# FILTERED TOPOLOGICAL HOCHSCHILD HOMOLOGY

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Abstract. In this paper we examine a certain filtration on topological Hochschild homology. This filtration has the virtue that it respects the cyclic structure of topological Hochschild homology, and therefore it is compatible with the cyclotomic structure used to define topological cyclic homology. As an example we show how the skeleton filtration of a simplicial ring gives rise to spectral sequences similar to the change of ring spectral sequences considered by Pirashvili and Waldhausen in [10].

## 1. Introduction

The aim of this paper is to examine a certain filtration of topological Hochschild homology of a functor with smash product equipped with a filtration. This filtration preserves the cyclic structure of topological Hochschild homology, and therefore it gives information about the fixed points of the topological Hochschild homology spectrum with respect to finite cyclic groups. Topological cyclic homology is constructed out of these fixed point spectra, and the motivation for the present work is to get information about topological cyclic homology. By a theorem of McCarthy topological cyclic homology is closely related to algebraic $K$-theory, and in some inter resting cases topological cyclic homology determines the $K$-groups. The present paper is closely related to a paper of Hesselholt and Madsen, where the $K$-groups for finite algebras over Witt vectors of perfect fields are computed [6]. One difference is that here general filtrations are considered, while the filtrations considered in [6] are split. In the paper [4] the filtration $p\mathbb{Z}/p^2 \subseteq \mathbb{Z}/p^2$ was used to compute topological Hochschild homology of the ring $\mathbb{Z}/p^2$, and this is the example that motivated writing the generality of present paper. In a continuation, which is joint work with M. Bökstedt, we will use this filtration of $\mathbb{Z}/p^2$ to compute $K_i(\mathbb{Z}/p^2)$ for $i < 2p$. It is the hope that we will become able to compute all the $K$-groups for $\mathbb{Z}/p^2$. It is worth mentioning that the author knows of no way to use a filtration of a ring to give information about algebraic $K$-theory directly without using the comparison to topological cyclic homology.

Only elementary properties of the filtrations of topological Hochschild homology are studied in this note, and no explicit computations are given. An example on how the filtrations can be used in computations is given in [4]. The focus in this paper is on a filtered version of the norm cofibration sequence for a cyclotomic spectrum in the sense of Madsen [3] and on topological Hochschild homology of simplicial rings. The role of the norm cofibration sequence is that it allows us to determine the fixed point spectra inductively. A simplicial ring together with its skeleton filtration is an example of a filtered ring (see 2.2 for the definition of a filtered ring). To such a filtered ring, there is an associated filtered functor with smash product (see 3.2 for the definition of a filtered functor with smash product), giving rise to a filtration of topological Hochschild homology. The associated spectral sequence has $E^2$-term completely determined by the Hochschild complex for the simplicial ring, and in this respect it is similar to a spectral sequence obtained by Pirashvili and Waldhausen in [10].

The paper is organized as follows: In section 2, generalities on filtrations of monoids in a symmetric monoidal category are given. It is noted that a filtered monoid is a monoid in the symmetric monoidal category of filtered objects, and therefore it fits into the Hochschild construction. In section 3.2 a filtered functor with smash product is defined to be a filtered monoid in the category of Gamma spaces, and fundamental properties of the topological Hochschild homology of a filtered functor with smash product are established. In section 4.1 it is shown that topological Hochschild homology of a filtered functor with smash product is a filtered version of a cyclotomic spectrum, and it is noted that this implies that there is a filtered version of the norm cofibration sequence. In section 5 the filtration of topological Hochschild associated to the skeleton filtration of a simplicial ring is examined. This filtration gives rise to a spectral sequence which is similar to the change of ring spectral sequences given by Pirashvili and Waldhausen in [10]. In appendix A some comments on Day's construction of a symmetric monoidal structure on functor categories are given.

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## 2. Filtered Hochschild Construction

In this section we shall study filtered objects in a category $\mathcal{C}$. Apart from in section 2.1 $\mathcal{C}$ is assumed to be equipped with a symmetric monoidal product $\otimes$.

### 2.1. Filtered Objects.

A *filtered object* in $\mathcal{C}$ is a functor from the category $\mathbb{Z}$, with exactly one morphism $n \to m$ if $m \geq n$, to $\mathcal{C}$. That is, a filtered object is a sequence

$$\cdots \to X(-1) \to X(0) \to X(1) \to \cdots \to X(n) \to X(n+1) \to \dots$$

of morphisms in $\mathcal{C}$. A morphism of filtered objects is simply a natural transformation. For some choices of $\mathcal{C}$, there is a functor $H$ from the category $\mathcal{C}^{\mathbb{Z}}$ of filtered objects in $\mathcal{C}$ to the category of exact couples of (graded) abelian groups.

**Example 2.1.1.** Functors from filtered objects to exact couples:

1. The category of chain complexes and injective chain homomorphisms together with the functor $H$ given by homology.
2. We can take $\mathcal{C}$ to be the category of topological spaces and cofibrations and let $H$ be given by (generalized) homology.
3. We can take $\mathcal{C}$ to be a triangulated category and let $H$ be a (co)homology functor.

### 2.2. Filtered objects in monoidal categories.

If $\mathcal{C}$ has all colimits, then there is a symmetric monoidal structure on the category $\mathcal{C}^{\mathbb{Z}}$ of filtered objects in $\mathcal{C}$. Given filtered objects $X$ and $Y$ in $\mathcal{C}$, their (monoidal) product is the filtered object $X \otimes Y$ in $\mathcal{C}$ with $(X \otimes Y)(k) = \operatorname*{colim}_{i+j \leq k} X(i) \otimes Y(j)$. The unit for this operation is the filtered object $I$ with $I(k)$ equal to the initial object in $\mathcal{C}$ for $k < 0$ and with $I(k)$ equal to the unit for the monoidal structure of $\mathcal{C}$ when $k \geq 0$. Day has proven that the operation $\otimes$ on $\mathcal{C}^{\mathbb{Z}}$ equippes $\mathcal{C}^{\mathbb{Z}}$ with a symmetric monoidal structure [5]. Some more comments on the work of Day are given in appendix A.

A *filtered monoid* in $\mathcal{C}$ is a monoid in the category $\mathcal{C}^{\mathbb{Z}}$. Explicitly, a filtered monoid in $\mathcal{C}$ is a sequence:

$$\cdots \to M(-1) \to M(0) \to M(1) \to \cdots \to M(n) \to M(n+1) \to \dots$$

of morphisms in $\mathcal{C}$ together with morphisms

$$\mu_{i,j} : M(i) \otimes M(j) \to M(i+j)$$
$$\eta : I \to M(0),$$

satisfying the following relations for associativity and unit:

$$\mu_{i+j,k} \circ (\mu_{i,j}, \mathrm{id}_{M(k)}) = \mu_{i,j+k} \circ (\mathrm{id}_{M(i)}, \mu_{j,k}),$$
$$\mu_{0,i} \circ (\eta, \mathrm{id}_{M(i)}) = \mathrm{id}_{M(i)},$$
$$\mu_{i,0} \circ (\mathrm{id}_{M(i)}, \eta) = \mathrm{id}_{M(i)} \,.$$

We shall call a filtered monoid in the category of abelian groups a *filtered ring*.

If $*$ is a terminal object of $\mathcal{C}$ and $X \to Y$ is a map in $\mathcal{C}$, we shall denote any choice of pushout of the diagram $* \leftarrow X \to Y$ by $X/Y$. We shall say that the product $\otimes$ of $\mathcal{C}$ commutes with quotients, if there is a natural isomorphism $(X_1/X_2) \otimes Y \cong (X_1 \otimes Y)/(X_2 \otimes Y)$.

**Lemma 2.2.1.** *If $\mathcal{C}$ is a cocomplete symmetric monoidal category with a terminal object, then given filtered objects $X$ and $Y$ of $\mathcal{C}$, there is an isomorphism:*

$$\frac{(X \otimes Y)(k)}{(X \otimes Y)(k-1)} \cong \bigoplus_{i+j=k} \frac{X(i) \otimes Y(j)}{X(i-1) \otimes Y(j) \oplus_{X(i-1)\otimes Y(i-1)} X(i) \otimes Y(j-1)}.$$

*If in addition $\otimes$ commutes with quotients, then there is an isomorphism:*

$$\frac{(X \otimes Y)(k)}{(X \otimes Y)(k-1)} \cong \bigoplus_{i+j=k} \frac{X(i)}{X(i-1)} \otimes \frac{Y(j)}{Y(j-1)}.$$

*Proof.* For the first part, it suffices to note that the following diagram in $\mathcal{C}$ is a pushout:

$$\begin{tikzcd}
\bigoplus_{i+j=k} X(i-1) \otimes Y(j) \oplus_{X(i-1)\otimes Y(i-1)} X(i) \otimes Y(j-1) \arrow[r] \arrow[d] & (X \otimes Y)(k-1) \arrow[d] \\
\bigoplus_{i+j=k} X(i) \otimes Y(j) \arrow[r] & (X \otimes Y)(k).
\end{tikzcd}$$

For the second part, we note that:

$$(X_1/X_0) \otimes (Y_1/Y_0) \cong \frac{(X_1/X_0) \otimes Y_1}{(X_1/X_0) \otimes Y_0} \cong \frac{(X_1 \otimes Y_1)/(X_0 \otimes Y_1)}{(X_1 \otimes Y_0)/(X_0 \otimes Y_0)}$$

for $X_0 \to X_1$ and $Y_0 \to Y_1$ maps in $\mathcal{C}$, and that given a map $B \oplus_D C \to A$ in $\mathcal{C}$ we have:

$$\frac{A/B}{C/D} \cong \frac{A}{B \oplus_D C}.$$

∎

2.3. **The Hochschild construction.** Let $\mathcal{C}$ denote a symmetric monoidal category. The Hochschild construction is a functor $Z$ from the category of monoids in $\mathcal{C}$ to the category of cyclic objects in $\mathcal{C}$. Given a monoid $M$ in $\mathcal{C}$, $Z(M)$ is defined as follows: It has $n$-simplices

$$Z_n(M) = M \otimes \cdots \otimes M \qquad (n+1) \text{ factors.}$$

The cyclic operator is given by the automorphism $t_n$ of $M \otimes \cdots \otimes M$ cyclically shifting the $(n+1)$ factors to the right. The face maps are given by the formula:

$$d_i = t_{n-1}^i \circ (\mu \otimes \mathrm{id}) \circ t_n^{-i}, \qquad 0 \leq i \leq n,$$

where $\mu : M \otimes M \to M$ is the multiplication in $M$. The degeneracies are given by the formula:

$$s_i = t_{n+1}^{(i+1)} \circ (\eta \otimes \mathrm{id}) \circ t_n^{-(i+1)}, \qquad 0 \leq i \leq n,$$

where $\eta : I \to M$ is the unit in $M$.

