LEIBNIZ n-ALGEBRAS

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Abstract. A *Leibniz n-algebra* is a vector space equipped with an n-ary operation which has the property of being a derivation for itself. This property is crucial in Nambu mechanics. For $n = 2$ this is the notion of Leibniz algebra. In this paper we prove that the free Leibniz $(n+1)$-algebra can be described in terms of the n-magma, that is the set of n-ary planar trees. Then it is shown that the n-tensor power functor, which makes a Leibniz $(n+1)$-algebra into a Leibniz algebra, sends a free object to a free object. This result is used in the last section, together with former results of Loday and Pirashvili, to construct a small complex which computes Quillen cohomology with coefficients for any Leibniz n-algebra .

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1. Introduction

Leibniz algebras were introduced by the second author in [4]. They play an important role in Hochschild homology theory [4], [5] as well as in Nambu mechanics ([6], see also [1]). Let us recall that *a Leibniz algebra* is a vector space g equipped with a bilinear map $[-,-] : \mathsf{g} \otimes \mathsf{g} \to \mathsf{g}$ satisfying the identity :

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]. \tag{1.1}$$

One easily sees that Lie algebras are exactly Leibniz algebras satisfying the relation $[x,x] = 0$. Hence Leibniz algebras are a non-commutative version of Lie algebras.

Recently there have been several works dealing with various generalization of Lie structures by extending the binary bracket to an n-bracket (see [1], [2], [9]).

In this paper we introduce the notion of a Leibniz n-algebra — a natural generalization of both concepts. For $n = 2$ one recovers Leibniz algebras. Any Leibniz algebra g is also a Leibniz n-algebra under the following n-bracket:

$$[x_1, x_2, \cdots, x_n] := [x_1, [x_2, \cdots, [x_{n-1}, x_n] \cdots]].$$

Conversely, if $\mathcal{L}$ is a Leibniz $(n+1)$-algebra, then on $\mathcal{D}_n(\mathcal{L}) = \mathcal{L}^{\otimes n}$ the following bracket

$$[a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n] := \sum_{i=1}^{n} a_1 \otimes \cdots \otimes [a_i, b_1, \cdots, b_n] \otimes \cdots \otimes a_n$$

makes $\mathcal{D}_n(\mathcal{L})$ into a Leibniz algebra. This construction goes back to Gautheron [2] and plays an important role in [1].

The main result of this paper is to show that if $\mathcal{L}$ is a free Leibniz n-algebra , then $\mathcal{D}_{n-1}(\mathcal{L})$ is a free Leibniz algebra too (on a different vector space). This result plays an essential role in the cohomological investigation of Leibniz n-algebras, which we consider in the last section.

We first introduce a notion of *representation* of a Leibniz n-algebra $\mathcal{L}$. This notion for $n = 2$ was already considered in [5]. One observes that if M is a representation of a Leibniz n-algebra $\mathcal{L}$, then $Hom(\mathcal{L}, M)$ can be considered as a representation of the Leibniz algebra $\mathcal{D}_{n-1}(\mathcal{L})$. The work of [2] and [1] suggests to define the cohomology of $\mathcal{L}$ with coefficients in M to be $HL^*(\mathcal{D}_{n-1}(\mathcal{L}), Hom(\mathcal{L}, M))$. We deduce from our main theorem that this theory is exactly the Quillen cohomology for Leibniz n-algebras.

2. Derivations

In the whole paper K is a field. All tensor products are taken over K. Let $\mathcal{A}$ be a vector space equipped with an n-linear operation $\omega : \mathcal{A}^{\otimes n} \to \mathcal{A}$. A map $f : \mathcal{A} \to \mathcal{A}$ is called a *derivation* with respect to ω if

$$f(\omega(a_1, \ldots, a_n)) = \sum_{i=1}^{n} \omega(a_1, \ldots, f(a_i), \cdots, a_n).$$

In this case we also say that f is an *ω-derivation*. We let $\mathcal{D}er_\omega(\mathcal{A})$ be the set of all ω-derivations. The following is well-known.

Proposition 2.1 i) *The subset $\mathcal{D}er_\omega(\mathcal{A})$ of the Lie algebra of endomorphisms $End(\mathcal{A})$ is a Lie subalgebra.*

ii) *If $f \in \mathcal{D}er_\omega(\mathcal{A})$ and $f \in \mathcal{D}er_\sigma(\mathcal{A})$, then $f \in \mathcal{D}er_{\omega+\sigma}(\mathcal{A})$. Here $\sigma : \mathcal{A}^{\otimes n} \to \mathcal{A}$ is also an n-linear operation.*

Proposition 2.2 *Let $[-,-] : \mathcal{A}^{\otimes 2} \to \mathcal{A}$ be a bilinear operation and let $\omega : \mathcal{A}^{\otimes n} \to \mathcal{A}$ be given by*

$$\omega(x_1, \cdots, x_n) := [x_1, [x_2, \cdots, [x_{n-1}, x_n] \cdots]].$$

If f is a derivation with respect to $[-,-]$, then $f \in \mathcal{D}er_\omega(\mathcal{A})$.

Proof. One has

$$f(\omega(x_1,\cdots,x_n)) = f([x_1,[x_2,\cdots,[x_{n-1},x_n]\cdots]])$$

$$= [f(x_1),[x_2,\cdots,[x_{n-1},x_n]\cdots]]) + [x_1, f([x_2,\cdots,[x_{n-1},x_n]\cdots])] =$$

$$= \sum_i [x_1,\cdots,[f(x_i),\cdots,x_n]] = \sum_i \omega(x_1,\cdots,f(x_i),\cdots,x_n).$$

The following is an immediate generalization of Proposition 2.2. Since it has the same proof, we omit it here.

Proposition 2.3 *Let $\omega_i : \mathcal{A}^{\otimes n_i} \to \mathcal{A}$ be n_i-ary operations for $i = 1,\ldots,k$ and let $\omega : \mathcal{A}^{\otimes k} \to \mathcal{A}$ be a k-ary operation. If f is a derivation with respect to $\omega_1,\ldots,\omega_n,\omega$, then it is also a derivation with respect to the composite $\sigma : \mathcal{A}^{\otimes n} \to \mathcal{A}$. Here $n = n_1 + \ldots + n_k$,*

$$\sigma(a_1,\ldots,a_n) := \omega(\omega_1(a_1,\ldots,a_{n_1}),\ldots,\omega_k(a_s,\ldots,a_n))$$

and $s = n - n_k + 1 = n_1 + \ldots + n_{k-1} + 1$.

