Higher finiteness properties of S-arithmetic groups in the function field case I

by Helmut Behr (Frankfurt a.M.)

It is well known that S-arithmetic subgroups of reductive algebraic groups over number fields have "all" finiteness properties (see [BS 2]). On the contrary there exist many counterexamples in the function field case.
Let F be a finite extension of $\mathbb{F}_q(t)$, G an almost simple algebraic group of F-rank r, 0_S an S-arithmetic subring of F with $\# S = s$, r_v the F_v-rank of G over the completion F_v of F for $v \in S$, and finally Γ a S-arithmetic subgroup of $G(F)$.

We are interested in the following question: Is it true that

$$\Gamma \quad \text{is of type} \quad F_{n-1} \quad \text{but not} \quad F_n \quad \text{iff} \quad r > 0 \quad \text{and} \quad \sum_{v \in S} r_v = n \ ?$$

(For the definition of finiteness properties, cf the introduction of [Ab].)
The answer is yes in the following cases:

(a) $G = SL_2$: see [St 2].

(b) $n = 1$ or 2 (finite generation or finite presentability): see [B 2].

(c) G classical, $0_S = \mathbb{F}_q[t]$ under the assumption that q is big enough compared with r: see [Ab] and [A] for SL_n.

In particular it is not known if the assumption in (c) is necessary. For $r = 0$, in the so-called cocompact case, Γ is of type F_∞ (cf. [BS 2]).

This paper is an attempt to attack this question with some new methods — old in other contexts. First of all, inspired by the work of Serre, Quillen, Stuhler and Grayson (cf. [G1,2]), we use *semi-stability for reduction theory*, and the idea to determine the homotopy type of the boundary of the unstable region by *retraction.*

In this part we only deal with Chevalley groups G and arithmetic rings O_S for $\# S = 1$. The groups $G(F)$ and Γ act on the Bruhat–Tits building $X = X_v$, corresponding to G and F_v; Γ leaves the unstable region X' invariant. X' has a cover whose nerve is given by the spherical Tits building X_0, so it is $(r-1)$-spherical. The retraction to its boundary is not possible as in the number field case, since the geodesic lines are branching (discretely).

Therefore we have to "split up" X' into apartments, thereby constructing a bigger complex $\widetilde{X}'$, which has a cover with nerve $\mathrm{Opp}X_0$, defined by an opposition relation in X_0. This complex was first considered by Charney for $G = GL_n$, by Lehrer and Rylands for classical groups who called it "split building", finally v. Heydebreck showed in the general case that this complex is also $(r-1)$-spherical — so is $\widetilde{X}'$.

$\widetilde{X}'$ can be retracted to its boundary $\widetilde{Y}$, but $\widetilde{Y}$ is not finite $\mathrm{mod}\Gamma$. Thus we have to consider a subcomplex $\widetilde{X}'_\Gamma$, where opposition is defined only with respect to Γ and to show that $\widetilde{X}'_\Gamma$ is a deformation retract of $\widetilde{X}'$. Now we obtain that $\widetilde{Y}_\Gamma$ is finite modulo Γ and can deduce the F_{n-1}-property of Γ. For the negative part, i.e. Γ is not of type F_n, one should come back to filtrations, the method used for the proofs of (a), (b) and (c) above, but for the moment I have no detailed argument. Thus we sketch the proof of the following

Theorem. *The S-arithmetic subgroup $\Gamma = G(O_S)$ of a simply connected almost simple Chevalley group G of rank r is for $s = 1$ of type F_{r-1}.*
Conjecture: Γ is not of type F_r.

I hope that this program will turn out to be useful even in more general situations: For coefficient rings which are defined by more than one prime or, on the other side, for non-split groups.

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1. Notations

Let us denote by

F	a finite extension of the field of rational functions $\mathbb{F}_q(t)$ in t with coefficients in the finite field $\mathbb{F}_q$, $q = p^m$;
$\widehat{F} = F_v$	the completion of F with respect to the valuation v of F;
O and $\widehat{O}$	the valuation rings with respect to v in F or $\widehat{F}$;
G	a simply connected almost simple Chevalley group, defined over F;
r	the F-rank of G, $I = \{1, \dots, r\}$;
T	a maximal (split) F-torus of G;
$\Delta = \{\alpha_i\}_{i \in I}$	a set of simple roots of G with respect to T;
P_{Δ_0}	a parabolic subgroup of G of cotype Δ_0, $\Delta_0 \subseteq \Delta$, which means that $\Delta - \Delta_0$ is a set of simple roots for the semi-simple part of P_{Δ_0}, especially
$B = P_\Delta$	the Borel subgroup, defined by Δ, and
P_α	the maximal parabolic subgroup for $\Delta_0 = \{\alpha\}$.
X	the Bruhat–Tits–building, corresponding to G and v with its simplicial structure and its metric topology;
$A = X_T$	the apartment of X corresponding to T, thus $A \sim \mathbb{R}^r$;
$\{\alpha_i(x)\}_{i \in I}$	the coordinates of $x \in A$ which means by abuse of notation the following: If $x = t \cdot x_0$, x_0 the "origin" of A, $t \in T(\widehat{F})$, then $\alpha_i(x) := -v(\alpha_i(t))$;
X_0	the spherical Tits building of $G(F)$;
Γ	the S-arithmetic subgroup of $G(F)$ for $S = \{v\}$.