Since $\mathcal{C}^{\mathbb{Z}}$ has a symmetric monoidal structure, we can also consider the Hochschild construction on monoids in $\mathcal{C}^{\mathbb{Z}}$, that is on filtered monoids in $\mathcal{C}$.

**Proposition 2.3.1.** *Let $M$ be a filtered monoid in a cocomplete symmetric monoidal category $\mathcal{C}$, where $\otimes$ commutes with quotients. Then for each $n \geq 0$ there is an isomorphism*

$$\frac{Z_n(M)(k)}{Z_n(M)(k-1)} \cong \bigoplus_{i_0+\cdots+i_n=k} \frac{M(i_0)}{M(i_0-1)} \otimes \cdots \otimes \frac{M(i_n)}{M(i_n-1)}.$$

*Proof.* This is a direct consequence of lemma 2.2.1 □

The above proposition can be reformulated in terms of the associated graded monoid $\mathcal{G}M$ for $M$. $\mathcal{G}M$ is the filtered object in $\mathcal{C}$ with

$$\mathcal{G}(M)(k) = \bigoplus_{i \leq k} M(i)/M(i-1),$$

and with multiplication induced by the maps

$$\frac{M(i)}{M(i-1)} \otimes \frac{M(j)}{M(j-1)} \cong \frac{M(i) \otimes M(j)}{M(i) \otimes M(j-1) \oplus_{M(i-1)\otimes M(j-1)} M(i-1) \otimes M(j)}$$
$$\to \frac{M(i+j)}{M(i+j-1)}.$$

The proposition says that the filtration quotients for $Z(M)$ and $Z(\mathcal{G}M)$ are isomorphic.

2.4. **Filtrations by cofibrations.** In this section we shall consider filtrations in monoidal model categories in the sense of Schwede and Shipley [12].

**Definition 2.4.1.** Suppose $\mathcal{C}$ is a closed monoidal category which is also a model category. Then $\mathcal{C}$ is a *monoidal model category* if the following pushout product axiom holds. If $f : A \to B$ and $g : K \to L$ are cofibrations, so is $f \otimes g : A \otimes L \oplus_{A \otimes K} B \otimes K \to B \otimes L$. If one of $f$ or $g$ in addition is a weak equivalence, so is $f \otimes g$.

Let $\mathcal{C}$ be a monoidal model category. We shall say that a a filtered object in $\mathcal{C}$ is *filtered by cofibrations* if it is a filtered object in the category of cofibrations in $\mathcal{C}$.

**Lemma 2.4.2.** *Let $\mathcal{C}$ be a monoidal model category. If $X$ and $Y$ are filtered objects in $\mathcal{C}$ filtered by cofibrations, then $X \otimes Y$ is filtered by cofibrations.*

*Proof.* This follows from the pushout diagram given in the proof of lemma 2.2.1. □

Quillen has shown that the category of (pointed) simplicial sets with monoidal structure given by (smash) products is a monoidal model category [11, Theorem 3, p. II.3.14]. He also showed that there is a model structure on the category of simplicial abelian groups with weak equivalences, fibrations and cofibrations defined as follows. A map is a weak equivalence if it induces a quasi isomorphism of associated normalized chain complexes. It is a fibration if it is degreewise surjective, and it is a cofibration if it degreewise is an injection having a free abelian groups as cokernel [11, p. II.4.11]. (This is a slight reformulated of what Quillen writes using that given a simplicial abelian groups $A$, there is an isomorphism $A_n \cong \bigoplus_{\beta:[n]\twoheadrightarrow[m]} \beta^* M_m$, where the sum is taken over all oder preserving epimorphisms and where $M_m$ consists of the non-degenerate $m$-simplices in $A$.)

**Lemma 2.4.3.** *The category of simplicial abelian groups with monoidal structure given by tensor product is a monoidal model category.*

*Proof.* Let us start by considering the category $\mathcal{D}$ of monomorphisms of abelian groups having a free cokernel. Given two morphisms $f : A \to B$ and $g : K \to L$ in $\mathcal{D}$, we can choose retractions of them because the cokernels are free abelian groups, and these retractions give us a retraction of the map

$$f \otimes g : A \otimes L \oplus_{A\otimes K} B \otimes K \to B \otimes L \oplus_{B\otimes L} B \otimes L \cong B \otimes L.$$

In particular $f \otimes g$ is a monomorphism. Its cokernel can be identified with $B/A \otimes L/K$. This is a free abelian group, and therefore $f \otimes g$ is in $\mathcal{D}$.

A map $f : A \to B$ in the category of simplicial abelian groups is a cofibration iff it degreewise is a morphism in the category $\mathcal{D}$. Given another cofibration $g : K \to L$ of simplicial abelian groups, it follows from the above discussion that $f \otimes g$ degreewise is a morphism in $\mathcal{D}$, and therefore it is a cofibration. Suppose now that either $f$ or $g$ is a weak equivalence. Let us consider the short exact sequence

$$A \otimes L \oplus_{A\otimes K} B \otimes K \to B \otimes L \to B/A \otimes L/K$$

of simplicial abelian groups. To see that $f \otimes g$ is a weak equivalence it suffices to show that the chain complex associated to $B/A \otimes L/K$ is acyclic. This is so by the Eilenberg Zilber theorem and the fact that the tensor product of two degreewise free acyclic chain complexes is acyclic. □

Note that if $\mathcal{C}$ is a cofibrantly generated model category, then there is a cofibrantly generated model category structure on the category $\mathcal{C}^{\mathbb{Z}}$ of filtered objects in $\mathcal{C}$ [7, chapter 13]. Fibrations and weak equivalences are levelwise fibrations and weak equivalences respectively. As a set of generating cofibrations for $\mathcal{C}^{\mathbb{Z}}$ we can take the morphisms of the form $F^m_X \to F^m_Y$, where $X \to Y$ is a generating cofibration in $\mathcal{C}$. Here $F^m_X(n)$ is the initial object of $\mathcal{C}$ when $n < m$, and $F^m_X(n) = X$ when $n \geq m$. Let us assume that $\otimes$ commutes with quotients. Then $F^m_X \otimes F^{m'}_{X'} \cong F^{m+m'}_{X\otimes X'}$. Let us further assume that the cofibrations in $\mathcal{C}$ satisfy the pushout product axiom. Then given generating cofibrations $A \to B$, $K \to L$ in $\mathcal{C}^{\mathbb{Z}}$, the map:

$$A \otimes L \oplus_{A\otimes K} B \otimes K \to B \otimes L$$

is a generating cofibration in $\mathcal{C}^{\mathbb{Z}}$. If we in addition assume that the product $\otimes$ in $\mathcal{C}$ preserves colimits, then $\mathcal{C}^{\mathbb{Z}}$ satisfies the pushout-product axiom.

2.5. **A filtration of a simplicial ring.** Given a simplicial ring $R$, the skeleton filtration can be used to construct a filtered ring:

$$0 \to \mathbb{Z} \to \operatorname{sk}^0 R \to \operatorname{sk}^1 R \to \dots$$

where $\operatorname{sk}^i R$ is in filtration degree $i+1$. Remember that a filtered ring is a filtered monoid in the category of simplicial abelian groups. Let us denote this filtered ring by $R$ too, and let us consider the filtered object $Z(R)$ in the category of cyclic simplicial abelian groups. For the use in section 5 we shall show that the homotopy of the filtration quotient

$$X = Z(R)(n)/Z(R)(n-1)$$

is concentrated in degrees $n$ and $n-1$. To show this we consider the push out diagram:

$$\begin{array}{ccc} \operatorname{nd} X_k \otimes \mathbb{Z}(\partial\Delta^k) & \longrightarrow & \operatorname{sk}^{k-1} X \\ \downarrow & & \downarrow \\ \operatorname{nd} X_k \otimes \mathbb{Z}(\Delta^k) & \longrightarrow & \operatorname{sk}^k X \end{array}$$

of simplicial abelian groups associated to the simplicial simplicial abelian group $[k] \mapsto X_k = Z_k(R)(n)/Z_k(R)(n-1)$, where $\operatorname{nd} X_k$ denotes the non-degenerate $k$-simplices. That is, by proposition 2.3.1, we have

$$\operatorname{nd} X_k \cong \bigoplus_{\substack{i_0+\dots+i_k=n \\ i_1,\dots,i_k>0}} \frac{R(i_0)}{R(i_0-1)} \otimes \dots \otimes \frac{R(i_k)}{R(i_k-1)}.$$

Since the homotopy of $R(i)/R(i-1)$ is concentrated in degree $i-1$ for $i>0$, it follows from the Künneth theorem, that the only possibly nonzero homotopy groups of $\operatorname{nd} X_k$ are $\pi_{n-k}$ and $\pi_{n-k-1}$. From the fact that the cofibres of the above push out diagram are isomorphic, it follows that the only possibly non-zero homotopy groups of the quotient $\operatorname{sk}^k X/\operatorname{sk}^{k-1} X$ are $\pi_n$ and $\pi_{n-1}$. Now assume by induction on $k$ that the homotopy of $\operatorname{sk}^{k-1} X$ is concentrated in degrees $n$ and $n-1$. From the long exact sequence of homotopy groups associated to the short exact sequence

$$\operatorname{sk}^{k-1} X \to \operatorname{sk}^k X \to \operatorname{sk}^k X/\operatorname{sk}^{k-1} X,$$

it follows that the homotopy of $\operatorname{sk}^k X$ is concentrated in degrees $n$ and $n-1$ too. Since homotopy (homology) commutes with filtered colimits, it follows that the homotopy of $X = \bigcup \operatorname{sk}^k X$ is concentrated in degrees $n$ and $n-1$.

