ON THE CENTRAL SERIES OF THE MULTIPLICATIVE GROUP OF DIVISION RINGS

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Abstract. Let D^* be the multiplicative group of a division ring D. In this note we study the descending central series of D^*. We show that the structure of subgroups which appear in a central series of D^* affect greatly the structure of D^*. For example it is shown that if every element of a subgroup in a central series of D^* is algebraic over the center of D then D is algebraic division algebra. As an application, some classical theorems of Kaplansky and Jacobson on commutativity of a division ring are generalized. Also the descending central series of D^* in the case of totally ramified and unramified valued division algebra is completely determined.

1. On central series of division algebras

Let D be a division ring and D^* be the multiplicative group of D. Put $G^0(D) = D^*$ and for any non-negative integer i, define $G^i(D) = [D^*, G^{i-1}(D)]$, i.e, the subgroup generated by mix-commutators of D^* and $G^{i-1}(D)$. The sequence

$$\cdots \subseteq G^2(D) \subseteq G^1(D) \subseteq G^0(D) = D^*$$

is called the *descending central series* of D^*. It is a classical result that the multiplicative group of D^* is not nilpotent, that is, the above series never reaches 1 [8].

The descending central series of D^* has been studied by U. Rehmann in [12] and P. Draxl in [2, Vortrag 7] in the case of division algebras over *local fields* where it is shown that this series becomes stationary and the quotients $G^i(D)/G^{i+1}(D)$ have been calculated. (Also see C. Riehm [14].)

In this note we study the subgroups which appear in a central series of a division ring. We will show that many properties of subgroups of a central series of D^* can actually be lifted in a natural way to the group D^*. For example it is shown that if a subgroup in a central series of D^* is algebraic over F then D is algebraic division ring (Theorem 1.4). This is then used to obtain a generalization for some

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commutativity theorems for division rings. In section 2 we concentrate on a valued division algebra. We show that in the case of unramified valued division algebra, the study of descending central series reduce to the case of residue division algebra. Also in the case of totally ramified division algebra, the descending central series of D^* is completely determined. Using this, we show that on the case of tame valued division algebra, the group $PSL_1(D)$ is not a simple group (Theorem 2.3).

Before stating our first Lemma, we fix some notation. If G is a group, denote by G^n the subgroup of G generated by all $n-th$ powers of element of G. If H and K are subgroups of G, denote by $[H,K]$ the subgroup of G generated by mix-commutators $[h,k]=hkh^{-1}k^{-1}$, where $h\in H$ and $k\in K$. Note that for convenience we sometimes denote $[D^*,D^*]$ by D'. We say that a subset S of D is algebraic over F if each element of S is algebraic over F. Also if S and T are subsets of D, then S is said to be radical over T, if for any element $x\in S$, there is an integer r such that $x^r\in T$.

Let us begin with the following Lemma which is based on Wedderburn's factorization theorem and is crucial in our study. Variants of the trick which is used in this Lemma is also employed in [3],[9],[16] and [17].

Lemma 1.1. *Let D be a division algebra with center F, of index n. Let N be a normal subgroup of D^*. Then $N^n\subseteq Nrd_{D/F}(N)[D^*,N]$.*

Proof. Suppose $a\in N$ with the minimal polynomial $f(x)\in F[x]$ of degree m. Then from the theory of central simple algebras, we have,

$$f(x)^{n/m}=x^n-Trd_{D/F}(a)x^{n-1}+\cdots+(-1)^nNrd_{D/F}(a), \tag{1}$$

where $Nrd_{D/F}:D^*\longrightarrow F^*$ is the reduced norm, $Trd_{D/F}$ is the reduced trace and the right hand side of the equality (1) is the reduced characteristic polynomial of a. (See [13], §9.) Using Wedderburn's factorization theorem for the minimal polynomial $f(x)$ of a, one obtains $f(x)=(x-d_1ad_1^{-1})\cdots(x-d_mad_m^{-1})$ where $d_i\in D$. Now from the equality (1), it follows that

$$Nrd_{D/F}(a)=(d_1a{d_1}^{-1}\cdots d_ma{d_m}^{-1})^{n/m}.$$

Since N is a normal subgroup of D^*, therefore $Nrd_{D/F}(a)\in N$. But

$$d_1a{d_1}^{-1}\cdots d_ma{d_m}^{-1}=[d_1,a]a[d_2,a]a\cdots[d_m,a]a=a^md_a$$

for some $d_a\in[D^*,N]$. Therefore $a^n=Nrd_{D/F}(a)d'_a$ where $d'_a\in[D^*,N]$. Thus $N^n\subseteq Nrd_{D/F}(N)[D^*,N]$. $\square$

Now let $N=G^1(D)$. Since for $a\in[D^*,D^*]$, $Nrd_{D/F}(a)=1$, the above lemma shows that ${G^1(D)}^n\subseteq G^2(D)$. So in this way one can observe the following interesting fact.