2. Reduction Theory and the Unstable Region

We shall use reduction theory for arithmetic groups over function fields in the version described by Harder in [H2], 1.4. He defines

$$\pi(x, P) := \text{vol } (K_x \cap U(\widehat{F}))$$

for a special point $x \in X$, corresponding to a maximal compact subgroup K_x of $G(\widehat{F})$ and a F-parabolic group P and its unipotent radical U; the volume vol comes from the adelic Tamagawa measure. The function

$$d_P(x) := \log_q \pi(x, P)$$

can be extended by linear interpolation to all points x in an apartment $A = X_T$, defined by a maximal split $\widehat{F}$-torus T, contained in P and thereby uniquely for all $x \in X$. We may consider d_P as a *co-distance* with respect to the simplex σ_P, given by P in the spherical building X_∞ at infinity (cf. [Br2], VI.9).
For the action of $T(\widehat{F})$ on X_T via $\operatorname{ad} T$ we have the formula

$$d_P(t \cdot x) = d_P(x) + \log_q |\delta_P(t)|$$

where δ_P is the character "sum of roots in U", which is a multiple of the dominant weight ω_P and the q-logarithm is the negative additive valuation $-v(\delta_P(t))$.
For each Borel group B over F and its set $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots (with respect to a F-torus T), the maximal parabolic groups P_α $(\alpha \in \Delta)$ containing B and their fundamental weights ω_{P_α}, one has

$$\alpha = \sum_{\beta \in \Delta} c_{\alpha,\beta} \omega_{P_\alpha} = \sum_{\beta \in \Delta} c'_{\alpha,\beta} \delta_{P_\alpha}$$

where $c_{\alpha,\beta}$ are the integral coefficients of the Cartan-matrix, such that $c'_{\alpha,\beta} \in \mathbb{Q}$; in particular, $c'_{\alpha,\alpha}$ is positive and $c'_{\alpha,\beta}$ for $\beta \neq \alpha$ is zero or negative (for at most 3 β's). Using these coefficients, Harder defines *numerical invariants*

$$n_\alpha(x, B) := \prod_{\beta \in \Delta} \pi(x, P_\beta).$$

Again we pass to the additive version, setting

$$c_{B,\alpha}(x) := \log_q \left[n_\alpha(x, B)\right]$$

and obtain for each $b \in B(F)$ the relation

$$\begin{aligned} c_{B,\alpha}(b \cdot x) &= c_{B,\alpha}(x) + \log_q |\alpha(b)| \\ &= c_{B,\alpha}(x) - v(\alpha(b)) \end{aligned}$$

$c_{B,\alpha}$ is an affine linear function on the apartment X_T; we define the origin O_B by $c_{B,\alpha}(O_B) = 0$ for all $\alpha \in \Delta$ and by abuse of notation $\alpha(t \cdot O_B) := -v(\alpha(t))$ for $t \in T(F)$, thus we get by linear interpolation a set of affine coordinates
$\{\alpha_1(x), \ldots, \alpha_r(x)\}$ for each point $x \in X_T$.

Now we are able to state the **main theorems of reduction theory** (for Chevalley groups):

(A) There exists a constant C_1 such that for all $x \in X$ there is a F-Borel group B with $c_{B,\alpha}(x) \geq C_1$ for all $\alpha \in \Delta$; then x is called *"reduced with respect to B"*.

(B) There exists a constant $C_2 \geq C_1$, such that for $x \in X$ reduced with respect to B and B', and $c_{B_\alpha}(x) \geq C_2$ for all $\alpha \in \Delta_0 \subseteq \Delta$, $P = P_{\Delta_0} \supseteq B$, it follows $P \supseteq B'$; then x is called *"close to P"*, P is uniquely determined.

(C) There exists a constant $C_3 \geq C_2$, depending on the arithmetic group Γ, such that for $x \in X$, reduced with respect to B and with $c_{B,\alpha}(x) \geq C_3$ for all $\alpha \in \Delta_0 \subseteq \Delta$, we have for the unipotent radical U of the parabolic group $P = P_{\Delta_0} \supseteq B$

$$U(\widehat{F}) = (U(\widehat{F}) \cap K_x)(U(F) \cap \Gamma);$$

x is then called *"very close to P"*.

(D) For each constant $C \geq C_1$ the set

$$X_C := \left\{ x \in X \;\middle|\; \begin{array}{l} c_{B,\alpha}(x) \leq C \text{ for all } \alpha \in \Delta \text{ and all } B \\ \text{for which } x \text{ is reduced with respect to } B \end{array} \right\}$$

is Γ-invariant and X_C/Γ is compact.

(E) The number of Borel subgroups over F of G belongs to finitely many classes under Γ-conjugation (see [B1], 8).

Remark. The constant C_1 can be chosen as $C_1 \leq -2g - 2(h-1)$ where g denotes the genus of F and h is a *"class-number"* (for the precise definition see [H1], 2.2.6). For example, if $\Gamma = SL_n(\mathbb{F}_q[t])$ we may use $C_1 = 0$, but in general C_1 is negative.

We define the cone or sector of points in X_T, reduced with respect to $B \supset T$ by

$$D_{B,T} := \{x \in X_T \,|\, \alpha_i(x) \geq C_1 \text{ for } i = 1, \ldots, r\}$$

Warning: For different Borel groups B and B', containing the same torus T, the origins O_B und $O_{B'}$ must not coincide and therefore the sectors $D_{B,T},\ B \supset T$ do not cover in general the apartment X_T: see example below.