## 3. Filtered Topological Hochschild Homology

3.1. **Topological Hochschild homology.** We briefly define topological Hochschild homology: Let $J(k)$ be the category whose objects are $k$-tuples of finite disjoint subsets of $\mathbb{N}$ and whose morphisms are $k$-tuples of injective maps. There is a cyclic object $[k] \mapsto J(k+1)$ in the category of (small) categories, where the simplicial structure maps are of the same type as for the Hochschild construction, with union as substitute for the product in a monoid and with the empty set as a substitute for the unit. In order to define the cyclic structure, we need to choose a free action of the cyclic group $C_{k+1}$ with $k+1$ elements on $\mathbb{N}$. Let $\alpha_{k+1}$ denote a generator for $C_{k+1}$, and let for $x$ a subset of $\mathbb{N}$, $\alpha_{k+1}\cdot x$ denote the set of all $\alpha_{k+1}\cdot i$ for $i\in x$. The cyclic structure on $[k]\mapsto J(k+1)$ is given by letting $tx=(\alpha_{k+1}\cdot x_k,\alpha_{k+1}\cdot x_0,\dots,\alpha_{k+1}\cdot x_{k-1})$ for $x=(x_0,\dots,x_k)\in J(k+1)$. The reason for choosing this complicated cyclic structure is that with this structure there is an isomorphism between the fixed points of $J(k+1)$ under the action of $C_{k+1}$ and $J(1)$ giving rise to fixed points in topological Hochschild homology. Let $L$ denote a functor with smash product. For each finite subset $i$ of $\mathbb{N}$, there is a sphere $S^i$ defined as a quotient of the space of maps from $i$ to the circle. $\mathrm{THH}(L)$ is the cyclic pointed simplicial set with $k$-simplices equal to the homotopy colimit

$$\operatorname*{hocolim}_{(i_0,\dots,i_k)\in J(k+1)} F(S^{i_0}\wedge\cdots\wedge S^{i_k}, L(S^{i_0})\wedge\cdots\wedge L(S^{i_k}))$$

and with structure maps of the same type as the structure maps for the Hochschild construction, where the cyclic operator acts on the indexing category as explained above. Details on how to make sense of this homotopy colimit can be found in [4]. The symbol $F$ denotes derived function space, that is, if $X$ and $Y$ are pointed simplicial sets, then $F(X,Y)=\mathcal{S}_*(X,\sin|Y|)$, where $\sin|Y|$ denotes the singular complex on the geometric realization of $Y$, and $\mathcal{S}_*$ denotes the internal function object in the category $\mathcal{S}_*$ of pointed simplicial sets.

For the purpose of this note, a functor with smash product is a monoid in the category $\Gamma\mathcal{S}_*$ of Gamma spaces. Let $\Gamma$ denote the category of pointed finite sets with one object $n^+=\{0,1,\dots,n\}$ for each $n\geq 0$, and with $\Gamma(m^+,n^+)$ the set maps from $m^+$ to $n^+$ fixing 0. A Gamma space is a pointed functor from $\Gamma$ to the category $\mathcal{S}_*$ of pointed simplicial sets. Given two Gamma spaces $X$ and $Y$, their smash product is the Gamma space $X\wedge Y$ with

$$(X\wedge Y)(n^+)=\operatorname*{colim}_{n_1^+\wedge n_2^+\to n^+} X(n_1^+)\wedge Y(n_2^+).$$

The unit for the operation $\wedge$ is the functor $\mathbb{S}$ with $\mathbb{S}(n^+)=n^+$. It follows from work of Day (see theorem A.1) that the category of Gamma spaces is a symmetric monoidal category. A functor with smash product $L$ is a monoid in the category $\Gamma\mathcal{S}_*$. Explicitly this means that $L$ is a pointed functor $L:\Gamma\to\mathcal{S}_*$ together with natural transformations

$$\mu: L(m^+)\wedge L(n^+)\to L(m^+\wedge n^+),$$
$$\eta: n^+\to L(n^+),$$

satisfying the following relations for associativity and unitality:

$$\mu\circ(\mu\wedge\mathrm{id})=\mu\circ(\mathrm{id}\wedge\mu),\quad \mu\circ(\mathrm{id}\wedge\eta)=\lambda,\quad \mu\circ(\eta\wedge\mathrm{id})=\rho,$$

where $\lambda: m^+\wedge L(n^+)\to L(m^+\wedge n^+)$ is adjoint to the map

$$m^+\to\Gamma(n^+,m^+\wedge n^+)\xrightarrow{L}\mathcal{S}_*(L(n^+),L(m^+\wedge n^+))$$

and $\rho: L(m^+)\wedge n^+\to L(m^+\wedge n^+)$ is adjoint to the map

$$n^+\to\Gamma(m^+,m^+\wedge n^+)\xrightarrow{L}\mathcal{S}_*(L(m^+),L(m^+\wedge n^+)).$$

Given a Gamma space $X$, we can extend it to a functor $X_1$ defined on the category of pointed sets by letting

$$X_1(K)=\operatorname*{colim}_{n^+\to K} X(n^+),$$

for $K$ a pointed set, and we can define an endofunctor $X_2$ on $\mathcal{S}_*$ by letting $(X_2(U))_k = (X_1(U_k))_k$ for $U$ a pointed simplicial set. From now on we shall not distinguish notationally between a gamma space and the induced endofunctor on $\mathcal{S}_*$.

Given a Gamma space $X$ and pointed simplicial sets $U$ and $V$, there is a map $X(U) \wedge V \to X(U \wedge V)$ obtained by applying the above map $\rho$ degreewise. The following lemma is given in [8, prop. 5.21]:

**Lemma 3.1.1.** *If $U$ is $m$-connected and $V$ is $n$-connected, then the map $X(U) \wedge V \to X(U \wedge V)$ is $2m + n - 1$-connected.*

The above lemma in particular implies that if $U$ is $n$-connected, then $X(U)$ is $n$-connected (provided that $U$ is at least 2-connected). It follows from the following special case of Bökstedt's approximation lemma that topological Hochschild homology is a presentation of the Hochschild construction in the category of spectra:

**Lemma 3.1.2.** *Given a Gamma space $X$ and $(j_0, \dots, j_k) \in J(k+1)$, then the map*

$$\begin{aligned} &F(S^{j_0} \wedge \cdots \wedge S^{j_k}, X(S^{j_0}) \wedge \cdots \wedge X(S^{j_k})) \to \\ &\qquad\qquad \operatorname*{hocolim}_{(i_0,\dots,i_k)\in J(k+1)} F(S^{i_0} \wedge \cdots \wedge S^{i_k}, X(S^{i_0}) \wedge \cdots \wedge X(S^{i_k})) \end{aligned}$$

*is $2j - 2$-connected, where $j$ denotes the minimum of the cardinalities of $j_1, \dots, j_k$.*

See either [1] of [4] for a proof of Bökstedt's approximation lemma.

Given a FSP $L$ and a finite pointed set $n^+$, we shall let $\mathrm{THH}(L; n^+)$ denote the cyclic pointed simplicial set with $k$-simplices equal to the homotopy colimit

$$\operatorname*{hocolim}_{J(k+1)} F(S^{i_0} \wedge \cdots \wedge S^{i_k}, L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}) \wedge n^+),$$

where $n^+$ acts as a dummy variable for the cyclic structure. There is an endofunctor $\mathrm{THH}(L, -)$ on $\mathcal{S}_*$ associated to the Gamma space $n^+ \mapsto \mathrm{THH}(L; n^+)$.

**Lemma 3.1.3.** *The map $\mathrm{THH}(L) \to \Omega^n \mathrm{THH}(L; S^n)$, adjoint to $\mathrm{THH}(S; S^0) \wedge S^n \to \mathrm{THH}(L; S^n)$, is a homotopy equivalence.*

*Proof.* By the work of Segal [13, prop 1.4] it suffices to show that the gamma space $n^+ \mapsto \mathrm{THH}(L; n^+)$ is *very special*, that is that the map

$$(\mathrm{pr}_{m^+ *}, \mathrm{pr}_{n^+ *}) : \mathrm{THH}(L; m^+ \vee n^+) \to \mathrm{THH}(L; m^+) \times \mathrm{THH}(L; n^+)$$

induced by the projections $\mathrm{pr}_{m^+} : m^+ \vee n^+ \to m^+ \vee 0^+ = m^+$ and $\mathrm{pr}_{n^+} : m^+ \vee n^+ \to 0^+ \vee n^+ = n^+$, is a homotopy equivalence, and that $\pi_0 \mathrm{THH}(L; 1^+)$ is a group. By lemma 3.1.4 it suffices to show that the gamma spaces $n^+ \mapsto \mathrm{THH}_k(L; n^+)$ are very special. To see that the map

$$\mathrm{THH}_k(L; m^+ \vee n^+) \to \mathrm{THH}_k(L; m^+) \times \mathrm{THH}_k(L; n^+)$$

is a homotopy equivalence, it suffices by the approximation lemma 3.1.2 to note that by the Whitehead theorem the map

$$\begin{aligned} &L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}) \wedge (m^+ \vee n^+) \\ &\cong (L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}) \wedge m^+) \vee (L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}) \wedge n^+) \\ &\to (L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}) \wedge m^+) \times (L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}) \wedge n^+) \end{aligned}$$

is $2(i_0 + \cdots + i_k) - 1$-connected.

To see that $\pi_0 \operatorname{THH}_k(L;1^+)$ is a group, it suffices to note that $\pi_0 F(S^{i_0} \wedge \cdots \wedge S^{i_k}, L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}))$ is a group. □

**Lemma 3.1.4.** *Let $X$ be a simplicial Gamma space, and assume that for each $k$, $X_k$ is a very special Gamma space. Then the Gamma space $|X|$ sending $n^+$ to the realization of $[k] \mapsto X_k(n^+)$ is very special.*

*Proof.* It follows from the realization lemma and the fact that realization commutes with products, that the resulting Gamma space is special, that is that the map $|X(m^+ \vee n^+)| \to |X(m^+)| \times |X(n^+)|$ induced by the projections $\mathrm{pr}_{m^+}$ and $\mathrm{pr}_{n^+}$ is a weak equivalence. A special Gamma space $Y$ is very special, when the monoid $\pi_0 Y(1^+)$ with multiplication induced by the composite

$$Y(1^+) \times Y(1^+) \xrightarrow{f} Y(2^+) \xrightarrow{Y(\mu)} Y(1^+)$$

is a group. Here $\mu : 2^+ \to 1^+$ is the fold map with $\mu(i) = 1$ for $i = 1, 2$ and where $f$ is a homotopy inverse to the homotopy equivalence $Y(2^+) \to Y(1^+) \times Y(1^+)$ induced by the projections $p_i : 2^+ \to 1^+$, $i = 1, 2$, with $p_i(i) = 1$ and $p_i(j) = 0$ for $j \neq i$. This is equivalent to the map $(Y(\mu), Y(\mathrm{pr}_2)) : Y(2^+) \to Y(1^+) \times Y(1^+)$ being a homotopy equivalence. It follows from the realization lemma that $|X|$ is very special. □

A *commutative FSP* is a FSP $L$ satisfying that $\mu \circ T \cong \mu$, where $T$ twists factors in a smash product. The following lemma is proved in [4].