Corollary 1.2. *Let D be a division ring with center F, of index n. For any $i > 0$, the quotient $G^i(D)/G^{i+1}(D)$ is a torsion abelian group of bounded exponent n.* $\square$

Note that the corollary is valid for $i > 0$. Using a Jacobson's Theorem [8, p. 219], it is an easy exercise to see that for $i = 0$, the group $G^0(D)/G^1(D)$, namely $K_1(D) = D^*/D'$ is not a torsion group.

Lemma 1.3. *Let D be a division ring with center F and N be a subgroup in a central series of D^*. If $a \in D$ is algebraic over F, then a is radical over F^*N.*

Proof. Since N is a subgroup in a central series of D^*, there is an integer i such that $G^i(D) \subseteq N$. Because a is algebraic over F, then from field theory, we have

$$f(x) = x^n - Tr_{F(a)/F}(a)x^{n-1} + \cdots + (-1)^n N_{F(a)/F}(a), \tag{1}$$

where $f(x)$ is the minimal polynomial of a over F. Now using Wedderburn's factorization theorem for $f(x)$ as in the Lemma 1.1 and repeating this procedure, one can obtain an integer r such that $a^r = fd$ where $f \in F^*$ and $d \in G^i(D)$. So the Lemma follows. $\square$

We are now in a position to show how the properties of a subgroup which appear in the central series of D^* can be lifted to D^*. The following Theorem shows that the algebracity of a division algebra is inherited from the algebracity of its subgroup in a central series. In contrast, note that a division algebra can be transcendental over its center and yet have maximal subfields which are algebraic (Example 1.5).

Theorem 1.4. *Let D be a division ring with center F and N be a subgroup in a central series of D^*. If N is algebraic over F, then D is algebraic division ring.*

Proof. Since N is a subgroup in a central series of D^*, there is an integer i such that $G^i(D) \subseteq N$. Consider the set $A = \{a \in G^{i-1}(D) |\ a \text{ is algebraic over } F\} \cup F^*$. Since D^* is not a nilpotent group, $G^i(D) \nsubseteq F$. On the other hand $G^i(D) \subseteq A$, so $F \subsetneq A$. Now suppose $a \in A$ and b is an algebraic element of D^*. Denote by $\bar{a}$ and $\bar{b}$ the images of a and b in the quotient group $D^*/F^*G^i(D)$. Since $a \in G^{i-1}(D) \cup F^*$, then $\bar{a}$ commutes with $\bar{b}$. By Lemma 1.3, $\bar{a}$ and $\bar{b}$ are torsion elements. Therefore $\overline{ab}$ is torsion. So it follows that ab is algebraic over F. Next consider the element $a+b = a(1+a^{-1}b)$. Since $a \in G^{i-1}(D)$ and b are algebraic, $1+a^{-1}b$ is algebraic. It follows that $a+b$ is algebraic. Now consider the ring $\langle A \rangle$ generated by elements of A. One can see that the ring $\langle A \rangle$ is algebraic over F, therefore it is a division ring. It is easy to see that A^* is a normal subgroup of D^*. Therefore by Cartan-Brauer-Hua, $A = D$. Hence D is algebraic division ring. $\square$

For the sake of completeness, let us give an example which shows that the algebracity of a maximal subfield of a division ring D, does not give rise to algebracity of D.

Example 1.5. Let L be a field which is algebraic over its prime subfield and $\sigma \in Aut(L)$ such that $ord(\sigma) = \infty$, e.g., take $L = \overline{\mathbb{Z}_p}$ and $\sigma(x) = x^{p^r}$ or $L = \bigcup_{i=1}^{\infty} F_{p^{2^i}}$ and $\sigma(x) = x^{p^r}$ where p is a prime number and r is an integer. Now let $D = L((x, \sigma))$ be a formal twisted Laurent series and K be a fixed field of σ. By Hilbert classical construction, D is a division algebra with center $Z(D) = K$. It is an easy exercise to show that L is a maximal subfield of D which is algebraic over $Z(D)$. But clearly D is not algebraic. We remark that L and $K((x))$ are two maximal subfields of D such that L is algebraic over K whereas $K((x))$ is purely transcendental over K.

We are now ready to generalize some commutativity theorems for a division ring. The following result may be considered as a generalization of Kaplansky's Theorem (See [8, p. 259] and [9]).

Corollary 1.6. *Let D be a division ring with center F. If a subgroup N in a central series of D^* is radical over F, then D is commutative.*

Proof. Since N is radical over F, by Theorem 1.4, we conclude that D is algebraic division algebra. But again by Lemma 1.3, it follows that D is radical over F^*N. On the other hand since N is radical over F, therefore D is radical over F. Now applying Kaplansky's Theorem, the proof is complete. □

Next application of Lemma 1.3 is to generalize Noether-Jacobson's Theorem which asserts that any noncommutative algebraic division ring D contains an element in $D\backslash F$ which is separable over F. (See [8], p.257.)

