perp \alpha_1 \perp \cdots \perp \alpha_h \perp \varphi_h$ with $\dim \varphi_h \le 1$.
ii) α_0 is the hyperbolic part of $q \otimes K'$. For every r with $1 \le r \le h$ the form $\alpha_0 \perp \cdots \perp \alpha_r$ is a generic splitting form of q of level r. □

The next proposition tells us that we can always pass from a generic splitting preparation of q to a generic splitting decomposition of q.

4.9. Proposition. *As before, let $(K'_r \mid 0 \le r \le h)$, $(\eta_r \mid 0 \le r \le h)$ be a generic splitting preparation of q. Then $(\eta_r \otimes K'_h \mid 0 \le r \le h)$ is a generic splitting decomposition of q over K'_h.*

Proof. Let $K' := K'_h$ and $\alpha_r := \eta_r \otimes K'_h \quad (0 \le r \le h)$. Since η_0 is the hyperbolic part of q and K'/k is purely transcendental, the form α'_0 is the hyperbolic part of $q \otimes K'$. For $1 \le r \le h$ we have – with the notations from above – $\varepsilon_r \otimes K' = \alpha_0 \perp \cdots \perp \alpha_r$. Let F_r be a regular total generic splitting field of ε_r (over K'_r), hence a partial generic splitting field of q (over k) of level r. The free composite $F_r \cdot K' = F_r \cdot_{K'_r} K'$ is a total generic splitting field of $\varepsilon_r \otimes K' = \alpha_r$ by Proposition 2.6. Now $F_r \cdot K'$ is purely transcendental over F_r. Thus $F_r \cdot K'$ is also a partial generic splitting field of q of level r. We have $q \otimes K' \cong \alpha_r \perp (\varphi_r \otimes K')$. The form $\varphi_r \otimes F_r$ is anisotropic. Since $F_r \cdot K'/F_r$ is purely transcendental, it follows that $(\varphi_r \otimes K') \otimes F_r \cdot K' = (\varphi_r \otimes F_r) \otimes F_r \cdot K'$ is anisotropic. Thus α_r is a generic splitting form of q of level r. □

Although generic splitting decompositions look simpler than generic splitting preparations, it is up to now not clear to us which of the two concepts is better to work with. See also our discussion below at the end of §6.

§5. A brief look at quadratic places

We need some more terminology.
If K is a field, we denote the group of square classes K^*/K^{*2} by $Q(K)$ and a single square class aK^{*2} by $\langle a\rangle$, identifying this class with the bilinear form $\langle a\rangle$ over K.
If $\lambda: K \to L \cup \infty$ is a place with associated valuation ring $\mathfrak{o} = \mathfrak{o}_\lambda$, we denote the image of the unit group $\mathfrak{o}^*$ in $Q(K)$ by $Q(\mathfrak{o})$. Notice that $Q(\mathfrak{o}) \cong \mathfrak{o}^*/\mathfrak{o}^{*2}$, and that λ gives us a homomorphism λ_*: $Q(\mathfrak{o}) \to Q(L)$, $\lambda_*(\langle\varepsilon\rangle) = \langle\lambda(\varepsilon)\rangle$. (The bilinear form $\langle\varepsilon\rangle$ has good reduction under λ and $\lambda_*(\langle\varepsilon\rangle)$ is the specialization of this form under λ.)

5.1. DEFINITION. A *quadratic place*, or *Q-place* for short, from a field K to a field L is a triple (λ, H, χ) consisting of a place $\lambda: K \to L \cup \infty$, a subgroup H of $Q(K)$ containing $Q(\mathfrak{o}_\lambda)$, and a homomorphism $\chi: H \to Q(L)$ (called a "character" in the following) extending the homomorphism λ_*: $Q(\mathfrak{o}_\lambda) \to Q(L)$. □

We often denote such a triple (λ, H, χ) by a capital Greek letter Λ and symbolically write $\Lambda: K \to L \cup \infty$ for the Q-place Λ.
Every place $\lambda: K \to L \cup \infty$ gives us a Q-place $\hat{\lambda} = (\lambda, Q(\mathfrak{o}_\lambda), \lambda_*)$: $K \to L \cup \infty$, where $\lambda_*: Q(\mathfrak{o}_\lambda) \to Q(L)$ is defined as above. We regard λ and $\hat{\lambda}$ essentially as the same object. In this sense Q-places are a generalization of the usual places.