**Lemma 3.1.5.** *If $L$ is a commutative FSP, then* $\operatorname{THH}(L, -)$ *is a FSP.*

3.2. **Filtered Topological Hochschild Homology.** To make a filtered version of topological Hochschild homology, we replace the category $\mathcal{S}_*$ of pointed simplicial sets by the category ${\mathcal{S}_*}^{\mathbb{Z}}$ of filtered pointed simplicial sets. By a *Gamma filtered space* we shall mean a pointed functor from $\Gamma$ to ${\mathcal{S}_*}^{\mathbb{Z}}$. The smash product of two Gamma filtered spaces $X$ and $Y$, given by the formula

$$(X \wedge Y)(n^+) = \operatorname*{colim}_{n_1^+ \wedge n_2^+ \to n^+} X(n_1^+) \wedge Y(n_2^+),$$

makes the category $\Gamma {\mathcal{S}_*}^{\mathbb{Z}}$ of Gamma filtered spaces into a symmetric monoidal category. A *filtered FSP* is a monoid in the category $\Gamma {\mathcal{S}_*}^{\mathbb{Z}}$. Explicitly a filtered FSP can be described as a functor $L : \Gamma \times \mathbb{Z} \to \mathcal{S}_*$ together with natural transformations

$$\mu : L(m^+, s) \wedge L(n^+, t) \to L(m^+ \wedge n^+, s + t)$$
$$\eta : n^+ \to L(n^+, 0)$$

satisfying the following relations:

$$\mu \circ (\mu \wedge \mathrm{id}) = \mu \circ (\mathrm{id} \wedge \mu), \quad \mu \circ (\mathrm{id} \wedge \eta) = \lambda, \quad \mu \circ (\eta \wedge \mathrm{id}) = \rho,$$

where $\lambda : m^+ \wedge L(n^+, s) \to L(m^+ \wedge n^+, s)$ is adjoint to the map

$$m^+ \to \Gamma(n^+, m^+ \wedge n^+) \xrightarrow{L(-,s)} \mathcal{S}_*(L(n^+, s), L(m^+ \wedge n^+, s))$$

and $\rho : L(m^+, s) \wedge n^+ \to L(m^+ \wedge n^+, s)$ is adjoint to the map

$$n^+ \to \Gamma(m^+, m^+ \wedge n^+) \xrightarrow{L(-,s)} \mathcal{S}_*(L(m^+, s), L(m^+ \wedge n^+, s)).$$

Note that the category of Gamma filtered spaces is isomorphic to the category of filtered Gamma spaces, and hence a filtered FSP also can be described as being a filtered monoid in the category of Gamma spaces.

The topological Hochschild homology of a filtered FSP $L$ is the filtered pointed simplicial set $\mathrm{THH}(L)$ with $k$-simplices of $\mathrm{THH}(L)(s)$ given by the homotopy colimit

$$\operatorname*{hocolim}_{J(k+1)} F(S^{i_0} \wedge \cdots \wedge S^{i_k}, (L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}))(s)),$$

where the smash product of the $L(S^{i_\alpha})$'s is a smash product of filtered pointed simplicial sets, and with cyclic structure of Hochschild type. We define cyclic spaces $\overline{\mathrm{THH}}(L, s)$ for $s \in \mathbb{Z}$ with $k$-simplices given by the homotopy colimit

$$\operatorname*{hocolim}_{J(k+1)} F(S^{i_0} \wedge \cdots \wedge S^{i_k}, \frac{(L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}))(s)}{(L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}))(s-1)}),$$

and with cyclic structure as for the Hochschild construction.

Of course there is also a filtered versions of the Gamma space $n^+ \mapsto \mathrm{THH}(L; n^+)$ with $k$-simplices of $\mathrm{THH}(L; n^+)(s)$ given by the homotopy colimit:

$$\operatorname*{hocolim}_{J(k+1)} F(S^{i_0} \wedge \cdots \wedge S^{i_k}, (L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}))(s) \wedge n^+),$$

and there is a Gamma space $n^+ \mapsto \overline{\mathrm{THH}}(L, s; n^+)$ with $k$-simplices of $\overline{\mathrm{THH}}(L, s; n^+)$ given by the homotopy colimit

$$\operatorname*{hocolim}_{J(k+1)} F(S^{i_0} \wedge \cdots \wedge S^{i_k}, \frac{(L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}))(s)}{(L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}))(s-1)} \wedge n^+).$$

**Lemma 3.2.1.** *The spectra $n \mapsto \mathrm{THH}(L; S^n)(s)$ and $n \mapsto \overline{\mathrm{THH}}(L, s; S^n)$ are $\Omega$-spectra.*

*Proof.* The proof is similar to the proof of lemma 3.1.3. □

**Lemma 3.2.2.** *Let $L$ be a filtered FSP, filtered by cofibrations. The map from $\mathrm{THH}(L)(s-1)$ to the homotopy fibre of the map $q : \mathrm{THH}(L)(s) \to \overline{\mathrm{THH}}(L, s)$ is a homotopy equivalence.*

*Proof.* Given $n \in \mathbb{N}$, we shall show that the map from the mapping cone of the map $\mathrm{THH}(L; S^n)(s-1) \to \mathrm{THH}(L; S^n)(s)$ to $\overline{\mathrm{THH}}(L, s; S^n)$ is $2(n-1)$-connected. Let $q(S^n)$ denote the map $\mathrm{THH}(L; S^n)(s) \to \overline{\mathrm{THH}}(L, s; S^n)$, and let $hFq(S^n)$ denote its homotopy fibre. It then follows from the Blakers-Massey theorem that the map $\mathrm{THH}(L; S^n)(s-1) \to hFq(S^n)$ is $2n-4$-connected. From the homotopy equivalence $F(S^n, hFq(S^n)) \simeq hFq$ it follows that the map $\mathrm{THH}(L)(s-1) \to hFq$ is $n-4$-connected. Since $n$ is arbitrary, it follows that this map is a homotopy equivalence.

Now the map from the mapping cone of the map

$$\mathrm{THH}_k(L; S^n)(s-1) \to \mathrm{THH}_k(L; S^n)(s)$$

to $\overline{\mathrm{THH}}_k(L; S^n, s)$ is $2(n-1)$-connected because if $X \to Y$ is a cofibration of $i+n$-connected pointed simplicial sets, then by applying the Blakers Massey theorem several times, it is seen that the map from the mapping cone of the map $F(S^i, X) \to F(S^i, Y)$ to $F(S^i, Y/X)$ is $2(n-1)$-connected. Since the mapping cone construction commutes with geometric realization, it follows that the induced map of geometric realizations is $2(n-1)$-connected, which is what we wanted to prove. □

## 4. Cyclotomic structure

In this section we shall describe how the filtration on topological Hochschild homology of a FSP filtered by cofibrations is compatible with topological cyclic homology. The description is centered around Madsen's concept of a cyclotomic spectrum. The present presentation is essentially a repetition of parts of section 2 of [6] and section 1 in [3] with a few extra comments about the filtrations and about the index categories $J(k+1)$ versus the index categories $I^{k+1}$.

4.1. **Cyclotomically filtered spectra.** Following the notation of [6], we shall let $G$ denote the circle group, and we shall let $\mathcal{U} = \bigoplus_{n\in\mathbb{Z},\alpha\in\mathbb{N}} \mathbb{C}(n)_\alpha$, where $\mathbb{C}(n)_\alpha = \mathbb{C}(n)$ is $\mathbb{C}$ with $G$ acting through the $n$th power map $g\cdot z = g^n \cdot z$. Given a finite subgroup $C_s \subseteq G$, we shall denote the isomorphism between $G$ and $G/C_r$ by $\rho_{C_r} : G \to G/C_r$. Notation about equivariant stable homotopy theory is taken from [6]. Given a rational number $q \in \mathbb{Q}$, we shall let $[q]$ denote the largest integer that is smaller than or equal to $q$.

**Definition 4.1.1.** A *c*yclotomically filtered spectrum is a filtered $G$-spectrum $T$ indexed on $\mathcal{U}$ together with a $G$-equivalence

$$r_{C_r} : \rho^\sharp_{C_r}\Phi^{C_r}T(s) \to T([s/r])$$

for every finite subgroup $C_r \subset G$ and for every $s \in \mathbb{Z}$ satisfying that the equivalence is an equivalence of $G$-spectra making the diagram

$$\begin{tikzcd}
\rho^\sharp_{C_r}\Phi^{C_r}\rho^\sharp_{C'_r}\Phi^{C'_r}T(s) \arrow[r, equal] \arrow[d, "\rho^\sharp_{C_r}\Phi^{C_r}r_{C_{r'}}"'] & \rho^\sharp_{C_{rr'}}\Phi^{C_{rr'}}T(s) \arrow[d, "r_{C_{rr'}}"'] \\
\rho^\sharp_{C_r}\Phi^{C_r}T([s/r']) \arrow[r, "r_{C_r}"] & T([s/(rr')])
\end{tikzcd}$$

commute.

Here $\Phi^{C_r}T(s)$ denotes the geometric fixed point spectrum of $T(s)$. Given a cyclotomically filtered spectrum there is a map $T(s)^{C_r} \to \Phi^{C_r}T(s)$. Together with the cyclotomic structure map $r_{C_r}$ we obtain a map

$$\rho^\sharp_{C_{rn}}T(s)^{C_{rn}} = \rho^\sharp_{C_n}(\rho^\sharp_{C_r}T(s)^{C_r})^{C_n} \to \rho^\sharp_{C_n}(\rho^\sharp_{C_r}\Phi^{C_r}T(s))^{C_n} \to \rho^\sharp_{C_n}T([s/r])^{C_n}.$$

We shall call the associated map $R_r : T(s)^{C_{rn}} \to T([s/r])^{C_n}$ of underlying non-equivariant spectra the *rth restriction* map.