Proposition 2.4 *Let $\omega : \mathcal{A}^{\otimes(n+1)} \to \mathcal{A}$ be an $(n+1)$-linear map and let $\mu_i : \mathsf{g}\otimes\mathsf{g} \to \mathsf{g}$ be the bilinear map given by*

$$\mu_i(a_1\otimes\ldots\otimes a_n, b_1\otimes\ldots\otimes b_n) = a_1\otimes\cdots\otimes\omega(a_i, b_1\otimes\ldots\otimes b_n)\otimes\cdots\otimes a_n.$$

Here $\mathsf{g} = \mathcal{A}^{\otimes n}$ and $1 \le i \le n$. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is an ω-derivation and $\varphi : \mathsf{g} \to \mathsf{g}$ is given by

$$\varphi(a_1\otimes\ldots\otimes a_n) = \sum_{j=1}^n a_1\otimes\ldots\otimes f(a_j)\otimes\cdots\otimes a_n.$$

Then φ is a derivation with respect to μ_i, for any $1 \le i \le n$.

Proof. One has

$$\varphi(\mu_i(a_1\otimes\ldots\otimes a_n, b_1\otimes\ldots\otimes b_n)) = \varphi(a_1\otimes\cdots\otimes\omega(a_i,b_1\otimes\ldots\otimes b_n)\otimes\cdots\otimes a_n).$$

Since f is an ω-derivation, we see that this expression is equal to

$$f(a_1)\otimes\cdots\otimes\omega(a_i,b_1\otimes\ldots\otimes b_n)\otimes\cdots\otimes a_n + \cdots +$$

$$+a_1\otimes\cdots\otimes\omega(f(a_i),b_1\otimes\ldots\otimes b_n)\otimes\cdots\otimes a_n +$$

$$+a_1 \otimes \cdots \otimes \omega(a_i, f(b_1) \otimes \ldots \otimes b_n) \otimes \cdots \otimes a_n + \cdots +$$
$$+a_1 \otimes \cdots \otimes \omega(a_i, b_1 \otimes \ldots \otimes f(b_n)) \otimes \cdots \otimes a_n + \cdots +$$
$$+a_1 \otimes \cdots \otimes \omega(a_i, b_1 \otimes \ldots \otimes b_n) \otimes \cdots \otimes f(a_n).$$

On the other hand the expression

$$\mu_i(\varphi(a_1 \otimes \ldots \otimes a_n), b_1 \otimes \ldots \otimes b_n) + \mu_i(a_1 \otimes \ldots \otimes a_n, \varphi(b_1 \otimes \ldots \otimes b_n)),$$

is clearly equal to the previous expression thanks to the definition of μ_i and φ. This proves that φ is a derivation with respect to μ_i.

3. Leibniz n-algebras

A Leibniz algebra of order n, or simply *a Leibniz n-algebra*, is a vector space $\mathcal{L}$ equipped with an n-linear operation $[-,\ldots,-] : \mathcal{L}^{\otimes n} \to \mathcal{L}$ such that for all $x_1,\ldots,x_{n-1}$ the map $ad(x_1,\ldots,x_{n-1}) : \mathcal{L} \to \mathcal{L}$ given by $ad(x_1,\cdots,x_{n-1})(x) = [x, x_1,\ldots,x_{n-1}]$ is a derivation with respect to $[-,\ldots,-]$. This means that the following *Leibniz n-identity* holds:

$$[[x_1,x_2,\ldots,x_n],y_1,y_2,\ldots,y_{n-1}] = \sum_{i=1}^{n} [x_1,\ldots,x_{i-1},[x_i,y_1,y_2,\ldots,y_{n-1}],x_{i+1},\cdots,x_n]. \tag{3.1}$$

We let ${}_n\mathbf{Lb}$ be the category of Leibniz n-algebras. Let us observe that for $n = 2$ the identity (3.1) is equivalent to (1.1). So a Leibniz 2-algebra is simply a Leibniz algebra in the sense of [4], and so Leibniz 2-algebras are called just Leibniz algebras, and we use $\mathbf{Lb}$ instead of ${}_2\mathbf{Lb}$.

Clearly a Lie algebra is a Leibniz algebra such that $[x,x] = 0$ holds. Similarly for $n \geq 3$ an *n-Lie* or an *n-Nambu-Lie algebra* is a Leibniz n-algebra such that $[x_1,\ldots,x_i,x_{i+1},\ldots,x_n] = 0$ as soon as $x_i = x_{i+1}$ for $1 \leq i \leq n-1$. Such algebras appear in the so called *Nambu mechanics* and there exists several interesting papers about them (see [1], [2] and references given there).

Another big class of Leibniz 3-algebras which were considered in the literature are the so called *Lie triple systems*. Let us recall that a Lie triple system [3] is a vector space equipped with a bracket $[-,-,-]$ that satisfies the same identity (3.1) and, instead of skew-symmetry, satisfies the conditions

$$[x,y,z] + [y,z,x] + [z,x,y] = 0$$

and

$$[x, y, y] = 0.$$

Proposition 3.2 *Let* g *be a Leibniz algebra. Then* g *is also a Leibniz* $(n+1)$*-algebra with respect to the operation* $\omega : \mathsf{g}^{\otimes(n+1)} \to \mathsf{g}$ *given by*

$$\omega(x_0, x_1, \cdots, x_n) := [x_0, [x_1, \cdots [x_{n-1}, x_n]].$$

Proof. From the definition of Leibniz algebra we know that

$$ad(x) = [-, x] : \mathsf{g} \to \mathsf{g}$$

is a derivation with respect to 2-bracket. By Proposition 2.2 we know that it is also a derivation with respect to ω. Since for all $x_1, x_2, \cdots, x_n \in \mathsf{g}$ one has $ad(x_1, x_2, \cdots, x_n) = [x_1, \cdots [x_{n-1}, x_n] \cdots]$ the Proposition follows. Here $ad(x_1, x_2, \cdots, x_n) = \omega(-, x_1, x_2, \cdots, x_n) : \mathsf{g} \to \mathsf{g}$.

Proposition 3.2 shows that there exists a "forgetful" functor

$$\mathbf{U}_n : \mathbf{Lb} \to \ {}_n\mathbf{Lb}.$$

Here are more examples of Leibniz 3-algebras.

