CLASSIFICATION OF QUADRATIC FORMS OVER SKEW FIELDS OF CHARACTERISTIC 2

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Abstract. Quadratic forms over division algebras over local or global fields of characteristic 2 are classified by an invariant derived from the Clifford algebra construction.

Quadratic forms over skew fields were defined by Tits in [14] to investigate twisted forms of orthogonal groups in characteristic 2, and by C.T.C. Wall [16] in a topological context. The purpose of this paper is to obtain a classification of these generalized quadratic forms—which we call simply quadratic forms—over finite-dimensional division algebras over local or global fields of characteristic 2 by means of "classical" invariants.

When the characteristic of the base field is different from 2, the corresponding classification theorem is due to Bartels [3, Satz 5] over global fields and Tsukamoto [13, p. 363] over local fields. We define in section 2 a relative invariant of quadratic forms which plays the same rôle as the invariant introduced by Bartels in characteristic different from 2. The methods are different, however: our definition is based on Tits' construction of Clifford algebras, whereas Bartels uses Galois cohomology with coefficients $\mu_2 = \{\pm 1\}$, which is not available in characteristic 2.

Another feature of the paper is that we systematically consider quadratic spaces from the viewpoint of their endomorphism algebra. We show in section 1 that every nonsingular quadratic form on a vector space V induces on its endomorphism algebra a quadratic pair, as defined in [10, (5.4)], so that quadratic pairs on $\operatorname{End} V$ correspond bijectively to quadratic forms up to a scalar factor. In section 2, we discuss Arf invariants and Clifford algebras of quadratic forms, and use them to define the analogue of the Bartels invariant mentioned above. Section 3 collects results in the literature about the Witt group of (ordinary) quadratic forms over a field, and its behaviour under scalar extension to a separable quadratic extension. The same theme is discussed for generalized quadratic forms in section 4, where a criterion for a (generalized) quadratic form to become hyperbolic over a separable quadratic extension is given in terms of the adjoint quadratic pair. This result is crucial for the classification theorems of section 5, since the main idea of the proof (as in the work [4] of Bayer–Fluckiger and Parimala, which was the main inspiration for this part) is to reduce the orthogonal case to the unitary case by a quadratic extension. Our main classification theorems are the following:

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Theorem A. *Let F be a local or global field of characteristic 2, and let q, q' be nonsingular quadratic forms of the same dimension over a central division F-algebra. If q, q' have the same Arf invariant and if the relative invariant $c(q,q')$ vanishes, then q and q' are isometric.*

Theorem B. *Let F be a local or global field of characteristic 2, and let (σ,f), (σ',f') be quadratic pairs on a central simple F-algebra A. If the Clifford algebras $C(A,\sigma,f)$ and $C(A,\sigma',f')$ are F-isomorphic, then (σ,f) and (σ',f') are conjugate.*

Our techniques can be applied also in characteristic different from 2, to yield similar results (except in section 3: the transfer map tr_* is not onto if $\operatorname{char} F \neq 2$). Indeed, it would be possible to give an exposition of our results valid in all characteristics; we refrained from this option for the sake of clarity, and because all these results are already known in characteristic different from 2: we refer to [10, (4.2)] for the relation between hermitian forms and their adjoint involution, to [5, Theorem 3.3] for the hyperbolicity criterion, to [4, Theorem 4.4.1] and [3, Satz 5] for the classification theorem for hermitian forms over division algebras and to [11, Proposition 6] for the classification of quadratic pairs. (In characteristic different from 2, a quadratic pair is uniquely determined by its orthogonal involution.)

Thus, we assume throughout the paper that the characteristic of the base field F is 2. We let $\wp(F) = \{x^2 - x \mid x \in F\}$ and for $\alpha \in F$, $\beta \in F^\times$ we denote by $[\alpha,\beta)$ the quaternion F-algebra generated by two elements i, j subject to $i^2 - i = \alpha$, $j^2 = \beta$ and $ji = ij + j$. Abusing notations, we also denote by $[\alpha,\beta)$ the image of this algebra in the Brauer group $\operatorname{Br}(F)$.

