On the computation of trace forms of algebras with involution

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Introduction: In this article, we determine up to isometry the trace form $\mathcal{T}_\sigma$ of central simple algebras with involution over a field k, which has at most one ordering and such that $I^3(k)$ is torsion free. This quadratic form has been introduced by Weil in [W], and has been studied by Lewis [L] and Quéguiner [Q2].

Definitions and notation: Let A be a central simple algebra of degree n over a field k of characteristic different from 2. An *involution* on A is a ring antiautomorphism of order at most 2. An involution σ is *of the first kind* if $\sigma|k = \mathrm{Id}_k$, and *of the second kind* if $\sigma|k$ is a non trivial involution on k, denoted by $\bar{}$. In the last case, k is a quadratic extension of the subfield k_0 fixed by $\bar{}$. So we have $k = k_0(\sqrt{\alpha}), \alpha \in k_0^*/k_0^{*2}$. The involution $\bar{}$ is then defined by $\overline{u + v\sqrt{\alpha}} = u - v\sqrt{\alpha}$, with $u, v \in k_0$. If A is split, the involutions of the first kind are exactly the involutions adjoint to symmetric or skew symmetric bilinear forms. In the case where A is a central simple algebra with an involution of the first kind, and if L is a splitting field of A, $A \otimes L$ is a split algebra and $\sigma \otimes Id_L$ is an involution on $A \otimes L$, then adjoint to a bilinear form b. We say that σ is of *orthogonal type* if b is symmetric , and of *symplectic type* if b is skew symmetric. This definition does not depend on the splitting field. Now consider the function $(x, y) \in A \times A \mapsto \mathrm{Trd}_A(\sigma(x)y)$. If the involution σ is of the first kind, this is a nondegenerate symmetric bilinear form of dimension n^2 over k, and we denote by $\mathcal{T}_\sigma$ the corresponding quadratic form. If σ is of the second kind, this is a hermitian form over k, denoted by $\mathcal{H}_\sigma$. Then we define $\mathcal{T}_\sigma$ by $\mathcal{T}_\sigma\ (x) = \mathcal{H}_\sigma\ (x, x)$. This is a nondegenerate quadratic form over k_0 of dimension $2n^2$. If σ is of the first kind, we set $Alt(\sigma)=\{\ x - \varepsilon\sigma(x), x \in A\ \}$, with $\varepsilon = 1$ if σ is orthogonal, and $\varepsilon = -1$ if σ is symplectic. Recall now the definition of the determinant of an involution of the first kind, given by Knus, Parimala and Sridharan:

Definition 1 (see [KPS]): Let A be an even dimensional central simple algebra with an involution σ of the first kind. The determinant of σ, denoted by $d(\sigma)$ is the square class of $\mathrm{Nrd}_A(u)$, where u is any element of $A^* \bigcap Alt(\sigma)$, and where Nrd_A is the reduced norm.

We have $d(\sigma) = 1$ if σ is symplectic (take $u = 1$). If A is split, and σ is adjoint to b, then $d(\sigma) = \det(b)$.

In this paper, we denote by $\mathbb{H}$ the two-dimensional hyperbolic plane, and by $I^3(k)$ the cube of the fundamental ideal of the Witt ring of k. Finally, we will often use the following formula, which can be easily deduced from the definition of the Hasse-Witt invariant (see also [S], lemma 12.6):

$w_2(q_1 \perp q_2) = w_2(q_1) + (\det(q_1), \det(q_2)) + w_2(q_2)$.

In the following, we determine the isomorphism class of $\mathcal{T}_\sigma$, when k or k_0 has at most one ordering and such that I^3 is torsion free. For this, we use the well known property that quadratic forms over these fields are classified by dimension, determinant, Hasse-Witt invariant and the signature (see [EL]). Then we use the results of A. Quéguiner, who computed in [Q2] the the determinant and the Hasse invariant of $\mathcal{T}_\sigma$. This last invariant has also been computed by Lewis in [L].

I. Case $I^3 = 0$

In this part, we will prove the following result:

Theorem 1: Let A be a central simple algebra of degree n over k, with an involution σ.

