The middle of the diagonal of a surface with $p_g = 0$ and $q = 1$

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Abstract

We prove that if X is a smooth projective complex surface with the invariants $p_g = 0$ and $q = 1$ then the middle Murre projector π_2 (see [Mu90] or [Sch94] for the definition of π_2) can be generated by two natural divisors on X whose cohomology classes form a basis for the second cohomology group $H^2(X, \mathbb{Q})$. As a consequence, this provides a second, in fact, Chow-motivic, proof of the triviality of the Albanese kernel for surfaces with $p_g = 0$ and $q = 1$ (the first proof was made in [BKL76]).

1 *Introduction*

All varieties will be smooth projective and defined over the field of complex numbers $\mathbb{C}$. If Y is a variety then let $A^t(Y)$ be the Chow-group of codimension t cycles on Y with coefficients in the field of rational numbers $\mathbb{Q}$ and let $A(Y) = \oplus_t A^t(Y)$ be the Chow-ring of Y.

For any smooth projective surface X by $A^2_0(X)$ denote the $\mathbb{Q}$-subspace in $A^2(X)$ consisting of zero-cycles of degree zero on X. Also let $\mathsf{A}(X)$ be the Albanese variety for the surface X. By definition, the Albanese kernel $T(X)$ is the kernel of the surjective homomorphism

$$A^2_0(X) \longrightarrow \mathsf{A}(X) \otimes \mathbb{Q}$$

induced by the Albanese mapping $X \to \mathsf{A}(X)$. The Bloch conjecture predicts that if $p_g = 0$, then $T(X) = 0$. Here $p_g = \dim \Gamma(X, \Omega^2_X)$ is the dimension of the space of global holomorphic 2-forms on X.

In [BKL76] Bloch, Kas and Lieberman proved that if $p_g = 0$ and $q = 1$, where q is the irregularity of X, then $T(X) = 0$. The purpose of this paper is to prove that the middle Murre projector π_2 for a such surface X (see [Mu90] or [Sch94] for the definition of π_2) is rationally equivalent to a sum of external products of two natural divisors on X whose cohomology classes form a basis for the cohomology group $H^2(X, \mathbb{Q})$. As a consequence this provides a second, in fact, Chow-motovic, proof of the triviality of $T(X)$ for surfaces with $p_g = 0$ and $q = 1$.

Recall the definition of the category $\mathcal{M}$ of Chow motives over $\mathbb{C}$ with coefficients in $\mathbb{Q}$. Objects in $\mathcal{M}$ are pairs (X, p) where X is variety and p is a projector of X, that is a class in $A(X \times X)$, such that $p \circ p = p$ in the sense of compositions of correspondences. If $M = (X, p)$ and $N = (Y, q)$ are two motives then a morphism $f : M \to N$ is a class $f \in A(X \times Y)$, such that $q \circ f = f \circ p$. If Δ_X is a diagonal of a variety X then Δ_X is a projector of X. The motive $h(X) = (X, \Delta_X)$ is called the motive of the variety X. At the same time Δ_X can be viewed as an identity morphism $1_M : M \to M$ for any motive $M = (X, p)$.

Let X be a smooth projective surface over $\mathbb{C}$. According to the Murre's results (see [Mu90] and [Mu93]), we have that there exist the projectors π_0, π_1, π_2, π_3 and π_4 of the surface X, such that

$$\Delta_X = \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4$$

in $A^2(X \times X)$ and the cohomology class of the correspondence π_j coincides with the $(4-j, j)$ Künneth component of the cohomology class of the diagonal Δ_X in the group $H^4(X \times X, \mathbb{Q})$. The projectors π_j are pairwise orthogonal, whence

$$h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X) \oplus h^3(X) \oplus h^4(X)$$

in the additive category $\mathcal{M}$, where $h^j(X) = (X, \pi_j)$.

This decomposition of Δ_X allows us to study zero cycles on X following the first Bloch's lecture in [Bloch80]. Indeed, correspondences act on Chow-groups: for any two varieties X and Y, if $\nu \in A^d(X \times Y)$, where $d = \dim(X)$, and if $\xi \in A^p(X)$, then one can define

$$\nu(\xi) = p_{Y*}(\nu \cdot p_X^*(\xi)) \ .$$

Here p_Y and p_X are projections, p_{Y*} and p_X^* are direct and inverse images respectively. In particular we have the homomorphisms

$$\pi_j : A^2(X) \longrightarrow A^2(X)$$

on zero cycles induced by the projectors π_j. Murre proved (see [Mu90], [Mu93] or [J94]) that π_0 and π_1 operate as zero on the whole Chow-group $A^2(X)$, the kernel $\ker(\pi_4)$ coincides with the group $A_0^2(X)$ and $\ker(\pi_3 \mid_{A_0^2(X)})$ coincides with the Albanese kernel $T(X)$. Consequently the action of the middle Murre projector π_2 on $T(X)$ is the same as the identical action of the diagonal Δ_X on $T(X)$. Therefore, if π_2 acts as zero on $T(X)$, then $T(X) = 0$.

Now let X be a smooth projective surface with $p_g = 0$ and $q = 1$. Since $q = 1$, it follows that the Albanese variety $\mathsf{A}(X)$ is an elliptic curve A over $\mathbb{C}$. Let F be a general fiber of the Albanese mapping

$$\alpha : X \longrightarrow A \ .$$

According to the classification of surfaces (see [Sh65], chapter IV) we have that, since $p_g = 0$ and $q = 1$, it follows that $X \cong (E \times F)/G$ where E is a curve and G is a finite abelian group acting on E and on F. The action of G on $E \times F$ is diagonal free. Let

$$\eta : E \times F \longrightarrow X$$

be the corresponding quotient map. The quotient curve E/G coincides with the elliptic curve A and the regular map $X \to E/G = A$ induced by the projection $p_E : E \times F \to E$ coincides with the Albanese mapping α.

