A PURITY THEOREM

FOR FUNCTORS WITH TRANSFERS

K. Zainoulline

Russian Academy of Sciences
Steklov Mathematical Institute
St. Petersburg Department
Fontanka 27, St. Petersburg 191011, Russia

e-mail: kirill@pdmi.ras.ru
URL: http://www.pdmi.ras.ru/~kirill/index.html

April 5, 2000

ABSTRACT

Let R be a local regular ring of geometric type and K be its field of fractions. Let $\mathfrak{F}$ be a covariant functor from the category of R-algebras to abelian groups satisfying some additional properties (continuity, existence of well behaving transfer map). We show that the following equality holds for the subgroups of the group $\mathfrak{F}(K)$:

$$\bigcap_{\mathfrak{p}\in Spec\, R, ht\mathfrak{p}=1} im\{\mathfrak{F}(R_{\mathfrak{p}})\to\mathfrak{F}(K)\} = im\{\mathfrak{F}(R)\to\mathfrak{F}(K)\},$$

where all maps are induced by the canonical inclusions.

Agreements

All rings are assumed to be commutative with unit.

Let A and S be any rings. By an A-algebra S we will mean the pair (S, i), where $i : A \to S$ is a ring homomorphism. Sometimes, we will write S instead of the pair (S, i) keeping in mind the homomorphism i. We will denote an A-algebra S by $A \xrightarrow{i} S$.

By a morphism $f : (S, i) \to (S', i')$ between two A-algebras we will mean such a ring homomorphism $f : S \to S'$ that $f \circ i = i'$.

Let $\mathfrak{F}$ be a covariant functor from the category of A-algebras to the category of abelian groups. Let $A \xrightarrow{i} R$ be an A-algebra. Since every R-algebra T has the A-algebra structure induced by the homomorphism i the functor $\mathfrak{F}$ may be considered as a functor given on the category of R-algebras. To make a difference from $\mathfrak{F}$ this functor is denoted by $\mathfrak{F}_R$ and is called by the restriction of the functor $\mathfrak{F}$ to the category of R-algebras via the homomorphism i. So on objects the functor $\mathfrak{F}_R$ is defined as $\mathfrak{F}_R(R \xrightarrow{t} T) = \mathfrak{F}(A \xrightarrow{i} R \xrightarrow{t} T)$, for any R-algebra $R \xrightarrow{t} T$.

§1. The Purity Theorem

Let A be a smooth algebra over an infinite field k. Let $R = A_{\mathfrak{q}}$ be the localization of the algebra A at some prime ideal $\mathfrak{q} \in Spec\, A$ and let K denote its field of fractions. So R is a local regular ring of geometric type. Such a ring will be the main object of our discussion.

Definition. Let $\mathfrak{G}$ be a covariant functor from the category of R-algebras to the category of abelian groups. We say that the purity holds for the functor $\mathfrak{G}$ if the following equality holds for the subgroups of $\mathfrak{G}(K)$:

$$\bigcap_{\substack{\mathfrak{p} \in Spec\, R, \\ ht\mathfrak{p}=1}} im\{\mathfrak{G}(R_{\mathfrak{p}}) \to \mathfrak{G}(K)\} = im\{\mathfrak{G}(R) \to \mathfrak{G}(K)\}, \qquad (*)$$

where all maps are induced by the canonical inclusions. □

This paper is devoted to the proof of the following theorem:

Theorem I. (The Purity Theorem) *Let R be a local regular ring of geometric type got by localizing a smooth k-algebra A. Let $\mathfrak{F}$ be a functor from the category of A-algebras to the category of abelian groups and $\mathfrak{F}_R$ denote its restriction to the category of R-algebras via the canonical inclusion $A \xhookrightarrow{i_R} R$. Let $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy properties* **C**, **T** *and* **E** *described at the end of this section.*

Then the purity holds for the functor $\mathfrak{F}_R$.

In the constant case, i.e. when the functor $\mathfrak{F}$ is given on the category of k-algebras, we have

Theorem II. *Let R be a local regular ring of geometric type got by localizing a smooth k-algebra A. Let $\mathfrak{F}$ be a functor from the category of k-algebras to the category of abelian groups and $\mathfrak{F}_R$ denote its restriction to the category of R-algebras via the canonical inclusion $k \xhookrightarrow{i_R} R$. Let $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy properties* **C** *and* **T** *described below.*

Then the purity holds for the functor $\mathfrak{F}_R$.

Below we restate these theorems in a slightly different form.

Under the hypotheses of the theorems above the main result of paper [Za] states that the map $\mathfrak{F}(R) \xrightarrow{i_K^*} \mathfrak{F}(K)$ induced by the canonical inclusion $R \xhookrightarrow{i_K} K$ is injective. Thus we may identify the group $\mathfrak{F}(R)$ with its image in $\mathfrak{F}(K)$. Since the localization $R_\mathfrak{p}$ at height 1 prime $\mathfrak{p}$ is a local regular ring we may rewrite the purity condition for the functor $\mathfrak{F}_R$ as

$$\bigcap_{\substack{\mathfrak{p}\in Spec\,R,\\ ht\mathfrak{p}=1}} \mathfrak{F}(R_\mathfrak{p}) = \mathfrak{F}(R), \qquad (*')$$

where the left and the right parts are considered as subgroups of $\mathfrak{F}(K)$.

Theorem III. *Under the hypotheses of Theorem I or II the following complex is exact:*

$$0 \longrightarrow \mathfrak{F}(R) \xrightarrow{i_K^*} \mathfrak{F}(K) \xrightarrow{\oplus \mathrm{can}_\mathfrak{p}} \bigoplus_{\substack{\mathfrak{p}\in Spec\,R,\\ ht\mathfrak{p}=1}} \mathfrak{F}(K)/\mathfrak{F}(R_\mathfrak{p}),$$

where $\mathrm{can}_\mathfrak{p} : \mathfrak{F}(K) \to \mathfrak{F}(K)/\mathfrak{F}(R_\mathfrak{p})$ *is the canonical map.*

Proof. By property **C** of the functor $\mathfrak{F}$ we have the canonical isomorphism $\varinjlim_{g\in R} \mathfrak{F}(R_g) \cong \mathfrak{F}(K)$. Thus there exists such a localization R_g that an element $\alpha \in \mathfrak{F}(K)$ comes from some element $\alpha_g \in \mathfrak{F}(R_g)$. Since R is smooth over k there are only finite number of prime ideals of height 1 lying over g. This means that the element α doesn't lie only in finite number of subgroups $\mathfrak{F}(R_\mathfrak{p})$ in $\mathfrak{F}(K)$. Hence the homomorphism $\oplus \mathrm{can}_\mathfrak{p}$ is well-defined.

The injectivity of the map i_K^* was already mentioned. Clearly the complex is exact at the $\mathfrak{F}(K)$-term if and only if the purity $(*')$ holds for the functor $\mathfrak{F}_R$. $\square$

Example 1. Consider a linear algebraic group G over the base field k which is rational as the k-variety. Let T be a torus over k and let $\mu : G \to T$ be a group morphism which is surjective over a separable closure $\bar{k}$ of the field k, i.e. the map $\mu(\bar{k}) : G(\bar{k}) \to T(\bar{k})$ is surjective.

Let $\mathfrak{F}$ be a functor from the category of k-algebras to the category of abelian groups defined as follows:

$$\mathfrak{F} : S \mapsto T(S)/\mu(G(S))$$

for any k-algebra S. Then the purity holds for the functor $\mathfrak{F}_R$ and by Theorem III we get

Theorem IV. *The following sequence of abelian groups is exact:*

$$0 \longrightarrow T(R)/\mu(G(R)) \xrightarrow{i_K^*} T(K)/\mu(G(K)) \xrightarrow{\oplus_\mathfrak{p} \mathrm{can}_\mathfrak{p}} \bigoplus_{\substack{\mathfrak{p}\in Spec\,R,\\ ht\mathfrak{p}=1}} T(K)/\mu(G(K))\cdot T(R_\mathfrak{p})$$

Indeed, by Theorem II we only have to check that the functor $\mathfrak{F}$ and its restriction $\mathfrak{F}_R$ satisfies properties **C** and **T**.

The property **C** follows immediately. To check the property **T** we consider the obvious norm map $N : T(S) \to T(R)$ for any R-algebra S finitely generated projective as the R-module.

To get the desired transfer map $\mathrm{Tr}_R^S : \mathfrak{F}_R(S) \to \mathfrak{F}_R(R)$ we have to verify the following inclusion, usually called Norm Principle: $N(\mu(G(S))) \subset \mu(G(R))$.

To do that we consider the following commutative diagram:

$$\begin{array}{ccccc}
T(S) & \xrightarrow{N} & T(R)/\mu(G(R)) & \xrightarrow{\partial} & H^1_{\text{ét}}(R, \ker\mu) \\
{\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\
T(S_K) & \xrightarrow[N]{} & T(K)/\mu(G(K)) & \xrightarrow[\partial_K]{} & H^1_{\text{ét}}(K, \ker\mu_K)
\end{array}$$

where $S_K = S \otimes_R K$ denotes the scalar extension.

Observe now that g has trivial kernel by the result of [CO]. Since the map ∂ has trivial kernel the map f has trivial kernel as well.

Let now $\alpha \in \mu(G(S)) \subset T(S)$ be an element. We have to show that $N(\alpha) \in \mu(G(R))$. By the commutativity of the left square of the diagram and the injectivity of f it suffices to verify that $N(f'(\alpha)) \in \mu(G(K))$. This is indeed the case by the Norm Principle of Merkurjev [Me] because the group G is rational.

Thus the norm map N induces a transfer map Tr_R^S. The properties $\mathbf{T}$.(a) , $\mathbf{T}$.(b1) and $\mathbf{T}$.(b2) of the transfer map follows immediately. □

Example 2. Suppose the group G from the previous example is a torus which is not necessary a rational k-variety. Then the purity holds for the functor $\mathfrak{F}_R$ from Example 1.