Examples 3.3 i) Let g be a Leibniz algebra with involution σ. This means that σ is an automorphism of g and $\sigma^2 = id$. Then

$$\mathcal{L} := \{x \in \mathsf{g} \mid x + \sigma(x) = 0\}$$

is a Leibniz 3-algebra with respect to the bracket

$$[x, y, z] := [x, [y, z]].$$

ii) Let V be a 4-dimensional vector space with basis i, j, k, l. Then we define $[x, y, z] := det(A)$, where A is the following matrix

$$\begin{pmatrix} i & j & k & l \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}$$

One sees that this gives rise to a Leibniz 3-algebra. Moreover it is a Nambu-Lie algebra. Here $x = x_1 i + x_2 j + x_3 k + x_4 l$ and so on. One easily generalizes this example to obtain an n-Nambu-Lie algebra starting with an

$(n+1)$-dimensional vector space. This example was a starting point for investigating n-Lie (or Nambu-Lie) algebras.

Let $\mathcal{L}$ be a Leibniz n-algebra. Thanks to Proposition 2.1 i) we know that

$$\mathcal{D}er(\mathcal{L}) = \{f : \mathcal{L} \to \mathcal{L} \mid \ f \text{ is a derivation}\}$$

is a Lie algebra.

Proposition 3.4. *Let $\mathcal{L}$ be a Leibniz $(n+1)$-algebra. Then $\mathcal{D}_n(\mathcal{L}) = \mathcal{L}^{\otimes n}$ is a Leibniz algebra with respect to the bracket*

$$[a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n] := \sum_{i=1}^{n} a_1 \otimes \cdots \otimes [a_i, b_1, \cdots, b_n] \otimes \cdots \otimes a_n$$

Moreover

$$ad : \mathcal{L}^{\otimes n} \to \mathcal{D}er(\mathcal{L}), \quad x_1 \otimes \cdots \otimes x_n \mapsto ad(x_1, x_2, \cdots, x_n)$$

is a homomorphism of Leibniz algebras.

Proof. Fix $x_1, \cdots, x_n \in \mathcal{L}$. We have to prove that

$$\varphi : \mathcal{D}_n(\mathcal{L}) \to \mathcal{D}_n(\mathcal{L})$$

given by

$$\varphi(a_1 \otimes \ldots \otimes a_n) = \sum_{j=1}^{n} \ a_1 \otimes \ldots \otimes f(a_j) \otimes \cdots \otimes a_n,$$

is a derivation with respect to $[-,-]$. Here $f = [-, x_1, \cdots, x_n] : \mathcal{L} \to \mathcal{L}$. Thanks to Proposition 2.4 we know that φ is a derivation with respect of all $\mu_i, \ 1 \leq i \leq n$, where

$$\mu_i(a_1 \otimes \ldots \otimes a_n, b_1 \otimes \ldots \otimes b_n) = a_1 \otimes \cdots \otimes \omega(a_i, b_1 \otimes \ldots \otimes b_n) \otimes \cdots \otimes a_n.$$

Then φ is also a derivation with respect to $[-,-] = \sum_{i=1}^{n} \mu_i$ thanks to Proposition 2.1 ii) and the first part of the Proposition follows. Let us show that ad is a homomorphism of Leibniz algebras. Indeed, one sees that

$$ad([a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n])(a) = \sum_{i=1}^{n} [a, [a_1, \cdots, [a_i, b_1, \cdots, b_n], \cdots, a_n]].$$

On the other hand

$$[ad(a_1 \otimes \cdots \otimes a_n), ad(b_1 \otimes \cdots \otimes b_n)](a) =$$
$$= ad(a_1 \otimes \cdots \otimes a_n) ad(b_1 \otimes \cdots \otimes b_n)(a) - ad(b_1 \otimes \cdots \otimes b_n) ad(a_1 \otimes \cdots \otimes a_n)(a)$$
$$= [[a, [b_1, \cdots, b_n]], a_1 \otimes \cdots \otimes a_n] - [[a, a_1 \otimes \cdots \otimes a_n], b_1, \cdots, b_n].$$

Therefore (3.1) shows that

$$ad([a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n]) = [ad(a_1 \otimes \cdots \otimes a_n), ad(b_1 \otimes \cdots \otimes b_n)].$$

Hence $ad : \mathcal{D}_n\mathcal{L} \to \mathcal{D}er(\mathcal{L})$ is a homomorphism of Leibniz algebras.

Remark 3.5 One can prove that if $\mathcal{A}$ is a Leibniz $(kn+1)$-algebra, then $\mathcal{A}^{\otimes k}$ is a Leibniz $(n+1)$-algebra with respect to the following bracket

$$[x_{01} \otimes x_{02} \ldots \otimes x_{0k},\ \ldots\ , x_{n1} \otimes x_{n2} \ldots \otimes x_{nk}] :=$$
$$[x_{01}, x_{11}, \ldots, x_{1k}, \ldots, x_{n1},\ \ldots\ , x_{nk}] \otimes x_{02} \otimes \ldots \otimes x_{0k} +$$
$$\cdots + x_{01} \otimes \ldots \otimes x_{0k-1} \otimes [x_{0k}, x_{11}, \ldots, x_{nk}].$$

By Proposition 3.4 the map $\mathcal{L} \mapsto \mathcal{D}_n(\mathcal{L})$ from Leibniz $(n+1)$-algebras to Leibniz algebras is a functor that we denote by $\mathcal{D}_n$. More generally, by Remark 3.5, there exist functors $\mathcal{D}^n_{kn} :\ _{kn+1}\mathbf{Lb} \to\ _{n+1}\mathbf{Lb}$ (so $\mathcal{D}_n = \mathcal{D}^1_n$) and we have $\mathcal{D}^1_n \circ \mathcal{D}^n_{kn} = \mathcal{D}^1_{kn}$.

4. The main theorem

The goal of this section is to prove that the functor $\mathcal{D}_n :\ _{n+1}\mathbf{Lb} \to \mathbf{Lb}$ sends free objects to free objects. For more specific statements see Theorem 4.4 and Theorem 4.8 below. Since $\mathcal{D}_1$ is nothing but the identity functor, we have to consider the case $n \geq 2$. To avoid long formulas we will first restrict ourself to the case $n = 2$ and, second, we indicate how to modify the argument for $n \geq 3$.