There are two different cases: either genus of the curve F is greater than one or F is a curve of genus one. In the first case E is an elliptic curve, i.e. a curve of genus one with fixed zero-point $e \in E$. The group G is a finite subgroup in E acting on E by translations. Therefore the quotient map $\eta_E : E \longrightarrow A$ is unramified finite covering. We take $o = \eta_E(e)$ as a zero point on A. Then η_E is in fact an isogeny of elliptic curves. We fix any point $f \in F$ and take $x_0 = \eta(e \times f)$ as a fixed point on X. If genus of the general fiber F is one then the quotient morphism η_E can be ramified. In this case we choose a zero-point $o \in A$ in such a way that any point $e \in \eta_E^{-1}(o)$ is unramified over o. Let e be a fixed point on E lying in $\eta_E^{-1}(o)$ and let f be a fixed point on F. Then let $x_0 = \eta(e, f)$ be a fixed point on the surface X. So in both cases e is unramified point of the quotient map η_E. Also note that $p_g = 0$ and $q = 1$ imply that genus of the curve F/G is zero (see [Sh65], ch. IV, § 8), so that $F/G = \mathbb{P}^1$.

Since $p_q = 0$ and $q = 1$, it can be shown that the second Betti number of X is 2. Let n be the order of the group G. Consider the divisors

$$D_1 = \eta_*[e \times F] \qquad \text{and} \qquad D_2 = \frac{1}{n}\,\eta_*[E \times f]$$

on X. Its cohomology classes form a basis for the $\mathbb{Q}$-vector space $H^2(X, \mathbb{Q})$. The Poincaré dual basis is formed by the cohomology classes of divisors D_2 and D_1 (the same divisors but in inverse order).

Theorem 1.1 *Let X be a smooth projective surface with $p_g = 0$ and $q = 1$. Let D_1 and D_2 be the above divisors on X. Then the middle Murre projector π_2 of the surface X is rationally equivalent (with coefficients in $\mathbb{Q}$) to the correspondence $D_1 \times D_2 + D_2 \times D_1$.*

Now one can imitate the proof of Proposition 1.11 in [Bloch80] and deduce the triviality of $T(X)$ in the Chow-motivic way. Namely, we know that the middle projector π_2 acts on $T(X)$ identically. At the same time, one can move the divisors D_1 and D_2 from points on X and obtain that π_2 acts as zero on $T(X)$. Hence $T(X) = 0$.

For any cycle Z on X of codimension t its class in $A^t(X)$ will be denoted by $[Z]$. Let $f : Y \to X$ be a morphism of varieties and $d = \dim(X)$. Then the graph Γ_f is either a closed subvariety $\{(f(y), y) \in X \times Y |\ y \in Y\}$ or its class in $A^d(X \times Y)$ (depending from a context). The diagonal Δ_X of a variety X can be viewed as a graph of the identical map $1_X : X \to X$. All computations with algebraic cycles are based on the book [F84].

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2 *Linear section*

Let $\tilde{Y}$ be a variety and let G be a finite group of order n acting on $\tilde{Y}$. Let $Y = \tilde{Y}/G$ be the quotient variety of $\tilde{Y}$ by G and let $\tau : \tilde{Y} \to Y$ be the corresponding finite covering. It is well known that the composition $\tau_*\tau^* : A^t(Y) \to A^t(Y)$ coincides with the multiplication-by-n homomorphism for any codimesion t. By

$$N_{\tilde{Y}/G} : A^t(\tilde{Y}) \to A^t(\tilde{Y})$$

denote the norm-homomorphism

$$w \mapsto \sum_{g \in G} g(w)\ .$$

Lemma 2.1 *The composition $\tau^*\tau_* : A^t(\tilde{Y}) \to A^t(\tilde{Y})$ coincides with the norm homomorphism $N_{\tilde{Y}/G} : A^t(\tilde{Y}) \to A^t(\tilde{Y})$.*

Proof. This lemma is a well known fact and it can be easily deduced, for example, from the considerations in [B62]. Recall the proof for the convenience of the reader. Let V be a prime cycle on $\tilde{Y}$ and $W = \tau(V)$. Then $\tau_*(V) = dW$ where $d = [\mathbb{C}(V) : \mathbb{C}(W)]$ is the degree of V over

W. Let $V_1, \ldots, V_s$ be the distinct irreducible components of $\tau^{-1}(W)$. If l is a number of elements in the inertia group $\{g \in G|\ g(y) = y \text{ for all } y \in V\}$, then $\tau^*(W) = l \sum_j V_j$. Consequently, $\tau^*\tau_*(V) = dl \sum_j V_j$. But $sdl = n$, where n is the order of G, and $dl = \frac{n}{s}$ is the order of the splitting group $\{g \in G|\ g(V) = V\}$ (loc. cit.). Thus we obtain that $\tau^*\tau_*(V) = \frac{n}{s} \sum_j V_j = N_{\tilde{Y}/G}(V)$. □

Let X be a surface with $p_g = 0$ and $q = 1$ and let $X = (E \times F)/G$ be the presentation of X described in Introduction. Since a point on an algebraic curve is an ample divisor, it follows that for any $g \in G$ the divisor

$$[g(e) \times F] + [E \times g(f)]$$

is ample on $E \times F$. Then

$$\eta^*\eta_*([e \times F] + [E \times f]) = N_{(E\times F)/G}([e \times F] + [E \times f]) = \sum_{g\in G}([g(e) \times F] + [E \times g(f)])$$

is an ample divisor on $E \times F$. Since η is a finite surjective morphism of varieties, it follows that the divisor

$$\eta_*([e \times F] + [E \times f])$$

is ample on X (see [H70], p. 25). Hence there exists a positive integer m such that

$$m\eta_*([e \times F] + [E \times f])$$

is very ample.

Fix an embedding $X \hookrightarrow \mathbb{P}^N$ induced by this very ample divisor. Using Bertini theorem one can choose a hyperplane section C of X, such that the following properties hold: (i) C is a smooth projective curve; (ii) $x_0 \in C$; (iii) the curve C has a positive genus. Let $i : C \hookrightarrow X$ be the corresponding closed embedding of the curve C into X.