Indeed, in the previous notation the following diagram commutes

$$\begin{array}{ccc} G(S) & \xrightarrow{N_G} & G(R) \\ {\scriptstyle \mu(S)}\downarrow & & \downarrow{\scriptstyle \mu(R)} \\ T(S) & \xrightarrow[N]{} & T(R) \end{array}$$

Therefore the norm map N induces a well-defined transfer map $\mathrm{Tr}_R^S : \mathfrak{F}_R(S) \to \mathfrak{F}_R(R)$. Again the properties $\mathbf{T}$.(a) , $\mathbf{T}$.(b1) and $\mathbf{T}$.(b2) of the transfer map hold and the continuity of the functor $\mathfrak{F}$ holds as well. □

Example 3. Let $\mathcal{A}$ be an Azumaya algebra over the local regular ring R. Let $\mathrm{Nrd} : \mathcal{A}^* \to R^*$ denote the reduced norm homomorphism. For any R-algebra S let $\mathcal{A}_S = \mathcal{A} \otimes_R S$ be the extended Azumaya algebra over S.

Let $\mathfrak{F}$ be a functor from the category of R-algebras to the category of abelian groups defined as follows:

$$\mathfrak{F} : S \mapsto S^*/\mathrm{Nrd}(\mathcal{A}_S^*)$$

for any R-algebra S. Then the purity holds for the functor $\mathfrak{F}_R$ and by Theorem III we get

Theorem V. *The following sequence of abelian groups is exact:*

$$0 \longrightarrow R^*/\mathrm{Nrd}(\mathcal{A}^*) \xrightarrow{i_K^*} K^*/\mathrm{Nrd}(\mathcal{A}_K^*) \xrightarrow{\oplus_{\mathfrak{p}} \mathrm{can}_{\mathfrak{p}}} \bigoplus_{\substack{\mathfrak{p} \in Spec\, R,\\ ht\mathfrak{p}=1}} K^*/\mathrm{Nrd}(\mathcal{A}_K^*) \cdot R_{\mathfrak{p}}^*$$

Observe that this is the result of I.Panin and A.Suslin (Theorem I, [PS]) proved by completely different way.

Indeed, all necessary properties $\mathbf{C}$, $\mathbf{T}$ and $\mathbf{E}$ of the functor $\mathfrak{F}$ and its restriction $\mathfrak{F}_R$ were checked in 3.2 of [Za]. □

Example 4. Under the notation of the previous example assume that there is the additional structure on our Azumaya algebra:

Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with unitary involution over the local regular ring R (*char* $k \neq 2$). It means that there is a tower $\mathcal{A}/C/R$, where C is the center of $\mathcal{A}$ and C/R is an etale quadratic extension with restricted involution σ. Therefore, C_K/K is a separable quadratic extension of the corresponding fields of fractions.

Let $\mathrm{U}(\mathcal{A}_S) = \{a \in \mathcal{A}_S \mid aa^\sigma = 1\}$ be the unitary group of an algebra $(\mathcal{A}_S, \sigma)$ for any R-algebra S. And let $\mathrm{U}(C_S)$ denote the unitary group of the center C_S of the algebra $\mathcal{A}_S$.

Let $\mathfrak{F}$ be a functor from the category of R-algebras to the category of abelian groups defined as follows:

$$\mathfrak{F} : S \mapsto \mathrm{U}(C_S)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_S))$$

for any R-algebra S. Then the purity holds for the functor $\mathfrak{F}_R$ and by Theorem III we get

Theorem VI. *The following sequence of abelian groups is exact:*

$$0 \to \mathrm{U}(C)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A})) \xrightarrow{i_K^*} \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K)) \xrightarrow{\oplus_{\mathfrak{p}} \mathrm{can}_{\mathfrak{p}}} \bigoplus_{\substack{\mathfrak{p} \in Spec\, R, \\ ht\mathfrak{p}=1}} \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K)) \cdot \mathrm{U}(C_{\mathfrak{p}})$$

Indeed, all necessary properties **C**, **T** and **E** of the functor $\mathfrak{F}$ and its restriction $\mathfrak{F}_R$ were checked in 3.4 of [Za]. $\square$

Now we describe the properties **C**, **T** and **E** of the functors $\mathfrak{F}$ and $\mathfrak{F}_R$ mentioned above (compare with sect. 1 and 2, [Za]):

C. (continuity) For any A-algebra S essentially smooth over k and for any multiplicative system M in S the canonical map $\varinjlim_{g \in M} \mathfrak{F}(S_g) \to \mathfrak{F}(M^{-1}S)$ is an isomorphism, where $M^{-1}S$ is the localization of S with respect to M.

T. (transfer) For any R-algebra T finitely generated projective as the R-module there is given a homomorphism $\mathrm{Tr}_R^T : \mathfrak{F}_R(T) \to \mathfrak{F}_R(R)$ called transfer map such that

(a) (additivity and normalization) $\mathrm{Tr}_R^{R\times T}(x) = \mathrm{pr}_R^*(x) + \mathrm{Tr}_R^T(\mathrm{pr}_T^*(x))$ for every $x \in \mathfrak{F}(R \times T)$, where pr_R^* and pr_T^* are induced by the canonical projections;

(b1) (scalar extension and homotopy invariance) for an $R[t]$-algebra S finitely generated projective as the $R[t]$-module (thus, $S/(t)$ and $S/(t-1)$ are finitely generated projective as the R-modules) the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{F}_R(S) & \xrightarrow{\mathrm{can}_0^*} & \mathfrak{F}_R(S/(t)) \\ {\scriptstyle \mathrm{can}_1^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_0} \\ \mathfrak{F}_R(S/(t-1)) & \xrightarrow[\mathrm{Tr}_1]{} & \mathfrak{F}_R(R) \end{array}$$

where can_0^*, can_1^* are induced by the canonical projections and Tr_0, Tr_1 denote the corresponding transfer maps $\mathrm{Tr}_R^{S/(t)}$ and $\mathrm{Tr}_R^{S/(t-1)}$.

(b2) the following diagram induced by extension of scalars via the canonical inclusion $R \hookrightarrow K$ commutes:

$$\begin{array}{ccc} \mathfrak{F}_R(T) & \longrightarrow & \mathfrak{F}_R(T \otimes_R K) \\ {\scriptstyle \mathrm{Tr}_R^T}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_K^{T\otimes_R K}} \\ \mathfrak{F}_R(R) & \longrightarrow & \mathfrak{F}_R(K) \end{array}$$

E. (extension property) Given an R-algebra $R \xrightarrow{i} S$ essentially smooth over k, given an A-algebra $A \xrightarrow{i_S} S$ and an augmentation $S \xrightarrow{\varepsilon} R$ of the inclusion i, such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{i_S} & S \\ \| & & \downarrow{\scriptstyle \varepsilon} \\ A & \xrightarrow[i_R]{} & R \end{array}$$

and given a multiplicative system M with respect to a finite set $\{\mathfrak{m}_i\}_{i\in\mathfrak{I}}$ of maximal ideals in S with the property $\varepsilon^{-1}(\mathfrak{m}_R) \subset \cup_{i\in\mathfrak{I}}\mathfrak{m}_i$, there exist:

(a) A localization S_g and a finite etale extension $e : S_g \to \tilde{S}$ for a certain $g \in M$;

(b) An augmentation $\tilde{\varepsilon} : \tilde{S} \to R$ for the inclusion $R \xrightarrow{i} S_g \xrightarrow{e} \tilde{S}$ such that the following diagram commutes:

$$\begin{array}{ccc} S_g & \xrightarrow{e} & \tilde{S} \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle \tilde{\varepsilon}} \\ R & = & R \end{array}$$

(c) A natural transformation $\Phi : \mathfrak{F} \to \mathfrak{F}_R$ between two functors $\mathfrak{F}$ and $\mathfrak{F}_R$ restricted to the category of $\tilde{S}$-algebras via the maps $A \xrightarrow{i_S} S_g \xrightarrow{e} \tilde{S}$ and $R \xrightarrow{i} S_g \xrightarrow{e} \tilde{S}$ correspondingly, such that the morphism

$$\mathfrak{F}(\tilde{S} \xrightarrow{\tilde{\varepsilon}} R \to T) \xrightarrow{\Phi(\tilde{S} \xrightarrow{\tilde{\varepsilon}} R \to T)} \mathfrak{F}_R(\tilde{S} \xrightarrow{\tilde{\varepsilon}} R \to T)$$

is the identity. In particular, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{F}(\tilde{S}) & \xrightarrow{\Phi(\tilde{S})} & \mathfrak{F}_R(\tilde{S}) \\ {\scriptstyle \tilde{\varepsilon}^*}\downarrow & & \downarrow{\scriptstyle \tilde{\varepsilon}^*} \\ \mathfrak{F}(R) & = & \mathfrak{F}_R(R) \end{array}$$

where R is considered as the $\tilde{S}$-algebra via the augmentation $\tilde{\varepsilon}$. □

The present paper is organized as follows:

Section 2 is devoted to the proof of the Purity Theorem. This proof is based on a further development of the technique offered in the papers of K.Zainoulline, [Za], and I.Panin, M.Ojanguren, [PO], [PO_1].

In section 3 we prove the Divisor Theorem — the key point of the proof of the Purity Theorem. It was motivated by the work of V.Voevodsky [Vo].

We finish by proving a stronger version of Geometric Presentation Lemma (Lemma 5.2, [PO]).

Acknowledgements. I am grateful to Ivan Panin for the useful discussions and ideas led to the proof of the Purity Theorem. I thank INTAS project YSF-99-3966 for the financial support.

§2. The Proof of the Purity Theorem

Let $R = A_{\mathfrak{q}}$ be the localization of a smooth k-algebra A at some prime ideal $\mathfrak{q} \in Spec\, A$ and let K be its field of fractions.

Let $\mathfrak{F}$ be a covariant functor from the category of A-algebras to abelian groups and let $\mathfrak{F}_R$ denote its restriction to the category of R-algebras via the canonical inclusion $A \overset{i_R}{\hookrightarrow} R$. By the very assumption functors $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy properties **C**, **T** and **E**.

Let $\mathfrak{F}_{nr}(K)$ denote the left part of the equality $(*')$, sect.1, i.e

$$\mathfrak{F}_{nr}(K) = \bigcap_{\substack{\mathfrak{p}\in Spec\, A \\ ht\mathfrak{p}=1, \mathfrak{p}\subset\mathfrak{q}}} \mathfrak{F}(A_{\mathfrak{p}}) = \bigcap_{\substack{\mathfrak{p}\in Spec\, R, \\ ht\mathfrak{p}=1}} \mathfrak{F}(R_{\mathfrak{p}}) \subset \mathfrak{F}(K).$$

We have to prove that any element $\alpha \in \mathfrak{F}_{nr}(K)$ lies in the image of the map $\mathfrak{F}(R) \overset{i_K^*}{\to} \mathfrak{F}(K)$ induced by the canonical inclusion $R \overset{i_K}{\hookrightarrow} K$.