Corollary 1.7. *Let D be non-commutative algebraic division ring with center F. Then for any subgroup N in a central series of D^*, there exist $a \in N\backslash F$ which is separable over F.*

Proof. Suppose this is not the case. Then all elements in N are purely inseparable over F. This means that N becomes radical over F. Now apply Corollary 1.6 to get a contradiction. □

The following can be viewed as a generalization of a Jacobson's Theorem. (See [8], p.219.)

Corollary 1.8. *Let D be noncommutative algebraic division ring with center F. If a subgroup N in a central series of D^* is algebraic over a finite subfield of F, then D is commutative.*

Proof. Exercise. □

2. On descending central series of valued division algebra

In this section we study the descending central series of a Henselian valued division algebra. Theorem 2.1 determines completely this series in the case of totally ramified case. In the case of unramified case we show that the quotient group $G^i(D)/G^{i+1}(D)$ is *stable under reduction*, namely

$$\frac{G^i(D)}{G^{i+1}(D)} \simeq \frac{G^i(\overline{D})}{G^{i+1}(\overline{D})}.$$

In order to describe the descending central series of a valued division algebra, we need to recall some concepts from valuation theory. Let D be a finite dimensional division algebra with center a Henselian field F. Recall that a valuation v on a field F is called *Henselian* if and only if v has a unique extension to each field algebraic over F. Therefore v has a unique extension denoted also by v to D ([16]). Denote by V_D, V_F the valuation rings of v on D and F respectively and let M_D, M_F denote their maximal ideals and $\overline{D}, \overline{F}$ their residue division algebra and residue field, respectively. We let Γ_D, Γ_F denote the value groups of v on D and F, respectively and U_D, U_F the groups of units of V_D, V_F respectively. Furthermore, we assume that D is a *tame* division algebra, i.e., $Z(\overline{D})$ is separable over $\overline{F}$ and $Char\overline{F}$ does not divide $i(D)$, the index of D. The quotient group Γ_D/Γ_F is called the *relative value group* of the valuation. D is said to be *unramified* over F if $[\Gamma_D : \Gamma_F] = 1$. At the other extreme D is said to be *totally ramified* if $[D : F] = [\Gamma_D : \Gamma_F]$.

Theorem 2.1. *Let D be a tame and totally ramified division algebra over a henselian field F with index n. Then*
(i) $G^1(D)/G^2(D) = \mathbb{Z}_e$, where $e = exp(\Gamma_D/\Gamma_F)$.
(ii) $G^i(D) = G^{i+1}(D)$ where $i \geq 2$.

Proof. (i). Since $\overline{D} = \overline{F}$, then $U_D = U_F(1 + M_D)$. This shows that $[D^*, D^*] \subseteq U_F(1 + M_D)$. Therefore it follows that

$$G^2(D) \subseteq [D^*, 1 + M_D]. \tag{1}$$

Now applying Lemma 1.1 to the normal subgroup $1 + M_D$ we have

$$(1 + M_D)^n \subseteq (1 + M_F)[D^*, 1 + M_D]. \tag{2}$$

Let $a \in 1 + M_D$. Consider the field $F(a)$ and $a \in 1 + M_{F(a)}$. Since F is a henselian field, so is $F(a)$. Applying Hensel lemma to the polynomial $f(x) = x^n - a$, we obtain an element $b \in 1 + M_{F(a)}$ such that $b^n = a$. This shows that $a \in (1 + M_D)^n$. Therefore $(1 + M_D)$ is a *n-divisible group*, namely $(1 + M_D) = (1 + M_D)^n$. So the equation (2) takes the form,

$$1 + M_D = (1 + M_F)[D^*, 1 + M_D]. \tag{3}$$

In particular $1 + M_D \subseteq (1 + M_F)D'$. This shows that

$$[D^*, 1 + M_D] \subseteq G^2(D). \tag{4}$$

Now (1) and (4) imply that $G^2(D) = [D^*, 1 + M_D]$. On the other hand it is not difficult to see that $1 + M_F \cap D' = 1$. Therefore $1 + M_D \cap D' = [D^*, 1 + M_D]$. So

$$G^2(D) = [D^*, 1 + M_D] = D' \cap 1 + M_D. \tag{5}$$

Now consider the reduction map $U_D \longrightarrow \overline{D}^*$. Restriction of this map to D' gives rise to the following isomorphism,

$$\frac{D'}{D' \cap (1 + M_D)} \xrightarrow{\simeq} \overline{D'}.$$

Since $G^2(D) = D' \cap 1 + M_D$, therefore $D'/G^2(D) \simeq \overline{D'}$. On the other hand $\overline{D'} \simeq \mathbb{Z}_e$ where $e = exp(\Gamma_D/\Gamma_F)$. (See the proof of Theorem 3.1 in [15].) Therefore

$$\frac{D'}{G^2(D)} = \mathbb{Z}_e$$

and the proof is complete.