Let us recall (see [8]) that a *magma* $\mathcal{M}$ is a set together with a map (binary operation)

$$\mathcal{M} \times \mathcal{M} \to \mathcal{M}\ ,\ (x, y) \mapsto x \star y.$$

Let Y be the free magma with one generator e. We recall from [8] the construction of Y. First one defines the sequence of sets $(Y_m)_{m \geq 1}$ as follows:

$$Y_1 = \{e\},\quad Y_m = \coprod_{p+q=m} Y_p \times Y_q,\ (m \geq 2;\ p, q \geq 1).$$

We let Y be the disjoint union

$$Y = \coprod_{m \geq 1} Y_m.$$

One defines $\star : Y \times Y \to Y$ by means of

$$Y_p \times Y_q \to Y_{p+q} \subseteq Y.$$

Then Y is a magma, which is freely generated by e. Let C_m be the number of elements of Y_{m+1}. Clearly $C_0 = 1, C_1 = 1$ and

$$C_{m+1} = \sum_{i+j=m} C_i C_j.$$

Hence the function $f(t) = \sum_{m=0}^{\infty} C_m t^m$ satisfies the functional equation

$$f(t) - 1 = tf^2(t). \tag{4.1}$$

Of course this equation is well known, as well as the fact that C_m is equal to the Catalan number, that is

$$C_m = \frac{(2m)!}{m!(m+1)!}.$$

So one has

$$C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14,\ C_5 = 42,\ C_6 = 132, \ldots$$

If $\omega \in Y_m$, then we say that ω is of length m and we write $l(\omega) = m$. Clearly if $\omega \in Y$, then $\omega = e$ or $\omega = \omega_1 \star \omega_2$, with unique $(\omega_1, \omega_2) \in Y \times Y$. Moreover $l(\omega) = l(\omega_1) + l(\omega_2)$. Recall that the elements of Y_n can be interpreted as planar binary trees with n leaves. Under this interpretation the operation $\star$ is simply the grafting operation (join the roots to a new vertex and add a new root).

The following proposition is the analogue for Leibniz 3-algebras of Lemma 1.3 in [5] concerning Leibniz algebras.

Proposition 4.2 *Let $K[Y]$ be the vector space spanned by Y. Then there exists a unique structure of Leibniz 3-algebra on $K[Y]$ such that*

$$[\omega_1, \omega_2, e] = \omega_1 \star \omega_2.$$

Moreover $K[Y]$ with this structure is a free Leibniz 3-algebra generated by e.

Proof. We use the method devised in [5] for the case of Leibniz algebras. Let us observe that (3.1) for Leibniz 3-algebras is equivalent to

$$[a,b,[c,x,y]] = [[a,b,c],x,y] - [[a,x,y],b,c] - [a,[b,x,y],c]. \tag{4.2.1}$$

The 3-bracket $[\omega_1,\omega_2,\omega_3]$ has been already defined for $\omega_3 = e$. If $\omega_3 \neq e$, then it is of the form $\omega \star \omega'$ for some elements ω and ω'. Hence

$$[\omega_1,\omega_2,\omega_3] = [\omega_1,\omega_2,\omega\star\omega'] = [\omega_1,\omega_2,[\omega,\omega',e]],$$

and one can use (4.2.1) to rewrite it with 3-brackets whose last variable is either ω or ω'. Since $l(\omega)$ and $l(\omega')$ are less than $l(\omega_3)$, we get, by recursivity, the element $[\omega_1,\omega_2,\omega_3]$ as a unique algebraic sum of elements in Y.

We now have to prove that, with this definition, the 3-bracket satisfies the Leibniz 3-identity (4.2.1). Clearly it holds when $y = e$, since it is precisely this formula which was used to compute the left part. So we can work by induction with respect to $l(y)$. If $l(y) \geq 2$ then $y = y_1 \star y_2$ and therefore

$$\begin{gathered}
[a,b,[c,x,y]] = [a,b,[c,x,[y_1,y_2,e]]] = \\
[a,b,[[c,x,y_1],y_2,e]] - [a,b,[[c,y_2,e],x,y_1]] - [a,b,[c,[x,y_2,e],y_1]] \\
= [[a,b,[c,x,y_1]],y_2,e] - [[a,y_2,e],b,[c,x,y_1]] - [a,[b,y_2,e],[c,x,y_1]] - \\
[[a,b,[c,y_2,e]],x,y_1] + [[a,x,y_1],b,[c,y_2,e]] + [a,[b,x,y_1],[c,y_2,e]] - \\
[[[a,b,c],[x,y_2,e],y_1] + [[a,[x,y_2,e],y_1],b,c] - [a,[b,[x,y_2,c],y_1],c] = \\
[[a,b,c],x,y_1],y_2,e] - [[[a,x,y_1],b,c],y_2,e] - [[a,[b,x,y_1],c],y_2,e] \\
-[[[a,y_2,e],b,c],x,y_1] + [[[a,y_2,e],x,y_1],b,c] + [[a,y_2,e],[b,x,y_1],c] \\
-[[a,[b,y_2,e],c],x,y_1] + [[a,x,y_1],[b,y_2,e],c] + [a,[[b,y_2,e],x,y_1],c] \\
-[[[a,b,c],y_2,e],x,y_1] + [[[a,y_2,e],b,c],x,y_1] + [[a,[b,y_2,e],c],x,y_1] \\
+[[[a,x,y_1],b,c],y_2,e] - [[[a,x,y_1],y_2,e],b,c] - [[a,x,y_1],[b,y_2,e],c] \\
+[[a,[b,x,y_1],c],y_2,e] - [[a,y_2,e],[b,x,y_1],c] - [a,[[b,x,y_1],y_2,e],c] \\
-[[a,b,c],[x,y_2,e],y_1] + [[a,[x,y_2,e],y_1],b,c] + [a,[b,[x,y_2,e],y_1],c].
\end{gathered}$$

One sees that 2^{nd} and 13^{th}, as well as 3^{rd} and 16^{th}, 4^{th} and 11^{th}, 6^{th} and 17^{th}, 7^{th} and 12^{th}, 8^{th} and 15^{th} terms cancel. Hence we have

$$[a, b, [c, x, y]] =$$

$$[[[a, b, c], x, y_1], y_2, e] + [[[a, y_2, e], x, y_1], b, c] + [a, [[b, y_2, e], x, y_1], c]$$
$$-[[[a, b, c], y_2, e], x, y_1] - [[[a, x, y_1], y_2, e], b, c] - [a, [[b, x, y_1], y_2, e], c]$$
$$-[[a, b, c], [x, y_2, e], y_1] + [[a, [x, y_2, e], y_1], b, c] + [a, [b, [x, y_2, e], y_1], c].$$

On the other hand we have

$$[[a, b, c], x, y] =$$

$$[[[a, b, c], x, y_1], y_2, e] - [[[a, b, c], y_2, e], x, y_1] - [[a, b, c], [x, y_2, e], y_1].$$

Similarly

$$-[[a, x, y], b, c] =$$

$$-[[[a, x, y_1], y_2, e], b, c] + [[[a, y_2, e], x, y_1], b, c] + [[a, [x, y_2, e], y_1], b, c]$$

and

$$-[a, [b, x, y], c] =$$

$$-[a, [[b, x, y_1], y_2, e], c] + [a, [[b, y_2, e], x, y_1], c] + [a, [b, [x, y_2, e], y_1], c].$$

One checks that after substitution in (3.1) all terms cancel and therefore $K[Y]$ has a well defined structure of Leibniz 3-algebra. If $\mathcal{L}$ is any Leibniz 3-algebra and $x \in \mathcal{L}$, then by induction one can check that there exists a unique homomorphism

$$f : K[Y] \to \mathcal{L}$$

such that $f(e) = x$ and Proposition 4.2 is proved.