5.2. DEFINITION. If $\Lambda = (\lambda, H, \chi)$: $K \to L \cup \infty$ is a Q-place then an *expansion of* Λ is a Q-place $\Lambda' = (\lambda, H', \chi'): K \to L \cup \infty$ with the same first component λ as Λ, a subgroup H' of $Q(K)$ containing H, and a character $\chi': H' \to Q(L)$ extending χ. □

Usually Λ allows many expansions, and Λ itself is an expansion of $\hat{\lambda}$.
In the following $\Lambda = (\lambda, H, \chi)$: $K \to L \cup \infty$ a Q-place and $\mathfrak{o} := \mathfrak{o}_\lambda$.

5.3. DEFINITIONS. Let k be a subfield of K.

a) The *restriction* $\Lambda|k$ of the Q-place Λ to k is the Q-place (ρ, E, σ) where $\rho = \lambda|k: k \to L \cup \infty$ denotes the restriction of the place $\lambda: K \to L \cup \infty$ in the usual sense, E denotes the preimage of H in $Q(k)$ under the natural map $j: Q(k) \to Q(K)$, and σ denotes the character $\chi \circ (j|D)$ from D to $Q(L)$.
b) If $\Gamma: k \to L \cup \infty$ is any Q-place from k to L then we say that Λ *extends* Γ (or, that Λ is an extension of Γ), if $\Lambda|k$ is an expansion of Γ. □

At the first glance one might think that this notion of extension is not strong enough. One should demand $\Lambda|k = \Gamma$, in which case we say that Λ is a *strict extension* of Γ. But it will become clear below (cf. 5.8.ii, 5.12, 6.1) that the weaker notion of extension as above is the one needed most often.
As before, we stay with a Q-place $\Lambda = (\lambda, H, \chi)$: $K \to L \cup \infty$.

5.4. DEFINITION. Let φ be a regular quadratic form over K. We say that φ has *good reduction* (abbreviated: GR) *under* Λ, if there exists an orthogonal decomposition

$$(*) \qquad \varphi = \perp_{h \in H} h\psi_h$$

with forms ψ_h over K which all have GR under λ. Here $h\psi_h$ denotes the product $\langle h\rangle \otimes \psi_h$ of the bilinear form $\langle h\rangle$ with ψ_h. {This amounts to scaling ψ_h by a representative of the square class $\langle h\rangle$.} Of course, $\psi_h \neq 0$ for only finitely many $h \in H$.

Alternatively we say in this situation that φ is Λ-*unimodular*. In harmony with this speaking we call a form ψ over K, which has GR under λ, also a λ-*unimodular form.* □

5.5. REMARK. We may choose a subgroup U of H such that $H = U \times Q(\mathfrak{o})$. If φ has GR under Λ then we can simplify the decomposition $(*)$ to a decomposition

$$\varphi \cong \mathop{\perp}_{u \in U} u\varphi_u$$

with λ-unimodular forms φ_u. □

5.6. PROPOSITION. *If φ has GR under Λ, and a decomposition $(*)$ is given, then the form $\mathop{\perp}_{h \in H} \chi(h)\lambda_*(\psi_h)$ is up to isometry independent of the choice of the decomposition $(*)$.*

This has been proved in [K_4] if char$L \neq 2$. A proof in general (which is rather different) will be contained in [K_5].

5.7. DEFINITION. If φ has GR under Λ then we denote the form $\mathop{\perp}_{h \in H} \chi(h)\lambda_*(\psi_h)$ (cf. 5.6) by $\Lambda_*(\varphi)$, and we call $\Lambda_*(\varphi)$ the *specialization of φ under λ*. □

5.8. REMARKS.

i) *If ψ is a second form over K such that $\varphi \perp \psi$ is regular, and if both φ and ψ have GR under Λ then $\varphi \perp \psi$ has GR under Λ and*

$$\Lambda_*(\varphi \perp \psi) \cong \Lambda_*(\varphi) \perp \Lambda_*(\psi).$$

ii) *Assume that k is a subfield of K and $\Gamma: k \to L \cup \infty$ is a Q-place such that Λ extends Γ (cf.5.3.b). Let q be a regular form over k which has GR under Γ. Then $q \otimes K$ has GR under Λ and $\Lambda_*(q \otimes K) \cong \Gamma_*(q)$.*

We omit the easy proofs. □

It seems that quadratic places come up in connection with generic splitting forms (cf. 4.1) in a natural way. We illustrate this by a little proposition, which will also serve us to indicate some of the difficulties we have to face if we want to make good use of quadratic places in generic splitting business.

