Trace forms of central simple algebras over a local field or a global field

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Introduction: Let A be a central simple algebra over a field k of characteristic different from 2.
The quadratic form $x \in A \mapsto \mathrm{Trd}_A(x^2) \in k$ is called *the trace form of* A, and is denoted by $\mathcal{T}_A$. This trace form has been studied by many authors (cf.[L], [LM], [Ti] et [Se], Annexe §5 for example). In particular, these classical invariants are well-known (*loc.cit.*). In [B], we have determined a diagonalization of the trace form of cyclic algebras over fields such that the cube of the fundamental ideal is trivial. In particular, this gives all trace forms of central simple algebras over local fields and non formally real global fields, since any algebra over such a field is cyclic. In this article, we give a characterization of the trace form of a central simple algebra over a local field or a global field in terms of determinant and signatures. We also show that a quadratic form over a global field is isomorphic to a trace form if and only if it is true locally. Then we give a necessary and sufficient condition on trace forms to be isomorphic. Finally, we apply these results to describe explicitly the elements of $Br_2(k)$ which can be written $\dfrac{n}{2}[A]$, where A is a central simple algebra of even degree n, when k is a local field or a global field of characteristic different from 2.

Recalls and notation: If A is a central simple algebra over k, *the exponent of* A, denoted by $\exp A$, is the order of $[A]$ in $Br(k)$ and *the index of* A, denoted by $\mathrm{ind} A$, is the degree of the division algebra which corresponds to A. We know that $\exp A$ divides the degree of A. If $a, b \in k^*$, we denote by $(a,b)_k$ the corresponding quaternion algebra, or simply (a,b) if no confusion is possible. We also use the same notation to design its class in the Brauer group. If L/k is any field extension, set $[A]_L = [A \otimes L]$. Then we have $[(a,b)_k]_L = (a,b)_L$. Recall now the definition of a cyclic algebra. Let L/k be a cyclic extension of degree n, σ a generator of $\mathrm{Gal}(L/k)$ and

$a \in k^*$. The ring $(a, L/k, \sigma) = \bigoplus_{i=0}^{n-1} Le^i$ with the multiplication law $e^n = a$ and $e\lambda = \sigma(\lambda)e, \lambda \in L$ is a central simple algebra, called a *cyclic algebra*.
If k is a local field, we know that ind$A$ =exp$A$. Moreover, we get an isomorphism $\mathrm{inv}_k : Br(k) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$. Under this isomorphism, $\frac{s}{n}$ (where s and n are not necessarily relatively prime) corresponds to the class of the cyclic algebra $(\pi^s, L/k, \sigma)$, where π is a uniformizing parameter, L/k is the unique unramified extension (then cyclic) of degree n and σ the associated Frobenius automorphism. Moreover $(\pi^s, L/k, \sigma)$ is a division algebra if and only if $(s, n) = 1$. We also set $\mathrm{inv}_{\mathbb{C}} \equiv 0$. Finally we define $\mathrm{inv}_{\mathbb{R}} : Br(\mathbb{R}) \to \mathbb{Q}/\mathbb{Z}$ by $\mathrm{inv}_{\mathbb{R}}(-1,-1) = \frac{1}{2}$. If k is a global field, we write $\mathrm{inv}_{\mathfrak{p}}$ instead of $\mathrm{inv}_{k_{\mathfrak{p}}}$ for all prime $\mathfrak{p}$ of k. Then we get the following exact sequence $0 \to Br(k) \to \sum Br(k_{\mathfrak{p}}) \overset{\sum \mathrm{inv}_{\mathfrak{p}}}{\to} \mathbb{Q}/\mathbb{Z} \to 0$. Moreover, we know that exp$A$ =ind$A$ =lcm(ind$A \otimes k_{\mathfrak{p}}$). For more details, we refer to [W] or [CS]. Concerning central simple algebras over general fields, we refer to [D], [J] or [Sc].
In the following, all the quadratic forms are non singular. If q is a quadratic form over k, dim$q$ is the dimension, $\det q \in k^*/k^{*2}$ is the determinant and $\mathrm{sign}_P q \in \mathbb{Z}$ is the signature of q relatively to the ordering P, that is the difference between the number of positive elements and the number of negative elements in any diagonalization of q. If k is a global field, each real prime (if there is any) corresponds to a real embedding, then to an ordering and vice versa. We will denote by $\mathrm{sign}_{\mathfrak{p}} q$ the signature of q relatively to the ordering corresponding to this prime. If $q \simeq < a_1, \cdots, a_n >$, the *Hasse-Witt invariant of* q is given by $w_2(q) = \sum_{i<j} (a_i, a_j) \in Br_2(k)$. Thus, if L/k is any field extension, we get $w_2(q)_L = w_2(q \otimes L)$. Finally, we denote by $\mathbb{H}$ the hyperbolic plane.