1. If σ is of the second kind, set $k = k_0(\sqrt{\alpha})$, and assume that $I^3(k_0) = 0$. Then $\mathcal{T}_\sigma \simeq < (-\alpha)^n > \perp (2n^2 - 1) < 1 >$
2. If σ is of the first kind, assume that $I^3(k) = 0$. Then:
   (a) If n is odd, A is a split algebra, σ is adjoint to a nondegenerate symmetric bilinear form, and we have $\mathcal{T}_\sigma \simeq n^2 < 1 >$.
   (b) If $n \equiv 0$ [4], we have $\mathcal{T}_\sigma \simeq (n^2 - 2) < 1 > \perp < d(\sigma), d(\sigma) >$.
   (c) If $n \equiv 2$ [4], then A is Brauer equivalent to a quaternion algebra Q. Moreover:
      i. If σ is symplectic, set $Q = (a, b)$. Then we get $\mathcal{T}_\sigma \simeq < 1, -a, -b, ab > \perp (n^2 - 4) < 1 >$.
      ii. If $n = 2$, that is $A = Q$, and if σ is orthogonal, there exists $b \in k^*$ such $Q = (a, b)$, with $a = -d(\sigma)$, and such that the

associated standard basis is orthogonal with respect to $\mathcal{T}_\sigma$. Then we have $\mathcal{T}_\sigma \simeq < 1, -a, b, -ab >$.

iii. If $n > 2$ and if σ is orthogonal, set $Q = (a, b)$. Then $\mathcal{T}_\sigma \simeq < -d(\sigma), -d(\sigma), 1, -a, -b, ab > \perp (n^2 - 6) < 1 >$.

We will divide the proof of this theorem into five lemmas.

A. Case of involutions of the second kind

Let σ be an involution on A of the second kind and $k = k_0(\sqrt{\alpha})$. Then we have the following result (see [Q2]):

Proposition 1: We have $\det(\mathcal{T}_\sigma) = (-\alpha)^n$ and $w_2(\mathcal{T}_\sigma) = 0$.

Assume now that $I^3(k_0) = 0$. The quadratic form $\mathcal{T}_\sigma$ is then uniquely determined by its determinant, dimension and Hasse invariant (see [EL]). Then we get:

Lemma 1: $\mathcal{T}_\sigma \simeq < (-\alpha)^n > \perp (2n^2 - 1) < 1 >$.

B. Case of involutions of the first kind

Assume first that n is odd. Then it is well known that, if A is an algebra with an involution σ of the first kind, then A is split and σ is orthogonal (see [KMRT] Corollary 2.8). Thus σ is adjoint to a symmetric bilinear form b over k, which is uniquely determined up to similarity. Since n is odd, we have $\det(< \lambda > \otimes b) = \lambda \det b$, so we can assume that $\det(b) = 1$, after multiplying b by a scalar. Moreover, we know that $\mathcal{T}_\sigma \simeq b \otimes b$ (see [KMRT], Proposition 11.4). Since we have $w_2(b \otimes b) = (-1, \det(b))$ if b is symmetric (see [Q2]), we get $w_2(\mathcal{T}_\sigma) = (-1, 1) = 0$. Finally, the two quadratic forms $\mathcal{T}_\sigma$ and $n^2 < 1 >$ have the same invariants, and we get:

Lemma 2: Let A be a central simple algebra over k of odd degree, with an involution σ of the first kind. Then $\mathcal{T}_\sigma \simeq n^2 < 1 >$, where $n = \deg A$.

Assume now that $n = 2m$. Then $\det(\mathcal{T}_\sigma) = 1$ for any involution (see [Q2]). Recall also that there exists an involution of the first kind on A if and only if $A \otimes A$ is split (see [J] Theorem 5.2.1). The class of A in the Brauer group of k, denoted by $[A]$, is then an element of $Br_2(k)$, which justifies the following statement (see [Q2]):

Proposition 2: We have $w_2(\mathcal{T}_\sigma) = \begin{cases} m(-1,-1) + m[A] \text{ if } \sigma \text{ is symplectic} \\ (-1, d(\sigma)) + m[A] \text{ if } \sigma \text{ is orthogonal} \end{cases}$

Assume that m is even.
Since $d(\sigma) = 1$ if σ is symplectic, we get $w_2(\mathcal{T}_\sigma) = (-1, d(\sigma))$ in all cases. Since the invariants of $\mathcal{T}_\sigma$ coincide with the invariants of the quadratic form $(n^2-2) < 1 > \perp < d(\sigma), d(\sigma) >$, we have:

Lemma 3: Let A be a central simple algebra of degree $n \equiv 0$ [4], with an involution σ of the first kind. Then $\mathcal{T}_\sigma \simeq (n^2-2) < 1 > \perp < d(\sigma), d(\sigma) >$.

Assume now that m is odd.
We know that the index of a central simple algebra of even degree with an involution of the first kind is a power of 2 (see [KMRT], Corollary 2.8). In our case, A is then Brauer equivalent to an algebra of degree 1 or 2. So A is equivalent to k or to a division quaternion algebra. Since k is equivalent to $M_2(k) \simeq (1,1)$, we have $[A] = (a,b)$ in all cases. In the symplectic case, we get $w_2(\mathcal{T}_\sigma) = (-1,-1) + (a,b)$. An easy computation shows that $w_2(< 1, -a, -b, ab >) = (-1,-1) + (a,b)$. Since
$w_2(< 1, -a, -b, ab > \perp (n^2-4) < 1 >) = w_2(< 1, -a, -b, ab >)$, and the determinant of this form is trivial, we get:

Lemma 4: Let $A \simeq M_m(Q)$, where m is odd and Q is the quaternion algebra (a,b), with a symplectic involution σ.
Then $\mathcal{T}_\sigma \simeq < 1, -a, -b, ab > \perp (n^2-4) < 1 >$.