5.9. PROPOSITION. *Let q be an anisotropic regular form over a field k,* $\dim q > 2$. *Let $k(q)$ denote the function field of the projective quadric $q = 0$. Let $L \supset k$ be a field extension such that $q_L = q \otimes L$ is isotropic. Then there is a purely transcendental subextension k'/k of $k(q)/k$, such that $k(q)/k'$ is separable quadratic, and a quadratic place $\Lambda: k' \to L \cup \infty$ over k, such that the form α described in 3.3 $\{q_{k'} \cong \alpha \perp q', \alpha_{k(q)} \sim 0, q'_{k(q)}$ anisotropic$\}$ has GR under Λ and $\Lambda_*(\alpha) \sim 0$.*

PROOF. Since q_L is isotropic we have a place $\lambda: k(q) \to L \cup \infty$ over k. Let $\mathfrak{o}$ denote the valuation ring of λ, let V denote the underlying k-vector space of q. We take a decomposition $V \cong \mathop{\perp}_{i=1}^{r} (ke_i \oplus kf_i) \perp V'$ with $ke_i \oplus kf_i = [a_i, b_i]$, and $V' = \langle a_0 \rangle$ or $V' = 0$, and $a_i, b_i, a_0 \in k^*$.

We choose a primitive isotropic vector $x \in V_{\mathfrak{o}} = \mathfrak{o} \otimes_k V$ with $(x, V_{\mathfrak{o}}) = \mathfrak{o}$. By rearranging coordinates over k we may assume that $x = Xe_1 + Yf_1 + z$, and $X \in \mathfrak{o} \setminus 0$, $Y \in \mathfrak{o}^*$, $z \in \mathop{\perp}_{i=2}^{r} (\mathfrak{o}e_i \oplus k\mathfrak{o}f_i) \perp \mathfrak{o}V'$, and after dividing by Y, we

may assume that $Y = 1$. We take the coordinates of $\overset{r}{\underset{i=2}{\perp}} (\mathfrak{o}e_i \oplus \mathfrak{o}f_i) \ \perp \mathfrak{o}V'$ as independent variables over k, which generate a purely transcendental subfield k' of $k(q)$. The equation $0 = q_{\mathfrak{o}}(x) = a_1X^2 + X + b$ with $b = b_1 + q_{k(q)}(z) \in \mathfrak{o} \cap k'$ defines the field $k(q)$ as a separable quadratic extension of k'.
By construction we have $\lambda(b) \neq \infty$. Hence the quadratic form $[a_1, b]$ has good reduction under this place. We write $\alpha = [a_1, b] \otimes \langle c_1, \dots, c_i \rangle$ with $c_\nu \in \Im(q_{k'}) \setminus 0$ according to 3.2. We denote the restriction of λ to k' by λ', and the associated valuation ring by $\mathfrak{o}'(= \mathfrak{o} \cap k')$.
Let H denote the subgroup of $Q(k')$ which is generated by $Q(\mathfrak{o}')$ and the classes $\langle c_1 \rangle, \dots, \langle c_i \rangle$. We choose some extension $\chi \colon H \to Q(L)$ of the character $\lambda'_* \colon Q(\mathfrak{o}') \to Q(L)$. The quadratic place $\Lambda = (\lambda', H, \chi)$ has the desired properties. Alternatively we may choose for Λ the restriction of $\hat{\lambda}$ to k'. □

This proposition leaves at least two things to be desired. Firstly, it would be nice and much more useful, to have the subextension k'/k to be chosen in advance, independently of the place λ in the proof. Secondly it would be pleasant if also the form q' has GR under Λ. This is by no means guaranteed by our proof. We see no reason, why the analogue of Lemma 1.15.b for quadratic places instead of ordinary places should be true. The main crux here is the case char $k = 2$. Then usually many forms over k' do not admit λ'-modular decompositions for a given place $\lambda' \colon k' \to L \cup \infty$ over k. {It is easy to give counterexamples in a sufficiently general situation, cf. [K$_5$]}. On the other hand, if we can achieve in Proposition 5.9 in addition that q' has GR under Λ, then the equation

$$q \otimes L = \Lambda_*(\alpha) \perp \Lambda_*(q'),$$

which follows from the remarks 5.8 above, together with $\Lambda_*(\alpha) \sim 0$ would "explain" that $q \otimes L$ is isotropic, and how the anisotropic part of $q \otimes L$ is connected with the generic splitting form α of q of level 1.
If char $k \neq 2$ it is still not evident that the analogue of 1.15.b holds for the quadratic place Λ' but now at least every form over k' has a λ'-modular decomposition. Indeed, this trivially holds for forms of dimension 1, hence for all forms. In [K$_4$] a way has been found, to force an analogue of 1.15.b for quadratic places to be true, by relaxing the notion "good reduction" to a slightly weaker – but still useful – notion "almost good reduction".
We can define "almost good reduction" without restriction to characteristic $\neq 2$ as follows.