For any variety Y denote by $\mathsf{P}(Y)$ and $\mathsf{A}(Y)$ the Picard and Albanese varieties of Y respectively. In fact we have the functors P and A on the category of connected varieties with fixed points. To be precise let $\mathcal{CV}_\bullet$ be the category whose objects are pairs (Y, y_0) where Y is a connected variety and $y_0 \in Y$ is a fixed point on Y. Morphisms in $\mathcal{CV}_\bullet$ are morphisms of varieties preserving fixed points. For short let $Hom_\bullet(Y, Z) = Hom_{\mathcal{CV}_\bullet}((Y, y_0), (Z, z_0))$. If Y is an abelian variety, then, automatically, y_0 is a zero point on Y. Now A is a functor from $\mathcal{CV}_\bullet$ into the category of abelian varieties and P is a functor from the opposite category $\mathcal{CV}_\bullet^{opp}$ into the category of abelian varieties. In particular, for the closed embedding $i : C \to X$ there exist the morphisms $\mathsf{P}(i) : \mathsf{P}(X) \to \mathsf{P}(C)$ and $\mathsf{A}(i) : \mathsf{A}(C) \to \mathsf{A}(X)$.

The Picard variety is an object presenting the Picard functor. Hence $\mathsf{P}(X)$ is defined up to an isomorphism. Since A is an elliptic curve, it follows that it isomorphic to its dual. Therefore we can take $\mathsf{P}(X) = A = \mathsf{A}(X)$. Identifying $\mathsf{P}(C)$ with $\mathsf{A}(C)$, $J = \mathsf{P}(C) = \mathsf{A}(C)$, we get an isogeny

$$A = \mathsf{P}(X) \xrightarrow{\mathsf{P}(i)} J = \mathsf{P}(C) = \mathsf{A}(C) \xrightarrow{\mathsf{A}(i)} \mathsf{A}(X) = A$$

(see [Sch94]).

Lemma 2.2 *Let n be the order of the group G and let m be the above natural number. Then the isogeny $\mathsf{A}(i) \circ \mathsf{P}(i)$ coincides with the multiplication-by-mn homomorphism of the abelian variety A.*

Proof. Let a be a point on A and let F_a be a set-theoretic fiber of the morphism α over the point a. Then for any point $t \in \eta_E^{-1}(a)$ we have that $F_a = \eta(t \times F)$. Suppose that $\eta_E : E \to A$ is unramified at $a \in A$. Then the restriction of the morphism η on the subvariety $t \times F \hookrightarrow E \times F$ is a closed embedding. Hence $F_a = \eta(t \times F) \cong F$ for any $t \in \eta_E^{-1}(a)$. Moreover,

$$\alpha^*[a] = \frac{1}{n}\, n\alpha^*[a] = \frac{1}{n}\, \eta_*\eta^*\alpha^*[a] = \frac{1}{n}\, \eta_* p_E^* \eta_E^*[a] =$$

$$= \frac{1}{n}\, \eta_* \left(\sum_{t \in \eta_E^{-1}(a)} [t \times F] \right) = \frac{1}{n} \sum_{t \in \eta_E^{-1}(a)} \eta_*[t \times F] = \frac{1}{n} \sum_{t \in \eta_E^{-1}(a)} [F_a] =$$

$$= \frac{1}{n}\, n[F_a] = [F_a] = \eta_*[t \times F] \ .$$

So we see that the fiber F_a is not multiply and $\alpha^*[a] = \eta_*[t \times F]$.

By a construction,

$$[C] = m\eta_*([E \times f + e \times F]) \ .$$

Then

$$[C] \cdot [F_a] = [C] \cdot \eta_*[t \times F] = (m\eta_*[E \times f] + m\eta_*[e \times F]) \cdot \eta_*[t \times F] =$$

$$= m\eta_*[E \times f] \cdot \eta_*[t \times F] + m\eta_*[e \times F] \cdot \eta_*[t \times F] = m\eta_*[E \times f] \cdot \eta_*[t \times F] + 0 =$$

$$= m\eta_*(\eta^*\eta_*[E \times f] \cdot [t \times F]) = m\eta_*(N_{E \times F/G}[E \times f] \cdot [t \times F]) =$$

$$= m\eta_* \left(\sum_{g \in G} [E \times g(f)] \cdot [t \times F] \right) = mn \ .$$

For any connected variety Y by $\mathrm{Pic}^0(Y)$ denote the group of Weil divisors on Y algebraically equivalent to zero. Consider the commutative diagram

$$\begin{tikzcd}
& \mathrm{Pic}^0(X) \arrow[r, "i^*"] & \mathrm{Pic}^0(C) & C \arrow[r, "i"] \arrow[dl] & X \arrow[dl, "\alpha"] \\
\mathrm{Pic}^0(A) \arrow[ur, "\alpha^*"] \arrow[r, "s", "\cong"'] & A \arrow[u, "\cong"] \arrow[r, "\mathsf{P}(i)"] & J \arrow[u, "\cong"] \arrow[r, "\mathsf{A}(i)"] & A &
\end{tikzcd}$$

Here s is an isomorphism defined by the formula $s^{-1}(a) = [a - o]$ for any point $a \in A$, $C \to J$ is the Albanese mapping for the curve C. The vertical isomorphisms are isomorphisms of functors presentations.

Let a be a point on A such that η_E is unramified at a. The fibre F_a is a prime cycle on X and $[F_a] \cdot [F_a] = 0$. Hence, by Nakai-Moishezon criterion, we have that C is not a fiber of α because $[C] \cdot [C] = 2nm^2 > 0$. It follows that the restriction $\alpha|_C \colon C \to A$ is a surjective map. Let $t \in \eta_E^{-1}(a)$. Note that e and t are unramified over o and a respectively. Therefore the multiplicity of fibers F_a and F_o are equal to 1. Then

$$i^*(\alpha^*(s^{-1}(a))) = i^*\alpha^*([a - o]) = i^*(\alpha^*[a]) - i^*(\alpha^*[o]) =$$

$$= i^*[F_a] - i^*[F_o] = [C] \cdot [F_a] - [C] \cdot [F_o] \ .$$

The first summand is

$$[C] \cdot [F_a] = [C \cdot \eta_*(t \times F)] = d_1 v_1 + \ldots + d_z v_z \ ,$$

where $v_1, \ldots, v_z$ are the points of intersection of C with $F_a = \eta(t \times F)$ and d_j is the multiplicity of this intersection at the point v_j. The second summand is

$$[C] \cdot [F_o] = [C \cdot \eta_*(e \times F)] = t_1 w_1 + \ldots + t_h w_h \ ,$$

where $w_1, \ldots, w_h$ are the points of intersection of C with $F_o = \eta(e \times F)$ and t_l is its multiplicity at the point w_l. Then we have

$$(\mathsf{A}(i) \circ \mathsf{P}(i))(a) = \sum_{j=1}^{z} d_j (\alpha \circ i)(v_j) + \sum_{l=1}^{h} t_l (\alpha \circ i)(w_l) \ .$$