To do this it is enought to prove the following two assertions:

1. For any regular function $f \in \mathfrak{q}$ there are two commutative diagrams:

$$\text{(i)} \quad \begin{array}{ccccc} \mathfrak{F}(A_f) & \xrightarrow{\phi} & \mathfrak{F}_R(S_f) & \xrightarrow{i_C^*} & \mathfrak{F}_R(C_f) \\ & & {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\Psi_K} \\ & & \mathfrak{F}_R(R) & \xrightarrow[i_K^*]{} & \mathfrak{F}_R(K) \end{array} \qquad \text{(ii)} \quad \begin{array}{ccc} \mathfrak{F}(A_f) & \xrightarrow{i_C^*\circ\phi} & \mathfrak{F}_R(C_f) \\ {\scriptstyle i_K^*}\downarrow & & \downarrow{\scriptstyle \varepsilon_K^*} \\ \mathfrak{F}(K) & = & \mathfrak{F}_R(K) \end{array}$$

for some algebras S_f, C_f and some morphisms $\phi, i_C^*, \Psi, \Psi_K, \varepsilon_K^*$;

2. There exists such a regular function $f \in A$ and an element $\alpha_f \in \mathfrak{F}(A_f)$ satisfying $i_K^*(\alpha_f) = \alpha$ that in the case of $f \in \mathfrak{q}$ the following relation holds:

$$\text{(iii)} \qquad \Psi_K(i_C^* \circ \phi(\alpha_f)) = \varepsilon_K^*(i_C^* \circ \phi(\alpha_f)),$$

where all morphisms are taken from the corresponding diagrams (i) and (ii) above. □

Indeed, let $f \in A$ be the regular function and $\alpha_f \in \mathfrak{F}(A_f)$ be the element from assertion 2.

If $f \notin \mathfrak{q}$ then there is the commutative diagram where all maps are induced by the canonical inclusions:

$$\begin{array}{ccc} \mathfrak{F}(A_f) & \xrightarrow{i_R^*} & \mathfrak{F}(R) \\ \| & & \downarrow{\scriptstyle i_K^*} \\ \mathfrak{F}(A_f) & \xrightarrow[i_K^*]{} & \mathfrak{F}(K) \end{array}$$

So $i_K^*(i_R^*(\alpha_f)) = i_K^*(\alpha_f) = \alpha$ and we get the required.

In the case of $f \in \mathfrak{q}$ we have the chain of equalities:

$$i_K^*(\Psi \circ \phi(\alpha_f)) \overset{\text{i}}{=} \Psi_K(i_C^* \circ \phi(\alpha_f)) \overset{\text{iii}}{=} \varepsilon_K^*(i_C^* \circ \phi(\alpha_f)) \overset{\text{ii}}{=} i_K^*(\alpha_f) = \alpha$$

and we get the required as well.

1. Let $f \in \mathfrak{q}$ be a regular function. Our aim is to construct the diagrams (i) and (ii).

We start by building up the diagram (i). More precisely we do this in three steps (see items 1.1-1.3 below): First (1.1) we construct the R-algebra S and the map $\phi : \mathfrak{F}(A_f) \to \mathfrak{F}_R(S_f)$. Then (1.2) we produce the homomorphism $\Psi : \mathfrak{F}_R(S_f) \to \mathfrak{F}_R(R)$ (1.2). On the last step (1.3) we construct the K-algebra C, the homomorphism $\Psi_K : \mathfrak{F}_R(C_f) \to \mathfrak{F}_R(K)$ and show the commutativity of the diagram (i).

We end up by producing the diagram (ii) (see item 1.4 below).

1.1. To construct the R-algebra S and the map $\mathfrak{F}(A_f) \xrightarrow{\phi} \mathfrak{F}_R(S_f)$ we use the following result (see Lemma 5.12, [Qu], or sect.7, [PO]):

Lemma. *Let A be a smooth finite type algebra of dimension d over a field k, let $f \in A$ be a regular function, and let $\mathfrak{I}$ be a finite subset of $\operatorname{Spec} A$. Then there exist functions $x_1, \dots, x_d$ in A algebraically independent over k and such that if $P = k[x_1, \dots, x_{d-1}] \overset{q}{\hookrightarrow} A$, then $A/(f)$ is finite over P; A is smooth over P at points of $\mathfrak{I}$; and the inclusion q factors as $q : P \hookrightarrow P[x_d] \xrightarrow{q_1} A$, where q_1 is finite.*

Apply this lemma to the given k-algebra A, the regular function $f \in \mathfrak{q}$ and the subset $\mathfrak{I} = \{\mathfrak{q}\}$.

Put $S = A \otimes_P R$ to be the tensor product from the canonical diagram:

$$\begin{array}{ccc} A & \xrightarrow{i_S} & A \otimes_P R \\ {\scriptstyle q}\uparrow & & \uparrow{\scriptstyle i} \\ P & \xrightarrow[r]{} & R \end{array}$$

where the map r is the composition $r : P \overset{q}{\hookrightarrow} A \overset{i_R}{\hookrightarrow} R$ and $i_S : a \mapsto a \otimes 1$, $i : r \mapsto 1 \otimes r$.

Consider the homomorphism $\mathfrak{F}(A_f) \xrightarrow{i_S^*} \mathfrak{F}(S_f)$ induced by the map $A_f \xrightarrow{i_S} S_f$.

In the constant case, i.e. when the functor $\mathfrak{F}$ is given on the category of k-algebras (see Theorem II), there is the identity transformation between the restrictions of the functors $\mathfrak{F}$ and $\mathfrak{F}_R$ to the category of S-algebras (see 1.3.1, [Za]). Thus we have $\mathfrak{F}(S_f) = \mathfrak{F}_R(S_f)$ and we put i_S^* to be the desired map ϕ. $\square$

In the general case (Theorem I) the homomorphism i_S^* takes values in the group $\mathfrak{F}(S_f)$ which is different from the group $\mathfrak{F}_R(S_f)$. To avoid this problem we use property **E** of the functor $\mathfrak{F}$. More precisely, following the steps 1-4 of the proof of Proposition, sect.2, [Za], we construct the 3×5 commutative diagram for the triple (A, R, f):

$$\begin{array}{ccccccccc} \mathfrak{F}(A_f) & \xrightarrow{i_g^* \circ i_S^*} & \mathfrak{F}(S_{gf}) & \xrightarrow{e^*} & \mathfrak{F}(\tilde{S}_f) & \xrightarrow{i_2^*} & \mathfrak{F}(\tilde{S}_{hf}) & \xrightarrow{\Phi(\tilde{S}_{hf})} & \mathfrak{F}_R(\tilde{S}_{hf}) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathfrak{F}(A) & \xrightarrow{i_g^* \circ i_S^*} & \mathfrak{F}(S_g) & \xrightarrow{e^*} & \mathfrak{F}(\tilde{S}) & \xrightarrow{i_2^*} & \mathfrak{F}(\tilde{S}_h) & \xrightarrow{\Phi(\tilde{S}_h)} & \mathfrak{F}_R(\tilde{S}_h) \\ {\scriptstyle i_R^*}\downarrow & & {\scriptstyle \varepsilon^*}\downarrow & & {\scriptstyle \tilde{\varepsilon}^*}\downarrow & & {\scriptstyle \tilde{\varepsilon}_h^*}\downarrow & & \downarrow{\scriptstyle \tilde{\varepsilon}_h^*} \\ \mathfrak{F}(R) & = & \mathfrak{F}(R) & = & \mathfrak{F}(R) & = & \mathfrak{F}(R) & = & \mathfrak{F}_R(R) \end{array}$$

Then we put $S = \tilde{S}_h$ to be the desired R-algebra and the composition $\phi = \Phi(\tilde{S}_{hf}) \circ i_2^* \circ e^* \circ i_g^* \circ i_S^*$ to be the desired map $\mathfrak{F}(A_f) \to \mathfrak{F}_R(S_f)$. $\square$

In the subsequent discussion we will use the following properties (see 1.2, [Za]) of the constructed R-algebra S:

S1. S is finite over the polynomial ring $R[t]$ and the quotient $S/(f)$ is finite over R (we identify f with $f \otimes 1$);

S2. There is the augmentation map $\varepsilon : S \to R$ for the R-algebra S defined as $\varepsilon(a \otimes r) = ar$;

S3. S is essentially smooth over the field k and $S/\mathfrak{m}S$ is smooth over the residue field $R/\mathfrak{m}$ at the maximal ideal $\varepsilon^{-1}(\mathfrak{m})/\mathfrak{m}S$, where $\mathfrak{m}$ denotes the maximal ideal of the local ring R.

In particular, there is the commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{i_S} & S \\ {\scriptstyle i_R}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ R & = & R \end{array}$$

where $\varepsilon : S \to R$ is the augmentation from S2 for the inclusion $i : R \to S$. And for the regular function $f \in S$ we have $f \notin \ker \varepsilon$ and $f \in \varepsilon^{-1}(\mathfrak{m})$.

Remark. From this point on we may assume that S is a domain that satisfy properties S1-3.

Indeed, since S is smooth over k it is the product of smooth domains (irreducible components), i.e., $S = \prod_i S_i$. Consider the domain S_j such that $\varepsilon(S_j) \neq 0$. Observe that for other domains S_i, $i \neq j$, we have $\varepsilon(S_i) = 0$. Replace the algebra S by the domain S_j and the map i_S by the composition $\mathrm{pr}_j \circ i_S$, where $\mathrm{pr}_j : S \to S_j$ is the canonical projection. Clearly properties S1-3 will still hold. □

1.2. Now let us produce the map $\Psi : \mathfrak{F}_R(S_f) \to \mathfrak{F}_R(R)$.

Consider the R-algebra S constructed above. Since S is a domain and it satisfy properties S1-3 we are under the hypotheses of Geometric Presentation Lemma (sect.4). Applying this Lemma we get a regular function $t' \in \ker \varepsilon$, where ε is the augmentation map from S2, such that:

G1. S is finite over $R[t']$;

G2. There is an ideal J coprime with $\ker \varepsilon$ and $\ker \varepsilon \cap J = (t')$;

G3. (f) is coprime with J and with $(t' - 1)$.