5.10. Definition. Let $\Lambda = (\lambda, H, \chi) \colon K \to L \cup \infty$ be a Q-place, $\mathfrak{o} := \mathfrak{o}_\lambda$, and let S be a subgroup of $Q(K)$ such that $Q(K) = S \times H$. A form φ over K has *almost good reduction* (abbreviated AGR) under Λ if φ has a decomposition

$$(**) \qquad \varphi \cong \underset{s \in S}{\perp} s\varphi_s$$

with Λ-unimodular forms φ_s and $\Lambda_*(\varphi_s) \sim 0$ for every $s \in S$, $s \neq 1$. In this case we call the form

$$\Lambda_*(\varphi) := \Lambda_*(\varphi_1) \perp (\dim \varphi - \dim \varphi_1)/2 \times H$$

the *specialization of* φ *under* Λ. □

The point here is that $\Lambda_*(\varphi)$ is independent of the choice of the decomposition $(**)$ and also of the choice of S. This has been proved in [K_4] in the case char $L \neq 2$. It also holds if char $L = 2$, cf. [K_5].
It now is almost trivial that the analogue of 1.15. holds for Q-places with AGR instead of GR, provided char $L \neq 2$, cf. [K_4, §2].

5.11. PROPOSITION. *Let $\Lambda: K \to L \cup \infty$ be a Q-place and let φ and ψ be regular forms over K. Assume that char $L \neq 2$.*

a) *If φ and ψ have AGR under Λ, then $\varphi \perp \psi$ has AGR under Λ and*

$$\Lambda_*(\varphi \perp \psi) \cong \Lambda_*(\varphi) \perp \Lambda_*(\psi).$$

b) *If φ and $\varphi \perp \psi$ have AGR under Λ, then ψ has AGR under Λ.* □

5.12. PROPOSITION. *Let $k \subset K$ be a field extension. Let $\Gamma: k \to L \cup \infty$ and $\Lambda: K \to L \cup \infty$ be Q-places with Λ extending Γ. Assume finally that q is a form over k with AGR under Γ. Then $q \otimes K$ has AGR under Λ and $\Lambda_*(q \otimes K) \cong \Gamma_*(q)$.*

The proof has been given in [K_4, §3] for char $L \neq 2$. The arguments are merely book keeping. They remain true if char $L = 2$. □
Now we can repeat the arguments in the proof of Proposition 1.16 for quadratic places and AGR instead of usual places and GR, provided char $L \neq 2$. We obtain the following.

5.13. PROPOSITION. *Let $\Lambda: K \to L \cup \infty$ be a Q-place and φ a form over K with AGR under Λ. Assume that char $L \neq 2$. Then $\varphi_0 := \ker(\varphi)$ has again AGR under Λ and $\Lambda_*(\varphi) \sim \Lambda_*(\varphi_0)$,* ind $\Lambda_*(\varphi) \geq$ ind (φ). *If* ind $\Lambda_*(\varphi) =$ ind (φ), *then* ker $\Lambda_*(\varphi) = \Lambda_*(\varphi_0)$. □

We finally state an important fact, proved in [K_4, §3], which has no counterpart on the level of ordinary places.

5.14. PROPOSITION. *Let again $\Lambda: K \to L \cup \infty$ be a Q-place with char $L \neq 2$. Let k be a subfield of K and $\Gamma := \Lambda|k$. Let q be a regular form over k. Then $q \otimes L$ has AGR under Λ if* and only if *q has AGR under Γ, and in this case $\Lambda_*(q \otimes L) \cong \Gamma_*(q)$.* □

§6. Control of the splitting behavior by use of quadratic places

If we stay with fields of characteristic $\neq 2$ then the propositions 5.11 – 5.13 indicate that it should be possible to obtain a complete analogue of the generic splitting theory displayed in §1 using quadratic places instead of ordinary places. Indeed such a theory has been developed in [K_4, §3]. We quote here the main result obtained there.
Let q be a form over a field k. We return to some notations from §1: $(K_r \mid 0 \leq r \leq h)$ is a generic splitting tower of q with higher indices $(i_r \mid 0 \leq r \leq h)$ and higher kernel forms $(\varphi_r \mid 0 \leq r \leq h)$. Further $(j_r \mid 0 \leq r \leq h)$, with $j_r = i_0 + \cdots + i_r$, is the splitting pattern SP(q) of q.