For a F parabolic group P of cotype $\Delta_0 \neq \emptyset$, we denote by X'_P the set of all points $x \in X$ which are close to P:

$$X'_P := \left\{ x \in X \;\middle|\; \begin{array}{l} c_{B,\alpha}(x) \geq C_1 \text{ for all } \alpha \in \Delta \setminus \Delta_0 \\ c_{B,\alpha}(x) \geq C_2 \text{ for all } \alpha \in \Delta_0 \end{array} \quad \text{for all } B \subseteq P \right\}$$

$$\text{or} \quad X'_P \; := \bigcup_{B\subseteq P} D_B := \bigcup_{B\subseteq P} \left(\bigcup_{T\subseteq B} D_{B,T} \cap X'_P \right)$$

and call

$$X' := \bigcup_P X'_P = \bigcup_{P\,\max} X'_P$$

the **unstable region** of X; the name is given in analogy to the description with vector bundles for the group $G = SL_n$ (cf. [G1], 4).

For a F-parabolic group Q let P run over all maximal F-parabolic groups which contain Q; then we have

$$X'_Q = \bigcap_{P\supseteq Q} X'_P \; .$$

We obtain a *polyhedral decomposition* of X', defining

$$X''_Q := \overline{X'_Q \setminus \bigcup_{Q_1 \subsetneqq Q} (X'_Q \cap X'_{Q_1})} \; .$$

In the special case, where $C_1 = 0$, we have in a fixed sector $D_{B,T}$ the following descriptions:

$$\begin{aligned} X'_Q \cap D_{B,T} &= \{x \in D_{B,T} \,|\, \alpha(x) \geq 0 \text{ for all } \alpha \in \Delta,\ \alpha(x) \geq C_2 \text{ for all } \alpha \in \Delta_0\} \\ X''_Q \cap D_{B,T} &= \{x \in D_{B,T} \,|\, 0 \leq \alpha(x) \leq C_2 \text{ for all } \alpha \in \Delta \setminus \Delta_0, \\ &\qquad\qquad \alpha(x) \geq C_2 \text{ for all } \alpha \in \Delta_0\} \end{aligned}$$

where $Q = P_{\Delta_0}$.
In particular for $Q = B$, which means $\Delta_0 = \Delta$, $X''_B \cap D_{B,T}$ is a cone inside $D_{B,T}$, for $Q = P$ maximal, i.e. $\Delta_0 = \{\alpha\}$, we get for $X''_P \cap D_{B,T}$ a cylindric convex set, furthermore infinite prisms etc.

Finally we have $X'_P = \bigcup_{Q\subseteq P} X''_Q$.

Remark. Assume we have an enumeration of the set of simple roots, given by a type function on the vertices of the spherical building X_0, then for $x \in X''_Q$ the set of maximal parabolic subgroups P containing Q defines a chain which generalizes the "canonical filtration" of vector bundles for $G = SL_n$ (cf. [G1]) or respectively lattices in the number field case (cf. [St1] and [G2]).

Above all we are interested in the *boundary* $Y := \partial X'$ of the unstable region, which can be described for a parabolic group Q of cotype $\Delta_0 \neq \emptyset$ as follows:

$$Y_Q := \partial X''_Q := \left\{ x \in X''_Q \;\middle|\; \begin{array}{ll} c_{\beta,\alpha}(x) \geq C_1 & \text{for all } \alpha \in \Delta \setminus \Delta_0 \text{ and all} \\ & B \subseteq Q \\ c_{\beta,\alpha}(x) \geq C_2 & \text{for all } \alpha \in \Delta_0 \text{ and equality for} \\ & \text{at least one } B \subseteq Q \end{array} \right\}$$

$$Y = \partial X' := \bigcup_Q \partial X''_Q \,.$$

In the next step we distinguish *geodesic lines* in X''_Q: A point $x \in X''_Q$ with coordinates $\alpha(x)$ for an appropriate B determines uniquely a boundary point $y \in Y_Q$ by setting $\alpha(y) = \alpha(x)$ for all $\alpha \in \Delta - \Delta_0$ and $\alpha(y) = C_2$ for all $\alpha \in \Delta_0$, the segment $\overline{xy}$ lies on a geodesic. The "geodesic action" on this line in the apartment X_T is given by the torus $T_{\Delta_0} := \{t \in T \mid \alpha(t) = 0$ for all $\alpha \in \Delta - \Delta_0\}$, contained in the radical of $Q = P_{\Delta_0}$, centralizing its semi-simple part. Along these geodesic lines we can define a retraction of X''_Q to its boundary Y_Q, for instance parametrized by the distance function d_Q. Therefore the local definitions fit together for X'_Q, but unfortunately they define no retraction from X'_Q to $\partial X'_Q$ since the geodesic lines are branching into different apartments.

We shall need a further retraction from the sets X'_P to "infinity" along geodesics of "type P_{Δ_0}", given by the action of T_{Δ_0}, see next section.

Example. $\quad G = SL_n,\ \Gamma = SL_n(\mathbb{F}_q[t])$

1. In this case Γ admits a strict simplicial fundamental domain D which is a sector $D_{B,T}$ for a fixed pair $T \subset B$: see [Ab], I.3); this result can also be deduced from reduction theory with Siegel sets. This corresponds to the fact that we can choose $C_1 = 0,\ C_2 = 1$ in Harder's theory for this case. One may then define the polyhedral decomposition locally in D and extend it to X by the action of Γ.