Since any point $i(w_l)$ lies in the fiber F_o, the point $\alpha(i(w_l))$ is equal to zero. Therefore,

$$(\mathsf{A}(i) \circ \mathsf{P}(i))(a) = \sum_{j=1}^{z} d_j (\alpha \circ i)(v_j) \ .$$

Since $i(v_j) \in F_a$, it follows that $\alpha(i(v_j)) = a$ for any j. Then

$$(\mathsf{A}(i) \circ \mathsf{P}(i))(q) = \sum_{j=1}^{z} d_j a \ .$$

But the number $\sum_{j=1}^{z} d_j$ coincides with the intersection number $[C] \cdot [F_a] = mn$. Consequently we obtain that

$$(\mathsf{A}(i) \circ \mathsf{P}(i))(a) = mna \ .$$

$\square$

3 *The Picard and Albanese projectors*

Recall (see [Mu90]) that for any smooth projective surface Y the projector π_1 is closely connected with the Picard variety $\mathsf{P}(Y)$ and so it is called the Picard projector of the surface Y. By definition, π_3 is a transpose of π_1. The projector π_3 is connected with the Albanese variety $\mathsf{A}(Y)$ and it is called the Albanese projector of Y. In this section we compute π_1 and π_3 for our surface X in terms of quotient structure on X.

Lemma 2.2 shows that the identical morphism $1_A : A \to A$ can be viewed as an inverse isogeny for the isogeny $A \xrightarrow{\mathsf{P}(i)} J \xrightarrow{\mathsf{A}(i)} A$. Then, according to the general method of constructing of π_1 (see [Mu90], [Sch94], [G99]), we can get the Picard projector in the form

$$\pi_1 = \frac{1}{mn} \, [C \times X] \cdot \varepsilon \ ,$$

where ε is a divisor on $X \times X$ that corresponds to the identical isogeny $1_A : A \to A$. Observe that since $[C] = m\eta_*([E \times f + e \times F])$, we have that

$$\frac{1}{mn} \, [C \times X] = \frac{1}{mn} \, [C] \times [X] = \frac{1}{n} \, (\eta_*[E \times f] + \eta_*[e \times F]) \times [X] \ .$$

Hence one can write

$$\pi_1 = \frac{1}{n} \, (\eta_*[E \times f] \times [X] + \eta_*[e \times F] \times [X]) \cdot \varepsilon \ . \qquad (1)$$

So to compute π_1 in terms of quotient structure on the surface X we have to compute ε.

For any variety Y with fixed point $y_0 \in Y$ let P_Y be the Poincaré divisor on $Y \times \mathsf{P}(Y)$. Also let $\alpha_Y : Y \longrightarrow \mathsf{A}(Y)$ be the Albanese map for Y sending y_0 into the zero point on $\mathsf{A}(Y)$. Then

$$P_Y = (\alpha_Y \times 1_{\mathsf{P}(Y)})^* P^t_{\mathsf{P}(Y)} \ ,$$

where $P^t_{\mathsf{P}(Y)}$ is a transpose of $P_{\mathsf{P}(Y)}$ (see [G99], Section 4.2, p. 20). In particular,

$$P_X = (\alpha \times 1_A)^* P^t_A \ .$$

Hence, the transpose P^t_X of P_X is

$$P^t_X = (1_A \times \alpha)^* P_A \ .$$

At the same time $\varepsilon = (\alpha \times 1_X)^*(P^t_X)$. To show this we need some more notation. For any connected variety $Y \in Ob(\mathcal{CV}_\bullet)$ one can consider two following functors, both from $\mathcal{CV}_\bullet^{opp}$ into the category of $\mathbb{Q}$-vector spaces. The first one is a functor $Z \mapsto Hom_\bullet(Z, \mathsf{P}(Y)) \otimes \mathbb{Q}$. The second functor sends any object $Z = (Z, z_0) \in Ob(\mathcal{CV}_\bullet)$ into the $\mathbb{Q}$-space $\{c \in A^1(Z \times Y) |\ c \circ z_{0*} = 0 = y_0^* \circ c\}$. Here $z_{0*} = [z_0 \times z_0]$ and $y_0^* = [y_0 \times y_0]$ are correspondences induced by the fixed points and compositions are compositions of correspondences. There exists an isomorphism of functors

$$\omega^Y_Z : Hom_\bullet(Z, \mathsf{P}(Y)) \otimes \mathbb{Q} \stackrel{\cong}{\longrightarrow} \{c \in A^1(Z \times Y) |\ c \circ z_{0*} = 0 = y_0^* \circ c\}$$

(see [Sch94] or [G99], Section 3.2). For any $Z \in Ob(\mathcal{CV}_\bullet)$, if $\alpha_Z : Z \to \mathsf{A}(Z)$ is the Albanese mapping for Z, then let Ω^Y_Z be the composition $\omega^Y_Z \circ (Hom_\bullet(\alpha_Z, \mathsf{P}(X)) \otimes \mathbb{Q})$. In fact

$$\Omega^Y_Z : Hom_\bullet(\mathsf{A}(Z), \mathsf{P}(Y)) \otimes \mathbb{Q} \stackrel{\cong}{\longrightarrow} \{c \in A^1(Z \times Y) |\ c \circ z_{0*} = 0 = y_0^* \circ c\}$$

is an usual one-to-one correspondence between isogenies and divisors normalised by fixed points.