Lemma 5: Let $A \simeq M_m(Q)$, where $m \geq 1$ is odd and Q is a quaternion algebra, with an orthogonal involution σ. If $m = 1$, there exists $b \in k^*$ such $Q = (a,b)$, with $a = -d(\sigma)$, and such that the associated standard basis is orthogonal with respect to $\mathcal{T}_\sigma$. Then we have $\mathcal{T}_\sigma \simeq < 1, -a, b, -ab >$. If $m > 1$, set $Q = (a,b)$. Then $\mathcal{T}_\sigma \simeq < -d(\sigma), -d(\sigma), 1, -a, -b, ab > \perp (n^2-6) < 1 >$.

Proof: If $m > 1$, set $Q = (a,b)$. We have $w_2(< -d(\sigma), -d(\sigma) >) = (d(\sigma), -1) + (-1,-1)$ and $w_2(< 1, -a, -b, ab >) = (-1,-1) + (a,b)$. These two quadratic forms have a trivial determinant, so we easily get
$w_2(< -d(\sigma), -d(\sigma), 1, -a, -b, ab > \perp (n^2-6) < 1 >) = (-1, d(\sigma)) + (a,b)$. Since this last quadratic form is n^2-dimensional and has a trivial determinant, we have $\mathcal{T}_\sigma \simeq < -d(\sigma), -d(\sigma), 1, -a, -b, ab > \perp (n^2-6) < 1 >$. The case $m = 1$ can be found in [Q2]. Note that in [Q2], it is shown that $\mathcal{T}_\sigma \simeq < 2, -2a, 2b, -2ab >$, but it is easy to see that this last form is isometric to $< 1, -a, b, ab >$ (it suffices to compare the invariants).

These five lemmas give the proof of theorem 1.

All these results are true in particular on local fields, function fields and non formally real number fields.

II. Case I^3 torsion free and k uniquely ordered.

Recall first some facts about the signature of the form $\mathcal{T}_\sigma$. If (k, P) is an ordered field, we denote by k_P the corresponding real closure, which is uniquely ordered by squares (see [S],Theorem 3.1.14). If q is a quadratic form over (k, P), then $\mathrm{sign}_P(q) = \mathrm{sign}(q_{k_P})$. Now let A be a central simple algebra over k. If σ is an involution of the first kind on A, then $\mathrm{sign}_P(\mathcal{T}_\sigma)$ is a square. Moreover, if $A \otimes k_P \sim 1$ and σ is symplectic, or if $A \otimes k_P \not\sim 1$ and σ is orthogonal, then $\mathrm{sign}_P(\mathcal{T}_\sigma) = 0$. See [KMRT] for more details. If σ is of the second kind, assume that k_0 is ordered by P, and $k = k_0(\sqrt{\alpha})$. Then $\mathrm{sign}_P(\mathcal{H}_\sigma)$ is a square and $\mathrm{sign}_P(\mathcal{T}_\sigma) = 2\mathrm{sign}_P(\mathcal{H}_\sigma)$ (see [Q1]). Moreover, this signature is equal to 0 if $\alpha > 0$. Now set $\mathrm{sign}_P(\sigma) = \sqrt{\mathrm{sign}_P(\mathcal{T}_\sigma)}$ or $\sqrt{\frac{1}{2}\mathrm{sign}_P(\mathcal{T}_\sigma)}$ respectively if σ is of the first or the second kind. In all cases, the signature and the degree of A have the same parity (see [KMRT] and [Q1]).

Before stating the results, recall that if the degree of A is odd, and if σ is an involution of the first kind on A, then A is split and $\sigma = \sigma_b$ where b is symmetric. Moreover, when the determinant is defined, we have $d(\sigma_b) = \det(b)$, and $d(\sigma) = 1$ if σ is symplectic.

In the following, we will show:

Theorem 2: Let A be a central simple algebra of degree n, with an involution σ. We keep the notation of Theorem 1.