G4. Algebras $S/(t')$ and $S/(t' - 1)$ are etale over R.

We construct the homomorphism $\Psi : \mathfrak{F}_R(S_f) \to \mathfrak{F}_R(R)$ according to the following definition:

Definition. Let we have an R-algebra S, an augmentation map $\varepsilon : S \to R$, regular functions $f \in S$ and $t' \in \ker \varepsilon$ that satisfy properties G1-3. Under these conditions we define the homomorphism $\Psi : \mathfrak{F}_R(S_f) \to \mathfrak{F}_R(R)$ by putting $\Psi(\beta) = \mathrm{Tr}_1({\mathrm{can}'_1}^*(\beta)) - \mathrm{Tr}_J({\mathrm{can}'_J}^*(\beta))$ for every $\beta \in \mathfrak{F}_R(S_f)$, where $\mathrm{can}'_1 : S_f \to S/(t' - 1)$ and $\mathrm{can}'_J : S_f \to S/J$ denote the canonical maps; $\mathrm{Tr}_1 = \mathrm{Tr}_R^{S/(t'-1)}$ and $\mathrm{Tr}_J = \mathrm{Tr}_R^{S/J}$ denote the transfer maps.

Observe that the canonical maps can'_1 and can'_J exist because of property G3 saying that the function f is invertible in $S/(t' - 1)$ and S/J. The transfer maps Tr_1 and Tr_J exist by property **T** since the R-algebras $S/(t' - 1)$ and S/J are finitely generated and projective as the R-modules. Indeed, since S and $R[t']$ are both regular property G1 and Grothendieck's theorem (Corollary 18.17, [Ei]) show that S is finitely generated and projective as the $R[t']$-module. Extending the scalars via $R[t'] \xrightarrow{t' \mapsto 0} R$ and $R[t'] \xrightarrow{t' \mapsto 1} R$ we get that $S/(t')$ and $S/(t' - 1)$ are finitely generated projective as the R-modules. By property G2 we get the canonical decomposition $S/(t') \cong S/\ker \varepsilon \times S/J \cong R \times S/J$. Thus S/J is finitely generated projective as the R-module as well. □

Remark. Let us show that the following diagram commutes (it means that the homomorphism Ψ can be considered as a lifting of the augmentation map ε^*):

$$\begin{array}{ccc} \mathfrak{F}_R(S) & \longrightarrow & \mathfrak{F}_R(S_f) \\ {\scriptstyle \varepsilon^*}\big\downarrow & & \big\downarrow{\scriptstyle \Psi} \\ \mathfrak{F}_R(R) & = & \mathfrak{F}_R(R) \end{array}$$

To do this we follow the proof of Specialization Lemma, sect.1.1, [Za]:

Let an element $\alpha_f \in \mathfrak{F}_R(S_f)$ comes from an element $\alpha \in \mathfrak{F}_R(S)$ via the map $\mathfrak{F}_R(S) \to \mathfrak{F}_R(S_f)$ induced by the canonical inclusion.

By property G3 there are two canonical compositions:

$$\mathrm{can}_J : S \to S_f \stackrel{\mathrm{can}'_J}{\to} S/J \quad \text{and} \quad \mathrm{can}_1 : S \to S_f \stackrel{\mathrm{can}'_1}{\to} S/(t'-1).$$

Thus we have

$$\Psi(\alpha_f) = \mathrm{Tr}_1(\mathrm{can}_1^*(\alpha)) - \mathrm{Tr}_J(\mathrm{can}_J^*(\alpha)).$$

By property $\mathbf{T}$.(b1) we have $\mathrm{Tr}_1(\mathrm{can}_1^*(\alpha)) = \mathrm{Tr}_0(\mathrm{can}_0^*(\alpha))$, where $\mathrm{Tr}_0 = \mathrm{Tr}_R^{S/(t')}$ denotes the transfer map and $S \stackrel{\mathrm{can}_0}{\longrightarrow} S/(t')$ is the canonical map.

By property $\mathbf{T}$.(a) applied to the decomposition $S/(t') \cong S/\ker\varepsilon \times S/J$ we get $\mathrm{Tr}_0(\mathrm{can}_0^*(\alpha)) = \varepsilon^*(\alpha) + \mathrm{Tr}_J(\mathrm{can}_J^*(\alpha))$. Thus we have

$$\Psi(\alpha_f) = \mathrm{Tr}_0(\mathrm{can}_0^*(\alpha)) - \mathrm{Tr}_J(\mathrm{can}_J^*(\alpha)) = \varepsilon^*(\alpha). \quad \square$$

1.3. To construct the K-algebra C and the homomorphism Ψ_K consider the scalar extension diagram induced by the canonical inclusion $R \stackrel{i_K}{\hookrightarrow} K$:

$$\begin{array}{ccc} S & \stackrel{i_C}{\longrightarrow} & C = S \otimes_R K \\ {\scriptstyle i}\big\uparrow & & \big\uparrow{\scriptstyle i} \\ R & \underset{i_K}{\longrightarrow} & K \end{array}$$

It is easy to see that the K-algebra C and the regular function $f \in C$ satisfy properties S1-3. Indeed, since $K[t \otimes 1] = R[t] \otimes_R K$ and $S \otimes_R K/(f \otimes 1) = S/(f) \otimes_R K$ the property S1 holds for the corresponding functions $t = t \otimes 1$ and $f = f \otimes 1$ in $S \otimes_R K$. Put the augmentation $\varepsilon_K : S \otimes_R K \to K$ to be the multiplication $\varepsilon_K(s \otimes k) = \varepsilon(s) \cdot k$. The property S3 follows immediately since it is compatible with the scalar extension.

Consider all corresponding data got by the scalar extension: the K-algebra C, the augmentation ε_K, the functions $t' = t' \otimes 1$, $f = f \otimes 1 \in C$ and the ideals $\ker\varepsilon_K = \ker\varepsilon \otimes 1$, $J = J \otimes 1 \subset C$. Clearly it satisfy properties G1-4. Let $\Psi_K : \mathfrak{F}_R(C_f) \to \mathfrak{F}_R(K)$ be the homomorphism defined for these data according to 1.2.

Lemma. *The following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{F}_R(S_f) & \stackrel{i_C^*}{\longrightarrow} & \mathfrak{F}_R(C_f) \\ {\scriptstyle \Psi}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_K} \\ \mathfrak{F}_R(R) & \underset{i_K^*}{\longrightarrow} & \mathfrak{F}_R(K) \end{array}$$

Proof. It follows immediately from the definition of homomorphism Ψ and from two commutative diagrams, where the upper squares are got by applying the functor $\mathfrak{F}_R$ to the corresponding scalar extension diagrams, the lower squares coincide with those from the property **T**.(b2) :

$$\begin{array}{ccc} \mathfrak{F}_R(S_f) & \xrightarrow{i_C^*} & \mathfrak{F}_R(C_f) \\ {\scriptstyle {\mathrm{can}'_1}^*}\big\downarrow & & \big\downarrow{\scriptstyle {\mathrm{can}'_1}^*} \\ \mathfrak{F}_R(S/(t'-1)) & \longrightarrow & \mathfrak{F}_R(C/(t'-1)) \\ {\scriptstyle \mathrm{Tr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Tr}_1} \\ \mathfrak{F}_R(R) & \xrightarrow[i_K^*]{} & \mathfrak{F}_R(K) \end{array} \qquad \begin{array}{ccc} \mathfrak{F}_R(S_f) & \xrightarrow{i_C^*} & \mathfrak{F}_R(C_f) \\ {\scriptstyle {\mathrm{can}'_J}^*}\big\downarrow & & \big\downarrow{\scriptstyle {\mathrm{can}'_J}^*} \\ \mathfrak{F}_R(S/J) & \longrightarrow & \mathfrak{F}_R(C/J) \\ {\scriptstyle \mathrm{Tr}_J}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Tr}_J} \\ \mathfrak{F}_R(R) & \xrightarrow[i_K^*]{} & \mathfrak{F}_R(K) \end{array} \quad \square$$

Finally taking together the map $\phi : \mathfrak{F}(A_f) \to \mathfrak{F}_R(S_f)$ constructed in 1.1 and the square diagram from the lemma above we get the desired diagram (i).

1.4. To get the diagram (ii) in the constant case (see 1.1) we apply the functor $\mathfrak{F}$ to the following commutative diagram:

$$\begin{array}{ccc} A_f & \xrightarrow{i_C \circ i_S} & C_f \\ {\scriptstyle i_K}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_K} \\ K & =\!= & K \end{array}$$

where $C = S \otimes_R K = A \otimes_P K$, $i_C \circ i_S : a \mapsto a \otimes 1$ and $\varepsilon_K : a \otimes k \mapsto ak$.