Now we can formulate the following

Theorem 4.3. *The vector space spanned by the set* $\bar{Y} = Y - \{e\}$ *has a unique Leibniz algebra structure such that*

$$[x \star y, z \star e] = (x \star z) \star y + x \star (y \star z).$$

It is a free Leibniz algebra over the set $Y' = \{x \star e \mid x \in Y\} \subset \bar{Y}$. *Moreover one has isomorphisms of Leibniz algebras*

$$\mathcal{D}_2(K[Y]) \cong K[\bar{Y}] \cong T(K[Y']).$$

Proof. 1^{st} *Step: Uniqueness.* Assume that such a Leibniz algebra structure exists. Then $ad(u) = [-, u]$ is uniquely determined when $u = x \star e$. We will prove by induction on $l(q)$ that $ad(u)$ is uniquely determined when $u = x \star q$. If $l(q) > 1$, then $q = y \star z$ and by assumption

$$u = [x \star y, z \star e] - (x \star z) \star y.$$

Therefore

$$ad(u) = -ad((x \star z) \star y) + ad[x \star y, z \star e] =$$

$$= -ad((x \star z) \star y) + ad(z \star e) \cdot ad(x \star y) - ad(x \star y) \cdot ad(z \star e)$$

and by induction assumption $ad(u)$ is uniquely determined.

2^{nd} *Step*: *Bijection.* There is a linear isomorphism $K[Y] \otimes K[Y] \cong K[\bar{Y}]$, which is given by

$$(x, y) \mapsto x \star y, \text{ for } x, y \in Y.$$

It yields indeed a bijection because

$$K[Y] \otimes K[Y] \cong \bigoplus_{p,q \geq 1} K[Y_p] \otimes K[Y_q] \cong \bigoplus_{p,q \geq 1} K[Y_p \times Y_q]$$

$$\cong \bigoplus_{m \geq 2} \bigoplus_{p+q=m} K[Y_p \times Y_q] \cong \bigoplus_{m \geq 2} K[Y_m] = K[\bar{Y}].$$

3^{rd} *Step: Algebra isomorphism* $\mathcal{D}_2(K[Y]) \cong K[\bar{Y}]$. Let us consider $K[\bar{Y}]$ as a Leibniz algebra induced by the linear isomorphism from Step 2. Since $x \star y$ and $z \star e$ are the images of $x \otimes y$ and $z \otimes e \in K[Y] \otimes K[Y]$ under the isomorphism of Step 2, we have to show that, in this algebra, the following identity

$$[x \star y, z \star e] = (x \star z) \star y + x \star (y \star z)$$

holds. By definition of the functor $\mathcal{D}_2$ one has

$$[x \otimes y, z \otimes e] = [x, z, e] \otimes y + x \otimes [y, z, e] = (x \star z) \otimes y + x \otimes (y \star z)$$

and this element goes to $(x \star z) \star y + x \star (y \star z)$ under the isomorphism of Step 2. This proves also the existence part of the Theorem.

4^{th} *Step: Y' generates $K[\bar{Y}]$.* Indeed let X be the subalgebra of $K[\bar{Y}]$ generated by Y'. We have to prove that $\bar{Y} \subset X$. Let $x \star y$ be an element in

$\bar{Y}$. We will show by induction on $l(y)$ that $x \star y \in X$. When $l(y) = 1$, then $x \star y = x \star e \in Y' \subset X$. If $l(y) > 1$ then $y = y_1 \star z$ and by the assumption

$$x \star y = -(x \star z) \star y_1 + [x \star y_1, z \star e].$$

But, by the induction assumption, one has $(x \star z) \star y_1, x \star y_1, z \star e \in X$. Therefore $x \star y \in X$ as well.

5^{th} *Step: $K[\bar{Y}]$ as a graded Leibniz algebra.* For $x \in \bar{Y}$ we let $d(x)$ to be $l(x) - 1$. Then $K[\bar{Y}]$ can be considered as a graded vector space by declaring that the degree of an element $x \in \bar{Y}$ is $d(x)$. We claim that under this grading $K[\bar{Y}]$ is a graded Leibniz algebra, that is, if $d(x) = k$ and $d(y) = m$, then $[x, y]$ is a linear combination of elements of degree $k + m$. The claim is clear when $y = x' \star y'$ and $l(y') = 1$ and it can be proved by the same induction arguments as in Step 1.

6^{th} *Step: Y' freely generates $K[\bar{Y}]$.* Let us recall from [5] that for a vector space U the free Leibniz algebra generated by U is the unique Leibniz algebra structure on

$$\bar{T}(U) = \bigoplus_{m \geq 1} U^{\otimes m}$$

such that for any $u \in U$ one has

$$[x, u] = x \otimes u,\ x \in \bar{T}(U).$$

Take $U = K[Y']$. Then we obtain the natural epimorphism

$$\varphi : \bar{T}(K[Y']) \to K[\bar{Y}].$$

We have to show that φ is injective. The vector space $K[Y']$ is a graded subspace of $K[\bar{Y}]$. Therefore $\bar{T}(K[Y'])$ is also graded and φ is a morphism of graded Leibniz algebras. Since each component is of finite dimension, it is enough to show that the degree p part of $\bar{T}(K[Y'])$ and $K[\bar{Y}]$ have the same dimension.