(ii). By (5), $G^2(D) = [D^*, 1 + M_D]$. On the other hand by (3), $1 + M_D = (1 + M_F)[D^*, 1 + M_D]$. Therefore $G^2(D) = [D^*, [D^*, 1 + M_D]] = G^3(D)$. $\square$

The calculation of $G^1(D)/G^2(D)$ in the above theorem was possible because we were able to calculate the exact amount of $1 + M_D \cap D'$ which not only gives rise to a short and elementary proof of Platonov's congruence theorem [11], but also compute the exact amount of it. (See [4].)

As it is shown in the course of the proof of Theorem 2.1, If D is tame and henselian division algebra, then

$$1 + M_D = (1 + M_F)[D^*, 1 + M_D].$$

This shows that $[D^*, 1 + M_D] \subseteq \bigcap_{i=0}^{\infty} G^i(D)$.

Theorem 2.2. *Let D be a tame and unramified division algebra over a henselian field F with index n. Then*

(i) $[D^*, 1 + M_D] \subsetneq G^i(D)$, for any $i \geq 1$.

(ii) $G^i(D)/G^{i+1}(D) \simeq G^i(\overline{D})/G^{i+1}(\overline{D})$, for any $i \geq 1$.

Proof. As in the proof of Theorem 2.1, the restriction of reduction map $U_D \longrightarrow \overline{D}$ to $[D^*, D^*]$ gives rise to ,

$$\frac{G^1(D)}{[D^*, 1 + M_D]} \xrightarrow{\simeq} \overline{[D^*, D^*]}.$$

But D is unramified, namely $[\Gamma_D : \Gamma_F] = 1$. Thus $Z(\overline{D}) = \overline{F}$ and $D^* = F^* U_D$. Therefore for $a, b \in D^*$, the element $c = aba^{-1}b^{-1}$ may be written in the form

$c = \alpha\beta\alpha^{-1}\beta^{-1}$ where α and $\beta \in U_D$. This shows that $\overline{[D^*, D^*]} = [\overline{D}^*, \overline{D}^*]$. Now by Corollary 1.6, $[\overline{D}^*, \overline{D}^*]$ is not a torsion group. Therefore $G^1(D)/[D^*, 1+M_D]$ is not torsion. On the other hand by Corollary 1.2, $G^1(D)/G^i(D)$ is a torsion group. This shows that $[D^*, 1+M_D] \subsetneq G^i(D)$.

(ii) Since the valuation is unramified, similar to the first part, it can be shown that, $G^i(\overline{D}) = \overline{G^i(D)}$. Now it is easy to see that the restriction of reduction map to the subgroup $G^i(D)$ give rises to

$$\frac{G^i(D)}{[D^*, 1+M_D]} \xrightarrow{\simeq} G^i(\overline{D}).$$

Therefore

$$\frac{G^i(D)}{G^{i+1}(D)} \simeq \frac{G^i(\overline{D})}{G^{i+1}(\overline{D})}$$

and we are done. $\square$

Dieudonne has shown that the projective special linear group

$$PSL_n(D) = \frac{SL_n(D)}{Z(SL_n(D))}$$

is a simple group where $n > 2$ or $n = 2$ and D has more than 3 elements [1, §21]. In the case of $n = 1$, there is also an example due to Dieudonne which shows that

$$PSL_1(D) = \frac{D'}{Z(D')}$$

is not a simple group [7, p.191]. The following theorem shows that, if a division algebra enjoys a tame valuation, then $PSL_1(D)$ is not a simple group. This Theorem also answers a question which is asked by B. Mirzaii in [6].

Theorem 2.3. *Let D be a tame valued division algebra over henselian field F. Then $PSL_1(D)$ is not a simple group.*

Proof. We consider two cases. If $\overline{D}$ is commutative, then it is known that the derived series $\cdots \subseteq D''' \subseteq D'' \subseteq D'$ never stops. (See [2], p. 58.) Therefore $U_F D'' \neq U_F D'$. Thus $U_F D''/U_F \lhd U_F D'/U_F$. But $PSL_1(D) = U_F D'/U_F$. This shows that $PSL_1(D)$ is not a simple group. If $\overline{D}$ is not commutative, then consider the normal subgroup $Z(D')(1 + M_D \cap D')$. It is not difficult now to show that $PSL_1(D)$ has a normal subgroup. $\square$

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