In the general case consider the 3×5 diagram for the triple (A, R, f) mentioned in 1.1. By extending the scalars via the canonical inclusion $R \overset{i_K}{\hookrightarrow} K$ and denoting $T = S \otimes_R K$ we get the following commutative diagram:

$$\begin{array}{ccccccccc} \mathfrak{F}(A_f) & \xrightarrow{i_g^* \circ i_T^*} & \mathfrak{F}(T_{gf}) & \xrightarrow{e^*} & \mathfrak{F}(\tilde{T}_f) & \xrightarrow{i_2^*} & \mathfrak{F}(\tilde{T}_{hf}) & \xrightarrow{\Phi(\tilde{T}_{hf})} & \mathfrak{F}_R(\tilde{T}_{hf}) \\ \big\uparrow & & \big\uparrow & & \big\uparrow & & \big\uparrow & & \big\uparrow \\ \mathfrak{F}(A) & \xrightarrow{i_g^* \circ i_T^*} & \mathfrak{F}(T_g) & \xrightarrow{e^*} & \mathfrak{F}(\tilde{T}) & \xrightarrow{i_2^*} & \mathfrak{F}(\tilde{T}_h) & \xrightarrow{\Phi(\tilde{T}_h)} & \mathfrak{F}_R(\tilde{T}_h) \\ {\scriptstyle i_K^*}\big\downarrow & & {\scriptstyle \varepsilon^*}\big\downarrow & & {\scriptstyle \tilde{\varepsilon}^*}\big\downarrow & & {\scriptstyle \tilde{\varepsilon}_h^*}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\varepsilon}_h^*} \\ \mathfrak{F}(K) & = \!= & \mathfrak{F}(K) & = \!= & \mathfrak{F}(K) & = \!= & \mathfrak{F}(K) & = \!= & \mathfrak{F}_R(K) \end{array}$$

One has the obvious homomorphisms from the first row of this diagram to the last row given by i_K and the extended augmentations. So there is the 2×5 commutative diagram:

$$\begin{array}{ccccccccc} \mathfrak{F}(A_f) & \xrightarrow{i_g^* \circ i_T^*} & \mathfrak{F}(T_{gf}) & \xrightarrow{e^*} & \mathfrak{F}(\tilde{T}_f) & \xrightarrow{i_2^*} & \mathfrak{F}(\tilde{T}_{hf}) & \xrightarrow{\Phi(\tilde{T}_{hf})} & \mathfrak{F}_R(\tilde{T}_{hf}) \\ {\scriptstyle i_K^*}\big\downarrow & & {\scriptstyle \varepsilon^*}\big\downarrow & & {\scriptstyle \tilde{\varepsilon}^*}\big\downarrow & & {\scriptstyle \tilde{\varepsilon}_h^*}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\varepsilon}_h^*} \\ \mathfrak{F}(K) & = \!= & \mathfrak{F}(K) & = \!= & \mathfrak{F}(K) & = \!= & \mathfrak{F}(K) & = \!= & \mathfrak{F}_R(K) \end{array}$$

Indeed, the middle squares commutes since the augmentations are compatible with each other. The first square commutes by definition of T. And the last square commutes by property **E**.(c).

In our notation the K-algebra C coincides with $\tilde{T}_h$, the composition $i_C^* \circ \phi$ coincides with the upper row $\Phi(\tilde{T}_{hf}) \circ i_2^* \circ e^* \circ i_g^* \circ i_T^*$ and the augmentation ε_K coincides with $\tilde{\varepsilon}_h$. So the outer contour of this diagram gives us the desired diagram (ii).

2. Let $\alpha \in \mathfrak{F}_{nr}(K)$ be the given element. Our goal is to check the assertion 2.

More precisely we have to construct a regular function $f \in A$ and an element $\alpha_f \in \mathfrak{F}(A_f)$ with the property $i_K^*(\alpha_f) = \alpha$ such that relation (iii) holds. To do this we use the following lemma:

2.1 Lemma. *There exist n regular functions $f_i \in A$, $i = 1 \dots n$, and n elements $\alpha_{f_i} \in \mathfrak{F}(A_{f_i})$ such that*

(a) *The elements α_{f_i} have the same image in $\mathfrak{F}(K)$ under the maps induced by the canonical inclusions $A_{f_i} \to K$ and this image coincides with the element α.*

(b) *The closed subset $Spec\, A \setminus (Spec\, A_{f_1} \cup \ldots \cup Spec\, A_{f_n})$ is of codimension at least 2 in $Spec\, A$ (a family of functions satisfying this condition will be called coprime in A in codimension 1).*

(c) *The elements α_{f_i} have the same image in $\mathfrak{F}(A_{f_1 \dots f_n})$ under the maps induced by the canonical inclusions $A_{f_i} \to A_{f_1 \dots f_n}$.*

Proof. First we prove the local version of the lemma got by writing R instead of A everywhere in the conditions.

By property **C** of the functor $\mathfrak{F}$ we have the canonical isomorphism $\varinjlim_{f \in \mathfrak{m}} \mathfrak{F}(R_f) \xrightarrow{\cong} \mathfrak{F}(K)$, where $\mathfrak{m}$ is the maximal ideal of the local ring R. So there exists a regular function $g \in \mathfrak{m}$ such that the element α comes from an element $\alpha_g \in \mathfrak{F}(R_g)$.

Let $(g) = \mathfrak{p}_1^* \mathfrak{p}_2^* \dots \mathfrak{p}_m^*$ be the factorization of the principal ideal (g) into the product of principle prime ideals of height 1 (here '$*$' means any natural number). It is unique and exists because R is a local regular ring and, thus, factorial [AK].

By property **C** we have m canonical isomorphisms $\varinjlim_{f \in \mathfrak{m} \setminus \mathfrak{p}_i} \mathfrak{F}(R_f) \xrightarrow{\cong} \mathfrak{F}(R_{\mathfrak{p}_i})$, $i = 1 \dots m$. So for every i there exists such a regular function $g_i \in \mathfrak{m} \setminus \mathfrak{p}_i$ that the element $\alpha \in \mathfrak{F}_{nr}(K) \subset \mathfrak{F}(R_{\mathfrak{p}_i})$ comes from an element $\alpha_{g_i} \in \mathfrak{F}(R_{g_i})$.

By the construction the elements $\alpha_g, \alpha_{g_1}, \dots, \alpha_{g_m}$ have the same image in $\mathfrak{F}(K)$ and this image coincides with α.

Since $(g) = \mathfrak{p}_1^* \mathfrak{p}_2^* \dots \mathfrak{p}_m^*$ and $g_i \notin \mathfrak{p}_i$, $i = 1 \dots m$, there is no such a prime ideal $\mathfrak{p} \in Spec\, R$ of height 1 that $g, g_1, \dots, g_m \in \mathfrak{p}$. This implies that the functions $g, g_1, \dots g_m$ are coprime in R in codimension 1.

Sublemma. *Let f, $g \in \mathfrak{m}$ be two regular functions. Let $\alpha_f \in \mathfrak{F}(R_f)$ and $\alpha_g \in \mathfrak{F}(R_g)$ be two elements that have the same image in $\mathfrak{F}(K)$.*

Then there exists such a regular function $h \in \mathfrak{m}$ coprime with f and g in R in codimension 1 that the elements α_f and α_g have the same image in $\mathfrak{F}(R_{fgh})$.

Proof of Sublemma. By property **C** for the fixed regular function $fg \in \mathfrak{m}$ we have the canonical isomorphism $\varinjlim_{h \in \mathfrak{m}} \mathfrak{F}(R_{fgh}) \xrightarrow{\cong} \mathfrak{F}(K)$.

Since the element $\alpha_f - \alpha_g$ has the trivial image in $\mathfrak{F}(K)$ there exists such a regular function $h' \in \mathfrak{m}$ that the element $\alpha_f - \alpha_g$ has the trivial image in $\mathfrak{F}(R_{fgh'})$.

Consider the factorizations of the ideals (fg) and (h') into the product of principle prime ideals of height 1. Let (d) be the greatest common divisor of (fg) and (h') in these factorizations. Then we have $h' = hd$ for some regular function h. Clearly h is coprime with fg in R in codimension 1 and $R_{fgh} = R_{fgh'}$. $\square$

Apply now Sublemma to the each pair of regular functions (g, g_i), $i = 1 \dots m$. We get m regular functions h_i.

Put $f_1 = g$, $f_2 = g_1 h_1$, $f_3 = g_2 h_2$, $\dots$, $f_{m+1} = g_m h_m$ and $n = m + 1$. Let $\alpha_{f_1}, \alpha_{f_2}, \dots, \alpha_{f_n}$ denote the corresponding images of the elements $\alpha_g, \alpha_{g_1}, \dots, \alpha_{g_m}$. We claim that f_i, $i = 1 \dots n$, are required functions and α_{f_i} are required elements.

Indeed, (a) holds for the elements α_{f_i} by definition. Since each function h_i is coprime with g in R in codimension 1 the functions $g, g_1h_1, \ldots, g_mh_m$ are coprime in codimension 1 as well and we get (b). Item (c) holds because the elements α_g and $\alpha_{g_ih_i}$ have the same image in $\mathfrak{F}(R_{gg_ih_i})$. Thus we have proved the local version of the lemma.

Let us lift the functions f_i and the elements α_{f_i} by the following way:

Since $\varinjlim_{f\notin\mathfrak{q}} A_f = R$ there exists such a regular function $g \notin \mathfrak{q}$ that the functions $f_i \in R$ are regular on the localization A_g. Thus replacing A_g by A we may assume $f_i \in A$.

The canonical isomorphism $\varinjlim_{f\notin\mathfrak{q}} \mathfrak{F}(A_{ff_i}) \stackrel{\cong}{\to} \mathfrak{F}(R_{f_i})$ implies that for every i there exists such a regular function $h_i \notin \mathfrak{q}$ that the element $\alpha_{f_i} \in \mathfrak{F}(R_{f_i})$ comes from $\mathfrak{F}(A_{h_if_i})$. Replacing $A_{h_1h_2\ldots h_n}$ by A we may assume $\alpha_{f_i} \in \mathfrak{F}(A_{f_i})$.

So we get n regular functions $f_i \in A$ and n elements $\alpha_{f_i} \in \mathfrak{F}(A_{f_i})$ that still satisfy conditions (a) and (c).

To see (b) let the closed subset $Spec\,A \setminus (Spec\,A_{f_1} \cup \ldots \cup Spec\,A_{f_n})$ contains a finite number of irreducible components $\mathfrak{p}_j \in Spec\,A$ of codimension 1. Since (b) holds for the local version of the lemma we may assume that $\mathfrak{p}_j \subsetneq \mathfrak{q}$ for all j. Take such a regular function $h \notin \mathfrak{q}$ that $h \in \mathfrak{p}_j$ for all j. Then replacing A_h by A we get the required.

And we finish the proof of Lemma. □

Applying this lemma we get the set of n regular functions $f_1, \ldots, f_n \in A$ and n elements $\alpha_{f_i} \in \mathfrak{F}(A_{f_i})$, $i = 1\ldots n$, that satisfy (a), (b) and (c).

Put $f = f_1f_2\ldots f_n$. According to (c) let α_f denote the image in $\mathfrak{F}(A_f)$ of the elements α_{f_i} under the map induced by the canonical inclusions $A_{f_i} \to A_f$. By (a) we have $i_K^*(\alpha_f) = \alpha$. We may assume that $f, f_1, f_2, \ldots, f_n \in \mathfrak{q}$ otherwise the Purity Theorem will follows immediately from the arguments concerning assertions 1 and 2 at the end of page 7.

We claim that the regular function $f \in \mathfrak{q}$ and the element $\alpha_f \in \mathfrak{F}(A_f)$ satisfy relation (iii). In other words f is the desired function and α_f is the desired element.