1. Quadratic forms and quadratic pairs

Throughout this section, we let D be a finite-dimensional central division algebra over a field F of characteristic 2, and let V be a finite-dimensional right vector space over D. We assume D carries an involution θ which is the identity on F and let

$$\operatorname{Sym}(D,\theta) = \{x \in D \mid \theta(x) = x\} (= \{x \in D \mid \theta(x) = -x\}),$$
$$\operatorname{Alt}(D,\theta) = \{x - \theta(x) \mid x \in D\} (= \{x + \theta(x) \mid x \in D\}).$$

Following [15], [16] (see also [9, Chapter 14]), we call *quadratic form* on V any pair (ψ, q) where

$$\psi\colon\ V \times V \to D$$

is a hermitian form with respect to θ, and

$$q\colon\ V \to D/\operatorname{Alt}(D,\theta)$$

is a map from V to the quotient of the additive group of D by $\operatorname{Alt}(D,\theta)$ subject to the following conditions:

(a) $q(x+y) - q(x) - q(y) = \psi(x,y) + \operatorname{Alt}(D,\theta)$ for x, $y \in V$;
(b) $q(x\lambda) = \theta(\lambda)q(x)\lambda$ for $x \in V$ and $\lambda \in D$.

1.1. Proposition. *Let (ψ, q) be a quadratic form on V.*

1. *The hermitian form ψ is uniquely determined by q through condition* (a).
2. *For all $x \in V$,*
$$\psi(x,x) = q(x) + \theta\big(q(x)\big),$$
where the right side is meant to be $\kappa + \theta(\kappa)$ for any representative $\kappa \in D$ of $q(x) \in D/\operatorname{Alt}(D,\theta)$.

In view of 1, we shall sometimes denote a quadratic form (ψ, q) simply by q.

Proof. 1. If ψ' is another hermitian form satisfying (a), then $\psi - \psi'$ is a hermitian form with values in $\mathrm{Alt}(D, \theta)$. Since the set of values of a hermitian form is either $\{0\}$ or D, and since $D \neq \mathrm{Alt}(D, \theta)$, we must have $\psi - \psi' = 0$.

2. Let $(e_1, \dots, e_n)$ be a basis of V and pick $g_i \in D$ such that $q(e_i) = g_i + \mathrm{Alt}(D, \theta)$ for all $i = 1, \dots, n$. There is a unique sesquilinear form $g\colon V \times V \to D$ with respect to θ such that

$$g(e_i, e_j) = \begin{cases} \psi(e_i, e_j) & \text{for } i < j; \\ g_i & \text{for } i = j; \\ 0 & \text{for } i > j. \end{cases}$$

Using (a) and (b), it is easily verified that

$$q(x) = g(x, x) + \mathrm{Alt}(D, \theta) \qquad \text{for all } x \in V, \tag{1}$$

hence for all x, $y \in V$,

$$\psi(x, y) \equiv g(x, y) + g(y, x) \equiv g(x, y) + \theta\bigl(g(y, x)\bigr) \mod \mathrm{Alt}(D, \theta).$$

By uniqueness of ψ (part 1 of the proof), it follows that

$$\psi(x, y) = g(x, y) + \theta\bigl(g(y, x)\bigr) \qquad \text{for all } x,\ y \in V. \tag{2}$$

Part 2 of the proposition follows from (1) and (2). □

Part 2 of the proposition shows that ψ is trace-valued (alternating, in the terminology of [10, §4.A]), i.e. $\psi(x, x) \in \mathrm{Alt}(D, \theta)$ for all $x \in V$.

A quadratic form (ψ, q) on V is called *nonsingular* if the hermitian form ψ is nonsingular, i.e. if $x = 0$ is the only vector such that $\psi(x, y) = 0$ for all $y \in V$. We may then consider the adjoint involution σ_ψ on $\mathrm{End}_D V$ defined by the equation

$$\psi\bigl(x, g(y)\bigr) = \psi\bigl(\sigma_\psi(g)(x), y\bigr) \qquad \text{for } x,\ y \in V \text{ and } g \in \mathrm{End}_D V.$$

1.2. Corollary. *If (ψ, q) is a nonsingular quadratic form on V, the adjoint involution σ_ψ is symplectic.*

Proof. This readily follows from the fact that ψ is trace-valued, by [10, (4.2)]. □

Since symplectic involutions exist only on even-degree algebras, it follows that $\deg \mathrm{End}_D V$ is even or, equivalently, $\deg D \dim_D V$ is even, if V carries a nonsingular quadratic form.