**Lemma 4.1.2.** *Let $t$ be a filtered good $G$-prespectrum and let $T = Lt$ ($L =$ "spectrification"). Then $T$ is a cyclotomically filtered spectrum if for each index space $V \subseteq \mathcal{U}$ and each finite subgroup $C_r \subseteq G$ there is a $G$-map*

$$r_{C_r}(V) : \rho^*_{C_r}(t(V)(s))^{C_r} \to t(\rho^*_{C_r}V^{C_r})([s/r])$$

*subject to the following three conditions:*

1. *For each pair $V \subseteq W \subset \mathcal{U}$ the diagram*

$$\begin{tikzcd}
S^{\rho^*_{C_r}(W-V)^{C_r}} \wedge (\rho^*_{C_r}t(V)(s))^{C_r} \arrow[r, "1\wedge r_{C_r}(V)"] \arrow[d, "\rho^*_{C_r}(\widetilde{\sigma})^{C_r}"'] & S^{\rho^*_{C_r}(W-V)^{C_r}} \wedge t(\rho^*_{C_r}V^{C_r})([r/s]) \arrow[d, "\widetilde{\sigma}"'] \\
(\rho^*_{C_r}t(W)(s))^{C_r} \arrow[r, "r_{C_r}(W)"] & t(\rho^*_{C_r}W^{C_r})([s/r])
\end{tikzcd}$$

*commutes.*

2. *For each pair of finite subgroups the diagram*

$$\begin{array}{ccc} \rho^*_{C_{rr'}}(t(V)(s))^{C_{rr'}} & \xrightarrow{\rho^*_{C_{r'}}(r_{C_r}(V))^{C_{r'}}} & \rho^*_{C_{r'}}(t(\rho^*_{C_r}V^{C_r})([s/r]))^{C_{r'}} \\ {\scriptstyle r_{C_{rr'}(V)}}\Big\downarrow & & {\scriptstyle r_{C_{r'}}(\rho^*_{C_r}V^{C_r})}\Big\downarrow \\ t(\rho^*_{C_{rr'}}V^{C_{rr'}})([s/(rr')]) & = & t(\rho^*_{C_{r'}}(\rho^*_{C_r}V^{C_r})^{C_{r'}})([s/(rr')]) \end{array}$$

*commutes.*

3. *For any $V \subset \mathcal{U}^{C_r}$ the induced map on colimits*

$$\operatorname*{colim}_{W\subset\mathcal{U}} \Omega^{\rho^*_{C_r}W^{C_r}-V}\rho^*_{C_r}(t(W)(s))^{C_r} \to \operatorname*{colim}_{W\subset\mathcal{U}} \Omega^{\rho^*_{C_r}W^{C_r}-V} t(\rho^*_{C_r}W^{C_r})(s/r)$$

*is a $G$-equivalence.*

Since it is similar to the proof of lemma 2.2 in [6], we omit the proof of the above lemma.

A $G$-prespectrum with the above structure will be called a *cyclotomically filtered prespectrum.* Following the discussion in [6] we obtain the following result:

**Theorem 4.1.3.** *For any cyclotomically filtered spectrum $T$ and any f.d. sub-inner product space $Z \subset \mathcal{U}$ there is a cofibration sequence of non-equivariant spectra:*

$$(T(s)_Z)_{hC_{p^n}} \xrightarrow{N} T(s)_Z^{C_{p^n}} \xrightarrow{R} T([s/p])^{C_{p^{n-1}}}_{\rho^*_{C_p}Z^{C_p}}$$

*where $(T(s)_Z)_{hC_{p^n}}$ is the homotopy orbit spectrum.*

4.2. **Topological Hochschild Homology is Cyclotomic.** In this section $L$ denotes a FSP filtered by cofibrations. We shall show that $\mathrm{THH}(L)$ is a cyclotomically filtered spectrum. The cyclotomic structure is defined through an application of edgewise subdivision. Given a simplicial set $X$, there is an action of $G$ on the filtered space $|\,\mathrm{THH}(L,X)|$ because it is the realization of a filtered cyclic space. If $X$ has a $G$-action, then $|\,\mathrm{THH}(L,X)|$ has an action of $G\times G$, and we can consider it as a filtered $G$-space through the diagonal $\Delta : G \to G\times G$.

Let us define a filtered $G$-prespectrum $t(L)$ whose 0th filtered space is $|\,\mathrm{THH}(L)|$. Let $V$ be a f.d. sub-inner product space of $\mathcal{U}$, and let $S^V$ be the one-point compactification. Then

$$t(L)(V) = |\,\mathrm{THH}(L;S^V)|$$

and the obvious maps

$$\sigma : t(L)(V) \to \Omega^{W-V}t(L)(W)$$

are $G$-equivariant and form a transitive system. Finally, we let $T(L)$ be the associated filtered $G$-spectrum of the filtered thickened $G$-prespectrum $t^\tau(L)$, that is

$$T(L)(V)(s) = \operatorname*{colim}_{W\subseteq\mathcal{U}} \Omega^{W-V}t^\tau(L)(W)(s).$$

The rest of this section will be devoted to proving that $T(L)$ is a cyclotomically filtered spectrum.

In order to define the cyclotomic structure maps we need the edgewise subdivision of [2, Section 1]. The realization of a cyclic space becomes a $G$-space upon identifying $G$ with $\mathbb{R}/\mathbb{Z}$ and hence $C = C_r$ may be identified with $r^{-1}\mathbb{Z}/\mathbb{Z}$. Edgewise subdivision associates with a cyclic space $Z$. a simplicial $C$-space $\mathrm{sd}_C\, Z$. with

$k$-simplices $\mathrm{sd}_C Z_k = Z_{r(k+1)-1}$. The generator $r^{-1} + \mathbb{Z}$ of $C$ acts as $t^{k+1}$. The diagonal $\Delta^k \to \Delta^k * \cdots * \Delta^k$ (join of $r$ factors) induces a natural (non-simplicial) homeomorphism

$$D : |\,\mathrm{sd}_C Z.| \to |Z.|$$

of the realizations. Finally, there is a natural $\mathbb{R}/r\mathbb{Z}$-action on $|\,\mathrm{sd}_C Z.|$ which extends the simplicial $C$-action, and the map $D$ is $G$-equivariant when $\mathbb{R}/r\mathbb{Z}$ is identified with $\mathbb{R}/\mathbb{Z}$ through division by $r$.

Let us consider the case of $\mathrm{THH}(L;X)$, and let us write $G^X_k(x_0,\ldots,x_k)(s)$ for the pointed mapping space

$$F(S^{x_0} \wedge \cdots \wedge S^{x_k}, (L(S^{x_0}) \wedge \cdots \wedge L(S^{x_k}))(s) \wedge X).$$

Then the $k$-simplices of the edgewise subdivision are the homotopy colimit

$$\mathrm{sd}_C \,\mathrm{THH}(L,X)_k(s) = \operatorname*{hocolim}_{J(r(k+1))} G^X_{r(k+1)-1}(s).$$

We are interested in the subspace of $C$-fixed points. If $X_\alpha$ is a diagram of $C$-spaces, then the homotopy colimit is again a $C$-space and its $C$-fixed set is the homotopy colimit of the spaces $X^C_\alpha$. However, the $C$-action on $\mathrm{sd}_C \,\mathrm{THH}(L,X)_k(s)$ does not arise this way. We consider instead the composite functor $G^X_{r(k+1)-1} \circ \Delta_r$ where $\Delta_r : J(k+1) \to J(r(k+1))$ is the functor sending $x = (x_0,\ldots,x_k)$ in $J(k+1)$ to $(x, \alpha^{k+1}_{r(k+1)} \cdot x, \ldots, \alpha^{(r-1)(k+1)}_{r(k+1)} \cdot x)$ in $J(r(k+1))$, where $\alpha_{r(k+1)}$ is a generator for $C_{r(k+1)}$ and $\alpha^i_{r(k+1)} \cdot x = (\alpha^i_{r(k+1)} \cdot x_0, \ldots, \alpha^i_{r(k+1)} \cdot x_k)$ and $C_{r(k+1)}$ acts freely on $\mathbb{N}$ as in the beginning of section 3.1. This composite functor is indeed a diagram of $C_r$-spaces, and the canonical map on homotopy colimits

$$\operatorname*{hocolim}_{J(k+1)} G^X_{r(k+1)-1}(s) \circ \Delta_r \to \operatorname*{hocolim}_{J(r(k+1))} G^X_{r(k+1)-1}(s)$$

is a $C_r$-equivariant inclusion which induces a homeomorphism on $C_r$-fixed sets.

We have

$$(\mathrm{sd}_{C_r} \,\mathrm{THH}(L;X)_k(s))^{C_r} \cong$$
$$\operatorname*{hocolim}_{J(k+1)} F\left(\left(S^{i_0} \wedge \cdots \wedge S^{i_k}\right)^{\wedge r}, \left(L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k})\right)^{\wedge r}(s) \wedge X\right)^{C_r}$$

with $C_r$ acting by cyclic permutation on the $r$th smash powers, and by conjugation on the mapping space. For any pointed $C_r$-spaces $X$ and $Y$ we have the obvious map

$$F(X,Y)^{C_r} \to F(X^{C_r}, Y^{C_r})$$

induced from the inclusion $X^C \subseteq X$ of the fixed set. In the case at hand this gives a map

$$\begin{gathered} F\left((S^{i_0} \wedge \cdots \wedge S^{i_k})^{\wedge r}, (L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}))^{\wedge r}(s) \wedge X\right)^{C_r} \\ \big\downarrow \\ F\left(S^{i_0} \wedge \cdots \wedge S^{i_k}, \left((L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}))^{\wedge r}(s)\right)^{C_r} \wedge X^{C_r}\right) \end{gathered}$$

We shall prove that there is an isomorphism:

$$(L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}))([s/r]) \xrightarrow{\cong} \left((L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}))^{\wedge r}(s)\right)^{C_r}.$$

Working degreewise, it suffices to show that given a filtered set $Y$, filtered by injections, there is a bijection

$$Y([s/r]) \xrightarrow{\cong} ((Y^{\wedge r})(s))^{C_r}.$$

In order to do this, we note that by lemma 2.4.2 the map $(Y^{\wedge r})(i) \to (Y^{\wedge r})(i+1)$ is an injection for all $i \in \mathbb{Z}$, and therefore we have an injection

$$Y^{\wedge r}(s) \hookrightarrow Y^{\wedge r}(\infty) \cong (Y(\infty))^{\wedge r},$$

with the convention that $Y^{\wedge r}(\infty) = \operatorname{colim}_i Y^{\wedge r}(i)$ and $Y(\infty) = \operatorname{colim}_i Y(i)$. There is a commutative diagram

$$\begin{array}{ccc} Y([s/r]) & \longrightarrow & (Y^{\wedge r}(s))^{C_r} \\ \big\downarrow & & \big\downarrow \\ Y(\infty) & \xrightarrow{\cong} & (Y(\infty)^{\wedge r})^{C_r} \end{array}$$

where the vertical arrows are injections. It follows from the diagram that the map $Y([s/r]) \to (Y^{\wedge r}(s))^{C_r}$ is injective. To see that it is onto, let us pick a representative $((a_1, \ldots, a_r), (y_1, \ldots, y_r))$ for a fixed point $y$ in

$$(Y^{\wedge r}(s)) = \operatorname*{colim}_{a_1+\cdots+a_r \leq s} Y(a_1) \wedge \cdots \wedge Y(a_r),$$

under the $C_r$-action. From the condition $a_1+\cdots+a_r \leq s$, it follows that there exists an $i$ such that $a_i \leq s/r$. Since the image of $y$ in $Y(\infty)^{\wedge r}$ is a fixed point, we must have that $(a_1, y_1), \ldots, (a_r, y_r)$ represent the same element in $Y(\infty)$. Since the map $Y^{\wedge r}(s) \to Y^{\wedge r}(\infty)$ is injective, it follows that $((a_i, \ldots, a_i), (y_i, \ldots, y_i))$ represents $y$, and we can conclude that the map $Y(s/r) \to (Y^{\wedge r}(s))^{C_r}$ is onto.