2.2. Consider the diagrams (i) and (ii) constructed (see item 1) for the regular function f. Let us rewrite relation (iii) in terms of the homomorphism $\Phi_{C_f} : \mathfrak{I}(C_f) \to Hom(\mathfrak{F}_R(C_f), \mathfrak{F}_R(K))$ defined in 1.3, sect.3.

Consider the data (see 1.3) used to construct the homomorphism $\Psi_K : \mathfrak{F}_R(C_f) \to \mathfrak{F}_R(K)$: the K-algebra C, the augmentation map ε_K, the regular functions f and t', the ideals $\ker\varepsilon_K$ and J. Recall that it satisfy properties G1-4.

Since $f \notin \ker\varepsilon_K$ (see 1.1) and $C/\ker\varepsilon_K = K$ the ideal (f) is coprime with $I = \ker\varepsilon_K$. Combining this fact with properties G2 and G3 we get that (f) is coprime with ideals I, J, $(t') = IJ$ and $(t'-1)$. By property G4 and G1 the algebras C/I, C/J, $C/(t')$ and $C/(t'-1)$ are etale over R.

According to Lemma 1.5, sect.3, the homomorphism Ψ_K can be rewritten as follows:

$$\Psi_K = \mathrm{Tr}_1 \circ \mathrm{can}_1'^{\,*} - \mathrm{Tr}_J \circ \mathrm{can}_J'^{\,*} = \Phi_{C_f}((t'-1)) - \Phi_{C_f}(J) =$$

$$= \Phi_{C_f}((t'-1)) - \Phi_{C_f}((t')) + \Phi_{C_f}(I) = \Phi_{C_f}((\tfrac{t'-1}{t'}) \cdot I),$$

and for the augmentation map $\varepsilon_K^* : \mathfrak{F}_R(C_f) \to \mathfrak{F}_R(K)$ we have $\varepsilon_K^* = \Phi_{C_f}(I)$. Thus in terms of the homomorphism Φ_{C_f} relation (iii) looks as follows:

$$\Phi_{C_f}((\tfrac{t'-1}{t'}) \cdot I)(i_C^* \circ \phi(\alpha_f)) = \Phi_{C_f}(I)(i_C^* \circ \phi(\alpha_f)).$$

Let $\mathcal{C} = Spec\,C$ denote the affine smooth curve over the field K and $\mathcal{C}^0 = Spec\,C_f$ be its principal open subset. Let Δ be the K-rational point corresponding to the augmentation ε_K and

(δ) be the divisor of the rational function $\delta = \frac{t'-1}{t'}$ on $\mathcal{C}^0$. Then using the geometric notation for the homomorphism Φ we may rewrite relation (iii) as

$$\Phi_{\mathcal{C}^0}(\Delta + (\delta))(\alpha^0) = \Phi_{\mathcal{C}^0}(\Delta)(\alpha^0), \quad \alpha^0 = i_C^* \circ \phi(\alpha_f). \qquad (*)$$

So to prove assertion 2 we have to show that relation $(*)$ holds. To do this we use Cor.2.2, sect.3.

2.3. Let us produce the data that satisfy the hypotheses of Theorem 2.1 of sect.3. They are given at the end of this section.

Recall that we have n regular functions $f_1, f_2, \dots, f_n \in \mathfrak{q} \subset A$ and n elements $\alpha_{f_i} \in \mathfrak{F}(A_{f_i})$, $i = 1 \dots n$, that satisfy conditions (a), (b) and (c) of Lemma 2.1.

For every $i = 1 \dots n$ let f_i denote the image in C of the function f_i under the composition $i_2 \circ e \circ i_g \circ i_S : A \to C$ (see the upper row of the 3×5 diagram from 1.1) or under the composition $i_C \circ i_S : A \to C$ in the constant case. Put $\mathcal{C}_i = Spec\, C_{f_i}$ and let $Z_i = \mathcal{C} \setminus \mathcal{C}_i$ denote the closed subset corresponding to the principal ideal (f_i) in C.

By condition (b) of Lemma 2.1 the closed subset $Spec\, A \setminus \cup_{i=1}^n Spec\, A_{f_i}$ is of codimension at least 2 in $Spec\, A$. Since the composition $A \to C$ above is flat the closed subset $\mathcal{C} \setminus \cup_{i=1}^n \mathcal{C}_i$ is of codimension at least 2 in $\mathcal{C}$ as well. Thus it is empty. In other words we have $\cap_{i=1}^n Z_i = \emptyset$.

Since $f = f_1 f_2 \dots f_n$ we have $\mathcal{C}^0 = \cap_{i=1}^n \mathcal{C}_i = \mathcal{C} \setminus \cup_{i=1}^n Z_i$.

For every $i = 1 \dots n$ let α_i denote the image in $\mathfrak{F}_R(\mathcal{C}_i)$ of the element $\alpha_{f_i} \in \mathfrak{F}(A_{f_i})$ under the map $\mathfrak{F}(A_{f_i}) \xrightarrow{i_C^* \circ \phi} \mathfrak{F}_R(C_{f_i})$ induced by the composition above. Since the following diagram commutes for every i:

$$\begin{array}{ccc} \mathfrak{F}(A_{f_i}) & \xrightarrow{i_C^* \circ \phi} & \mathfrak{F}_R(C_{f_i}) \\ \downarrow & & \downarrow \\ \mathfrak{F}(A_f) & \xrightarrow[i_C^* \circ \phi]{} & \mathfrak{F}_R(C_f) \end{array}$$

we get $\alpha_i|_{\mathcal{C}^0} = i_C^* \circ \phi(\alpha_{f_i})|_{C_f} = i_C^* \circ \phi(\alpha_{f_i}|_{A_f}) = i_C^* \circ \phi(\alpha_f) = \alpha^0$. $\square$

Consider now the regular function $t' \in \ker \varepsilon_K \subset C$ arisen in 2.2.

Let $\hat{\mathcal{C}} = \mathcal{C} \amalg \mathcal{C}_\infty$ be the smooth projective model of the affine curve $\mathcal{C}$. Let $\mathbb{A}_K^1 = Spec\, K[t']$ be the affine line and $\mathbb{P}_K^1 = Proj\, K[T_0, T_1]$ be the projective line with $T_0/T_1 = t'$. In other words we have $\mathbb{P}_K^1 = \mathbb{A}_K^1 \amalg \{\infty\}$, where $\infty = (1, 0)$.

By property S1 the morphism $\mathcal{C} \xrightarrow{t'} \mathbb{A}_K^1$ is finite. So there is a finite morphism $\hat{\mathcal{C}} \xrightarrow{\hat{t'}} \mathbb{P}_K^1$ such that $\mathcal{C}_\infty = \hat{t'}^{-1}(\infty)$ and the following diagram commutes, where the horizontal morphisms are the open immersions:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \hat{\mathcal{C}} \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle \hat{t'}} \\ \mathbb{A}_K^1 & \longrightarrow & \mathbb{P}_K^1 \end{array}$$

Consider the rational function $\hat{\delta} = \frac{T_0 - T_1}{T_0}$ on $\hat{\mathcal{C}}$.

Clearly $\hat{\delta}$ is regular in a neighbourhood of $\mathcal{C}_\infty$ and $\hat{\delta}|_{\mathcal{C}_\infty} \equiv \hat{\delta}|_{(1:0)} = 1$. Obviously we have $\hat{\delta}|_{\mathcal{C}} = \delta$ and $(\delta) = (\hat{\delta})$. As it was mentioned in 2.2 the element f is coprime with (t') and $(t' - 1)$. Therefore we have $supp\ (\delta) \subset \mathcal{C}^0$. Since the algebras $C/(t' - 1)$ and $C/(t')$ are etale over R the divisor (δ) is unramified (see Definition 1.4, sect.3). $\square$

Summarizing the discussion above we have the following data:

- the functor $\mathfrak{F}_R$ restricted to the category of K-algebras and the smooth affine curve $\mathcal{C}$ over K;
- the n closed subsets $Z_i \subset \mathcal{C}$, $i = 1 \dots n$, and n elements $\alpha_i \in \mathfrak{F}_R(\mathcal{C} \setminus Z_i)$ such that $\bigcap_{i=1}^n Z_i = \emptyset$, $\mathcal{C} \setminus \bigcup_{i=1}^n Z_i = \mathcal{C}^0$ and $\alpha_i|_{\mathcal{C}^0} = \alpha^0$;
- the rational function $\bar{\delta}$ on the smooth projective model $\hat{\mathcal{C}} = \mathcal{C} \amalg \mathcal{C}_\infty$ of the affine curve $\mathcal{C}$ such that $\hat{\delta}|_{\mathcal{C}_\infty} \equiv 1$ (the function $\hat{\delta}$ is regular in a neighbourhood of $\mathcal{C}_\infty$), $(\hat{\delta}) = (\delta)$ and the divisor (δ) is unramified with $supp\,(\delta) \subset \mathcal{C}^0$.

Observe that we are under the hypotheses of Theorem 2.1, sect.3 now. So applying Corollary 2.2, sect.3, we get relation $(*)$. And we finish the proof of assertion 2 and the proof of the Purity Theorem. $\square$

§3. The Divisor Theorem

1.1. Let K be an infinite field. Let $\mathfrak{G}$ be a covariant functor from the category of K-algebras to the category of abelian groups that satisfies property **T**. We also use the geometric notation for the functor $\mathfrak{G}$ as a contravariant functor from the category of affine K-schemes to the category of abelian groups.

1.2. Let $\mathcal{C}$ be a smooth affine curve over the field K. There is an isomorphism between the group $Div(\mathcal{C})$ of divisors of the curve $\mathcal{C}$ and the group $\mathfrak{I}(C)$ of fractional ideals of the Dedekind domain $C = K[\mathcal{C}]$. Let I_D denote the fractional ideal corresponding to divisor $D \in Div(\mathcal{C})$. Let $I_D = \mathfrak{p}_1^{n_1}\mathfrak{p}_2^{n_2}\dots\mathfrak{p}_m^{n_m}$, $n_i \in \mathbb{Z}$, be the factorization of I_D into the product of prime ideals $\mathfrak{p}_i \in Spec\,C$.