Consider now the left D-vector space ${}^\theta V$ defined by

$${}^\theta V = \{{}^\theta v \mid v \in V\}$$

with the operations

$${}^\theta(v + w) = {}^\theta v + {}^\theta w \qquad \text{and} \qquad d \cdot {}^\theta v = {}^\theta\bigl(v\theta(d)\bigr)$$

for v, $w \in V$ and $d \in D$. We may form the tensor product $V \otimes_D {}^\theta V$, which is just a vector space over the center F of D, and consider the F-linear involution ε on $V \otimes_D {}^\theta V$ defined by

$$\varepsilon(v \otimes {}^\theta w) = w \otimes {}^\theta v.$$

Let $\mathrm{Sym}(V \otimes_D {}^\theta V, \varepsilon)$ denote the F-vector space of symmetric tensors, i.e.

$$\mathrm{Sym}(V \otimes_D {}^\theta V, \varepsilon) = \{\xi \in V \otimes_D {}^\theta V \mid \varepsilon(\xi) = \xi\}.$$

For the following statement, we let Trd_D denote the reduced trace on D. Recall from [10, (2.3)] that $\operatorname{Sym}(D,\theta)$ and $\operatorname{Alt}(D,\theta)$ are orthogonal for the bilinear form $(x,y)\mapsto \operatorname{Trd}_D(xy)$ on D. Therefore, even though the map q takes its values in $D/\operatorname{Alt}(D,\theta)$, the scalar $\operatorname{Trd}_D\bigl(q(v)d\bigr)$ is well-defined for $v\in V$ and $d\in\operatorname{Sym}(D,\theta)$.

1.3. Proposition. *Let (ψ,q) be a quadratic form on V. There is a unique F-linear map*

$$\rho_q\colon\ \operatorname{Sym}(V\otimes_D{}^{\theta}V,\varepsilon)\to F$$

such that for all $v\in V$, $d\in\operatorname{Sym}(D,\theta)$,

$$\rho_q(vd\otimes{}^{\theta}v)=\operatorname{Trd}_D\bigl(q(v)d\bigr) \tag{3}$$

and for all u, $v\in V$,

$$\rho_q(u\otimes{}^{\theta}v+v\otimes{}^{\theta}u)=\operatorname{Trd}_D\bigl(\psi(u,v)\bigr). \tag{4}$$

Proof. Let $(e_1,\dots,e_n)$ be a basis of V. Every symmetric tensor ξ can be uniquely written in the form

$$\xi=\sum_{1\le i\le n}e_id_i\otimes{}^{\theta}e_i+\sum_{1\le i<j\le n}\bigl(e_id_{ij}\otimes{}^{\theta}e_j+e_j\theta(d_{ij})\otimes{}^{\theta}e_i\bigr)$$

for some $d_i\in\operatorname{Sym}(D,\theta)$, $d_{ij}\in D$. We let

$$\rho_q(\xi)=\sum_{1\le i\le n}\operatorname{Trd}_D\bigl(q(e_i)d_i\bigr)+\sum_{1\le i<j\le n}\operatorname{Trd}_D\bigl(\psi(e_id_{ij},e_j)\bigr).$$

Computation shows that ρ_q satisfies (3) and (4). Uniqueness is clear since elements of the form $vd\otimes{}^{\theta}v$, $u\otimes{}^{\theta}v+v\otimes{}^{\theta}u$ span $\operatorname{Sym}(V\otimes_D{}^{\theta}V,\varepsilon)$. $\square$

Using Proposition 1.3, we can define the adjoint quadratic pair of a nonsingular quadratic form. Recall from [10, (5.4)] that a *quadratic pair* on a central simple F-algebra A of degree n is a pair (σ,f) where $\sigma\colon A\to A$ is an involution of the first kind such that the F-vector space $\operatorname{Sym}(A,\sigma)$ has dimension $\frac12 n(n+1)$, and

$$f\colon\ \operatorname{Sym}(A,\sigma)\to F$$

is a linear map such that

$$f\bigl(x+\sigma(x)\bigr)=\operatorname{Trd}_A(x)\qquad\text{for all }x\in A. \tag{5}$$

This definition can be made in arbitrary characteristic. In characteristic 2, the condition on the dimension of $\operatorname{Sym}(A,\sigma)$ is satisfied by every involution of the first kind. However, in a quadratic pair (σ,f) the involution σ must be symplectic because (5) implies that the reduced trace of every symmetric element vanishes.