2. In order to show that origins O_B and $O_{B'}$ of different sectors in an apartment must not coincide, we use $n = 3$: Denote by B^+ the upper triangular matrices in SL_3, by $B^- = w\,B^+w^{-1}$ with $w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ the lower triangular matrices, define $B' = g \cdot B^- := g\,B^-g^{-1} = g\,w\,B^+w^{-1}g^{-1}$ with $g = \begin{pmatrix} 1 & 0 & t^{-n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $n \in \mathbb{N}$, such that B^+ and B' are opposite Borel groups,

defining an apartment A. We obtain an equation $gw = \gamma wb$ with $\gamma \in \Gamma,\ b \in B\left(\mathbb{F}_q(t)\right)$, explicitly

$$\begin{pmatrix} t^{-n} & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ t^n & 0 & -1 \end{pmatrix} \begin{pmatrix} t^{-n} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^n \end{pmatrix} \begin{pmatrix} 1 & 0 & t^{-n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute

$$\begin{aligned} c_{B',\alpha'}(O_B) &= c_{gw\cdot B^+, gw(\alpha)}(gw(O_B)) \\ &\quad \text{(since } w(O_B) = O_B \text{ and } g \text{ fixes a half-plane containg } O_B\text{)} \\ &= c_{\gamma w B^+, \gamma w(\alpha)}(\gamma w \cdot b(O_B)) \\ &= c_{B^+,\alpha}(b(O_B)) \\ &\quad \text{(by left-invariance of the measure)} \\ &= c_{B^+,\alpha}(O_B) + v(\alpha(b)) = 0 - n \end{aligned}$$

which is valid for $\alpha = \alpha_1$ and $\alpha = \alpha_2$, thus $O_B \neq O_{B'}$: to get $O_{B'}$, we have to shift O_B in "direction of B'", precisely: with the coordinates α_1, α_2 corresponding to B one has $O_{B'} = (-n, -n)$.

3. Compactification of the Bruhat–Tits Building

For the boundary at infinity of X we do not use the topologization of the building at infinity due to Borel–Serre; it is more convenient to have the compactification, constructed by *Landvogt* in [L], but we restrict it to the part defined over F.

For a local field $\widehat{F}$ and a reductive algebraic group H denote by $X(H)$ the Bruhat–Tits building for the pair $(H, \widehat{F})$, then define

$$\overline{X} := \overline{X}(G) := \bigcup_{P \in \mathcal{P}} X\left(P/R_u(P)\right),$$

where $\mathcal{P}$ is the set of all parabolic F-subgroups of G and $R_u(P)$ the unipotent radical of P (cf. [L], 14.21). $\overline{X}$ is equipped with a topology which comes from the $\widehat{F}$-analytic topology on $G(\widehat{F})$ and the compactification of apartments, described below, and it induces the metric topology on each of the buildings $X(P/R_u(P))$. Consequently we consider only the — incomplete, but good (cf. [Br2], VI.9) — apartment system $\mathcal{A}$, defined over F, which is in 1-1-correspondence with the apartment system $\mathcal{A}_0$ of the Tits building X_0 of $G(F)$.

For $A \in \mathcal{A}$ denote by V the underlying $\widehat{F}$-vectorspace, by Σ the Coxeter complex with respect to G in V, by C a chamber of Σ and by $\Delta(C)$ a set of simple roots, such that $C = \{x \in A \mid \alpha(x) \geq 0$ for all $\alpha \in \Delta(c)\}$. For an open face C' of C, set $\Delta(C') := \{\alpha \in \Delta(C) \mid \alpha_{|C'} > 0\}$ and denote by $\langle C' \rangle$ the subspace of V, generated by C'. $V^C := \bigcup_{\substack{C' \in \Sigma \\ C' \subseteq \overline{C}}} V/\langle C' \rangle$ is called the *corner* defined by C.

Provide $\widetilde{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ with its natural topology and topologize V^C in such a way that the map $f : V^C \longrightarrow \widetilde{\mathbb{R}}^n$, given by

$$f(x + \langle C' \rangle) := \begin{cases} \infty & \text{for } \alpha \in \Delta(C') \\ \alpha(x) & \text{for } \alpha \in \Delta(C) \setminus \Delta(C') \end{cases}$$

is a homeomorphism.

A set $U \subseteq \overline{V} := \bigcup_{C' \in \Sigma} V/\langle C' \rangle$ is called open if $U \cap V^C$ is open for all chambers $C \in \Sigma$; by that $\overline{V}$ becomes compact and is called the compactification of V. $\overline{A} := A \times {}^V V^\Sigma$ is then the *compactification* of A with corners A^C (cf. [L], §2).

We abbreviate in the following: $X(P) := X(P/R_u(P))$, and we define the **boundary** of $\overline{X}$ by

$$\partial\overline{X} := \overline{X} \setminus X = \bigcup_{P \neq G} X(P) .$$

The closure of $X(P)$ in $\overline{X}$ is given by $\bigcup_{Q \subseteq P} X(Q)$; we shall also need $X_P := X \cup X(P)$.