A. Trace forms of central simple algebras over a local field or a global field.

In this section, we first show the following results:

Theorem 1: Let k be a local field and $n \geq 2$ an even integer. Then a quadratic form q over k is isomorphic to the trace form of a central simple algebra of degree n if and only if the following conditions hold:

1. $\dim q = n^2$
2. $\det q = (-1)^{\frac{n(n-1)}{2}}$

Theorem 2: Let k be a global field of characterisitc different from 2, $n \geq 2$ an even integer, and q a quadratic form over k. Then the following conditions are equivalent:

(1) The quadratic form q is isomorphic to the trace form of a central simple algebra over k of degree n

(2) q is isomorphic to a trace form locally everywhere, that is for all prime $\mathfrak{p}$ of k, $q \otimes k_{\mathfrak{p}}$ is isomorphic to the trace form of a central simple algebra of degree n over $k_{\mathfrak{p}}$

(3) q satisfies the following conditions:

  (i) $\dim q = n^2$

  (ii) $\det q = (-1)^{\frac{n(n-1)}{2}}$

  (iii) $\mathrm{sign}_{\mathfrak{p}} q = \pm n$ for all real prime $\mathfrak{p}$ of k

These two theorems only deal with the case when n is even, because we know that $\mathcal{T}_A \simeq n < 1 > \perp \dfrac{n(n-1)}{2}\mathbb{H}$ if n is odd (cf.[Se], Annexe §5 for example).

Before proving these theorems, we recall some results about the classical invariants of trace forms of central simple algebras:

Theorem 3: Let k be a field of characteristic different from 2, and let A be a central simple algebra over k of degree n. Then we have:

1. $\dim \mathcal{T}_A = n^2$
2. $\det \mathcal{T}_A = (-1)^{\frac{n(n-1)}{2}}$
3. We have $\mathrm{sign}_P \mathcal{T}_A = \pm n$ for each ordering P, and $\mathrm{sign}_P \mathcal{T}_A = n$ if and only if $[A]_{k_P} = 0$, where k_P is the real closure of (k, P)
4. If $n = 2m \geq 2$, then $w_2(\mathcal{T}_A) = \dfrac{m(m-1)}{2}(-1,-1) + m[A]$

The three first statements can be found in [L], and the last one is proved in [LM] or [Ti] for example .

Lemma: Let k be a local field, $n = 2m \geq 2$ and A a central simple algebra of degree n. Let (a, b) be the unique non zero element of $Br_2(k)$. Then $m[A] = (a, b)$ if and only if A is a division algebra.

Proof of the lemma: We know that exp$A$ =ind A, then A is a division algebra if and only if $[A]$ has order n in $Br(k)$, which is equivalent to $m[A] \neq 0$, that is $m[A] = (a, b)$ since (a, b) is the unique non zero element of $Br_2(k)$.