1.3. Definition. We define a homomorphism

$$\Phi_C : \mathfrak{I}(C) \longrightarrow Hom(\mathfrak{G}(C), \mathfrak{G}(K))$$

or using the geometric notation

$$\Phi_{\mathcal{C}} : Div(\mathcal{C}) \longrightarrow Hom(\mathfrak{G}(\mathcal{C}), \mathfrak{G}(Spec\,K))$$

by the following way:

For a divisor $D = n_1x_1 + n_2x_2 + \dots n_mx_m$ and the corresponding fractional ideal $I_D = \mathfrak{p}_1^{n_1}\mathfrak{p}_2^{n_2}\dots\mathfrak{p}_m^{n_m}$, we put $\Phi_{\mathcal{C}}(D) = \Phi_C(I_D) = n_1\Psi_{\mathfrak{p}_1} + n_2\Psi_{\mathfrak{p}_2} + \dots + n_m\Psi_{\mathfrak{p}_m}$, where

$$\Psi_{\mathfrak{p}_i} : \mathfrak{G}(C) \xrightarrow{\mathrm{can}_i^*} \mathfrak{G}(C/\mathfrak{p}_i) \xrightarrow{\mathrm{Tr}} \mathfrak{G}(K), \quad i = 1 \dots m$$

is the composition of the morphism induced by the canonical map $C \stackrel{\mathrm{can}}{\to} C/\mathfrak{p}_i$ and the transfer map. The latter exists since the residue field $C/\mathfrak{p}_i$ is finite over K. $\square$

1.4. Definition. A divisor $D = n_1x_1 + \dots + n_mx_m$ of a smooth curve $\mathcal{C}$ is called unramified if $n_i = \pm 1$ for all $i = 1 \dots m$. Correspondingly a fractional ideal I is called unramified if $I = \mathfrak{p}_1^{\pm 1}\mathfrak{p}_2^{\pm 1}\dots\mathfrak{p}_m^{\pm 1}$.

It is easy to see that an effective divisor D is unramified iff the corresponding algebra C/I_D is etale over K. $\square$

1.5. Lemma. *Let $D = x_1 + \ldots + x_m$ be an unramified effective divisor of the curve $\mathcal{C}$. Then the homomorphism $\Phi_{\mathcal{C}}(D) = \Phi_C(I_D)$ coincides with the composition*

$$\Psi_{I_D} : \mathfrak{G}(C) \xrightarrow{\mathrm{can}^*_{I_D}} \mathfrak{G}(C/I_D) \xrightarrow{\mathrm{Tr}} \mathfrak{G}(K).$$

Proof. It is enough to check the case when $D = x_1 + x_2$.

By Chinees Remainder Theorem we have $C/I_D = C/\mathfrak{p}_1 \times C/\mathfrak{p}_2$. By additivity of the transfer **T**.(a) we get:

$$\mathrm{Tr}_K^{C/I_D}(\mathrm{can}^*_{I_D}(x)) = \mathrm{Tr}_K^{C/\mathfrak{p}_1}(\mathrm{can}^*_1(x)) + \mathrm{Tr}_K^{C/\mathfrak{p}_2}(\mathrm{can}^*_2(x)),$$

for all elements $x \in \mathfrak{G}(C)$. Thus $\Psi_{I_D} = \Psi_{\mathfrak{p}_1} + \Psi_{\mathfrak{p}_2} = \Phi_C(I_D)$. $\square$

1.6. Let $\hat{\mathcal{C}}$ be the smooth projective model of the affine curve $\mathcal{C}$, i.e., $\hat{\mathcal{C}} = \mathcal{C} \amalg \{\infty\}$ is the smooth projective curve over K, where $\{\infty\}$ denotes a closed subset of $\hat{\mathcal{C}}$ finite over K. Informally speaking the set $\{\infty\}$ consists from infinity points of the affine curve $\mathcal{C}$. Observe that $\mathcal{C}$ is the open affine subset of $\hat{\mathcal{C}}$.

2. The goal of this section is to prove the following result:

2.1. Theorem (The Divisor Theorem)**.** *Let there are given a functor $\mathfrak{G}$ and a smooth affine curve $\mathcal{C}$. Thus we get the homomorphism Φ and the smooth projective model $\hat{\mathcal{C}} = \mathcal{C} \amalg \{\infty\}$.*

Let there are given n closed subsets $Z_1, Z_2, \ldots, Z_n$ of the affine curve $\mathcal{C}$ with $\bigcap_{i=1}^n Z_i = \emptyset$. Denote $Z = \bigcup_{i=1}^n Z_i$, $\mathcal{C}^0 = \mathcal{C} \setminus Z$ and $\mathcal{C}_i = \mathcal{C} \setminus Z_i$, $i = 1 \ldots n$.

Let there are given n elements $\alpha_i \in \mathfrak{G}(\mathcal{C}_i)$, $i = 1 \ldots n$, and element $\alpha^0 \in \mathfrak{G}(\mathcal{C}^0)$ such that $\alpha_i|_{\mathcal{C}^0} = \alpha^0$ for all $i = 1 \ldots n$.

Let δ be a rational function on the projective curve $\hat{\mathcal{C}}$ such that $\delta|_{\{\infty\}} \equiv 1$ (we assume that the function δ is regular in a neighbourhood of infinity) and the divisor (δ) is unramified with $supp\,(\delta) \subset \mathcal{C}^0$.

Then $\Phi_{\mathcal{C}^0}((\delta))(\alpha^0) = 0$.

2.2. Corollary. *Let we are in conditions of the theorem above and there are given a rational point Δ on the affine curve $\mathcal{C}^0$. Then $\Phi_{\mathcal{C}^0}(\Delta + (\delta))(\alpha^0) = \Phi_{\mathcal{C}^0}(\Delta)(\alpha^0)$.*

2.3. First of all we introduce the equivalence relation on the set of divisors of the projective curve $\hat{\mathcal{C}}$. Let $\mathcal{C}^0$ be an open subset of the affine curve $\mathcal{C}$. Let

$$\Phi_{\mathcal{C}^0} : Div(\mathcal{C}^0) \longrightarrow Hom(\mathfrak{G}(\mathcal{C}^0), \mathfrak{G}(Spec\,K))$$

be the corresponding homomorphism. Clearly the following diagram commutes

$$\begin{array}{ccc} Div(\mathcal{C}^0) & \xrightarrow{\Phi_{\mathcal{C}_0}} & Hom(\mathfrak{G}(\mathcal{C}^0), \mathfrak{G}(Spec\,K)) \\ \downarrow & & \downarrow \\ Div(\mathcal{C}) & \xrightarrow[\Phi_{\mathcal{C}}]{} & Hom(\mathfrak{G}(\mathcal{C}), \mathfrak{G}(Spec\,K)) \end{array} \qquad (*)$$

where vertical arrows are induced by the open immersion $\mathcal{C}^0 \hookrightarrow \mathcal{C}$.

2.4. Definition. We say that divisors D_0 and D_1 of the projective curve $\hat{\mathcal{C}}$ are equivalent on $\mathcal{C}^0$ if $supp\,D_0$, $supp\,D_1 \subset \mathcal{C}^0$ and $\Phi_{\mathcal{C}^0}(D_0) = \Phi_{\mathcal{C}^0}(D_1)$. $\square$

2.5. Lemma. *Let Z be a closed subset of $\mathcal{C}$. Let D_0 and D_1 be two divisors of $\hat{\mathcal{C}}$ such that $supp\, D_0$, $supp\, D_1 \subset \mathcal{C} \setminus Z$ and there exists a rational function (δ) on $\hat{\mathcal{C}}$ with properties:*

- *$D_0 - D_1 = (\delta)$;* • *$\delta|_{\{\infty\}} \cong 1$, $\delta|_Z \cong 1$;* • *the principal divisor (δ) is unramified.*

Then divisors D_0 and D_1 are equivalent on $\mathcal{C} \setminus Z$.

Proof. Since $\Phi_{\mathcal{C}^0}$ is a homomorphism and the diagram $(*)$ commutes it is enough to prove that $\Phi_{\mathcal{C}^0}((\delta)) = 0$ for some open subset $\mathcal{C}^0$ of the curve $\hat{\mathcal{C}}$ such that $supp\,(\delta) \subset \mathcal{C}^0 \subset \mathcal{C} \setminus Z$.

The rational function (δ) defines a finite morphism $\delta : \hat{\mathcal{C}} \to \mathbb{P}^1_K$. Let $\mathcal{C}^0 = \hat{\mathcal{C}} \setminus \delta^{-1}(\{1\})$ be an open subset of $\hat{\mathcal{C}}$. Since $\delta|_{\{\infty\}} \equiv 1$ and $\delta|_Z \equiv 1$ we get that $\mathcal{C}^0$ is the open subset in $\mathcal{C} \setminus Z$. Consider the fibred product diagram

$$\begin{array}{ccccccc}
\hat{\mathcal{C}} & \longleftarrow & \mathcal{C}^0 & \supset & (\delta)_0 & & (\delta)_1 \\
{\scriptstyle \delta}\big\downarrow{\scriptstyle \text{finite}} & & {\scriptstyle \delta}\big\downarrow{\scriptstyle \text{finite}} & & \big\downarrow & & \big\downarrow \\
\mathbb{P}^1_K & \xleftarrow[\text{immersion}]{\text{open}} & \mathbb{P}^1_K \setminus \{1\} & & \{0\} & & \{\infty\} \\
 & & {\scriptstyle \alpha}\big\downarrow{\scriptstyle \cong} & & \big\downarrow & & \big\downarrow \\
 & & \mathbb{A}^1_K & & \{0\} & & \{1\}
\end{array}$$

where α is an isomorphism sending $\{0\}$ to $\{0\}$ and $\{\infty\}$ to $\{1\}$.

Look on the principal unramified divisor (δ). By definition $(\delta) = (\delta)_0 - (\delta)_1$, where $(\delta)_0$ and $(\delta)_1$ are corresponding preimages of $\{0\}$ and $\{1\}$ under the finite morphism $\alpha \circ \delta : \mathcal{C}^0 \to \mathbb{A}^1_K$. Obviously we have $supp\,(\delta)$, $supp\,(\delta)_0$, $supp\,(\delta)_1 \subset \mathcal{C}^0$ and $\Phi_{\mathcal{C}^0}((\delta)) = \Phi_{\mathcal{C}^0}((\delta)_0) - \Phi_{\mathcal{C}^0}((\delta)_1)$.