Our next aim is to determine the *homotopy type of the unstable region* X', using the cover with the sets X'_P, P a maximal parabolic F-group. The nerve of this cover is the spherical Tits building X_0 which is known to be $(r-1)$-spherical. For this purpose we have to show that the sets X'_P and their intersections X'_Q (Q an arbitrary F-parpabolic group) are contractible, and to prove this we construct *retractions to infinity*, more precisely to $X(Q)$, defined by the geodesic action of the torus T_{Δ_0} for $Q = P_{\Delta_0}$. To describe it in a sector $D_{B,T}$, $T \supseteq T_{\Delta_0}$, it is helpful not to use all local coordinates α for $D_{B,T}$ ($\alpha \in \Delta$), but only those α, lying in $\Delta - \Delta_0$ and to complete them with the functions d_P for all $P = P_\alpha$, $\alpha \in \Delta_0$ (this is admissible since the roots in $\Delta - \Delta_0$ and the fundamental weights for Δ_0 are linearly independent). Then we can define the map

$$r_{Q,B,T} : D_{Q,B,T} \times [0, \infty] \longrightarrow D_{Q,B,T} \ (Q \supseteq B)$$

for $D_{Q,B,T} := \overline{D_{B,T} \cap X'_Q} \cap X_Q$ where the closure is meant in $\overline{X}$, given by $r_{Q,B,T}(x,t) = x_t$ with

$$
\begin{array}{lcll}
\alpha(x_t) & = & \alpha(x) & \text{for all } \alpha \in \Delta - \Delta_0 \text{ and } x \in X \\
d_P(x_t) & = & d_P(x) + t & \text{for all } P = P_\alpha,\ \alpha \in \Delta_0 \text{ and } x \in X \\
\alpha(x) & = & x & \text{for all } \alpha \in \Delta,\ x \in \overline{X} \setminus X.
\end{array}
$$

For different tori T and T', containing T_{Δ_0}, points $x \in D_{B,T}$ and $x' \in D_{B',T'}$ can have the same image for $t = \infty$ in $X(Q)$, described by different systems of coordinates α, coming from the apartments X_T and $X_{T'}$ respectively, but the coordinates d_P for $P \supseteq Q$ are defined independently from these apartments. Thus the maps $r_{Q,B,T}$ fit together, defining for $t = \infty$ a retraction

$$r_Q : \overline{X'_Q} \cap X_Q \longrightarrow X(Q) \ .$$

The map r_Q is continuous since its restrictions to the sectors $D_{B,T}$ are fibrations. Moreover, the map r_Q is surjective: For each point $x \in X(Q)$ we find a point x' projecting to x for sufficiently large values $d_P(x')$ for all $P \supseteq Q$ such that x' is close to Q, and therefore exists $B \subseteq Q$ for which x' is reduced, so $x' \in D_{B,T}$ for some $T \subseteq B$ and $x' \in D_{B,T} \cap X'_Q$.
Finally the affine building $X(Q)$ is contractible, thus by the retraction r_Q the set $\overline{X'_Q} \cap X_Q$ is also contractible and as a metrizable manifold the same is true for its interior X'_Q (cf. [BS1], 8.3.1).

Proposition 1. *The unstable region X' is $(r-1)$-spherical.*

Proof: $X' = \bigcup\limits_{P \in \mathcal{P}_{\max}} X'_P$ with $\mathcal{P}_{\max} := \{P \text{ maximal } F\text{-parabolic in } G\}$, the non-empty intersections of the covering sets X'_P are of type X'_Q, Q F-parabolic, and we have seen above that alle this sets are contractible. The covering sets are closed and the cover is locally finite because X is a locally finite simplicial complex. Its nerve is given by the spherical Tits building X_0 as an abstract complex which is known to be $(r-1)$-spherical. Thus we obtain that X' is $(r-1)$-spherical, using the same theorem as Borel–Serre in [BS1], 8.2.

Remark. For the group $G = SL_n$ (or $G = GL_n$) proposition 1 was proved by Grayson with a similar argument using vector bundles (cf. [G1], thm. 4.1).
The same idea can be used for $\partial\overline{X} := \overline{X} - X = \bigcup\limits_{P \neq G} X(P)$. We have the natural cover $\partial\overline{X} = \bigcup\limits_{P \neq G} \overline{X(P)}$ with $\overline{X(P)} = \bigcup\limits_{Q \subseteq P} X(Q)$; all these sets are contractible as Bruhat–Tits buildings or closures of them and their intersection pattern is given again by X_0. So we get the

Corollary. *$\partial\overline{X}$ is $(r-1)$-spherical.*

4. Buildings with Opposition

(a) In each apartment A_0 of a *spherical building* X_0 there exists a natural opposition involution. If A_0 is described as an abstract Coxeter complex $\Sigma = \Sigma(W,S)$ with group W and generating set S, $W_J = \langle J \rangle$ for $J \subseteq S$, i.e. $\Sigma = \{wW_J \mid w \in W,\ J \subseteq S\}$ and w_0 denotes the element of maximal length in W, then define

$$\mathrm{op}_\Sigma(w\,W_J) := ww_0W_{w_oJw_0}\ ;$$

expecially if the Coxeter diagram has no non-trivial symmetry, then $w_0Jw_0 = J$ for all J.

If X_0 is the spherical *Tits building* of a group $G(F)$ (G reductive, F a field), the simplices of X_0 may be identified with the proper F-parabolic subgroups of $G(F)$. Each such group has a Levi decomposition $P = L \ltimes R_u(P)$, and two parabolics are called *opposite* if they have a common Levi subgroup, more precisely,

$$P \text{ op } P' :\Longleftrightarrow\ P \cap P' \text{ is a Levi subgroup of } P \text{ and } P'.$$

$[R_u(P)](F)$ acts simply transitive on the set of all parabolic subgroups opposite P (cf. [BT], § 4), thus we can identify them with the elements of this radical if we distinguish one opposite group.