Proof of theorem 1: According to the previous theorem, the two conditions are necessary. We show that there are sufficient. It is well-known that quadratic forms over k are determined up to isomorphism by dimension, determinant and Hasse-Witt invariant.
We have $\frac{m(m-1)}{2}(-1, -1) + w_2(q) = (c, d)$, with $(c, d) = 0$ or (a, b).
Assume first that $(c, d) = 0$. In this case, set $A = M_n(k)$.
Then the last statement of theorem 3 gives
$w_2(\mathcal{T}_A) = \frac{m(m-1)}{2}(-1, -1) = w_2(q)$. Thus the two quadratic forms q and $\mathcal{T}_A$ have the same invariants, so there are isomorphic.
Assume now that $(c, d) = (a, b)$, and set $A = (\pi, L/k, \sigma)$, where π is a uniformizing parameter of k, L/k is the unique unramified extension of degree n and σ is the Frobenius automorphism. Then A is a division algebra of degree n, then $m[A] = (a, b)$ by the lemma. Now conclude as previously.

Proof of theorem 2: We first show that (2) and (3) are equivalent. If $\mathfrak{p}$ is a complex prime, it is clear that a quadratic form over $k_\mathfrak{p}$ is isomorphic to a trace form if and only if (i) is satisfied. If $\mathfrak{p}$ is a real prime, quadratic forms are determined by dimension and signatures. By theorem 3, it follows that a quadratic form over $k_\mathfrak{p}$ is isomorphic to a trace form if and only if (i) et (iii) are satisfied. Finally, if $\mathfrak{p}$ is a finite prime, theorem 1 implies that a quadratic form over $k_\mathfrak{p}$ is isomorphic to a trace form if and only if (i) et (ii) are satisfied. So we get the desired equivalence.
Now we show that (1) and (3) are equivalent. The three conditions are necessary by theorem 3. In order to show that there are also sufficient, we consider the two cases $n \equiv 2$ [4] and $n \equiv 0$ [4] separately.

• Assume first that $n = 2m$, with m odd.
Since k is a global field, every central algebra of exponent 2 is similar to a quaternion algebra (it comes from the equality exp$A$ =ind$A$).
So we can write $\frac{m(m-1)}{2}(-1, -1)_k + w_2(q) = (c, d)_k$.
Let $\mathfrak{p}$ be a real prime of k.
If $\mathrm{sign}_\mathfrak{p} q = n$, we have $q \otimes k_\mathfrak{p} \simeq n < 1 > \perp \frac{n(n-1)}{2}\mathbb{H}$.

Then we easily get $w_2(q)_{k_\mathfrak{p}} = w_2(q\otimes k_\mathfrak{p}) = \dfrac{m(m-1)}{2}(-1,-1)_{k_\mathfrak{p}}$.

If $\mathrm{sign}_\mathfrak{p} q = -n$, we have $q\otimes k_\mathfrak{p} \simeq n < -1 > \perp \dfrac{n(n-1)}{2}\mathbb{H}$.

Then we have $w_2(q)_{k_\mathfrak{p}} = w_2(q\otimes k_\mathfrak{p}) = (\dfrac{m(m-1)}{2} + m)(-1,-1)_{k_\mathfrak{p}}$

$= \dfrac{m(m-1)}{2}(-1,-1)_{k_\mathfrak{p}} + (-1,-1)_{k_\mathfrak{p}}$, since m is odd.

So we have $\mathrm{sign}_\mathfrak{p} q = n$ if and only if $(c,d)_{k_\mathfrak{p}} = 0$.

Set $A = M_m((c,d)_k)$. Then $\mathrm{sign}_\mathfrak{p}\mathcal{T}_A = n$ if and only if $(c,d)_{k_\mathfrak{p}} = 0$, so q and $\mathcal{T}_A$ have the same signatures. Moreover we get $m[A] = (c,d)_k$ since m is odd, so the previous quadratic forms have the same Hasse-Witt invariants. By assumption, dimensions and determinants are equal, so q and $\mathcal{T}_A$ are isomorphic.

• Assume now that $n = 2m$, with $m \geq 2$ even. First we construct a central simple algebra over k, defining it locally everywhere and using the exact sequence recalled at the beginning.

As previously, set $\dfrac{m(m-1)}{2}(-1,-1)_k + w_2(q) = (c,d)_k$.