We obtain a simplicial map

$$r'_{C_r} : (\operatorname{sd}_{C_r} \mathrm{THH}(L; X)(s))^{C_r} \to \mathrm{THH}(L; X^{C_r})([s/r]),$$

and we define

$$r_{C_r} : \rho^*_{C_r} t(L)(V)^{C_r}(s) \to t(L)(\rho^*_{C_r} V^{C_r})([s/r])$$

to be the composite

$$\rho^* |\,\mathrm{THH}(L; S^V)(s)|^{C_r} \xrightarrow{D^{-1}} |\operatorname{sd}_{C_r} \mathrm{THH}(L; S^V)(s)| \xrightarrow{r'_{C_r}} |\,\mathrm{THH}(L; S^{\rho^*_{C_r} V^{C_r}})([s/r])|.$$

**Proposition 4.2.1.** *If $L$ is a FSP filtered by cofibrations, then $t^\tau(L)$ is a cyclotomically filtered prespectrum and $T(L)$ is a cyclotomically filtered spectrum.*

*Proof.* The proof is identical to the proof of proposition 2.5 in [6]. □

4.3. **Witt structure.** Let $L$ be a filtered FSP filtered by cofibrations. In this section we shall recall that there are maps $F_r : (T(L)(s))^{C_{rn}} \to (T(L)(s))^{C_n}$, $V_r : (T(L)(s))^{C_n} \to (T(L)(s))^{C_{rn}}$. Together with the map $R_r : (T(L)(s))^{C_{rn}} \to (T(L)([s/r]))^{C_n}$ induced by the cyclotomic structure of $T(L)$, these maps satisfy relations similar to the relations for the Frobenius, Verschiebung and Restriction maps on the Witt vectors on a commutative ring when $L$ is commutative.

The maps $F_r$ and $V_r$ are defined using only the fact that $T(L)(s)$ is a $G$-spectrum, where $G$ is the circle group. We shall refer to [6] for the construction of these maps as well as for the proof of the following lemma:

**Lemma 4.3.1.** *For any commutative FSP filtered by cofibrations the following relations hold on* $\pi_*(T(L)(s)^{C.})$*:*
1. $F_r(xy) = F_r(x)F_r(y)$,
2. $V_r(F_r(x)y) = xV_r(y)$,
3. $F_rV_r = r$, $V_rF_r = V_r(1)$,
4. $F_rV_n = V_nF_r$ *if* $(r, s) = 1$,
5. $R_rF_n = F_nR_r$, $R_rV_n = V_nR_r$.

## 5. Filtered Rings

5.1. **Hochschild homology.** Given a simplicial ring $R$, that is a monoid in the symmetric monoidal category of simplicial abelian groups, we can consider the Hochschild construction $Z(R)$. In this section we shall construct a cyclic pointed simplicial set $\mathrm{HH}(R)$, the *Hochschild homology* of $R$, which is homotopy equivalent to the underlying simplicial set of $Z(R)$, and a map from $\mathrm{THH}(R)$ to $\mathrm{HH}(R)$.

In order to construct Hochschild homology in its natural generality, we shall consider the category $\Gamma\mathcal{A}$ of pointed functors from the category $\Gamma$ of finite pointed sets to the category $\mathcal{A}$ of simplicial abelian groups. By theorem A.1, there is a symmetric monoidal structure on the category $\Gamma\mathcal{A}$, where the tensor product of two pointed functors $X, Y : \Gamma \to \mathcal{A}$ is the pointed functor $X \otimes Y : \Gamma \to \mathcal{A}$ with

$$(X \otimes Y)(n^+) = \operatorname*{colim}_{n_1^+ \wedge n_2^+ \to n^+} X(n_1^+) \otimes Y(n_2^+).$$

The unit for this operation is the functor taking $n^+$ to $\mathbb{Z}(n^+) = (\bigoplus_{i \in n^+} \mathbb{Z} \cdot i)/\mathbb{Z} \cdot 0$.

A *functor with tensor product* (FTP) is a monoid in the category $\Gamma\mathcal{A}$. Explicitly a FTP $A$ is given by a functor $A : \Gamma \to \mathcal{A}$ and natural transformations

$$\mu : A(m^+) \otimes A(n^+) \to A(m^+ \wedge n^+)$$
$$\eta : \mathbb{Z}(n^+) \to A(n^+)$$

satisfying the following relations:

$$\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu), \quad \mu \circ (\eta \otimes \mathrm{id}) = \lambda, \quad \mu \circ (\mathrm{id} \otimes \eta) = \rho,$$

where $\lambda : \mathbb{Z}(m^+) \otimes A(n^+) \to A(m^+ \wedge n^+)$ is adjoint to the map

$$\begin{aligned} \mathbb{Z}(m^+) \to \mathbb{Z}(\Gamma(n^+, m^+ \wedge n^+)) &\to \mathbb{Z}(\mathcal{A}(A(n^+), A(m^+ \wedge n^+))) \\ &\to \mathcal{A}(A(n^+), A(m^+ \wedge n^+)), \end{aligned}$$

and $\rho : A(m^+) \wedge \mathbb{Z}(n^+) \to A(m^+ \wedge n^+)$ is adjoint to the map

$$\mathbb{Z}(n^+) \to \mathbb{Z}(\Gamma(m^+, m^+ \wedge n^+)) \to \mathcal{A}(A(m^+), A(m^+ \wedge n^+)).$$

Given a FTP $A$, the Hochschild homology of $A$, denoted by $\mathrm{HH}(A)$, is the cyclic object in the category of pointed simplicial sets with $k$-simplices equal to the homotopy colimit

$$\operatorname*{hocolim}_{J(k+1)} F(S^{n_0} \wedge \cdots \wedge S^{n_k}, A(S^{n_0}) \otimes \cdots \otimes A(S^{n_k})),$$

and with Hochschild type structure maps. We define the Gamma space $n^+ \mapsto \mathrm{HH}(A; n^+)$ where $\mathrm{HH}(A; n^+)$ has $k$-simplices given by the homotopy colimit:

$$\operatorname*{hocolim}_{J(k+1)} F(S^{n_0} \wedge \cdots \wedge S^{n_k}, A(S^{n_0}) \otimes \cdots \otimes A(S^{n_k}) \wedge n^+).$$

**Lemma 5.1.1.** *The spectrum* $n \mapsto \mathrm{HH}(A; S^n)$ *is an* $\Omega$*-spectrum.*

*Proof.* The proof is similar to the proof of lemma 3.1.3. □

A *commutative FTP* is a FTP $A$ satisfying that $\mu \circ T \cong \mu$, where $T$ twists factors in a tensor product. The following lemma is proved in [4].

**Lemma 5.1.2.** *If $A$ is a commutative FTP, then* $\mathrm{HH}(A, -)$ *is a FSP.*

Let $\mathcal{A}^{\mathbb{Z}}$ denote the symmetric monoidal category of filtered simplicial abelian groups, and let $\Gamma\mathcal{A}^{\mathbb{Z}}$ denote the category of pointed functors from $\Gamma$ to $\mathcal{A}^{\mathbb{Z}}$. The category $\Gamma\mathcal{A}^{\mathbb{Z}}$ has a symmetric monoidal structure. A *filtered FTP* is a monoid in the category $\Gamma\mathcal{A}^{\mathbb{Z}}$. Explicitly a filtered FTP can be described as a functor $A : \Gamma \times \mathbb{Z} \to \mathcal{A}$ together with natural transformations

$$A(m^+, s) \otimes A(n^+, s) \xrightarrow{\mu} A(m^+ \wedge n^+, s+t)$$
$$\mathbb{Z}(n^+) \xrightarrow{\eta} A(n^+, 0)$$

satisfying the following relations:

$$\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu), \quad \mu \circ (\eta \otimes \mathrm{id}) = \lambda, \quad \mu \circ (\mathrm{id} \otimes \eta) = \rho,$$

where $\lambda : \mathbb{Z}(m^+) \otimes A(n^+, s) \to A(m^+ \wedge n^+, s)$ is adjoint to the map

$$\mathbb{Z}(m^+) \to \mathbb{Z}(\Gamma(n^+, m^+ \wedge n^+)) \to \mathcal{A}(A(n^+, s), A(m^+ \wedge n^+, s)),$$

and $\rho : A(m^+, s) \wedge \mathbb{Z}(n^+) \to A(m^+ \wedge n^+, s)$ is adjoint to the map

$$\mathbb{Z}(n^+) \to \mathbb{Z}(\Gamma(m^+, m^+ \wedge n^+)) \to \mathcal{A}(A(m^+, s), A(m^+ \wedge n^+, s)).$$

We define Hochschild homology of a filtered FTP $A$ to be the filtered cyclic object $\mathrm{HH}(A)$ in the category of pointed simplicial sets with $k$-simplices of $\mathrm{HH}(A)(s)$ given by the homotopy colimit

$$\operatorname*{hocolim}_{J(k+1)} F(S^{n_0} \wedge \cdots \wedge S^{n_k}, (A(S^{n_0}) \otimes \cdots \otimes A(S^{n_k}))(s)),$$

and with Hochschild type face and degeneracy maps. We define the cyclic object in the category of pointed simplicial sets $\overline{\mathrm{HH}}(A, s)$ with $k$-simplices given by the homotopy colimit

$$\operatorname*{hocolim}_{J(k+1)} F(S^{n_0} \wedge \cdots \wedge S^{n_k}, \frac{(A(S^{n_0}) \otimes \cdots \otimes A(S^{n_k}))(s)}{(A(S^{n_0}) \otimes \cdots \otimes A(S^{n_k}))(s-1)}),$$

and with Hochschild type structure maps, for each $s \in \mathbb{Z}$. Here the quotient is taken in the category of pointed simplicial sets. (Choosing to take the quotient in the category of simplicial abelian groups would not change the homotopy type.) There are obvious Gamma space versions of $\mathrm{HH}(A)(s)$ and $\overline{\mathrm{HH}}(A, s)$.