Let (ψ,q) be a nonsingular quadratic form on a vector space V as in the beginning of this section. We may use ψ to define an F-linear map $\varphi_\psi\colon\ V\otimes_D{}^{\theta}V\to\operatorname{End}_D V$ by

$$\varphi_\psi(u\otimes{}^{\theta}v)(w)=u\psi(v,w) \tag{6}$$

for u, v, $w\in V$. Since ψ is nonsingular, the map φ_ψ is a bijection (see [10, (5.1)]). Moreover, for u, $v\in V$ we have

$$\sigma_\psi\bigl(\varphi_\psi(u\otimes{}^{\theta}v)\bigr)=\varphi_\psi(v\otimes{}^{\theta}u)$$

and

$$\operatorname{Trd}_{\operatorname{End}_D V}\bigl(\varphi_\psi(u\otimes{}^{\theta}v)\bigr)=\operatorname{Trd}_D\bigl(\psi(v,u)\bigr), \tag{7}$$

see [10, (5.1)]. Therefore, φ_ψ restricts to a one-to-one correspondence

$$\operatorname{Sym}(V \otimes_D {}^\theta V, \varepsilon) \xrightarrow{\sim} \operatorname{Sym}(\operatorname{End}_D V, \sigma_\psi).$$

We may then define a linear map

$$f_q\colon \operatorname{Sym}(\operatorname{End}_D V, \sigma_\psi) \to F$$

by $f_q(s) = \rho_q\big(\varphi_\psi^{-1}(s)\big)$, where ρ_q is the map of Proposition 1.3.

1.4. Proposition. *With the notation above, (σ_ψ, f_q) is a quadratic pair on $\operatorname{End}_D V$.*

Proof. It suffices to show

$$f_q\big(g + \sigma_\psi(g)\big) = \operatorname{Trd}_{\operatorname{End}_D V}(g) \qquad \text{for } g \in \operatorname{End}_D V.$$

Using the bijection φ_ψ, this amounts to proving that for all $\xi = \sum_i u_i \otimes {}^\theta v_i \in V \otimes_D {}^\theta V$,

$$\rho_q\big(\xi + \varepsilon(\xi)\big) = \sum_i \operatorname{Trd}_D\big(\psi(v_i, u_i)\big).$$

This readily follows from property (4) of ρ_q , since

$$\xi + \varepsilon(\xi) = \sum_i (u_i \otimes {}^\theta v_i + v_i \otimes {}^\theta u_i).$$

$\square$

The quadratic pair (σ_ψ, f_q) is called the *adjoint quadratic pair* of the nonsingular quadratic form (ψ, q).

If (ψ, q) is a quadratic form and $\lambda \in F$, we may define a quadratic form $(\lambda\psi, \lambda q)$ in the natural way: for v, $w \in V$ we set

$$(\lambda\psi)(v, w) = \lambda\psi(v, w) \qquad \text{and} \qquad (\lambda q)(v) = \lambda q(v).$$

If (ψ, q) is nonsingular and $\lambda \neq 0$, it is clear from the definition of the adjoint involution that $\sigma_{\lambda\psi} = \sigma_\psi$. Moreover, the maps φ_ψ, $\varphi_{\lambda\psi}\colon V \otimes_D {}^\theta V \to \operatorname{End}_D V$ are related by $\varphi_{\lambda\psi} = \lambda\varphi_\psi$, while $\rho_{\lambda q} = \lambda\rho_q$, hence

$$f_{\lambda q} = \rho_{\lambda q} \circ \varphi_{\lambda\psi}^{-1} = \rho_q \circ \varphi_\psi^{-1} = f_q.$$

1.5. Theorem. *Each quadratic pair on $\operatorname{End}_D V$ is adjoint to a nonsingular quadratic form (ψ, q) on V, which is uniquely determined up to a scalar factor in $F^\times$.*