If $\mathfrak{p}$ is a complex prime, we have $(c,d)_{k_\mathfrak{p}} = 0$.

If $\mathfrak{p}$ is a real prime such that $\mathrm{sign}_\mathfrak{p} q = n$, we have already seen that

$w_2(q)_{k_\mathfrak{p}} = \dfrac{m(m-1)}{2}(-1,-1)_{k_\mathfrak{p}}$, so $(c,d)_{k_\mathfrak{p}} = 0$.

If $\mathfrak{p}$ is a real prime such that $\mathrm{sign}_\mathfrak{p} q = -n$, we have seen that

$w_2(q)_{k_\mathfrak{p}} = (\dfrac{m(m-1)}{2} + m)(-1,-1)_{k_\mathfrak{p}} = \dfrac{m(m-1)}{2}(-1,-1)_{k_\mathfrak{p}}$ since m is even in this case. So we have again $(c,d)_{k_\mathfrak{p}} = 0$. Finally, the primes for which $(c,d)_{k_\mathfrak{p}} \neq 0$ are finite. By Hilbert's reciprocity law, the number r of these primes is finite and even, and the set S of finite primes $\mathfrak{p}$ such that $(c,d)_{k_\mathfrak{p}} = 0$ is infinite. Let T be the set of real primes for which the signature of q is equal to $-n$, and let t be its cardinality.

If $r = 0$, set $B_{\mathfrak{p}_1} = (\pi_{\mathfrak{p}_1}^{tm}, L_{\mathfrak{p}_1}/k_{\mathfrak{p}_1}, \sigma_{\mathfrak{p}_1})$ for a given $\mathfrak{p}_1 \in S$ (so we have $B_\mathfrak{p} = M_n(k_{\mathfrak{p}_1})$ if $t = 0$) and $B_\mathfrak{p} = M_n(k_\mathfrak{p})$ for all finite prime $\mathfrak{p} \neq \mathfrak{p}_1$.

Suppose that $r > 0$ and choose $\mathfrak{p}_1, \mathfrak{p}_2 \in S$. Now set $B_{\mathfrak{p}_1} = (\pi_{\mathfrak{p}_1}^{tm}, L_{\mathfrak{p}_1}/k_{\mathfrak{p}_1}, \sigma_{\mathfrak{p}_1})$ and $B_{\mathfrak{p}_2} = (\pi_{\mathfrak{p}_2}^{\frac{n-r}{n}}, L_{\mathfrak{p}_2}/k_{\mathfrak{p}_2}, \sigma_{\mathfrak{p}_2})$. These algebras are not division algebras because r and n are even, and tm and n are both divisible by m. Then set $B_\mathfrak{p} = M_n(k_\mathfrak{p})$ for all primes in $S - \{\mathfrak{p}_1, \mathfrak{p}_2\}$.

So we have $\mathrm{inv}_{\mathfrak{p}_1}[B_{\mathfrak{p}_1}] = \dfrac{t}{2}$ in all cases and $\mathrm{inv}_{\mathfrak{p}_2}[B_{\mathfrak{p}_2}] = \dfrac{n-r}{n}$.

If $\mathfrak{p}$ is a finite prime, $\mathfrak{p} \notin S$, set $B_\mathfrak{p} = (\pi, L_\mathfrak{p}/k_\mathfrak{p}, \sigma_\mathfrak{p})$. The algebra $B_\mathfrak{p}$ is then a division algebra and $\mathrm{inv}_\mathfrak{p}[B_\mathfrak{p}] = \dfrac{1}{n}$ for each finite prime $\mathfrak{p} \notin S$.