Thus our aim is to prove $\Phi_{\mathcal{C}^0}((\delta)_0) = \Phi_{\mathcal{C}^0}((\delta)_1)$ for unramified effective divisors $(\delta)_0 = \sum_k x^0_k$ and $(\delta)_1 = \sum_k x^1_k$.

Turn now to the algebraic language. Let $C = K[\mathcal{C}^0]$ be a coordinate ring of the smooth affine curve $\mathcal{C}^0$. The fact that $\mathcal{C}^0$ is finite over $\mathbb{A}^1_K$ via the morphism $t = \alpha \circ \delta$ means that C is finitely generated $K[t]$-module. Moreover, C is projective $K[t]$-module because C is smooth over K.

It is easy to see that divisors $(\delta)_0$ and $(\delta)_1$ of the curve $\mathcal{C}^0$ correspond to the principal ideals (t) and $(t-1)$ of the ring C. By the previous lemma we get

$$\Phi_{\mathcal{C}^0}((\delta)_i) = \Phi_C((t-i)) = \Psi_{(t-i)} : \mathfrak{G}(C) \xrightarrow{\text{can}^*} \mathfrak{G}(C/(t-i)) \xrightarrow{\text{Tr}_i} \mathfrak{G}(K), \quad i = 0, 1.$$

And the equality $\Phi_{\mathcal{C}^0}((\delta)_0) = \Phi_{\mathcal{C}^0}((\delta)_1)$ follows from the axiom **T**.(b1). □

Proof of the Theorem. Apply the following procedure inductively n times starting with the given rational function $\delta = \delta_0$.

Procedure. For the rational function δ_i, $i = 0 \ldots n-1$, we find a rational function δ_{i+1} such that

- $\delta_{i+1}|_{\{\infty\}} \equiv 1$;
- $\delta_{i+1}|_{Z \setminus Z_{i+1}} \equiv 1$ and $\delta_{i+1}|_{Z_{i+1}} \equiv \delta_i|_{Z_{i+1}}$;
- $supp\,(\delta_{i+1}) \subset \mathcal{C}^0$ and $supp\,(\delta_{i+1}) \cap supp\,(\delta_i) = \emptyset$;
- the divisor (δ_{i+1}) is unramified.

Then we have

$$\Phi_{\mathcal{C}^0}((\delta_i))(\alpha^0) = \Phi_{\mathcal{C}_{i+1}}((\delta_i))(\alpha_{i+1}) = \Phi_{\mathcal{C}_{i+1}}((\delta_{i+1}))(\alpha_{i+1}) = \Phi_{\mathcal{C}^0}((\delta_{i+1}))(\alpha^0), \qquad (**)$$

where the left and the right equalities hold because of the diagram $(*)$ and the middle equality holds because of the previous lemma applied to divisors (δ_i) and (δ_{i+1}) on $\mathcal{C}_{i+1}$. Indeed, we have unramified divisor $(\delta_i) - (\delta_{i+1}) = (\delta_i/\delta_{i+1})$ and $(\delta_i/\delta_{i+1})|_{Z_{i+1}\cup\{\infty\}} \equiv 1$ which means that the divisor (δ_i) is equivalent to (δ_{i+1}) on $\mathcal{C}_{i+1}$.

Thus we get $n+1$ rational functions $\delta = \delta_0, \delta_1, \dots, \delta_n$ and n equalities $(**)$.

Taking all these equalities together we get $\Phi_{\mathcal{C}^0}((\delta))(\alpha^0) = \Phi_{\mathcal{C}^0}((\delta_n))(\alpha^0)$, where restriction $\delta_n|_{Z\setminus Z_n \cup Z\setminus Z_{n-1}\cup\ldots\cup Z\setminus Z_1} \equiv 1$. Since $\bigcap_{i=1}^n Z_i = \emptyset$ we have $\delta_n|_Z \equiv 1$.

The previous lemma shows that divisors (δ_n) and $(1) = 0$ are equivalent on $\mathcal{C}^0 = \mathcal{C} \setminus Z$. Thus $\Phi_{\mathcal{C}^0}((\delta_n))(\alpha^0) = \Phi_{\mathcal{C}^0}((1))(\alpha^0) = 0$. And we have finished. $\square$

§4. The Geometric Presentation Lemma

In this section we prove a stronger version of Lemma 5.2, [PO]:

Lemma. *Let R be a local essentially smooth algebra over an infinite field k, $\mathfrak{m}$ its maximal ideal and S an essentially smooth k-algebra (S is a domain), which is finite over the polynomial algebra $R[t]$. Suppose that $\varepsilon : S \to R$ is an R-augmentation and let $I = \ker\varepsilon$. Assume that $S/\mathfrak{m}S$ is smooth over the residue field $R/\mathfrak{m}$ at the maximal ideal $\varepsilon^{-1}(\mathfrak{m})/\mathfrak{m}S$. Given regular function $f \in S$ such that $S/(f)$ is finite over R we can find a $t' \in I$ such that*

G1. *S is finite over $R[t']$;*
G2. *There is an ideal J coprime with I and $I \cap J = (t')$;*
G3. (a) *(f) is coprime with J;* (b) *(f) is coprime with $(t'-1)$;*
G4. (a) *$S/(t')$ is etale over R;* (b) *$S/(t'-1)$ is etale over R.*

Proof. The proof bases up on the proof of Lemma 5.2, [PO] The only difference is the property G4: By construction from [PO] we have some freedom in the choice of the element t'. Showing the algebra $S/(t')$ is etale over R for almost all elements t', we get the required.

1. Replacing t by $t - \varepsilon(t)$ we may assume that $t \in I$. We denote by "bar" the reduction modulo $\mathfrak{m}$. By assumption made on S the quotient $\bar{S} = S/\mathfrak{m}S$ is smooth over the residue field $\bar{R} = R/\mathfrak{m}$ at its maximal ideal $\bar{I} = \varepsilon^{-1}(\mathfrak{m})/\mathfrak{m}S$. Choose an $\alpha \in R$ such that $\bar{\alpha}$ is a local parameter of the localization $\bar{S}_{\bar{I}}$ of $\bar{S}$ at $\bar{I}$. By Chinees Remainder's theorem we may assume that $\bar{\alpha}$ doesn't vanish at zeros of $\bar{f}$ different from $\bar{I}$. By Bertini's theorem we may assume that the algebra $\bar{S}/(\bar{\alpha})$ is etale over $\bar{R}$. Without changing $\bar{\alpha}$ we may replace α by $\alpha - \varepsilon(\alpha)$ and assume that $\alpha \in I$. Since S is integral over $R[t]$ there exists a relation of integral dependence

$$\alpha^n + p_1(t)\alpha^{n-1} + \ldots + p_n(t) = 0.$$

For any $r \in k^*$ and any N larger than the degree of each $p_i(t)$, putting $t' = \alpha - rt^N$ we see from the equation above that t is integral over $R[t']$. Hence S, which is integral over $R[t]$, is integral over $R[t']$ and property G1 holds.

2. Clearly $t' \in I$. To insure that $\bar{t}'$ is also a local parameter of $\bar{S}_{\bar{I}}$ it suffices to take $N \geq 2$. By assumption S and $R[t']$ are smooth and since S is finite over $R[t']$, S is finitely generated projective as the $R[t']$-module (see Corollary 18.17 of [Ei]) and hence $S/(t')$ is free as the R-module. Since $\bar{t}'$ is a local parameter of $\bar{S}_{\bar{I}}$, $\bar{S}/(\bar{t}')$ is etale over $\bar{R}$ at the augmentation ideal $\bar{I}$ and so we can find a $g \notin I + \mathfrak{m}S$ such that $(S/(t'))_g$ is etale over R. By Sublemma 5.3 of [PO] we get property G2.

3. Consider the following fibred product diagram:

$$\begin{array}{ccc} Spec\,\bar{S}[u]/(\bar{\alpha} - u\bar{t}^N) & \longleftarrow & Spec\,\bar{S}/(\bar{t}') \\ \pi\big\downarrow & & \big\downarrow \\ \mathbb{A}^1_k & \longleftarrow & Spec\,k \end{array}$$

which maps $Spec\, k$ to the rational point $\{r\} \in \mathbb{A}^1_k$.

Since the fibre of π at the rational point $\{r = 0\} \in \mathbb{A}^1_k$ is etale over $Spec\, k$ (it coincides with $Spec\, \bar{S}/(\bar{\alpha})$) the fibres of π at almost all rational points of $\mathbb{A}^1_k$ are etale over $Spec\, k$. It means that the algebra $\bar{S}/(\bar{t}') = \bar{S}/(\bar{\alpha} - r\bar{t}^N)$ is etale over k for almost all $r \in k^*$.

Consider the scalar extension diagram induced by the canonical map $R \to \bar{R}$

$$\begin{array}{ccc} S/(t') & \longrightarrow & \bar{S}/(\bar{t}') \\ \uparrow & & \uparrow \\ R & \longrightarrow & \bar{R} \end{array}$$

As it was mentioned in 2 the $S/(t')$ is free as the R-module. Therefore it is flat over R. If $S/(t')$ is flat over R and $\bar{S}/(\bar{t}')$ is etale over $\bar{R}$ then $S/(t')$ is etale over R. Thus the algebra $S/(t')$ is etale over R for almost all $r \in k^*$, i.e., property G4.(a) holds for almost all $r \in k^*$.

To check property G3.(a) we refer to the end of the proof of Lemma 5.2, [PO]. It turns out that G3.(a) holds for almost any choice of $r \in k^*$.

Hence we may choose the element t' (the corresponding $r \in k^*$) such that properties G3.(a) and G4.(a) hold.

4. It remains to check G3.(b) and G4.(b). Since $S/(t')$ is already etale over R the algebra $S/(t' - \lambda)$ is etale over R for almost all $\lambda \in k^*$. Since $S/(f)$ is semilocal the element $(t' - \lambda)$ is invertible in $S/(f)$ for almost all $\lambda \in k^*$. Hence we may choose the element $\lambda \in k^*$ such that the algebra $S/(t' - \lambda)$ is etale over R and $(t' - \lambda)$ is coprime with (f).

Without perturbing conditions G1, G2, G3.(a) and G4.(a) we may replace t' by $\frac{1}{\lambda}t'$ and thus satisfy G3.(b) and G4.(b). □

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