Given a FTP $A$, there is an underlying FSP $\widetilde{A}$ with $\widetilde{A}(U)$ the underlying simplicial set of the simplicial abelian group $A(U)$. Using the map

$$\widetilde{\rho} : \widetilde{\mathbb{Z}}(U) \wedge \widetilde{A}(V) \to \mathbb{Z}(U) \otimes A(V) \to \widetilde{A}(U \wedge V),$$

we obtain a map

$$\mathrm{THH}(\widetilde{\mathbb{Z}}; m^+) \wedge \mathrm{THH}(\widetilde{A}; n^+) \to \mathrm{THH}(\widetilde{A}; m^+ \wedge n^+),$$

which on $k$-simplices is induced by the map

$$\begin{array}{c} \widetilde{\mathbb{Z}}(S^{i_0}) \wedge \cdots \wedge \widetilde{\mathbb{Z}}(S^{i_k}) \wedge m^+ \wedge \widetilde{A}(S^{j_0}) \wedge \cdots \wedge \widetilde{A}(S^{j_k}) \wedge n^+ \\ \big\downarrow \\ \widetilde{\mathbb{Z}}(S^{i_0}) \wedge \widetilde{A}(S^{j_0}) \wedge \cdots \wedge \widetilde{\mathbb{Z}}(S^{i_k}) \wedge \widetilde{A}(S^{j_k}) \wedge m^+ \wedge n^+ \\ \big\downarrow \\ \widetilde{A}(S^{i_0} \wedge S^{j_0}) \wedge \cdots \wedge \widetilde{A}(S^{i_k} \wedge S^{j_k}) \wedge m^+ \wedge n^+. \end{array}$$

In this way the spectrum $n \mapsto \mathrm{THH}(\widetilde{A}; S^n)$ becomes a module spectrum over the ring spectrum $n \mapsto \mathrm{THH}(\widetilde{\mathbb{Z}}; S^n)$. See [4] for details. In the case where $A$ is a filtered FTP, then $n \mapsto \mathrm{THH}(\widetilde{A}; S^n)(s)$ is a filtered module over the spectrum $n \mapsto \mathrm{THH}(\widetilde{\mathbb{Z}}; S^n)$. There is a map from $\mathrm{THH}(\widetilde{A})$ to $\mathrm{HH}(A)$. On $k$-simplices it is induced by the map

$$\widetilde{A}(S^{n_0}) \wedge \cdots \wedge \widetilde{A}(S^{n_k}) \to (A(S^{n_0}) \otimes \cdots \otimes A(S^{n_k})).$$

If $A$ is a filtered FTP, then there is a filtered map $\mathrm{THH}(\widetilde{A}) \to \mathrm{HH}(A)$, and for each $s$ there is a map $\overline{\mathrm{THH}}(\widetilde{A}, s) \to \overline{\mathrm{HH}}(A, s)$.

**Lemma 5.1.3.** *If $A$ is a commutative FTP, then the map $\mathrm{THH}(\widetilde{A}, -) \to \mathrm{HH}(A, -)$ is a map of FSP's, i.e. it commutes with unit and multiplication.*

It follows from the above lemma that for any FTP $A$ the map from the spectrum $n \mapsto \mathrm{THH}(\widetilde{A}; S^n)$ to the spectrum $n \mapsto \mathrm{HH}(A; S^n)$ is a map of module spectra over the spectrum $n \mapsto \mathrm{THH}(\widetilde{\mathbb{Z}})$.

**Lemma 5.1.4.** *Let $A$ be a filtered FTP, filtered by cofibrations. The map from $\mathrm{HH}(A)(s-1)$ to the homotopy fibre of the map $q : \mathrm{HH}(A)(s) \to \overline{\mathrm{HH}}(A, s)$ is a homotopy equivalence.*

*Proof.* Since cofibrations of simplicial abelian groups satisfy the pushout product axiom discussed in section 2.4, it follows from lemma 2.4.2 that the map

$$(A(S^{i_0}) \otimes \cdots \otimes A(S^{i_k}))(s-1) \to (A(S^{i_0}) \otimes \cdots \otimes A(S^{i_k}))(s)$$

is a cofibration of simplicial abelian groups. In particular the underlying map of pointed simplicial sets is a cofibration. The proof now proceeds as the proof of lemma 3.2.2. $\square$

**Example 5.1.5.** Given a simplicial ring $R$, there is a FTP $R = R \otimes \mathbb{Z}$ associated to $R$. The map $Z(R) \to \mathrm{Hom}(\mathbb{Z}(S^n), Z(R) \otimes \mathbb{Z}(S^n)) \to F(S^n, Z(R) \otimes \mathbb{Z}(S^n))$ induces a map $Z(R) \to \mathrm{HH}(R)$. This map is a homotopy equivalence because there are homotopy equivalences $\sin |Z(R) \otimes \mathbb{Z}(S^n)| \simeq Z(R) \otimes \mathbb{Z}(S^n)$; $\mathbb{Z} \xrightarrow{\simeq} \mathrm{Hom}(\mathbb{Z}(S^n), \mathbb{Z}(S^n))$ and a commutative diagram

$$\begin{array}{ccc} Z(R) & \longrightarrow & \mathrm{Hom}(\mathbb{Z}(S^n), Z(R) \otimes \mathbb{Z}(S^n)) \\ {\scriptstyle\cong}\big\downarrow & & {\scriptstyle\cong}\big\downarrow \\ \mathbb{Z} \otimes Z(R) & \longrightarrow & \mathrm{Hom}(\mathbb{Z}(S^n), \mathbb{Z}(S^n)) \otimes Z(R). \end{array}$$

5.2. **Topological Hochschild homology of monoid rings.** In this section we shall recall why, given a pointed simplicial monoid $\Pi$, the map

$$\pi_* \operatorname{THH}(\mathbb{Z}(\Pi)) \to \pi_* \operatorname{HH}(\mathbb{Z}(\Pi))$$

is onto.

Given a FTP $A$ and a FSP $L$, we can consider the $FTP$ $A \circ L$ with $A \circ L(X) = A(L(X))$. The spectrum $n \mapsto \widetilde{A}(L(S^n))$ represented by $A \circ L$ is equivalent to the (derived) smash product of the spectra represented by $A$ and $L$ respectively. This is because by lemma 3.1.1 the map $\widetilde{A}(S^m) \wedge L(S^n) \to \widetilde{A}(S^m \wedge L(S^n)) \to \widetilde{A}(L(S^m \wedge S^n))$ is $m+n+\min\{m,n\}-1$-connected. A similar argument shows that the spectra represented by $\operatorname{THH}(\widetilde{A} \circ L, -)$ and $\operatorname{HH}(A \circ L, -)$ are equivalent to the smash products of $\operatorname{THH}(\widetilde{A}, -)$ and $\operatorname{THH}(L, -)$ and the smash product of $\operatorname{HH}(A, -)$ and $\operatorname{THH}(L, -)$ respectively. In fact, there is a commutative diagram:

$$\begin{array}{ccc}
\operatorname{THH}(\widetilde{A}; S^m) \wedge \operatorname{THH}(L; S^n) & \longrightarrow & \operatorname{THH}(\widetilde{A} \circ L; S^m \wedge S^n) \\
\big\downarrow & & \big\downarrow \\
\operatorname{HH}(A; S^m) \wedge \operatorname{THH}(L; S^n) & \longrightarrow & \operatorname{HH}(A \circ L; S^m \wedge S^n)
\end{array}$$

where the horizontal maps are $m+n+\min\{m,n\}-1$-connected. Let us indicate how the lower horizontal map in the above diagram is constructed. First since $\operatorname{HH}(A, -)$ is a Gamma space, there is a $m+n+\min\{m,n\}-1$-connected map

$$\begin{aligned}
\operatorname{HH}(A; S^m) \wedge \operatorname{THH}(L; S^n) &\to \operatorname{HH}(A; S^m \wedge \operatorname{THH}(L; S^n)) \\
&\to \operatorname{HH}(A; \operatorname{THH}(L; S^m \wedge S^n)).
\end{aligned}$$

Let us write $G_k^X(i_0, \dots, i_k)$ for the pointed mapping space

$$F(S^{i_0} \wedge \cdots \wedge S^{i_k}, L(S^{i_0}) \wedge \cdots \wedge L(S^{i_k}) \wedge X).$$

Let us now give a map of $k$-simplices of cyclic pointed simplicial sets:

$$\begin{array}{c}
\operatorname*{hocolim}_{i \in J(k+1)} F(S^i, A(S^{i_0}) \otimes \cdots \otimes A(S^{i_k}) \wedge \operatorname*{hocolim}_{j \in J(k+1)} G_k^{S^m \wedge S^n}(j)) \\
\big\downarrow \\
\operatorname*{hocolim}_{J(k+1) \times J(k+1)} F(S^i \wedge S^j, A(S^{i_0}) \otimes \cdots \otimes A(S^{i_k}) \wedge L(S^{j_0}) \wedge \cdots \wedge L(S^{j_k}) \wedge S^m \wedge S^n) \\
\big\downarrow \\
\operatorname*{hocolim}_{J(k+1)} \operatorname*{hocolim}_{J(k+1)} F(S^i \wedge S^j, A(L(S^{i_0} \wedge S^{j_0})) \otimes \cdots \otimes A(L(S^{i_k} \wedge S^{j_k})) \wedge S^m \wedge S^n) \\
\big\downarrow \\
\operatorname*{hocolim}_{J(k+1)} F(S^x, A(L(S^{x_0})) \otimes \cdots \otimes A(L(S^{x_k})) \wedge S^m \wedge S^n),
\end{array}$$

where the last map is induced by the functor $J(k+1) \times J(k+1) \to J(k+1)$ taking a pair $(i,j)$ of tuples of finite sets to the disjoint union $i \amalg j$. This map is a homotopy equivalence by the approximation lemma 3.1.2. Assembling these maps into a simplicial map, we get a weak equivalence

$$\operatorname{HH}(A; \operatorname{THH}(L; S^m \wedge S^n)) \xrightarrow{\simeq} \operatorname{HH}(A \circ L; S^m \wedge S^n).$$

Hence we have produced the lower horizontal map in the above digram with connectivity as stated. The upper horizontal map goes the same way replacing tensor products in the above formulas by smash products.