(b) Pairs of opposite simplices of a spherical building with incidence in both components provide again a simplicial complex. It was introduced by R. Charney (see [C]) for $G = GL_n$, even over Dedekind domains in the language of flags; she showed that it has the same homotopy type as the spherical building of GL_n itsself. Lehrer and Rylands (see [LR]) defined such a complex for reductive groups G — they called it the *"split building"* of G — and proved the corresponding homological result for types A_n and C_n. A. von Heydebreck (see [vH]) considered this complex for arbitrary spherical buildings and showed that it is also $(n-1)$-spherical in dimension n. We use the definition

$$\mathrm{Opp}\, X_0 := \{(P,P') \mid P\,\mathrm{op}\,P'\}.$$

(c) Moreover, we need a subcomplex of OppX_0, where the opposition relation is defined with respect to Γ.

As a first step we distinguish an apartment $A_1 = X_{T_1}$ of X, T_1 a maximal split F-torus such that $N(T_1) \cap \Gamma$ contains (a copy of) the Weyl group W of X_0 (for instance, A_1 could contain a vertex with stabilizer $G(\widehat{O}) \supset G(\mathbb{F}_q) \supset W$). We fix a Borel group $B_1 \supset T_1$ and its opposite B_1' in A_1. The choice of B_1' defines an identification of $\operatorname{Opp} B_1 := \{B' \mid B' \operatorname{op} B_1\}$ with $U_{B_1}(F)$, and we can consider the subset $\operatorname{Opp}_\Gamma B_1$, corresponding to $U_{B_1}(F) \cap \Gamma =: U_1 \cap \Gamma$ such that

$$\operatorname{Opp}_\Gamma B_1 := \{B' = \gamma_1 B_1' \gamma_1^{-1} \mid \gamma_1 \in U_1 \cap \Gamma\}.$$

We extend this notion Γ-invariant: For $B = \gamma B_1 \gamma^{-1}$ with $\gamma \in \Gamma$, the element γ is determined up to $B_1(F) \cap \Gamma$, so we obtain different opposite Borel groups $B' = \delta B_1' \delta^{-1}$ with $\delta \in \gamma \cdot (U_1 \cap \Gamma)$ — neglecting the torus component in $T_1 \subset B_1$ since it fixes also B_1'. Consequently the identification of $\operatorname{Opp} B$ with $U_B(F)$ depends on the choice of δ, but this has no influence on the definition

$$\operatorname{Opp}_\Gamma B := \{B' = \gamma' B' (\gamma')^{-1} \mid \gamma' \in U_B(F) \cap \Gamma\}$$

because $U_B = \gamma U_1 \gamma^{-1}$, which implies with $u, u' \in U_1 \cap \Gamma$:

$$\gamma' B' (\gamma')^{-1} = \gamma u' \gamma^{-1} \gamma u B_1' (\gamma u)^{-1} (\gamma u' \gamma^{-1})^{-1} = \gamma u' u B_1' (\gamma u' u)^{-1}$$

thus $\operatorname{Opp}_\Gamma B = \gamma \cdot \operatorname{Opp}_\Gamma B_1 \gamma^{-1}$.

In general, not all F-Borel groups are conjugate under Γ; there exist finitely many Γ-conjugacy classes (see part E of reduction theory). We fix a set
$B_1, B_2 = g_2 B_1 g_2^{-1}, \ldots, B_h = g_h B_1 g_h^{-1}$ $(g_i \in G(F))$ of repesentatives and also of their opposite groups $B_1', B_2' = g_2 B_1' g_2^{-1}, \ldots, B_h = g_h B_1' g_h^{-1}$, and define in the same way as above

$$\operatorname{Opp}_\Gamma B_i := \{B' = \gamma_i B_i' \gamma_i^{-1} \mid \gamma_i \in U_{B_i}(F) \cap \Gamma\},\ i = 1, \ldots, h$$

and for $B = \gamma B_i \gamma^{-1},\ B' = \gamma B_i' \gamma^{-1}$

$$\operatorname{Opp}_\Gamma B := \{B' = \gamma' B (\gamma')^{-1} \mid \gamma' \in U_B(F) \cap \Gamma\}$$

which does not depend on the special choice of B' (but we don't have
$g_i \operatorname{Opp}_\Gamma B_1 g_i^{-1} = \operatorname{Opp}_\Gamma B_i$ in general).

Finally we can make the same procedure with parabolic groups, starting with the set of standard parabolic groups Q_1 containing B_1 and their oposites $Q_1' \supseteq B_1'$. Since Q_1 and Q_1' have a Levi subgroup in common, we obtain all Γ-opposites of Q_1 by conjugation of Q_1' with elements from $U_{Q_1}(F) \cap \Gamma$ and we have to restrict in all definitions above the groups $U_B(F) \cap \Gamma$ to its subgroups $U_Q(F) \cap \Gamma$ for $Q \supseteq B$. We denote this relation by $\operatorname{Opp}_\Gamma$ and define

$$\operatorname{Opp}_\Gamma X_0 := \{Q, Q') \mid Q \operatorname{op}_\Gamma Q'\}.$$

5. Proof of the theorem (sketch)

In order to define a retraction from the unstable region to its inner boundary, we have to split it up into apartments, thereby constructing a bigger complex (part of an *"affine split building"*) as follows:

Denote by $\mathcal{T}, \mathcal{B}, \mathcal{Q}$ and $\mathcal{P}$ the sets of maximal tori, Borel groups, parabolic and maximal parabolic groups in G, all defined over F (for other notations cf. section 2)

$$\begin{aligned} &Z := \{(x,T) \in X' \times \mathcal{T} \mid \exists\, B \in \mathcal{B} : x \in D_{B,T}\}, \\ &\text{by definition } D_{B,T} \subset X_T \text{ and } T \subset B. \end{aligned}$$