Proof. Using a basis of V, we may identify V with D^n and $\operatorname{End}_D V$ with the matrix algebra $M_n(D)$. Let $*$ be the involution on $M_n(D)$ defined by

$$(a_{ij})^*_{1\leq i,j\leq n} = \big(\theta(a_{ij})\big)^t_{1\leq i,j\leq n}.$$

As pointed out in[1] [10, (5.8)], for every quadratic pair (σ, f) on $M_n(D)$, there is a matrix $a \in M_n(D)$ such that $a + a^*$ is invertible and

$$\sigma(g) = (a + a^*)^{-1} g^* (a + a^*) \qquad \text{for all } g \in M_n(D),$$

$$f(s) = \operatorname{Trd}_{M_n(D)}\big((a + a^*)^{-1} a s\big) \qquad \text{for all } s \in \operatorname{Sym}\big(M_n(D), \sigma\big).$$

Define a hermitian form $\psi\colon D^n \times D^n \to D$ by

$$\psi(v, w) = \theta(v)^t \cdot (a + a^*) \cdot w$$

[1] It is assumed in [10, (5.8)] that the involution $*$ is orthogonal. This hypothesis is not necessary in characteristic 2.

for v, $w \in D^n$ (viewed as column vectors) and a map $q\colon\ D^n \to D/\operatorname{Alt}(D,\theta)$ by

$$q(v) = \theta(v)^t \cdot a \cdot v + \operatorname{Alt}(D,\theta).$$

It is easily seen that (ψ, q) is a nonsingular quadratic form on D^n, and that σ is the adjoint involution σ_ψ with respect to ψ. To show that $f = f_q$, we use the bijection

$$\varphi_\psi\colon\ D^n \otimes_D {}^\theta(D^n) \xrightarrow{\sim} \operatorname{End}_D D^n = M_n(D).$$

Since $\operatorname{Sym}\bigl(M_n(D),\sigma\bigr)$ is spanned by elements of the form $\varphi_\psi(vd \otimes {}^\theta v)$ and $\varphi_\psi(u \otimes {}^\theta v + v \otimes {}^\theta u)$ with u, $v \in V$ and $d \in \operatorname{Sym}(D,\theta)$, it suffices to show

$$\rho_q(vd \otimes {}^\theta v) = \operatorname{Trd}_{M_n(D)}\bigl((a+a^*)^{-1} a \varphi_\psi(vd \otimes {}^\theta v)\bigr) \tag{8}$$

for $v \in V$ and $d \in \operatorname{Sym}(D,\theta)$, and

$$\rho_q(u \otimes {}^\theta v + v \otimes {}^\theta u) = \operatorname{Trd}_{M_n(D)}\bigl((a+a^*)^{-1} a \varphi_\psi(u \otimes {}^\theta v + v \otimes {}^\theta u)\bigr) \tag{9}$$

for u, $v \in V$.

It is easily verified that $m\varphi_\psi(u \otimes {}^\theta v) = \varphi_\psi\bigl((mu) \otimes {}^\theta v\bigr)$ for $m \in M_n(D)$ and u, $v \in D^n$. Therefore, by (7) it follows that the right side of (8) is

$$\operatorname{Trd}_{M_n(D)}\bigl(\varphi_\psi\bigl((a+a^*)^{-1} avd \otimes {}^\theta v\bigr)\bigr) = \operatorname{Trd}_D\bigl(\psi(v, (a+a^*)^{-1} avd)\bigr).$$

By definition of ψ and q, this last expression is equal to

$$\operatorname{Trd}_D\bigl(\theta(v)^t avd\bigr) = \operatorname{Trd}_D\bigl(q(v)d\bigr),$$

proving (8). Similarly, the right side of (9) is

$$\operatorname{Trd}_D\bigl(\psi(v, (a+a^*)^{-1} au) + \psi(u, (a+a^*)^{-1} av)\bigr) = \operatorname{Trd}_D\bigl(\theta(v)^t au + \theta(u)^t av\bigr).$$

Since $\theta(v)^t au = \theta\bigl(\theta(u)^t a^* v\bigr)$ and $\operatorname{Trd}_D \circ\theta = \operatorname{Trd}_D$, the right side is equal to

$$\operatorname{Trd}_D\bigl(\theta(u)^t (a+a^*) v\bigr) = \operatorname{Trd}_D\bigl(\psi(u,v)\bigr),$$

hence (9) is proved.