Since $\mathrm{THH}(\widetilde{A})$ is a product of Eilenberg MacLane spectra, we have that

$$\pi_n \mathrm{THH}(\widetilde{A} \circ L) \cong \bigoplus_{s+t=n} H_s(\mathrm{THH}(L); \pi_t \mathrm{THH}(\widetilde{A})).$$

If we take $A = \mathbb{Z}$ then we know that the map $\pi_0 \mathrm{THH}(\widetilde{\mathbb{Z}}) \to \pi_0 \mathrm{HH}(\mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism, and that $\pi_n \mathrm{HH}(\mathbb{Z}) = 0$ for $n \neq 0$. Therefore the map

$$\bigoplus_{s+t=n} H_s(\mathrm{THH}(L); \pi_t \mathrm{THH}(\widetilde{\mathbb{Z}})) \cong \pi_n \mathrm{THH}(\widetilde{\mathbb{Z}} \circ L)$$
$$\to \pi_n \mathrm{HH}(\mathbb{Z} \circ L) \cong H_n(\mathrm{THH}(L); \mathbb{Z})$$

is isomorphic to the projection onto the factor $s = n$, $t = 0$.

Finally note that all maps given in this section preserve filtrations in the case where $A$ and $L$ are equipped with filtrations.

5.3. **Topological Hochschild homology of filtered rings.** In this section $A$ shall denote a filtered FTP satisfying that the associated graded FTP $\mathcal{G}A$ is on the form $\mathbb{Z} \circ L$ for some filtered FSP $L$. Then we have isomorphisms:

$$\pi_n \overline{\mathrm{THH}}(\widetilde{A}, s) \cong \pi_n \overline{\mathrm{THH}}(\widetilde{\mathbb{Z}} \circ L, s) \cong \bigoplus_{s+t=n} H_s(\overline{\mathrm{THH}}(L, s); \pi_t \mathrm{THH}(\widetilde{\mathbb{Z}}))$$

and

$$\pi_n \overline{\mathrm{HH}}(A, s) \cong \pi_n \overline{\mathrm{HH}}(\mathbb{Z} \circ L, s) \cong H_n(\overline{\mathrm{THH}}(L, s); \mathbb{Z}),$$

fitting into the commutative diagram:

$$\begin{tikzcd}
\pi_n \overline{\mathrm{THH}}(\widetilde{A}, s) \arrow[r, "\cong"] \arrow[d] & \bigoplus_{s+t=n} H_s(\overline{\mathrm{THH}}(L, s); \pi_t \mathrm{THH}(\widetilde{\mathbb{Z}})) \arrow[d] \\
\pi_n \overline{\mathrm{HH}}(A, s) \arrow[r, "\cong"] & H_n(\overline{\mathrm{THH}}(L, s); \mathbb{Z}),
\end{tikzcd}$$

where the right hand vertical map is induced by the map $\pi_* \mathrm{THH}(\widetilde{\mathbb{Z}}) \to \pi_* \mathrm{HH}(\mathbb{Z})$, that is, it is projection onto the summand corresponding to $s = n$ and $t = 0$.

If $R$ is a simplicial ring such that the filtration quotients in

$$0 \to \mathbb{Z} \to \mathrm{sk}^0 R \to \mathrm{sk}^1 R \to\to \mathrm{sk}^n R \to \dots$$

are free simplicial abelian groups, then the associated graded ring $\mathcal{G}R$ is on the form $\mathbb{Z}(\Pi)$ for some pointed simplicial monoid $\Pi$. In example 2.5 we found that $\pi_n \overline{\mathrm{HH}}(R, s) = 0$ for $n \neq s, s-1$. Using that $\mathrm{THH}(\widetilde{R})$ is filtered by modules over $\mathrm{THH}(\widetilde{\mathbb{Z}})$, there is a good chance of determining the $E^3$-terms in the spectral sequence associated to the filtration of $\mathrm{THH}(\widetilde{R})$.

**Example 5.3.1.** Let $R$ be the subring $\mathbb{Z}(S^0) + n\mathbb{Z}(\Delta^1)$ of the pointed monoid ring $\mathbb{Z}(\Delta^1)$. Here $\Delta^1_k$ is the pointed set of order preserving functions form $\{0, 1, \dots, k\}$ to $\{0, 1\}$ with the constant function 0 as base point and with monoid structure given by pointwise product of functions. From the short exact sequence

$$n\mathbb{Z}(\Delta^1) \to R \to \mathbb{Z}/n$$

we see that $R$ is homotopy equivalent to $\mathbb{Z}/n$. The above filtration of $R$ takes the form:

$$0 \to \mathbb{Z} \to \mathbb{Z} \to R \to R \to \dots.$$

An easy calculation, for example using the normalized chain complex, shows that $\pi_* \overline{\mathrm{HH}}(R, s) = \pi_* \overline{\mathrm{HH}}(\mathcal{G}R, s)$ can be described as follows:

$$\pi_{s+t}\overline{\mathrm{HH}}(R,s) \cong \begin{cases} \mathbb{Z} & \text{if } s=t=0 \\ \mathbb{Z} & \text{if } s>0 \text{ is even and } t=-1,0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the $E^1$-term in the spectral sequence for $\mathrm{THH}(\widetilde{R})$ can be described as follows:

$$E^1_{s,t} \cong \begin{cases} \pi_t \,\mathrm{THH}(\widetilde{\mathbb{Z}}) & \text{if } s=0 \\ \pi_t \,\mathrm{THH}(\widetilde{\mathbb{Z}}) \oplus \pi_{t-1} \,\mathrm{THH}(\widetilde{\mathbb{Z}}) & \text{if } s>0 \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Comparing with the spectral sequence for $\mathrm{HH}(R)$ we find that

$$E^3_{s,t} \cong \begin{cases} \pi_t \,\mathrm{THH}(\widetilde{\mathbb{Z}})/n & \text{if } s=0 \\ \pi_t \,\mathrm{THH}(\widetilde{\mathbb{Z}})/n \oplus {}_n\pi_t \,\mathrm{THH}(\widetilde{\mathbb{Z}}) & \text{if } s>0 \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$$

where ${}_n\pi_t \,\mathrm{THH}(\widetilde{\mathbb{Z}})$ denotes the kernel of multiplication by $n$ on $\pi_t \,\mathrm{THH}(\widetilde{\mathbb{Z}})$.

**Remark 5.3.2.** *We can consider bimodules over filtered simplicial rings. If $R$ is a simplicial ring filtered as above, and $M$ is a bimodule over $R$ filtered by*

$$0 \to \mathrm{sk}^0 M \to \mathrm{sk}^1 M \to \dots \to \mathrm{sk}^n M \to \dots$$

*with $\mathrm{sk}^n M$ in filtration degree $n$, then the only non-vansihing homotopy group of the filtration quotient $\overline{\mathrm{HH}}(R, M, s)$ is in degree $s$. This implies that the spectral sequence for $\mathrm{THH}(\widetilde{R}, \widetilde{M})$ has $E^2$-terms on the following form:*

$$E^2_{s,t} = \pi_s \,\mathrm{HH}(R, M \otimes \pi_t \,\mathrm{THH}(\widetilde{\mathbb{Z}})),$$

*where the action of $R$ on $M \otimes \pi_t \,\mathrm{THH}(\widetilde{\mathbb{Z}})$ is induced by the action of $R$ on $M$. This spectral sequence is similar to a spectral sequence first considered by Pirashvili and Waldhausen in* [10].

## Appendix A. Closed monoidal functor categories

A.1. In this appendix we shall collect some results of Day [5] for the ease of reference: Let $\mathcal{C}$ be a closed symmetric monoidal category, and let $\mathcal{D}$ be a $\mathcal{C}$-symmetric-monoidal-category, that is a $\mathcal{C}$-category with a $\mathcal{C}$-functor $\overline{\otimes} : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}$, an object $\overline{I} \in \mathcal{D}$, and $\mathcal{C}$-natural isomorphisms $\overline{a} : (A \overline{\otimes} B) \overline{\otimes} C \cong A \overline{\otimes} (B \overline{\otimes} C)$, $\overline{r} : A \overline{\otimes} \overline{I} \cong A$, $\overline{l} : \overline{I} \overline{\otimes} A \cong A$, and $\overline{c} : A \overline{\otimes} B \cong B \overline{\otimes} A$, satisfying the usual coherence axioms for a symmetric monoidal category. Given $\mathcal{C}$-functors $X$ and $Y$ from $\mathcal{D}$ to $\mathcal{C}$, Day constructs a $\mathcal{C}$-functor $X * Y$ from $\mathcal{D}$ to $\mathcal{C}$. On an object $d$ of $\mathcal{D}$ its value is given by the following coend:

$$(X * Y)(d) = \int^{(d_1, d_2)} X(d_1) \otimes Y(d_2) \otimes \mathcal{D}(d_1 \otimes d_2, d).$$

The functor $d \mapsto \mathcal{D}(1,d)$ is a right- and left unit for the operation $*$.

**Theorem A.1** (Day). *Provided the above coends exist, there is a closed symmetric monoidal structure on the category $[\mathcal{D},\mathcal{C}]$ of $\mathcal{C}$-functors from $\mathcal{D}$ to $\mathcal{C}$ with product $(X,Y) \mapsto X * Y$.*

Note that the monoidal (pointed) normalization functor $C : \mathcal{C} \to \mathrm{Set}_{(*)}$ has a monoidal closed left adjoint $F : \mathrm{Set}_{(*)} \to \mathcal{C}$ which sends a set $U$ to the copower $\bigoplus_U I$ (resp. $\bigoplus_{u \in U} I \cdot u / I \cdot *$) , in the underlying (pointed) category $\mathcal{C}_0$ of $\mathcal{C}$, of $U$ copies of the unit $I$. Given a $\mathrm{Set}_{(*)}$-category $\mathcal{A}$, there is a $\mathcal{C}$-category $F_\sharp \mathcal{A}$ whose objects are those of $\mathcal{A}$ and whose morphism objects are given by $F_\sharp \mathcal{A}(A,B) = F(\mathcal{A}(A,B))$ in $\mathcal{C}$. Now let $\mathcal{D} = F_\sharp \mathcal{A}$ for some $\mathrm{Set}_{(*)}$-category $\mathcal{A}$. Then the category $[\mathcal{D},\mathcal{C}]$ of $\mathcal{C}$-functors from $\mathcal{D}$ to $\mathcal{C}$ is isomorphic to the category $\mathcal{C}^{\mathcal{A}}$ of (pointed) functors from $\mathcal{A}$ to $\mathcal{C}$, and the formula for $(X * Y)(d)$ can be rewritten as

$$(X * Y)(d) = \operatorname*{colim}_{d_1 \otimes d_2 \to d} X(d_1) \otimes X(d_2),$$

where the colimit is taken over the comma category $\otimes \downarrow d$ (cf. MacLane [9]). The unit for $*$ is the functor $d \mapsto F(\mathcal{A}(1,d))$.

Taking $\mathcal{A}$ to be the category $\mathbb{Z}$, with exactly one morphism $n \to m$ if and only if $m \geq n$, we obtain the the construction of the tensor product in $\mathcal{C}^{\mathbb{Z}}$ given in section 2.2.

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