Now set $B_{\mathfrak{p}} = M_m((-1,-1)_{k_{\mathfrak{p}}})$ for all $\mathfrak{p} \in T$ (in this case, we have $\mathrm{inv}_{\mathfrak{p}}[B_{\mathfrak{p}}] = \dfrac{1}{2}$) and set $B_{\mathfrak{p}} = M_n(k_{\mathfrak{p}})$ for the other infinite primes.
By construction, $\sum \mathrm{inv}_{\mathfrak{p}}[B_{\mathfrak{p}}] = 0$ in $\mathbb{Q}/\mathbb{Z}$. Thus there exists some $[B] \in Br(k)$, where B is a division algebra, such that $[B]_{k_{\mathfrak{p}}} = [B_{\mathfrak{p}}]$ for all primes of k. Since the algebras $B_{\mathfrak{p}}$ have degree n, each index is a divisor of n, so the index of B divides n (because it is the least common multiple of the indices). Since B is a division algebra, we have $\mathrm{ind}B = \mathrm{deg}B$, so $\mathrm{deg}B$ divides n.
Then we set $A = M_j(B)$, with $j = \dfrac{n}{\mathrm{deg}B}$.

Now we show that $q \simeq \mathcal{T}_A$. It suffices to prove that $q \otimes k_{\mathfrak{p}} \simeq \mathcal{T}_{A\otimes k_{\mathfrak{p}}}$ for all primes of k. Then we get $q \otimes k_{\mathfrak{p}} \simeq \mathcal{T}_A \otimes k_{\mathfrak{p}}$ for all primes of k, and we obtain $q \simeq \mathcal{T}_A$. Notice that we have $[A]_{k_{\mathfrak{p}}} = [B_{\mathfrak{p}}]$.
If $\mathfrak{p}$ is a complex prime, the isomorphism is clear. If $\mathfrak{p}$ is a real prime, then we have by construction $[A]_{k_{\mathfrak{p}}} = [B_{\mathfrak{p}}] = 0$ if and only if $\mathrm{sign}_{\mathfrak{p}} q = n$.
By theorem 3, $q \otimes k_{\mathfrak{p}}$ and $\mathcal{T}_{A\otimes k_{\mathfrak{p}}}$ have the same signature, so there are isomorphic since the dimensions are equal.
Now we consider the case of finite primes. By construction, for every finite prime $\mathfrak{p} \in S$, $B_{\mathfrak{p}}$ is not a division algebra, so $m[A]_{k_{\mathfrak{p}}} = m[B_{\mathfrak{p}}] = 0$ by the lemma. Since $\mathfrak{p} \in S$, we have $(c,d)_{k_{\mathfrak{p}}} = 0$, so $q \otimes k_{\mathfrak{p}}$ and $\mathcal{T}_{A\otimes k_{\mathfrak{p}}}$ have the same Hasse-Witt invariant. Since their dimensions and their determinants are equal, there are isomorphic. Finally if $\mathfrak{p}$ is a finite prime such that $\mathfrak{p} \notin S$, we have $(c,d)_{k_{\mathfrak{p}}} \neq 0$ and $B_{\mathfrak{p}}$ is a division algebra. If $(a_{\mathfrak{p}}, b_{\mathfrak{p}})$ denotes the unique non zero element of $Br_2(k_{\mathfrak{p}})$, we have by the lemma $m[A]_{k_{\mathfrak{p}}} = m[B_{\mathfrak{p}}] = (a_{\mathfrak{p}}, b_{\mathfrak{p}}) = (c,d)_{k_{\mathfrak{p}}}$. Now we conclude as previously and this finishes the proof of theorem 2.

Remark: By theorem 2, if q is $k_{\mathfrak{p}}$-isomorphic to the trace form of a central simple algebra $A_{\mathfrak{p}}$ over $k_{\mathfrak{p}}$ of degree n for all primes, then q is k-isomorphic to the trace form of a central simple algebra A over k of degree n. Nevertheless, it does not mean that $A \otimes k_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$ for all $\mathfrak{p}$. Indeed , it can happens that $\sum \mathrm{inv}_{\mathfrak{p}} A_{\mathfrak{p}} \neq 0$, so there is no central simple algebra over k which is similar to $A_{\mathfrak{p}}$ in $Br(k_{\mathfrak{p}})$ for all primes, according to the exact sequence recalled at the beginning. Since the degrees are equal, this is equivalent to say that the isomorphism $A \otimes k_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$ does not hold for all $\mathfrak{p}$.