To complete the proof, suppose a quadratic pair (σ, f) on $\operatorname{End}_D V$ is adjoint to two quadratic forms (ψ, q) and (ψ', q'). Since the involution σ is adjoint to a unique hermitian form up to a scalar factor, by [10, (4.2)], we may assume $\psi' = \psi$, and it remains to show $q = q'$. Using the bijection φ_ψ and (3), we have for $v \in V$ and $d \in \operatorname{Sym}(D,\theta)$

$$\operatorname{Trd}_D\bigl(q(v)d\bigr) = f\bigl(\varphi_\psi(vd \otimes {}^\theta v)\bigr) = \operatorname{Trd}_D\bigl(q'(v)d\bigr).$$

Therefore, if κ (resp. κ') is a representative in D of $q(v) \in D/\operatorname{Alt}(D,\theta)$ (resp. of $q'(v) \in D/\operatorname{Alt}(D,\theta)$), we have $\operatorname{Trd}_D\bigl((\kappa - \kappa')d\bigr) = 0$ for all $d \in \operatorname{Sym}(D,\theta)$, hence $\kappa - \kappa' \in \operatorname{Alt}(D,\theta)$ by [10, (2.3)]. Therefore, $q = q'$. $\square$

To complete this section, we compare the Witt index of a quadratic form (as defined for instance in [15, p. 125]) and the Witt index of the adjoint quadratic pair, defined in [10, §6.A].

Let (ψ, q) be a nonsingular quadratic form on V, and let (σ_ψ, f_q) be the adjoint quadratic pair on $\operatorname{End}_D V$. Recall from [10, (1.12)] that every right ideal $I \subset \operatorname{End}_D V$ is of the form

$$I = \operatorname{Hom}_D(V,U) = \{g \in \operatorname{End}_D V \mid g(V) \subset U\}$$

for some uniquely determined subspace $U \subset V$. The *reduced dimension* of a right ideal I is defined by

$$\operatorname{rdim} I = \frac{\dim_F I}{\deg D},$$

so $\operatorname{rdim} \operatorname{Hom}_D(V, U) = \deg D \dim_D U$.

The following proposition readily follows from [10, (6.2)]:

1.6. Proposition. *The subspace $U \subset V$ is totally isotropic for ψ (i.e. $\psi(u, u') = 0$ for all u, $u' \in U$) if and only if the right ideal $\operatorname{Hom}_D(V, U)$ is isotropic for σ_ψ, i.e. $\sigma_\psi(g)h = 0$ for all g, $h \in \operatorname{Hom}_D(V, U)$.*

Let $w(V, \psi)$ be the Witt index of ψ and $w(\operatorname{End}_D V, \sigma_\psi)$ be the Witt index of σ_ψ, i.e.

$$w(V, \psi) = \max\{\dim_D U \mid U \subset V \text{ totally isotropic subspace for } \psi\},$$
$$w(\operatorname{End}_D V, \sigma_\psi) = \{\operatorname{rdim} I \mid I \subset \operatorname{End}_D V \text{ totally isotropic right ideal for } \sigma_\psi\}.$$

From Proposition 1.6, it follows that

$$w(\operatorname{End}_D V, \sigma_\psi) = \{k \deg D \mid 0 \leq k \leq w(V, \psi)\}.$$

There are corresponding results for quadratic forms:

1.7. Proposition. *The subspace $U \subset V$ is totally isotropic for (ψ, q) (i.e. $q(u) = 0$ for all $u \in U$) if and only if the right ideal $\operatorname{Hom}_D(V, U)$ is isotropic for (σ_ψ, f_q), i.e. $\sigma_\psi(g)h = 0$ for all g, $h \in \operatorname{Hom}_D(V, U)$ and $f_q(g) = 0$ for all $g \in \operatorname{Hom}_D(V, U)$ such that $\sigma_\psi(g) = g$.*

Proof. Let $\varphi_\psi \colon V \otimes_D {}^\theta V \to \operatorname{End}_D V$ be the bijection (6). Suppose first $\operatorname{Hom}_D(V, U)$ is isotropic for (σ_ψ, f_q). For $u \in U$ and $d \in \operatorname{Sym}(D, \theta)$ we have $\varphi_\psi(ud \otimes {}^\theta u) \in \operatorname{Hom}_D(V, U) \cap \operatorname{Sym}(\operatorname{End}_D V, \sigma_\psi)$, hence

$$f_q\bigl(\varphi_\psi(ud \otimes {}^\theta u)\bigr) = 0.$$

However, by definition of f_q and (3), we have

$$f_q\bigl(\varphi_\psi(ud \otimes {}^\theta u)\bigr) = \rho_q(ud \otimes {}^\theta u) = \operatorname{Trd}_D\bigl(q(u)d\bigr),$$

hence $\operatorname{Trd}_D\bigl(q(u)d\bigr) = 0$ for all $d \in \operatorname{Sym}(D, \theta)$, and therefore $q(u) = 0$ in $D/\operatorname{Alt}(D, \theta)$, proving that U is isotropic for (ψ, q).