The dimension of the degree p part of $K[\bar{Y}]$ is equal to C_p (Catalan number), while the dimension of the degree p part of $\bar{T}(K[Y'])$ is equal to the coefficient of t^p in the expansion of

$$g(t) = \sum_{m=1}^{\infty} (\sum_{k=1}^{\infty} C_{k-1} t^k)^m,$$

because the degree m part of $K[Y']$ is of dimension C_{m-1}. Hence we have to prove that $g(t) = f(t) - 1$. But

$$g(t) = \sum_{m=1}^{\infty} (tf(t))^m = \frac{tf(t)}{1 - tf(t)}$$

and it is equal to $f(t) - 1$, thanks to (4.1).

Let us now show the following parametrized version of Theorem 4.3.

Theorem 4.4. *Let V be a vector space, and put*

$$F(V) := \bigoplus_{m \geq 1} K[Y_m] \otimes V^{\otimes 2m-1}.$$

(i) There exists a unique Leibniz 3-algebra structure on $F(V)$ such that

$$[\omega_1 \otimes x_1, \omega_2 \otimes x_2, e \otimes x] = (\omega_1 \star \omega_2) \otimes x_1 \otimes x_2 \otimes x,$$

where $\omega_1 \in K[Y_p], x_1 \in V^{\otimes(2p-1)}, \omega_2 \in K[Y_q], x_2 \in V^{\otimes(2q-1)}$ and $x \in V$.

(ii) Equipped with this structure $F(V)$ is a free Leibniz 3-algebra generated by V.

(iii) The Leibniz algebra $\mathcal{D}_2(F(V))$ is isomorphic to the free Leibniz algebra generated by the vector space

$$E = \bigoplus_{m \geq 1} K[Y_m] \otimes V^{\otimes 2m}.$$

Proof. The proof is similar to the proof of Theorem 4.2. At the end, in order to show that the vector spaces $F(V)$ and $T(E)$ have the same dimension we use the following identity of formal power series:

$$(x + x^3 + 2x^5 + 5x^7 + 14x^9 + \cdots)^2 = \sum_{m=1}^{\infty} (x^2 + x^4 + 2x^6 + 5x^8 + 14x^{10} + \cdots)^m,$$

which is an immediate consequence of the functional equation (4.1).

Let us now state the results for $(n+1)$-Leibniz algebras. Since the proofs follow the same pattern as in the case $n + 1 = 2 + 1$, we mention only the main modifications. By definition an n-*magma* is a set $\mathcal{M}$ together with a map (n-ary operation)

$$(-, \cdots, -) : \underbrace{\mathcal{M} \times \cdots \times \mathcal{M}}_{n \text{ copies}} \to \mathcal{M}\ .$$

Let Z be the free n-magma with one generator e. It can be described as follows. The sequence of sets $(Z_m)_{m\geq 1}$ is given by:

$$Z_1 = \{e\}, \quad Z_m = \coprod_{p_1+\cdots+p_n=m} Z_{p_1} \times \cdots \times Z_{p_n}, \ (m \geq 2;\ p_i \geq 1).$$

Observe that $Z_m = \emptyset$ unless $m = (n-1)k+1$ for some $k \geq 0$. We let Z be the disjoint union

$$Z = \coprod_{m\geq 1} Z_m.$$

One defines $(-, \cdots, -) : Z \times \cdots \times Z \to Z$ by means of

$$Z_{(n-1)k_1+1} \times \cdots \times Z_{(n-1)k_n+1} \to Z_{(n-1)(k_1+\cdots+k_n+1)+1} \subseteq Z.$$

Then Z is a n-magma, which is freely generated by e. Let D_k be the number of elements of $Z_{(n-1)k+1}$. Clearly $D_0 = 1, D_1 = 1$ and

$$D_{k+1} = \sum_{k_1+\cdots+k_n=k} D_{k_1} \cdots D_{k_n}.$$

Hence the function $f(t) = \sum_{k=0}^{\infty} D_k t^k$ satisfies the functional equation

$$f(t) - 1 = tf(t)^n. \tag{4.5}$$

If $\omega \in Z_{(n-1)k+1}$, then we say that ω is of length k and we write $l(\omega) = k$. Clearly if $\omega \in Z$, then $\omega = e$ or $\omega = (\omega_1, \cdots, \omega_n)$, for some unique elements $\omega_1, \cdots, \omega_n$ in Z. Moreover $l(\omega) = l(\omega_1) + \cdots + l(\omega_n) + 1$. Recall that the elements of Z_m can be interpreted as n-ary planar trees, that is each vertex has one root and n leaves. Under this interpretation the operation $(-, \cdots, -)$ is simply the grafting operation. Observe that the number of vertices (resp. edges) of a tree in $Z_{(n-1)k+1}$ is k (resp. $kn+1$).

Proposition 4.6 *Let $K[Z]$ be the vector space spanned by Z. Then there exists a unique structure of $(n+1)$-Leibniz algebra on $K[Z]$ such that*

$$[\omega_1, \cdots, \omega_n, e] = (\omega_1, \cdots, \omega_n).$$

Moreover $K[Z]$ with this structure is a free $(n+1)$-Leibniz algebra generated by e.

Theorem 4.7. *The vector space spanned by the set $\bar{Z} = Z - \{e\}$ has a unique Leibniz algebra structure such that*

$$[(\omega_1, \cdots, \omega_n), (\omega'_1, \cdots, \omega'_{n-1}, e)] = \sum_{i=1}^{n} (\omega_1, \cdots, (\omega_i, \omega'_1, \cdots, \omega'_{n-1}), \cdots, \omega_n).$$

As a Leibniz algebra it is free over the $Z' = \{(\omega_1, \cdots, \omega_{n-1}, e) \mid \omega_i \in Z\}$. *Moreover one has isomorphisms of Leibniz algebras:*

$$\mathcal{D}_n(K[Z]) \cong K[\bar{Z}] \cong T(K[Z']).$$

Proof. Let us just mention the computation of the dimensions of the vector spaces. Let E_k be the number of n-ary trees with k vertices which are of the form $(\omega_1, \cdots, \omega_{n-1}, e)$. One has

$$E_k = \sum_{k_1 + \cdots + k_{n-1} + 1 = k} D_{k_1} \times \cdots \times D_{k_{n-1}}.$$

Hence we get $\sum E_k t^k = t \sum D_{k_1} t^{k_1} \cdots D_{k_{n-1}} t^{k_{n-1}} = t f(t)^{n-1}$. Therefore the generating series for $T(K[Z'])$ is $\frac{tf(t)^{n-1}}{1 - tf(t)^{n-1}}$. By the functional equation (4.6) this is equal to $f(t) - 1$, which is the generating series for $K[\bar{Z}]$.