6.1. THEOREM. *[K_4, Th. 3.7]. Let $\Gamma: k \to L \cup \infty$ be a Q-place into a field L of characteristic $\neq 2$. Assume that q has AGR under Γ. We choose a Q-place $\Lambda: K_m \to L \cup \infty$ extending Γ such that either $m = h$ or $m < h$ and Λ does not extend to a Q-place from K_{m+1} to L. Then* ind $(\Gamma_*(q)) = j_m$, *the form φ_m has GR under Λ and* $\ker(\Gamma_*(q)) \cong \Lambda_*(\varphi_m)$. □

A small point here – which we will not really need below – is that φ_m has GR under Λ, not just AGR.

6.2. REMARK.
The theorem shows that the generic splitting tower $(K_r \mid 0 \le r \le h)$ "controls" the splitting behavior of $\Gamma_*(q)$. Indeed, if L'/L is any field extension then we obtain from Γ a Q-place $j \circ \Gamma: k \to L' \cup \infty$ in a rather obvious way (cf. 6.4.iii below). The form q has also AGR under $j \circ \Gamma$, and $(j \circ \Gamma)_*(q) = \Gamma_*(q) \otimes L'$. We can apply the theorem to $j \circ \Gamma$ and q instead of Γ and q. In particular we see that ind $(\Gamma_*(q) \otimes L')$ is one of the numbers j_r. Thus the splitting pattern $\mathrm{SP}(\Gamma_*(q))$ is a subset of $\mathrm{SP}(q)$.□

We now aim at a result similar to Theorem 6.1, where the field K_m is replaced by an arbitrary partial generic splitting field for q. (This is not covered by [K4].) For that reason we briefly discuss the "composition" of Q-places.

6.3. DEFINITION. Let $\Lambda = (\lambda, H, \chi): K \to L \cup \infty$ and $M = (\mu, D, \psi): L \to F \cup \infty$ be Q-places. The *composition* $M \circ \Lambda$ of M and Λ is the Q-place

$$(M \circ \Lambda) = (\mu \circ \lambda, H_0, \psi \circ (\chi | H_0))$$

with $H_0 := \{\alpha \in H \mid \chi(\alpha) \in D\}$. □

6.4. REMARKS.
- i) If $N: F \to E \cup \infty$ is a third Q-place then $N \circ (M \circ \Lambda) = (N \circ M) \circ \Lambda$, as is easily checked.
- ii) Let $i: k \hookrightarrow K$ be a field extension, regarded as a trivial place. This gives us a "trivial" Q-place $\hat{i} = (i, Q(k), i_*: Q(k) \to Q(K))$ from k to K. If $\Lambda = (K, H, \chi): K \to L \cup \infty$ is any Q-places starting at K, then $\Lambda \circ \hat{i}$ is the restriction $\Lambda | k: k \to L \cup \infty$.
- iii) The Q-place $j \circ \Gamma$ alluded to in 6.2 is $\hat{j} \circ \Gamma$. □

In all the following $\Gamma: k \to L \cup \infty$ *is a Q-place into a field of characteristic* $\neq 2$ *such that q has AGR under* Γ.

6.5. PROPOSITION. *Let F/k be a generic splitting field of the form q of some level $r \in [0, h]$. Assume that* ind $\Gamma_*(q) \ge j_r$. *Then there exists a Q-place $\Lambda: F \to L \cup \infty$ extending Γ. For any such Q-place Λ the anisotropic part $\varphi = \ker(q \otimes F)$ of $q \otimes F$ has AGR under Λ and $\Lambda_*(\varphi) \sim \ker \Gamma_*(q)$. if* ind $\Gamma_*(q) = j_r$, *then* $\Lambda_*(\varphi) = \ker \Gamma_*(q)$.

PROOF. We only need to prove the existence of a Q-place $\Lambda: F \to L \cup \infty$ extending Γ. The other statements are clear from §5 (cf. 5.12, 5.13). By Theorem 6.1 we have a Q-place $\Lambda': K_r \to L \cup \infty$ extending Γ. We further have a place $\rho: F \to K_r$ over k. It now can be checked in a straightforward way that the Q-place $\Lambda = \Lambda' \circ \hat{\rho}$ from F to L extends Γ. □

6.6. THEOREM. *Let K/k be an inessential field extension and $q \otimes K \cong \eta \perp \varphi$ with η a generic splitting form of q of some level $r \in [0, h]$. Assume that* ind $\Gamma_*(q) \ge j_r$. *Then there exists a Q-place $\Lambda: K \to L \cup \infty$ extending Γ such that η has AGR under Λ and $\Lambda_*(\eta) \sim 0$. For every such Q-place Λ the form φ has AGR under Λ and $\Lambda_*(\varphi) \sim \Gamma_*(q)$. If* ind $\Gamma_*(q) = j_r$, *then* $\Lambda_*(\varphi) = \ker \Gamma_*(q)$.