Now applying the general method of a construction of the divisor ε in our case we have that

$$\varepsilon = \Omega^X_X(1_A)$$

(loc. cit.). Then the commutative diagram

$$\begin{tikzcd}
Hom_\bullet(A, A) \otimes \mathbb{Q} \arrow[r, "\Omega^X_X"] \arrow[d, "Hom_\bullet(\alpha, A)\otimes\mathbb{Q}"'] & \{c \in A^1(X \times X) |\ c \circ x_{0*} = 0 = x_0^* \circ c\} \\
Hom_\bullet(X, A) \otimes \mathbb{Q} \arrow[ur, "\omega^X_X"] & \{c \in A^1(A \times X) |\ c \circ 0_* = 0 = x_0^* \circ c\} \arrow[u, "(\alpha\times 1_X)^*"'] \\
Hom_\bullet(A, A) \otimes \mathbb{Q} \arrow[u, "Hom_\bullet(\alpha, A)\otimes\mathbb{Q}"] \arrow[ur, "\omega^X_A"] &
\end{tikzcd}$$

$$\begin{tikzcd}
1_A \arrow[r, mapsto] \arrow[d, mapsto] & \varepsilon \\
\alpha \arrow[ur, mapsto] & P^t_X \arrow[u, mapsto] \\
1_A \arrow[u, mapsto] \arrow[ur, mapsto] &
\end{tikzcd}$$

shows that

$$\varepsilon = (\alpha \times 1_X)^*(P_X^t)\ .$$

Therefore, we can write

$$\varepsilon = (\alpha \times 1_X)^*(1_A \times \alpha)^*(P_A) = (\alpha \times \alpha)^*(P_A)\ .$$

But it is very well known that the Poincaré divisor for the curve A is

$$P_A = \Delta_A - o \times A - A \times o\ .$$

Hence,

$$\varepsilon = (\alpha \times \alpha)^*(P_A) = (\alpha \times \alpha)^*(\Delta_A) - \eta_*[o \times F] \times [X] - [X] \times \eta_*[o \times F]\ .$$

Now we will calculate the divisor $(\alpha \times \alpha)^*(P_A)$. Let

$$l : E \times E \times F \times F \to E \times F \times E \times F$$

$$(e_1, e_2, f_1, f_2) \mapsto (e_1, f_1, e_2, f_2)$$

be the isomorphism transposing points in the second and third factors. Also for any $g \in G$ let Γ_g be the graph of the morphism $E \to E$, $q \mapsto g(q)$. Consider the commutative square

$$\begin{tikzcd}
E \times F \times E \times F \arrow[r, "\eta \times \eta"] \arrow[d, "l^{-1}"] & X \times X \arrow[d, "\alpha \times \alpha"] \\
E \times E \times F \times F \arrow[d, "p_{E \times E}"] & A \times A \arrow[d, equal] \\
E \times E \arrow[r, "\eta_E \times \eta_E"] & E/G \times E/G
\end{tikzcd}$$

Since

$$(\eta \times \eta)_*(\eta \times \eta)^*(\alpha \times \alpha)^*(\Delta_A) = n^2(\alpha \times \alpha)^*(\Delta_A)\ ,$$

one can write

$$(\alpha \times \alpha)^*(\Delta_A) = \frac{1}{n^2}\,(\eta \times \eta)_*(\eta \times \eta)^*(\alpha \times \alpha)^*(\Delta_A) = \frac{1}{n^2}\,(\eta \times \eta)_*(l^{-1})^* p_{E \times E}^*(\eta_E \times \eta_E)^*(\Delta_A) =$$

$$= \frac{1}{n^2}\,(\eta \times \eta)_*(l^{-1})^* p_{E \times E}^* \left[\sum_{g \in G} \Gamma_g\right] = \frac{1}{n^2}\,(\eta \times \eta)_* l_* \sum_{g \in G} [\Gamma_g \times F \times F]\ .$$

But for any $g \in G$ we have that

$$(\eta \times \eta)_* l_*[\Gamma_g \times F \times F] = (\eta \times \eta)_* l_*[\Delta_E \times F \times F]\ .$$

Therefore,

$$(\alpha \times \alpha)^*(\Delta_A) = \frac{1}{n^2}\,(\eta \times \eta)_* l_* \sum_{g \in G} [\Gamma_g \times F \times F] = \frac{1}{n}\,(\eta \times \eta)_* l_*[\Delta_E \times F \times F]\ .$$

So we have that

$$\varepsilon = \frac{1}{n}\,(\eta \times \eta)_* l_*[\Delta_E \times F \times F] - \eta_*[e \times F] \times [X] - [X] \times \eta_*[e \times F]\ .$$

Now one can substitute this formula for ε into the formula (1) and obtain that the projector π_1 is the intersection of the correspondences

$$\frac{1}{n}\left(\eta_*[E\times f]\times[X]+\eta_*[e\times F]\times[X]\right)$$

and

$$\frac{1}{n}(\eta\times\eta)_* l_*(\Delta_E\times[F\times F])-\eta_*[e\times F]\times[X]-[X]\times\eta_*[e\times F]\ .$$

Intersecting summands we have:

$$\begin{aligned}
\pi_1 \;=\; & \tfrac{1}{n}\left(\eta_*[E\times f]\times[X]\right)\cdot\tfrac{1}{n}(\eta\times\eta)_* l_*[\Delta_E\times F\times F]\\
& -\tfrac{1}{n}\left(\eta_*[E\times f]\times[X]\right)\cdot\left(\eta_*[e\times F]\times[X]\right)\\
& -\tfrac{1}{n}\left(\eta_*[E\times f]\times[X]\right)\cdot\left([X]\times\eta_*[e\times F]\right)\\
& +\tfrac{1}{n}\left(\eta_*[e\times F]\times[X]\right)\cdot\tfrac{1}{n}(\eta\times\eta)_* l_*[\Delta_E\times F\times F]\\
& -\tfrac{1}{n}\left(\eta_*[e\times F]\times[X]\right)\cdot\left(\eta_*[e\times F]\times[X]\right)\\
& -\tfrac{1}{n}\left(\eta_*[e\times F]\times[X]\right)\cdot\left([X]\times\eta_*[e\times F]\right)\;=
\end{aligned}$$

$$\begin{aligned}
=\; & \tfrac{1}{n^2}(\eta\times\eta)_*[E\times f\times E\times F])\cdot\tfrac{1}{n}(\eta\times\eta)_* l_*[\Delta_E\times F\times F]\\
& -\tfrac{1}{n^2}(\eta\times\eta)_*[E\times f\times E\times F])\cdot\tfrac{1}{n}(\eta\times\eta)_*[e\times F\times E\times F])\\
& -\tfrac{1}{n^2}(\eta\times\eta)_*[E\times f\times E\times F])\cdot\tfrac{1}{n}(\eta\times\eta)_*[E\times F\times e\times F])\\
& +\tfrac{1}{n^2}(\eta\times\eta)_*[e\times F\times E\times F])\cdot\tfrac{1}{n}(\eta\times\eta)_* l_*[\Delta_E\times F\times F]\\
& -\tfrac{1}{n^2}(\eta\times\eta)_*[e\times F\times E\times F])\cdot\tfrac{1}{n}(\eta\times\eta)_*[e\times F\times E\times F])\\
& -\tfrac{1}{n^2}(\eta\times\eta)_*[e\times F\times E\times F])\cdot\tfrac{1}{n}(\eta\times\eta)_*[E\times F\times e\times F])\;=
\end{aligned}$$