Theorem 4.8. *Let* V *be a vector space, and put*

$$F(V) := \bigoplus_{k \geq 0} K[Z_{(n-1)k+1}] \otimes V^{\otimes nk+1}.$$

(i) There exists a unique $(n+1)$*-Leibniz algebra structure on* $F(V)$ *such that*

$$[\omega_1 \otimes x_1, \cdots, \omega_n \otimes x_n, e \otimes x] = (\omega_1, \cdots, \omega_n) \otimes x_1 \otimes \cdots \otimes x_n \otimes x,$$

where $\omega_i \in K[Z_{(n-1)k_i+1}], x_i \in V^{\otimes nk_i+1}$ *and* $x \in V$.

(ii) Equipped with this structure $F(V)$ *is a free* $(n+1)$*-Leibniz algebra generated by* V.

(iii) The Leibniz algebra $\mathcal{D}_n(F(V))$ *is isomorphic to the free Leibniz algebra generated by the vector space*

$$E = \bigoplus_{k \geq 0} K[Z_{(n-1)k+1}] \otimes V^{\otimes n(k+1)}.$$

Remark 4.9 By the same kind of argument one can show that the functor $\mathcal{D}^k_{kn} :_{kn+1} \mathbf{Lb} \to_{k+1} \mathbf{Lb}$ sends free objects to free objects.

5. Cohomology of Leibniz n-algebras

An *abelian extension of Leibniz n-algebras*

$$0 \to M \to \mathcal{K} \to \mathcal{L} \to 0 \tag{5.1}$$

is an exact sequence of Leibniz n-algebras such that $[a_1, \cdots, a_n] = 0$ as soon as $a_i \in M$ and $a_j \in M$ for some $1 \leq i \neq j \leq n$. Here $a_1, \cdots, a_n \in \mathcal{K}$. Clearly then M is an *abelian* Leibniz n-algebra, that is the bracket vanishes on M. Let us observe that the converse is true only for $n = 2$.

If (5.1) is an abelian extension of Leibniz n-algebras, then M is equipped with n actions

$$[-, \cdots, -] : \mathcal{L}^{\otimes i} \otimes M \otimes \mathcal{L}^{\otimes n-1-i} \to M, \ 0 \leq i \leq n-1$$

satisfying $(2n-1)$ equations, which are obtained from (3.1) by letting exactly one of the variables $x_1, \cdots, x_n, y_1, \cdots y_{n-1}$ be in M and all the others in $\mathcal{L}$.

By definition a *representation* of the Leibniz n-algebra $\mathcal{L}$ is a vector space M equipped with n actions of $[-, \cdots, -] : \mathcal{L}^{\otimes i} \otimes M \otimes \mathcal{L}^{\otimes n-1-i} \to M$ satisfying these $(2n-1)$ axioms. For example $\mathcal{L}$ is a representation of $\mathcal{L}$. The notion of representation of a Leibniz n-algebra for $n = 2$ coincides with the corresponding notion given in [5]. Let $\mathcal{L}$ be a Leibniz n-algebra and let M be a representation of $\mathcal{L}$. Let

$$0 \to M \to \mathcal{K} \to \mathcal{L} \to 0 \tag{$\mathcal{K}$}$$

be an abelian extension, such that the induced structure of representation of $\mathcal{L}$ on M induced by the extension is the prescribed one. If this condition holds, then we say that we have an abelian extension of $\mathcal{L}$ by M. Two such extensions $(\mathcal{K})$ and $(\mathcal{K}')$ are isomorphic when there exists a Leibniz n-algebra map from $\mathcal{K}$ to $\mathcal{K}'$ which is compatible with the identity on M and on $\mathcal{L}$. One denotes by $\mathrm{Ext}(\mathcal{L}, M)$ the set of isomorphism classes of extensions of $\mathcal{L}$ by M.

Let $f : \mathcal{L}^{\otimes n} \to M$ be a linear map. We define an n-bracket on $\mathcal{K} = M \oplus \mathcal{L}$ by

$$[(m_1, x_1), (m_2, x_2) \cdots, (m_n, x_n)] :=$$

$$(\sum_{i=1}^{n} [x_1,\cdots,m_i,\cdots,x_n] + f(x_1,\cdots,x_n), [x_1,\cdots,x_n]).$$

Then $\mathcal{K}$ is a Leibniz n-algebra if and only if

$$f([x_1,\cdots,x_n],y_1,\cdots,y_{n-1}]) + [f(x_1,\cdots,x_n),y_1,\cdots,y_{n-1}] = \tag{5.2}$$

$$\sum_{i=1}^{n} (f(x_1,\cdots,[x_i,y_1\cdots,y_{n-1}],\cdots,x_n] + [x_1,\cdots,f(x_i,y_1\cdots,y_{n-1}),\cdots,x_n])$$

for all $x_1,\cdots,x_n,y_1,\cdots,y_{n-1} \in \mathcal{L}$. If this condition holds, then we obtain an extension

$$0 \to M \to \mathcal{K} \to \ \mathcal{L} \to 0$$

of Leibniz n-algebras. Moreover this extension is split in the category of Leibniz n-algebras if and only if there exists a linear map $g : \mathcal{L} \to M$ such that

$$f(x_1,\cdots,x_n) = \sum_{i=1}^{n} [x_1,\cdots,g(x_i),\cdots,x_n] - g([x_1,\cdots,x_n]). \tag{5.3}$$

An easy consequence of these facts is the following natural bijection :

$$\mathrm{Ext}(\mathcal{L},M) \cong Z(\mathcal{L},M)/B(\mathcal{L},M). \tag{5.4}$$

Here $Z(\mathcal{L},M)$ is the set of all linear maps $f : \mathcal{L}^{\otimes n} \to M$ satisfying (5.2) and $B(\mathcal{L},M)$ is the set of such f which satisfy (5.3) for some k-linear map $g : \mathcal{L} \to M$.

Let $\mathcal{L}$ be a Leibniz n-algebra and let M be a representation of $\mathcal{L}$. A map $f : \mathcal{L} \to M$ is called a *derivation* if

$$f([a_1,\cdots,a_n]) = \sum_{i=1}^{n} [a_1,\cdots,f(a_i),\cdots,a_n].$$

We let $Der(\mathcal{L},M)$ be the vector space of all derivations from $\mathcal{L}$ to M.