Conversely, suppose U is isotropic for (ψ, q), and let $(u_1, \dots, u_r)$ be a basis of U. Every $g \in \operatorname{Hom}_D(V, U)$ can be written

$$g = \varphi_\psi\Bigl(\sum_{i=1}^r u_i \otimes {}^\theta v_i\Bigr)$$

for some $v_1, \dots, v_r \in V$. Then

$$\sigma_\psi(g) = \varphi_\psi\Bigl(\sum_{i=1}^r v_i \otimes {}^\theta u_i\Bigr),$$

hence the image of $\sigma_\psi(g)$ is the span of $v_1, \dots, v_r$. Therefore, if $\sigma_\psi(g) = g$ we must have $v_1, \dots, v_r \in U$, and it follows that g can be written in the form

$$g = \varphi_\psi\Bigl(\sum_{1 \leq i \leq r} u_i d_i \otimes {}^\theta u_i + \sum_{1 \leq i < j \leq r} (u_i d_{ij} \otimes {}^\theta u_j + u_j \theta(d_{ij}) \otimes {}^\theta u_i)\Bigr)$$

for some $d_i \in \operatorname{Sym}(D,\theta)$ and some $d_{ij} \in D$. In order to prove that $f_q(g) = 0$ for all $g \in \operatorname{Hom}_D(V,U) \cap \operatorname{Sym}(\operatorname{End}_D V, \sigma_\psi)$, it thus suffices to show that

$$f_q\bigl(\varphi_\psi(ud \otimes {}^\theta u)\bigr) = 0$$

for all $u \in U$, $d \in \operatorname{Sym}(D,\theta)$, and

$$f_q\bigl(\varphi_\psi(u \otimes {}^\theta u' + u' \otimes {}^\theta u)\bigr) = 0$$

for all u, $u' \in U$. These equalities readily follow from the hypothesis that U is isotropic for (ψ, q), since

$$f_q\bigl(\varphi_\psi(ud \otimes {}^\theta u)\bigr) = \operatorname{Trd}_D\bigl(q(u)d\bigr)$$

and

$$f_q\bigl(\varphi_\psi(u \otimes {}^\theta u' + u' \otimes {}^\theta u)\bigr) = \operatorname{Trd}_D\bigl(\psi(u,u')\bigr).$$

The condition that $\sigma_\psi(g)h = 0$ for all g, $h \in \operatorname{Hom}_D(V,U)$ follows from Proposition 1.6, since U is isotropic for ψ. □

Let $w(V,q)$ be the Witt index of q and $w(\operatorname{End}_D V, \sigma_\psi, f_q)$ be the Witt index of (σ_ψ, f_q), defined by

$$w(V,q) = \max\{\dim_D U \mid U \subset V \text{ totally isotropic subspace for } q\},$$
$$w(\operatorname{End}_D V, \sigma_\psi, f_q) = \{\operatorname{rdim} I \mid I \subset \operatorname{End}_D V \text{ totally isotropic right ideal for } (\sigma_\psi, f_q)\}.$$

From Proposition 1.6, it follows that

$$w(\operatorname{End}_D V, \sigma_\psi, f_q) = \{k \deg D \mid 0 \le k \le w(V, \psi, f_q)\}.$$

1.8. Corollary. *For a nonsingular quadratic form (ψ, q) on a vector space V, the following conditions are equivalent:*

(a) *(ψ, q) is hyperbolic, i.e. V contains a totally isotropic subspace U with $\dim_D U = \frac{1}{2}\dim_D V$;*
(b) *(σ_ψ, f_q) is hyperbolic, i.e. $\operatorname{End}_D V$ contains an isotropic right ideal I with $\operatorname{rdim} I = \frac{1}{2}\deg \operatorname{End}_D V$;*
(c) *$\operatorname{End}_D V$ contains an idempotent e such that*

$$f_q(s) = \operatorname{Trd}_{\operatorname{End}_D V}(es) \qquad \textit{for all } s \in \operatorname{Sym}(\operatorname{End}_D V, \sigma_\psi).$$