$$\begin{aligned}
=\; & \tfrac{1}{n^3}(\eta\times\eta)_*(\ N_{(E\times F\times E\times F)/(G\times G)}[E\times f\times E\times F]\cdot l_*[\Delta_E\times F\times F]\\
& -N_{(E\times F\times E\times F)/(G\times G)}[E\times f\times E\times F]\cdot[e\times F\times E\times F]\\
& -N_{(E\times F\times E\times F)/(G\times G)}[E\times f\times E\times F]\cdot[E\times F\times e\times F]\\
& +N_{(E\times F\times E\times F)/(G\times G)}[e\times F\times E\times F]\cdot l_*[\Delta_E\times F\times F]\\
& -N_{(E\times F\times E\times F)/(G\times G)}[e\times F\times E\times F]\cdot[e\times F\times E\times F]\\
& -N_{(E\times F\times E\times F)/(G\times G)}[e\times F\times E\times F]\cdot[E\times F\times e\times F]\,)\;=
\end{aligned}$$

$$=\frac{1}{n^3}(\eta\times\eta)_*\left(n\sum_{g\in G}[E\times g(f)\times E\times F]\cdot l_*[\Delta_E\times F\times F]-n\sum_{g\in G}[E\times g(f)\times E\times F]\cdot[e\times F\times E\times F]-\right.$$

$$-n\sum_{g\in G}[E\times g(f)\times E\times F]\cdot[E\times F\times e\times F]+n\sum_{g\in G}[g(e)\times F\times E\times F]\cdot l_*[\Delta_E\times F\times F]-$$

$$-n\sum_{g\in G}[g(e)\times F\times E\times F]\cdot[e\times F\times E\times F]-n\sum_{g\in G}[g(e)\times F\times E\times F]\cdot[E\times F\times e\times F]\Bigg) =$$

$$= \frac{1}{n^2}\,(\eta\times\eta)_*l_*\Bigg(\sum_{g\in G}[E\times E\times g(f)\times F]\cdot[\Delta_E\times F\times F]-\sum_{g\in G}[E\times E\times g(f)\times F]\cdot[e\times E\times F\times F]-$$

$$-\sum_{g\in G}[E\times E\times g(f)\times F]\cdot[E\times e\times F\times F]+\sum_{g\in G}[g(e)\times E\times F\times F]\cdot[\Delta_E\times F\times F]-$$

$$-\sum_{g\in G}[g(e)\times E\times F\times F]\cdot[e\times E\times F\times F]-\sum_{g\in G}[g(e)\times E\times F\times F]\cdot[E\times e\times F\times F]\Bigg)\ .$$

Now one can simplify every summand in the obtained expression for the correspondence π_1. For any $g\in G$ we have that

$$(\eta\times\eta)_*l_*[E\times E\times g(f)\times F]=(\eta\times\eta)_*l_*[E\times E\times f\times F]\ ,$$

whence

$$(\eta\times\eta)_*l_*\Bigg(\sum_{g\in G}[E\times E\times g(f)\times F]\cdot[\Delta_E\times F\times F]\Bigg)=n(\eta\times\eta)_*l_*([E\times E\times f\times F]\cdot[\Delta_E\times F\times F])\ .$$

The other summands can be simplified similarly. Therefore we proceed the computation of the correspondence π_1:

$$\pi_1=\frac{1}{n}\,(\eta\times\eta)_*l_*([E\times E\times f\times F]\cdot[\Delta_E\times F\times F]-[E\times E\times f\times F]\cdot[e\times E\times F\times F]-$$

$$-[E\times E\times f\times F]\cdot[E\times e\times F\times F]+[e\times E\times F\times F]\cdot[\Delta_E\times F\times F]-$$

$$-[e\times E\times F\times F]\cdot[e\times E\times F\times F]-[e\times E\times F\times F]\cdot[E\times e\times F\times F])=$$

$$=\frac{1}{n}\,(\eta\times\eta)_*l_*([\Delta_E\times f\times F]-[e\times E\times f\times F]-[E\times e\times f\times F]+$$

$$+[e\times e\times F\times F]-0-[e\times e\times F\times F])=$$

$$=\frac{1}{n}\,(\eta\times\eta)_*l_*([\Delta_E\times f\times F]-[e\times E\times f\times F]-[E\times e\times f\times F])\ .$$

So we have that the Picard projector of the surface X can be expressed as

$$\pi_1=\frac{1}{n}\,(\eta\times\eta)_*l_*([\Delta_E\times f\times F]-[e\times E\times f\times F]-[E\times e\times f\times F])\ . \tag{2}$$

Then

$$\pi_3=\pi_1^t=\frac{1}{n}\,(\eta\times\eta)_*l_*([\Delta_E\times F\times f]-[E\times e\times F\times f]-[e\times E\times F\times f]) \tag{3}$$

is the Albanese projector of X. Also let

$$\pi_0=[x_0\times X]\qquad\text{and}\qquad\pi_4=[X\times x_0]\ .$$

As had shown in [Sch94], in the general situation the relations $\pi_i\circ\pi_j=\delta_{ij}\cdot\pi_i$ hold for any $i,j\in\{0,1,3,4\}$ except for, maybe, the relation $\pi_1\circ\pi_3=0$. This is the reason for the replacement π_1 by $\pi_1-\frac{1}{2}\pi_1\circ\pi_3$ in loc. cit. But in our case the last relation holds as well:

Lemma 3.1 $\pi_1\circ\pi_3=0$

Proof. Let $\tilde{Y}$ be a variety, let G be a finite abelian group acting on $\tilde{Y}$ and let $\tau : \tilde{Y} \to Y = \tilde{Y}/G$ be the quotient map. If S is a set then by S^t denote the product $S^t = S \times \ldots \times S$ with t factors. Let μ and ν be correspondences on $\tilde{Y} \times \tilde{Y}$. It is easy to show that

$$(\tau\times\tau)_*(\mu)\circ(\tau\times\tau)_*(\nu) = (\tau\times\tau)_* q_{13*}(N_{\tilde{Y}^3/G^3}\, q_{23}^*(\mu)\cdot q_{12}^*(\nu)) = (\tau\times\tau)_* q_{13*}(q_{23}^*(\mu)\cdot N_{\tilde{Y}^3/G^3}\, q_{12}^*(\nu))$$

where q_{12}, q_{23} and q_{13} are the projections

$$\begin{tikzcd}
 & \tilde{Y}\times\tilde{Y} & \\
 & \tilde{Y}\times\tilde{Y}\times\tilde{Y} \arrow[u, "q_{13}"] \arrow[dl, "q_{12}"'] \arrow[dr, "q_{23}"] & \\
\tilde{Y}\times\tilde{Y} & & \tilde{Y}\times\tilde{Y}
\end{tikzcd}$$

In particular, we can write

$$\pi_1\circ\pi_3 = \frac{1}{n}\,(\eta\times\eta)_* l_*[\Delta_E\times f\times F - e\times E\times f\times F - E\times e\times f\times F]\circ$$

$$\circ\,\frac{1}{n}\,(\eta\times\eta)_* l_*[\Delta_E\times F\times f - E\times e\times F\times f - e\times E\times F\times f] =$$

$$= \frac{1}{n^2}\,(\eta\times\eta)_* q_{13*}(N_{(E\times F)^3/G^3}\, q_{23}^* l_*[\Delta_E\times f\times F - e\times E\times f\times F - E\times e\times f\times F]\cdot$$

$$\cdot\, q_{12}^* l_*[\Delta_E\times F\times f - E\times e\times F\times f - e\times E\times F\times f]) =$$

$$= \frac{1}{n^2}\,(\eta\times\eta)_* q_{13*}\Bigg(n[E\times F]\times l_*\Bigg[\sum_{g\in G}\Gamma_g\times\sum_{g\in G}(g(f)\times F) - n\sum_{g\in G}(g(e)\times E\times g(f)\times F) -$$

$$-\sum_{g\in G}(E\times g(e))\times\sum_{g\in G}(g(f)\times F)\Bigg]\cdot l_*[\Delta_E\times F\times f - E\times e\times F\times f - e\times E\times F\times f]\times[E\times F]\Bigg) =$$

$$= \frac{1}{n^2}\,(\eta\times\eta)_* q_{13*}\Bigg(n[E\times F]\times l_*\Bigg[\sum_{g\in G}\Gamma_g\times\sum_{g\in G} g(f)\times F - n\sum_{g\in G}(g(e)\times E\times g(f)\times F) -$$

$$-E\times\sum_{g\in G} g(e)\times\sum_{g\in G} g(f)\times F\Bigg]\cdot l_*[\Delta_E\times F\times f - E\times e\times F\times f - e\times E\times F\times f]\times[E\times F]\Bigg) =$$

$$= \frac{1}{n^2}\,(\eta\times\eta)_* q_{13*}\sum_{g\in G}\sum_{g\in G}(n[E\times F]\times l_*[\Gamma_g\times g(f)\times F - g(e)\times E\times g(f)\times F -$$

$$-E\times g(e)\times g(f)\times F]\cdot l_*[\Delta_E\times F\times f - E\times e\times F\times f - e\times E\times F\times f]\times[E\times F])\,.$$

Now observe that we can move the divisor $[f]\in A^1(F)$ from the finite number of points $\{g(f)|\ g\in G\}$. In other words, there exist points $\{f_1,\ldots,f_s\}$ on the curve F, such that

$$[f] = \left[\sum_{j=1}^{s} r_j f_j\right]\ ,$$

where r_j are multiplicities, and $f_j \notin \{g(f)\mid\ g\in G\}$ for any j. Then the intersection of any summand in

$$[E\times F]\times l_*[\Gamma_g\times g(f)\times F] - [E\times F]\times[g(e)\times g(f)\times E\times F] - [E\times F]\times[E\times g(f)\times g(e)\times F]$$

with any summand in

$$l_*[\Delta_E \times F \times f] \times [E \times F] - [E \times F \times e \times f] \times [E \times F] - [e \times F \times E \times f] \times [E \times F]$$

is zero. Hence $\pi_1 \circ \pi_3 = 0$. $\square$

So we see that π_0, π_1, π_3 and π_4 are pairwise orthogonal idempotences in the ring of correspondences $A^2(X \times X)$.

4 *The middle projector*

By definition,

$$\pi_2 = \Delta_X - \pi_0 - \pi_1 - \pi_3 - \pi_4$$

(see [Mu90]). Hence, tautologically, we have the decomposition $\Delta_X = \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4$. For any variety Y let

$$cl_Y^t : A^t(Y) \longrightarrow H^{2t}(Y, \mathbb{Q})$$

be the cycle map for codimension t cycles. As it was mentioned in Introduction,

$$cl^2_{X \times X}(\pi_j) = \Delta(4-j, j)$$

is the $(4-j, j)$ Künneth component of the cohomology class of the diagonal Δ_X. Since the classes $cl^1_X([D_1])$, $cl^1_X([D_2])$ form a basis for $H^2(X, \mathbb{Q})$ and $cl^1_X([D_2])$, $cl^1_X([D_1])$ form a Poincaré dual basis (see Introduction), it follows that

$$\Delta(2,2) = cl^2_{X \times X}([D_1 \times D_2] + [D_2 \times D_1]) \ .$$

Therefore the correspondences π_2 and $[D_1 \times D_2] + [D_2 \times D_1]$ are the same modulo cohomological equivalence. Theorem 1.1 asserts, that, in fact, these correspondences are the same modulo rational equivalence. Hence to prove Theorem 1.1 we have to prove that the difference of correspondences

$$\Xi = [D_1 \times D_2] + [D_2 \times D_1] - \pi_2$$

is equal to zero in $A^2(X \times X)$.