Since a maximal torus T is uniquely determined by a pair of opposite Borel groups (B, B'), say $T = T_{B,B'}$, there exists an equivalent description

$$Z = \{(x,B') \in X' \times \mathcal{B} \mid \exists\, B \in \mathcal{B} : B \operatorname{op} B',\ x \in D_{B,T} \text{ for } T = T_{B,B'}\}$$

In Z we need an equivalence relation, according to the structure of $\mathrm{Opp}X_0$, so we define

$$(x_1,T_1) \sim (x_2,T_2) \iff \begin{cases} x_1 = x_2 =: x \in D_{B_1,T_1} \cap D_{B_2,T_2} \\ \exists\, Q \in \mathcal{Q} : Q \supseteq B_1,\ Q \supseteq B_2,\ x \in X''_Q \end{cases}$$

The group Q is uniquely determined by reduction theory and this fact implies the transitivity of the relation. We can define the equivalence also using the second description of Z:

$$(x_1,B'_1) \sim (x_2,B'_2) \iff \begin{cases} x_1 = x_2 =: x \\ \exists\,(Q,Q') \in \mathrm{Opp}X_0 : B_i \subseteq Q,\ B'_i \subseteq Q' \text{ for } i = 1,2 \\ x \in X''_Q \end{cases}$$

In this situation the common Levi subgroup L of Q and Q' is the centralizer of a torus T_L (not necessarily maximal), contained in $T_1 \cap T_2$. Let us denote by

$$\begin{aligned} &[x,B'] \text{ the class of } (x,B') \quad \text{and by} \\ &\widetilde{X}' := Z/\sim\, = \{[x,B'] \mid (x,B') \in Z\} \quad \text{and} \\ &\widetilde{X}'_{Q,Q'} := \{[x,B'] \in \widetilde{X}' \mid x \in X'_Q, B' \subseteq Q'\} \text{ for } (Q,Q') \in \mathrm{Opp}\, X_0\ , \end{aligned}$$

and finally the analogous definition for $\widetilde{X}''_{Q,Q'}$ with $x \in X''_Q$.
The *topology* of $\widetilde{X}'$ is given as follows: We choose for X' the metric topology as a subspace of the affine building X, for $\mathcal{T}$ and $\mathcal{B}$ the $\widehat{F}$-analytic topology induced from $G(\widehat{F})$, since all maximal tori in $\mathcal{T}$ or all Borel groups in $\mathcal{B}$ are conjugate

under $G(F)$; finally we have the product topology on Z and the quotient topology on $\widetilde{X}'$.
One should emphasize that every point (x, B') has an open neighbourhood in Z of the form $U \times V$, where U is the disjoint union of open sets U_T in X_T, because the complex X is locally finite, so we can avoid ramification inside U_T. For a point $[x, B']$ in $\widetilde{X}''_{Q,Q'} \subset \widetilde{X}'$ there exists a neighbourhood $U \times V$, where U is the union of segments of geodesic lines in $\widetilde{X}''_{Q,Q'}$, defined by the torus $T = T_{\Delta_0}$ if Q and Q' are both of cotype Δ_0.

We want moreover to define a *boundary at infinity* for $\widetilde{X}'$, generalizing the construction of Landvogt. There the Bruhat–Tits buildings $X(Q) := X(Q_{/R_u(Q)})$, which contribute to the boundary $\partial\overline{X}$ are defined only by quotient groups. For a pair (Q, Q') of opposite parabolic groups, the common Levi group $L = Q \cap Q'$ is isomorphic to $Q_{/R_u(Q)}$, so we may consider $X(L)$ instead of $X(Q)$, defined by a subgroup of G. For $\widetilde{X}'$ it is more convenient to split up also $\partial\overline{X}$, using the different buildings $X(L)$ instead of a single $X(Q)$. Therefore we set

$$\begin{array}{rcl} \partial_\infty \widetilde{X}' & := & \bigcup\limits_L X(L) \text{ , where } L = Q \cap Q',\ (Q, Q') \in \operatorname{Opp} X_0. \\ \overline{X}' & := & \widetilde{X}' \cup \partial_\infty \widetilde{X}' \ . \end{array}$$

The details are the same as in Landvogt's construction, but let us remark that for a point of $\partial_\infty \widetilde{X}'$ each neighbourhood meets infinitely many "apartments" $\widetilde{X}'_T := \{[x, T] \in \widetilde{X}' \mid x \in X_T\}$.
Now we can imitate the proof of proposition 1, in order to determine the *homotopy type* of $\widetilde{X}'$. We have a cover

$$\widetilde{X}' = \bigcup_{\operatorname{Opp} X_0} \widetilde{X}'_{Q,Q'} = \bigcup_{(P,P')} \widetilde{X}'_{P,P'} \text{ with } (P, P') \in \operatorname{Opp} X_0 \cap (\mathcal{P} \times \mathcal{P})$$

with closed sets; their intersections are given by

$$\widetilde{X}'_{Q,Q'} = \bigcap \left\{ \widetilde{X}'_{P,P'} \,\middle|\, (P, P') \supseteq (Q, Q') \right\}$$

thus this cover has the nerve $\operatorname{Opp} X_0$.