Example: Assume that $n \equiv 0$ [8]. For all primes $\mathfrak{p}$ of k, set $q_{\mathfrak{p}} = \mathcal{T}_{A_{\mathfrak{p}}}$, where $A_{\mathfrak{p}} = M_n(k_{\mathfrak{p}})$ except for two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ for which we set $A_{\mathfrak{p}_i} = (\pi_{\mathfrak{p}_i}, L_{\mathfrak{p}_i}/k_{\mathfrak{p}_i}, \sigma_{\mathfrak{p}_i})$. Then we have $w_2(q_{\mathfrak{p}}) = 0$ if $\mathfrak{p} \neq \mathfrak{p}_1, \mathfrak{p}_2$ and $w_2(q_{\mathfrak{p}_i}) = (a_{\mathfrak{p}_i}, b_{\mathfrak{p}_i})$ (it suffices to apply theorem 3 and the lemma).

For all primes $\mathfrak{p}$, we have $\det q_{\mathfrak{p}} = (-1)^{\frac{n(n-1)}{2}}$ and the number of primes $\mathfrak{p}$ for which $w_2(q_{\mathfrak{p}}) \neq 0$ is finite and even. So there exists a quadratic form q over k such that $q \otimes k_{\mathfrak{p}} \simeq q_{\mathfrak{p}}$. (cf.[Sc], theorem 6.6.10). Thus this form q satisfies $q \otimes k_{\mathfrak{p}} \simeq \mathcal{T}_{A_{\mathfrak{p}}}$. So we have $q \simeq \mathcal{T}_A$, but we cannot have $A \otimes k_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$ because $\sum \mathrm{inv}_{\mathfrak{p}}[A_{\mathfrak{p}}] = \dfrac{2}{n}$.

We can explain the fact that it is always possible to find a suitable A observing that over a local field, there are many central simple algebras with the same trace form. So we can choose some algebras $B_{\mathfrak{p}}$ such that $\mathcal{T}_{B_{\mathfrak{p}}} \simeq \mathcal{T}_{A_{\mathfrak{p}}}$ for all primes and satisfying $\sum \mathrm{inv}_{\mathfrak{p}}[B_{\mathfrak{p}}] = 0$. More precisely, we have the following proposition:

Proposition 1: Let k be a field of characteristic different from 2, and let A and B be two central simple algebras of even degree n.

(1) If k is a local field, we have $\mathcal{T}_A \simeq \mathcal{T}_B$ if and only if (A is division algebra $\iff$ B is a division algebra)

(2) If $k = \mathbb{R}$, we have $\mathcal{T}_A \simeq \mathcal{T}_B$ if and only if $A \simeq B$

(3) If k is a global field, we have $\mathcal{T}_A \simeq \mathcal{T}_B$ if and only if $A \otimes k_{\mathfrak{p}}$ and $B \otimes k_{\mathfrak{p}}$ satisfy the previous conditions over $k_{\mathfrak{p}}$ for all finite or real primes

Proof: If k is a local field, the Hasse-Witt invariants of $\mathcal{T}_A$ et $\mathcal{T}_B$ are equal if and only if A and B satisfy the condition (1) by the lemma. Since their dimensions and their determinants are equal, this is equivalent to $\mathcal{T}_A \simeq \mathcal{T}_B$. If $k = \mathbb{R}$, it suffices to notice that $M_n(\mathbb{R})$ and $M_m((-1,-1)_{\mathbb{R}})$ are the only central simple algebras of degree n and that their trace forms are not isomorphic (since the signatures differ by theorem 3). If k is a global field , the condition (3) simply expresses that $\mathcal{T}_A$ and $\mathcal{T}_B$ are isomorphic if and only if it is true locally everywhere (using the fact that these forms are automatically $k_{\mathfrak{p}}$-isomorphic for all complex primes).