PROOF. Again it suffices to prove the existence of a Q-place Λ extending Γ such that η has AGR under Λ and $\Lambda_*(\eta) \sim 0$, the other statements being covered by §5 (cf. 5.11, 5.12).
Let E be a total generic splitting field of η. Then E is a partial generic splitting field of q (cf. 4.1). By the preceding proposition 6.5 there exists a Q-place $M: E \to L \cup \infty$

extending Γ. Let $\Lambda: K \to L \cup \infty$ denote the restriction of M to K (cf. 5.3). Of course, also Λ extends Γ. The form $\eta \otimes E$ is hyperbolic, hence certainly has AGR under M and $M_*(\eta \otimes E) \sim 0$. Now Proposition 5.14 tells us that η has AGR under Λ and $\Lambda_*(\eta) \sim 0$. $\square$

6.7. REMARK. Suppose that $\Gamma = \hat{\gamma}$ with $\gamma: k \to L \cup \infty$ a place and that q has GR under γ. Then Theorem 6.1 and Proposition 6.5 give us nothing more than we know from the generic splitting theory in §1. Indeed, if $\Lambda = (\lambda, H, \chi)$ is a Q-place as stated there, then the form φ_m in 6.1, resp. φ in 6.5, automatically has GR under λ.
This is different with Theorem 6.6. Even in the case $\Gamma = \hat{j}$ for $j: k \hookrightarrow L$ the inclusion map into an overfield L of k (actually the case which is perhaps the most urgent at present), the Q-place Λ will be different from $\hat{\lambda}$. Thus, in 6.6, Q-places instead of usual places are needed even in the case $\Gamma = \hat{j}$. $\square$

We now choose again a generic splitting preparation $(K'_r \mid 0 \le r \le h)$, $(\eta_r \mid 0 \le r \le h)$ of q with residual forms φ_r and generic splitting forms $\varepsilon_r = \perp_{j=0}^{r} \eta_j \otimes K'_j \quad (0 \le r \le h)$, cf. 4.3.

6.8. DISCUSSION. Assume that ind $\Gamma_*(q) \ge j_m$ for some $m \in [0, h]$. Theorem 6.6 tells us that there exists a Q-place $\Lambda: K'_m \to L \cup \infty$ extending Γ such that ε_m has AGR under Λ and $\Lambda_*(\varepsilon_m) \sim 0$. Moreover, for every such Q-place λ the form φ_m has AGR under Λ and $\Lambda_*(\varphi_m) \sim \Gamma_*(q)$. If ind $\Gamma_*(q) = j_m$, it follows that $\Lambda_*(\varphi_m)$ is the anisotropic part of $\Gamma_*(q)$.
Assume now that ind $\Gamma_*(q) > j_m$ and $\Lambda: K'_m \to L \cup \infty$ is a Q-place as just described. Then by analogy with 1.18 one might suspect at first glance that Λ extends to a place $M: K'_{m+1} \to L \cup \infty$ such that ε_{m+1} has AGR under M and $M_*(\varepsilon_{m+1}) \sim 0$. {N.B. Since $\Lambda_*(\varepsilon_m) \sim 0$, this is equivalent to the property that η_{m+1} has AGR under M and $M_*(\eta_{m+1}) \sim 0$.} But this is too much to be hoped for. Indeed, let us consider the special case that $K'_1 = \cdots = K'_h =: K'$, i.e., we are given a generic splitting preparation $(\eta_0, \eta_1, \dots, \eta_h)$ over K'. If Λ: $K' = K'_m \to L \cup \infty$ is as above, then we can expand $\Lambda = (\lambda, H, \chi)$ to a Q-place $\Lambda' = (\lambda, Q(K'), \psi'): K' \to L \cup \infty$. Also Λ' extends Γ, further ε_m has AGR under Λ' and $\Lambda'_*(\varepsilon_m) = \Lambda_*(\varepsilon_m) \sim 0$ (cf. 5.12). But since $K'_{m+1} = K'$, the only extension of Λ' to K'_{m+1} is Λ' itself, and there is no reason why η_{m+1} should have AGR under Λ' and $\Lambda'_*(\eta_{m+1}) \sim 0$. $\square$

Nevertheless, if ind $\Gamma_*(q) > j_{m+1}$, there exists a somewhat natural procedure to obtain from a Q-place $\Lambda: K'_m \to L \cup \infty$ as above a Q-place $M: K'_{m+1} \to L \cup \infty$ extending Γ such that both $\varepsilon_m \otimes K'_{m+1}$ and ε_{m+1} have AGR under M and we have $M_*(\varepsilon_m \otimes K'_{m+1}) \sim 0$ as well as $M_*(\varepsilon_{m+1}) \sim 0$. This runs as follows.