Let us compute the cycle Ξ. Observe that

$$\pi_0 = [x_0 \times X] = [x_0] \times [X] = \eta_*[e \times f] \times \frac{1}{n}\, \eta_*[E \times F]$$

$$= \frac{1}{n}\, (\eta \times \eta)_*[e \times f \times E \times F] = \frac{1}{n}\, (\eta \times \eta)_* l_*[e \times E \times f \times F] \ .$$

Respectively,

$$\pi_4 = \frac{1}{n}\, (\eta \times \eta)_* l_*[E \times e \times F \times f] \ .$$

Then it is easy to compute:

$$\Xi = [D_1 \times D_2] + [D_2 \times D_1] + \pi_0 + \pi_1 + \pi_3 + \pi_4 - \Delta_X =$$

$$= \frac{1}{n}\, (\eta \times \eta)_* l_*([e \times E \times F \times f] + [E \times e \times f \times F] + [e \times E \times f \times F] + [\Delta_E \times f \times F] -$$

$$-[e \times E \times f \times F] - [E \times e \times f \times F] + [\Delta_E \times F \times f] - [E \times e \times F \times f] -$$

$$-[e \times E \times F \times f] + [E \times e \times F \times f] - [\Delta_E \times \Delta_F]) =$$

$$= \frac{1}{n}\,(\eta\times\eta)_* l_*([\Delta_E \times f \times F] + [\Delta_E \times F \times f] - [\Delta_E]\times[\Delta_F])\ .$$

So we obtain that

$$\Xi = \frac{1}{n}\,(\eta\times\eta)_* l_*([\Delta_E \times f \times F] + [\Delta_E \times F \times f] - [\Delta_E]\times[\Delta_F])\ .$$

Now, again, let $Y = \tilde{Y}/G$ be the quotient variety of a variety $\tilde{Y}$ by a finite abelian group G. Let $\tau : \tilde{Y} \to Y$ be the corresponding quotient map. Let $\theta \in A^t(\tilde{Y})$ be any cycle of codimension t on $\tilde{Y}$. Then $\tau_*(\theta)$ is zero if and only if $N_{\tilde{Y}/G}(\theta)$ is zero. Indeed, if $\tau_*(\theta) = 0$ then $N_{\tilde{Y}/G}(\theta) = \tau^*\tau_*(\theta) = 0$. Conversely, if $N_{\tilde{Y}/G}(\theta) = 0$ then $\tau_*\, N_{\tilde{Y}/G}(\theta) = \tau_*\tau^*\tau_*(\theta) = n\tau_*(\theta) = 0$ where n is order of the group G. Dividing by n we obtain that $\tau_*(\theta) = 0$.

In particular $\Xi = 0$ if and only if the cycle

$$\Upsilon = N_{(E\times F\times E\times F)/(G\times G)}(l_*([\Delta_E \times f \times F] + [\Delta_E \times F \times f] - [\Delta_E]\times[\Delta_F]))$$

is zero. We omit $\frac{1}{n}$ here because we can multiply and divide by n. For any $g \in G$ let Θ_g be the graph of the regular map $F \to F$, $q \mapsto g(q)$. Then we compute:

$$\Upsilon = l_*\, N_{E\times E\times F\times F/G\times G}[\Delta_E \times f \times F + \Delta_E \times F \times f - \Delta_E \times \Delta_F] =$$

$$= l_*\left[\sum_{g_1,g_2\in G} \Gamma_{g_1g_2^{-1}} \times g_1(f) \times F + \sum_{g_1,g_2\in G} \Gamma_{g_1g_2^{-1}} \times F \times g_2(f) - \sum_{g_1,g_2\in G} \Gamma_{g_1g_2^{-1}} \times \Theta_{g_1g_2^{-1}}\right] =$$

$$= l_*\left[\sum_{g\in G}\Gamma_g \times \sum_{g\in G} g(f) \times F + \sum_{g\in G}\Gamma_g \times F \times \sum_{g\in G} g(f) - n\sum_{g\in G}\Gamma_g \times \Theta_g\right] =$$

$$= l_*\left(\sum_{g\in G}[\Gamma_g] \times \sum_{g\in G}[g(f)\times F + F\times g(f)] - n\sum_{g\in G}[\Gamma_g \times \Theta_g]\right)\ .$$

Let

$$\lambda = (\eta_E, \eta_E, \eta_F, \eta_F) : E\times E\times F\times F \longrightarrow A\times A\times\mathbb{P}^1\times\mathbb{P}^1$$

where $\eta_F : F \to \mathbb{P}^1$ is a quotient map. Then we have that

$$\lambda^*\lambda_* l^*\Upsilon = N_{(E\times E\times F\times F)/(G\times G\times G\times G)}\, l^*\Upsilon =$$

$$= N_{(E\times E\times F\times F)/(G\times G\times G\times G)}\left(\sum_{g\in G}[\Gamma_g] \times \sum_{g\in G}[g(f)\times F + F\times g(f)] - n\sum_{g\in G}[\Gamma_g\times\Theta_g]\right) =$$

$$= \sum_{g_1,g_2\in G}\sum_{g\in G}[\Gamma_g] \times \sum_{g_3,g_4\in G}\sum_{g\in G}[g(f)\times F + F\times g(f)] - n\sum_{g_1,g_2,g_3,g_4\in G}\sum_{g\in G}[\Gamma_g\times\Theta_g] =$$

$$= n^2\sum_{g\in G}[\Gamma_g] \times n^2\sum_{g\in G}[g(f)\times F + F\times g(f)] - n^5\sum_{g\in G}[\Gamma_g\times\Theta_g] =$$

$$= n^4 l^*\Upsilon\ .$$

Hence, if the cycle $\lambda_* l^*\Upsilon$ is trivial, then the cycle $l^*\Upsilon$ is trivial too. We have that

$$\lambda_* l^*\Upsilon = n^4([\Delta_A]\times[pt\times\mathbb{P}^1 + \mathbb{P}^1\times pt] - [\Delta_A]\times[\Delta_{\mathbb{P}^1}]) =$$

$$= n^4[\Delta_A]\times[pt\times\mathbb{P}^1 + \mathbb{P}^1\times pt - \Delta_{\mathbb{P}^1}] =$$

$$= n^4[\Delta_A]\times 0 = 0\ .$$

So we obtain that

$$\lambda_* l^*\Upsilon = 0 \Rightarrow \lambda^*\lambda_* l^*\Upsilon = 0 \Rightarrow l^*\Upsilon = 0 \Rightarrow \Upsilon = 0 \Rightarrow \Xi = 0\ .$$

Theorem 1.1 is proved.

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