The covering sets and their intersections can be surjectively contracted to $X(L) \subset \partial_\infty \widetilde{X}$ along geodesic lines defined by the torus T_L in the center of $L = Q \cap Q'$ and $X(L)$ is a contractible space, so $\widetilde{X}'_{Q,Q'}$ is also contractible. Using the result of v. Heydebreck, cited in section 4, we know that $\operatorname{Opp} X_0$ is $(r-1)$-spherical and therefore we have

Proposition 2. *$\widetilde{X}'$ is $(r-1)$-spherical.*

But in contrast to X' it is now possible to retract $\widetilde{X}'$ to its "inner boundary" (cf. section 2)

$$\widetilde{Y} := \partial_0 \widetilde{X}' := \{[x, B'] \in \widetilde{X}' \mid x \in Y\}$$

along geodesic lines in $\widetilde{X}''_{Q,Q'}$, which do not ramify in $\widetilde{X}'$, because we identified different apartments only in these sets $\widetilde{X}''_{Q,Q'}$, and the geodesics coincide in their intersections. Thus we have

Corollary. *$\widetilde{Y}$ is $(r-1)$-spherical.*

We need the analogous results for a subcomplex $\widetilde{X}'_\Gamma$ of $\widetilde{X}'$, replacing in the definitions the relation "op" by "op_Γ", consequently we have to admit only pairs of Borel groups (B, B') with $B \,\mathrm{op}_\Gamma\, B'$ and tori $T_{B,B'}$ for $(B, B') \in \mathrm{Opp}_\Gamma X_0$. For this purpose we require that also $\mathrm{Opp}_\Gamma X_0$ is $(r-1)$-spherical witch is true for $G = SL_n$ by the proof of Charney (see [C]), for the general case see the appendix. Then we obtain

Proposition 3. *$\widetilde{X}'_\Gamma$ and $\widetilde{Y}_\Gamma := \widetilde{Y} \cap \widetilde{X}'_\Gamma$ are $(r-1)$-spherical.*

The next step is to show that $\widetilde{Y}_\Gamma$ *is modulo* Γ *a finite complex* — this is the only point where we need $\widetilde{X}'_\Gamma$ instead of $\widetilde{X}'$. For the points of $\widetilde{Y}_\Gamma$ the numerical invariants of reduction theory are bounded from above (and below by definition), so part D of the "main theorem" says that $\widetilde{Y}_\Gamma/\Gamma$ is compact. Moreover, by part (E) there exist only finitely many conjugacy classes of Borel groups, therefore in a set of representatives $[y, B']$ for $\widetilde{Y}_\Gamma/\Gamma$ with $y \in D_{B,T}$ only finitely many Borel groups B occur, and since $B' \,\mathrm{op}_\Gamma\, B$, there is only one B' modulo Γ for each B : $\widetilde{Y}_\Gamma/\Gamma$ is a finite complex. Since all stabilizers in Γ are finite, we can apply the finiteness criterion of K. Brown (see [Br1], 1.1 and 3.1) to get

Proposition 4. *Γ is of type F_{r-1}.*

Remark to the conjecture "Γ is not of type F_r":
Construct an infinite series of $(r-1)$-spheres S_k in $Y = \partial_0 X'$, which are contractible only in growing parts X_k, defined by a (rough) filtration of X; then $\{\pi_{r-1}(X_k)\}$ is not "essentially trivial" in the sense of K. Brown (see [Br1], 2).

6. Appendix

For the group $G = SL_n$ the complex $\mathrm{Opp}_\Gamma X_0$ is also $(r-1)$-spherical by [C] and so are $\widetilde{X}'_\Gamma$ and $\widetilde{Y}_\Gamma$. It is not true that $\mathrm{Opp}_\Gamma X_0$ is a deformation retract of $\mathrm{Opp}\, X_0$, as was shown by Abramenko, who constructed a counter-example. But we have the following

Lemma. *$\widetilde{X}'_\Gamma$ is a deformation retract of $\widetilde{X}'$.*

Proof: We wish to map a point $[x, B']$ of $\widetilde{X}'$ with $x \in D_{B,T} \subset X_T$, $B \,\mathrm{op}\, B'$, $T = T_{B,B'}$ to $[x, B'_0]$ with the same $x \in X'$ and $B'_0 \,\mathrm{op}_\Gamma\, B$, obtaining a new torus $T_0 := T_{B,B'_0}$.
Identifying the Borel groups opposite to B with elements of $U(F)$ (the unipotent radical of B), for $[x, B']$ the group B' corresponds to an element of $U(F) \cap K_x$ with $K_x = \mathrm{Stab}_{G(F)}(x)$ since $x \in X_{T_{B,B'}}$. This compact group contains finitely many elements of the discrete group $U(F) \cap \Gamma$; we have to make a choice: There is one element, defining B'_0 and T_0, such that $X_{T_0} \cap X_T$ is maximal because the intersection is given as the intersection of half-apartments, defined by root groups, and for a Chevalley group, U is the semi-direct product of its root-groups. This definition is compatible with the equivalence relation in Z and the map induces the identity on $\widetilde{X}'_\Gamma$.
This map is also continuous: The topology in the second component is induced by the analytic topology of the group $G(\widehat{F})$; an element of $U(F) \cap K_x$ has a neighbourhood which contains only one element of $U(F) \cap \Gamma$, due to its discreteness.

Remark. Since $\mathrm{Opp}_\Gamma X_0$ is the nerve of a cover of $\widetilde{X}'_\Gamma$, we proved indirectly that it is $(r-1)$-spherical. A direct proof for groups over Dedekind rings would be of interest.

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