The next goal is to construct a cochain complex for Leibniz n-algebras so that the derivations and the elements of Z are cocycles in this complex. It turns out that this problem reduces to the case $n = 2$, that is for Leibniz algebras, which was the subject of the paper [5]. Let us recall the main construction of [5]. Let g be a Leibniz algebra and let M be a representation of g. We let $CL^*(\mathsf{g},M)$ be a cochain complex given by

$$CL^m(\mathsf{g},M) := \mathrm{H}om(\mathsf{g}^{\otimes m},M), \ m \geq 0,$$

where the coboundary operator $d^m : CL^m(\mathsf{g}, M) \to CL^{m+1}(\mathsf{g}, M)$ is defined by

$$(d^m f)(x_1, \cdots, x_{m+1}) := [x_1, f(x_2, \cdots, x_{m+1})] + \sum_{i=2}^{m+1} (-1)^i [f(x_1, \cdots, \hat{x}_i, \cdots, x_{m+1}), x_i] + \sum_{1 \le i < j \le m} (-1)^{j+1} f(x_1, \cdots, x_{i-1}, [x_i, x_j], x_{i+1}, \cdots, \hat{x}_j, \cdots, x_m).$$

According to [5] *cohomology of the Leibniz algebra* g with coefficients in the representation M is defined by

$$HL^*(\mathsf{g}, M) := H^*(CL^*(\mathsf{g}, M), d).$$

In order to generalize this notion to Leibniz n-algebras for $n \geq 3$ we need the following Proposition. Let us recall that if $\mathcal{L}$ is an $(n+1)$-Leibniz algebra, then the Leibniz algebra $\mathcal{D}_n(\mathcal{L})$ was defined in Section 3. Let $\mathcal{L}$ be an $(n+1)$-Leibniz algebra and let M be a representation of $\mathcal{L}$. One defines the maps

$$[-,-] : \mathrm{Hom}(\mathcal{L}, M) \otimes \mathcal{D}_n(\mathcal{L}) \to \mathrm{Hom}(\mathcal{L}, M)$$
$$[-,-] : \mathcal{D}_n(\mathcal{L}) \otimes \mathrm{Hom}(\mathcal{L}, M) \to \mathrm{Hom}(\mathcal{L}, M)$$

by

$$[f, x_1 \otimes \cdots \otimes x_n](x) := [f(x), x_1, \cdots, x_n] - f([x, x_1, \cdots, x_n]),$$

$$[x_1 \otimes \cdots \otimes x_n, f](x) := f([x, x_1, \cdots, x_n]) - [f(x), x_1, \cdots, x_n] - \cdots - [x, x_1, \cdots, f(x_n)].$$

The proof of the next result is a straightforward (but somehow tedious) calculation.

Proposition 5.5 *Let $\mathcal{L}$ be an $(n+1)$-Leibniz algebra and let M be a representation of $\mathcal{L}$. Then the above homomorphisms define a structure of representation of $\mathcal{D}_n(\mathcal{L})$ on* $\mathrm{Hom}(\mathcal{L}, M)$.

Let $\mathcal{L}$ be a Leibniz n-algebra and let M be a representation of $\mathcal{L}$. One defines the cochain complex ${}_nCL^*(\mathcal{L}, M)$ to be $CL^*(\mathcal{D}_{n-1}(\mathcal{L}), \mathrm{Hom}(\mathcal{L}, M))$. We also put

$${}_nHL^*(\mathcal{L}, M) := H^*({}_nCL^*(\mathcal{L}, M))$$

Thus, by definition one has ${}_nHL^*(\mathcal{L}, M) \cong HL^*(\mathcal{D}_{n-1}(\mathcal{L}), \mathrm{Hom}(\mathcal{L}, M))$. Let us observe that for $n = 2$, one has ${}_2CL^m(\mathcal{L}, M) \cong CL^{m+1}(\mathcal{L}, M)$ for all $m \geq 0$. Thus

$$ {}_2HL^m(\mathcal{L}, M) \cong HL^{m+1}(\mathcal{L}, M), \ m \geq 1 $$

and ${}_2HL^0(\mathcal{L}, M) \cong Der(\mathcal{L}, M)$. Comparison of the definitions shows that

$$ {}_nHL^0(\mathcal{L}, M) \cong Der(\mathcal{L}, M) $$

holds for any Leibniz n-algebras $\mathcal{L}$. Similarly one has

$$ Ker\ (d :\ {}_nCL^1(\mathcal{L}, M) \to\ {}_nCL^2(\mathcal{L}, M)) \cong\ Z(\mathcal{L}, M) $$

and therefore

$$ \mathrm{Ext}(\mathcal{L}, M) \cong\ {}_nHL^1(\mathcal{L}, M). \tag{5.6} $$

Proposition 5.7. *Let $\mathcal{L}$ be a free n-Leibniz algebra and let M be a representation of $\mathcal{L}$. Then*

$$ {}_nHL^m(\mathcal{L}, M) = 0, \ m \geq 1. $$

Proof. The main result of Section 4 shows that $\mathcal{D}_{n-1}(\mathcal{L})$ is a free Leibniz algebra. Thanks to Corollary 3.5 of [5] we have $HL^i(\mathcal{D}_{n-1}(\mathcal{L}), -) = 0$ for $i \geq 2$ and thus ${}_nHL^m(\mathcal{L}, -) = 0, \ m \geq 1$.

Let us recall that in [7] Quillen developped the cohomology theory in a very general framework. This theory can be applied to Leibniz n-algebras. It has the following description. Let $\mathcal{L}$ be a Leibniz n-algebra and let M be a representation of $\mathcal{L}$. Then Quillen cohomology of $\mathcal{L}$ with coefficients in M is defined by

$$ H^*_{Quillen}(\mathcal{L}, M) := H^*(Der(P_*, M)). $$

Here $P_* \to \mathcal{L}$ is an augmented simplicial n-Leibniz algebra, such that $P_* \to \mathcal{L}$ is a weak equivalence and each component of P_* is a free Leibniz n-algebra.

Corollary 5.8. *Let $\mathcal{L}$ be a Leibniz n-algebra and let M be a representation of $\mathcal{L}$. Then*

$$ H^*_{Quillen}(\mathcal{L}, M) \cong\ {}_nHL^*(\mathcal{L}, M). $$

Proof. Since $P_* \to \mathcal{L}$ is a weak equivalence, we obtain from the Künneth theorem that ${}_nCL^m(P_*, M) \to {}_nCL^m(\mathcal{L}, M)$ is also a weak equivalence. This fact, together with Proposition 5.7, shows that both spectral sequences for the bicomplex ${}_nCL^*(P_*, M)$ degenerate and give the expected isomorphism.

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