1. If σ is of the second kind, set $k = k_0(\sqrt{\alpha})$ and $\mathrm{sign}(\sigma) = s$ (recall that $s = 0$ if $\alpha > 0$). Assume that k_0 is uniquely ordered and $I^3(k_0)$ is torsion free. Then
$$\mathcal{T}_\sigma \simeq \begin{cases} <(-\alpha)^n> \perp (n^2+s^2-1) <1> \perp (n^2-s^2) <-1> \text{ if } \alpha < 0 \\ n^2\mathbb{H} \text{ if } \alpha > 0 \text{ and } n \text{ is even} \\ <1, -\alpha> \perp (n^2-1)\mathbb{H} \text{ if } \alpha > 0 \text{ and } n \text{ is odd} \end{cases}$$

2. If σ is of the first kind, assume that k is uniquely ordered and that $I^3(k)$ is torsion free, and set $\mathrm{sign}(\sigma) = s$.

(a) If n is odd, then σ is adjoint to a symmetric bilinear form and we have $\mathcal{T}_\sigma \simeq \frac{n^2+s^2}{2} < 1 > \perp \frac{n^2-s^2}{2} < -1 >$.

(b) If $n \equiv 0$ [4], then we have:

i. If $d(\sigma) > 0$, we have $s \equiv 0$ [4] and $\mathcal{T}_\sigma \simeq (\frac{n^2+s^2}{2} - 2) < 1 > \perp \frac{n^2-s^2}{2} < -1 > \perp < d(\sigma), d(\sigma) >$

ii. If $d(\sigma) < 0$, we have $s \equiv 2$ [4] and $\mathcal{T}_\sigma \simeq \frac{n^2+s^2}{2} < 1 > \perp (\frac{n^2-s^2}{2} - 2) < -1 > \perp < d(\sigma), d(\sigma) >$

(c) Assume now that $n \equiv 2$ [4].

i. Assume that σ is symplectic.

A. If $A \otimes k_P \sim 1$, then we have $s = 0$ and $\mathcal{T}_\sigma \simeq < 1, -a, -b, ab > \perp \frac{n^2-4}{2} \mathbb{H}$.

B. If $A \otimes k_P \not\sim 1$, then $s \equiv 2$ [4] and $\mathcal{T}_\sigma \simeq < 1, -a, -b, ab > \perp (\frac{n^2+s^2}{2} - 4) < 1 > \perp \frac{n^2-s^2}{2} < -1 >$.

ii. If σ is orthogonal and $n = 2$, then $\mathcal{T}_\sigma \simeq < 1, -a, b, -ab >$ and $s = 0$ or 2 depending of the signs of a and b.

iii. If σ is orthogonal and $n > 2$, then

A. If $A \otimes k_P \not\sim 1$, then $s = 0$, $d(\sigma) > 0$ and we have $\mathcal{T}_\sigma \simeq < -d(\sigma), -d(\sigma), 1, -a, -b, ab > \perp (\frac{n^2}{2} - 4) < 1 > \perp (\frac{n^2}{2} - 2) < -1 >$.

B. If $A \otimes k_P \sim 1$ and $d(\sigma) > 0$, then $s \equiv 2$ [4], and we have $\mathcal{T}_\sigma \simeq < -d(\sigma), -d(\sigma), 1, -a, -b, ab > \perp (\frac{n^2+s^2}{2} - 2) < 1 > \perp (\frac{n^2-s^2}{2} - 4) < -1 >$.

C. If $A \otimes k_P \sim 1$ and $d(\sigma) < 0$, then $s \equiv 0$ [4], and we have $\mathcal{T}_\sigma \simeq < -d(\sigma), -d(\sigma), 1, -a, -b, ab > \perp (\frac{n^2+s^2}{2} - 4) < 1 > \perp (\frac{n^2-s^2}{2} - 2) < -1 >$.

These results are true in particular over euclidean fields and over $\mathbb{Q}$.

Proof: It suffices to show that the quadratic forms have the correct invariants. Note that there exist a unique division quaternion algebra on k_P, namely $(-1, -1)$. Then $(a, b) \otimes k_P$ is a division algebra if and only if $a < 0$ and $b < 0$. Then $\mathrm{sign}_P(< 1, -a, -b, ab >) = 0$ or 4 if $A \otimes k_P$ is a split algebra or a division algebra respectively. Then it is easy to verify that the signatures are the correct ones. Now we justify the conditions on s and $d(\sigma)$: if q is a quadratic form over k, we have $\mathrm{sign}(q \otimes k_P)=\mathrm{sign}_P(q)$, $w_2(q \otimes k_P) = w_2(q) \otimes k_P$. Now quadratic forms over k_P are the forms $u < 1 > \perp v < -1 >$, and the unique quadratic form with dimension n^2

and signature s^2 is the form $\frac{n^2+s^2}{2} < 1 > \perp \frac{n^2-s^2}{2} < -1 >$. If we compute the Hasse invariant of this form and if we compare it with the Hasse invariant of $\mathcal{T}_\sigma \otimes k_P$, we get the announced conditions.

Acknowledgments: I would thank the referee for his remarks, which have permitted to simplify the statements of the main results.

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