B. Application to the study of the elements of $Br_2(k)$.

We are now interested in the following problem: if A is a central simple algebra of even degree n, we know that $\dfrac{n}{2}[A] \in Br_2(k)$. Conversely, which classes $[B] \in Br_2(k)$ can be written $[B] = \dfrac{n}{2}[A]$, where A is a central simple algebra of degree n ?
If k is a local field or a global field of characteristic different from 2, the answer is given by the following proposition:

Proposition 2: Let k be a field of characteristic different from 2 and $n \geq 2$ an even integer.

(1) If k is a local field, then for all $[B] \in Br_2(k)$ there exists a central simple algebra A over k of degree n such that $\frac{n}{2}[A] = [B]$.

(2) If k is a global field, then we have:

  (i) If $n \equiv 2\ [4]$, then for all $[B] \in Br_2(k)$ there exists a central simple algebra A over k of degree n such that $\frac{n}{2}[A] = [B]$

  (ii) If $n \equiv 0\ [4]$, then for all $[B] \in Br_2(k)$, there exists a central simple algebra A over k of degree n such that $\frac{n}{2}[A] = [B]$ if and only if $[B]_{k_\mathfrak{p}} = 0$ for all real primes

Proof: If k is a local field, we can take $A = M_n(k)$ if $[B] = 0$ and A is a division algebra if $[B] = (a, b)$ by the lemma. Now assume that k is a global field. We can assume that B is a quaternion algebra $(c, d)_k$. If $n = 2m$ with m odd, set $A = M_m((c, d)_k)$. So we can assume that $n = 2m$, where m is even. If $[B] = m[A]$, we have $[B]_{k_\mathfrak{p}} = m[A]_{k_\mathfrak{p}} = 0$ for all real primes since m is even and $[A]_{k_\mathfrak{p}} = 0$ or $(-1, -1)_{k_\mathfrak{p}}$. Conversely, assume that $[B]_{k_\mathfrak{p}} = 0$ for all real primes. For each infinite prime $\mathfrak{p}$, set $q_\mathfrak{p} = \mathcal{T}_{M_n(k_\mathfrak{p})}$. If $\mathfrak{p}$ is a finite prime, set $q_\mathfrak{p} = \mathcal{T}_{A_\mathfrak{p}}$, where $A_\mathfrak{p} = M_n(k_\mathfrak{p})$ if $(c, d)_{k_\mathfrak{p}} = 0$ and $A_\mathfrak{p}$ is a division algebra otherwise. Then we get $w_2(q_\mathfrak{p}) = \dfrac{m(m-1)}{2}(-1, -1)_{k_\mathfrak{p}} + (c, d)_{k_\mathfrak{p}}$ for all primes. Indeed, this is true by assumption if $\mathfrak{p}$ is real, it is trivial if $\mathfrak{p}$ is complex, and this is verified by choice of $A_\mathfrak{p}$ and by the lemma if $\mathfrak{p}$ is finite. So we have $w_2(q_\mathfrak{p}) = (\alpha, \beta)_{k_\mathfrak{p}}$, where $(\alpha, \beta)_k = \dfrac{m(m-1)}{2}(-1, -1)_k + (c, d)_k$. Then by Hilbert's reciprocity law, the number of primes $\mathfrak{p}$ for which $w_2(q_\mathfrak{p}) = 0$ is finite and even. Since $\det q_\mathfrak{p} = (-1)^{\frac{n(n-1)}{2}}$ for all primes, there exists a quadratic form q over k such that $q \otimes k_\mathfrak{p} \simeq q_\mathfrak{p}$. By choice of $q_\mathfrak{p}$ and by theorem 2, we have $q \simeq \mathcal{T}_A$ where A is a central simple algebra over k of degree n. Comparing Hasse-Witt invariants, we get $m[A]_{k_\mathfrak{p}} = (c, d)_{k_\mathfrak{p}}$ for all primes. Then we get $m[A] = (c, d)_k = [B]$ (It is a consequence of the Hasse principle for the elements of $Br(k)$. We refer to [CS] or [W] for more details for example).

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