6.9. PROCEDURE. Assume that ind $\Gamma_*(q) > j_m$. Let $\Lambda: K'_m \to L \cup \infty$ be a Q-place extending Γ such that ε_m has AGR under Λ and $\Lambda_*(\varepsilon_m) \sim 0$. We choose a regular total generic splitting field F of ε_m. By Proposition 6.5 the Q-place Λ extends to a Q-place $\tilde{\Lambda}: F \to L \cup \infty$. The form $\varphi_m \otimes F$ is anisotropic. We now invoke Corollary 4.7, which tells us that the tower $F \subset F \cdot K'_{m+1} \subset \cdots \subset F \cdot K'_h$ together with the sequence of forms $(\eta_r \otimes F \cdot K'_r \mid m < r \le h)$ is a generic splitting preparation of $\varphi_m \otimes F$. Here $F \cdot K'_r$ denotes the free composite of the fields F and K'_r over k. In particular $\eta_{m+1} \otimes F \cdot K'_r$ is a generic splitting form of $\varphi_m \otimes F$ over $F \cdot K'_r$ of level 1. We have $\tilde{\Lambda}_*(\varphi_m \otimes F) \sim \tilde{\Lambda}_*(q \otimes F) = \Gamma_*(q)$. Since ind $\Gamma_*(q) > j_m$, we

conclude that $\tilde{\Lambda}_*(\varphi_m \otimes F)$ is isotropic. Thus, again by Proposition 6.5, $\tilde{\Lambda}$ extends to a Q-place $\tilde{M}: F \cdot K'_{m+1} \to L \cup \infty$ such that $\eta_{m+1} \otimes F \cdot K'_{m+1}$ has AGR under $\tilde{M}$ and $\tilde{M}_*(\eta_{m+1} \otimes F \cdot K'_{m+1}) \sim 0$. Let $M: K'_{m+1} \to L \cup \infty$ denote the restriction of $\tilde{M}$ to K'_{m+1}. By 5.14, the form η_{m+1} has AGR under M and $M_*(\eta_{m+1}) \sim 0$.
Now we need a delicate argument to prove that also $\varepsilon_m \otimes K'_{m+1}$ has AGR under M and $M_*(\varepsilon_m \otimes K'_{m+1}) \sim 0$. The problem is that the diagram of field embeddings

$$\begin{tikzcd} F \arrow[r] & F \cdot K'_{m+1} \\ K'_m \arrow[u] \arrow[r] & K'_{m+1} \arrow[u] \end{tikzcd}$$

does not commute, since the field composite $F \cdot K'_{m+1}$ is built over k instead of K'_m. Thus M probably does not extend the Q-place Λ.
The argument runs as follows. Also $\varphi_{m+1} \otimes F \cdot K'_{m+1}$ has AGR under $\tilde{M}$, hence φ_{m+1} has AGR under M, and

$$M_*(\varphi_{m+1}) \cong \tilde{M}_*(\varphi_{m+1} \otimes F \cdot K'_{m+1}) \sim \tilde{\Lambda}_*(\varphi_m \otimes F) \sim \tilde{\Lambda}_*(q \otimes F) \cong \Gamma_*(q).$$

Since $q \otimes K'_{m+1} \cong \varepsilon_m \otimes K'_{m+1} \perp \eta_{m+1} \perp \varphi_{m+1}$ and both, η_{m+1} and φ_{m+1}, have AGR under M, also $\varepsilon_m \otimes K'_{m+1}$ has AGR under M, cf. 5.11, and

$$\Gamma_*(q) \cong M_*(q \otimes K'_{m+1}) \cong M_*(\varepsilon_m \otimes K'_{m+1}) \perp M_*(\eta_{m+1}) \perp M_*(\varphi_{m+1}).$$

Since $M_*(\eta_{m+1}) \sim 0$ and $M_*(\varphi_{m+1}) \sim \Gamma_*(q)$, we conclude that $M_*(\varepsilon_m \otimes K'_{m+1}) \sim 0$. □

We have ind $\Gamma_*(q) = j_r$ for some $r \in [0, h]$ (cf. 6.2). Iterating the procedure with $m = 0, 1, \ldots, r-1$, we obtain the following theorem.

6.10. THEOREM. *Let* ind $\Gamma_*(q) = j_r$. *Then there exists a Q-place $\Lambda: K'_r \to L \cup \infty$ extending Γ such that $\eta_m \otimes K'_r$ has AGR under Λ and $\Lambda_*(\eta_m \otimes K'_r) \sim 0$ for every $m \in [0, r]$. If Λ is any such Q-place then φ_r has AGR under Λ and $\Lambda_*(\varphi_r) =$* ker $\Gamma_*(q)$.

We briefly discuss the case that L is a field extension of k and $\Gamma = \hat{j}$ with $j: k \hookrightarrow L$ the inclusion map.

6.11. DEFINITION/REMARK. Let K and L be field extensions of k. A Q-place from K to L *over* k is a Q-place $\Lambda = (\lambda, H, \chi): K \to L \cup \infty$ such that the first component λ is a place over k. It is evident from Definitions 5.3 that this condition just means that Λ extends the quadratic place $\hat{j}: k \to L \cup \infty$, and also that $\Lambda | k = \hat{j}$. □

6.12. SCHOLIUM. *Let L/k be a field extension,* ind $(q \otimes L) = j_r$. *Then there exists a Q-place $\Lambda: K'_r \to L \cup \infty$ over k such that $\eta_m \otimes K'_r$ has AGR under Λ and $\Lambda_*(\eta_m \otimes K'_r) \sim 0$ for every $m \in [0, r]$. For any such Q-place Λ the form φ_r has AGR under Λ and $\Lambda_*(\varphi_r) =$* $\ker(q \otimes L)$. □

We return to an arbitrary Q-place $\Gamma: k \to L \cup \infty$ such that q has AGR under Γ.

6.13. DEFINITION. We call the generic splitting preparation $(K'_r \mid 0 \le r \le h)$, $(\eta_r \mid 0 \le r \le h)$ of q *tame*, if there exists a generic splitting tower $(K_r \mid 0 \le r \le h)$ of q such that K'_r is a subfield of K_r and $\eta_r \otimes K_r \sim 0$ for every $r \in [0, h]$. □

Notice that a generic splitting preparation $(K'_r \mid 0 \le r \le h)$, $(\eta_r \mid 0 \le r \le h)$ as described by 3.4 and 3.5 is tame. Thus every form admits tame generic splitting preparations. On the other hand we suspect that there exist generic splitting preparations which are wild (= not tame), although we did not look for examples. If our given generic splitting preparation of q is tame then there exists a much simpler procedure than the one described above, to obtain a Q-place $\Lambda: K'_r \to L \cup \infty$ with the properties stated in Theorem 6.10.

6.14. Procedure. Assume that ind $\Gamma_*(q) = j_r$, and that $(K_i \mid 0 \le i \le h)$ is a generic splitting tower of q such that K'_i is a subfield of K_i and $\varepsilon_i \otimes K_i \sim 0$ for every $i \in [0, h]$. By 6.1 there exists a Q-place $\tilde{\Lambda}: K_r \to L \cup \infty$ extending Γ. Let Λ denote the restriction of $\tilde{\Lambda}$ to K'_r. Then Λ again extends Γ. For every $i \in [0, r]$ we have $(\varepsilon_i \otimes K'_r) \otimes K_r = \varepsilon_i \otimes K_r \sim 0$. Certainly $\varepsilon_i \otimes K_r$ has AGR under $\tilde{\Lambda}$ and $\tilde{\Lambda}_*(\varepsilon_i \otimes K_r) \sim 0$. By 5.14 the form $\varepsilon_i \otimes K'_r$ has AGR under Λ and $\Lambda_*(\varepsilon_i \otimes K'_r) \sim 0$. Since this holds for every $i \in [0, r]$ we conclude (using 5.11) that $\eta_i \otimes K'_r$ has AGR under Λ and $\Lambda_*(\eta_i \otimes K'_r) \sim 0$ for every $i \in [0, r]$.

Thus it seems that life is easier if we have a tame generic splitting preparation at our disposal than an arbitrary one. Up to now this is an argument in favor of working with generic splitting preparations instead of the more special generic splitting decompositions (cf. 4.8), in spite of Proposition 4.9, for we do not know whether every form admits a *tame* generic splitting decomposition.

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