Proof. The equivalence of (a) and (b) follows from Proposition 1.7. The equivalence of (b) and (c) is proved in [10, (6.14)]. (To see that (c) ⇒ (a), one can take for U the image of e.) □

For later use, we also mention the corresponding statement for hermitian forms (which can be found in [10, (6.7)]):

1.9. Corollary. *For a nonsingular hermitian form ψ on a vector space V, the following conditions are equivalent:*

(a) *ψ is hyperbolic, i.e. V contains a totally isotropic subspace U with $\dim_D U = \frac{1}{2}\dim_D V$;*
(b) *σ_ψ is hyperbolic, i.e. $\operatorname{End}_D V$ contains an isotropic right ideal I with $\operatorname{rdim} I = \frac{1}{2}\deg \operatorname{End}_D V$;*
(c) *$\operatorname{End}_D V$ contains an idempotent e such that*

$$\sigma_\psi(e) = 1 - e.$$

2. Invariants of quadratic forms

Throughout this section, we use the same notation as in section 1. Thus, D is a finite-dimensional central division algebra over a field F of characteristic 2, θ is an involution on D which is the identity on F, and V is a finite-dimensional right vector space over D.

The Arf invariant $\operatorname{Arf}(q)$ and the (even) Clifford algebra $\operatorname{Cl}(q)$ of any nonsingular quadratic form q on V are defined by Tits in [14, §4]. We have $\operatorname{Arf}(q) \in F/\wp(F)$, and the center of the algebra $\operatorname{Cl}(q)$ is a quadratic étale algebra isomorphic to $F[t]/(t^2 - t - \delta)$, where $\delta \in F$ is a representative of $\operatorname{Arf}(q)$, see [14, Corollaire 5]. On the other hand, the discriminant $\operatorname{disc}(\sigma, f)$ and the Clifford algebra $C(A, \sigma, f)$ of a quadratic pair (σ, f) on a central simple F-algebra A are defined in [10, (7.7)].

2.1. Proposition. *Let (ψ, q) be a nonsingular quadratic form on V and let (σ_ψ, f_q) be the adjoint quadratic pair on $\operatorname{End}_D V$. Then*

$$\operatorname{Arf}(q) = \operatorname{disc}(\sigma_\psi, f_q) \quad \textit{and} \quad \operatorname{Cl}(q) \simeq C(\operatorname{End}_D V, \sigma_\psi, f_q).$$

Proof. It suffices to consider the Clifford algebras, since their centers yield the Arf invariant and the discriminant respectively. Using a basis of V, we identify V with D^n and $\operatorname{End}_D V$ with the matrix algebra $A = M_n(D)$. As in the proof of Theorem 1.5, let $*$ be the involution on A defined by $u^* = \theta(u)^t$ for $u \in A$, i.e.,

$$(u_{ij})^*_{1\le i,j\le n} = \big(\theta(u_{ij})\big)^t_{1\le i,j\le n}.$$

The quadratic form q on D^n has the form

$$q(x) = \theta(x)^t \cdot a \cdot x$$

for some $a \in A$ (where $x \in D^n$ is viewed as a column vector). The polar bilinear form ψ is then

$$\psi(x, y) = \theta(x)^t \cdot g \cdot y$$

where $g = a + a^*$, and the adjoint quadratic pair (σ, f) is defined by

$$\sigma(u) = g^{-1}u^*g \qquad \text{for } u \in A,$$
$$f(s) = \operatorname{Trd}_A(g^{-1}as) \qquad \text{for } s \in \operatorname{Sym}(A, \sigma),$$

see Theorem 1.5.

Let $\operatorname{Sand}\colon A \otimes_F A \to \operatorname{End}_F A$ be the linear map such that

$$\operatorname{Sand}(u \otimes v)(w) = uwv \qquad \text{for } u,\ v,\ w \in A.$$

According to Tits [14, §4], the algebra $\operatorname{Cl}(q)$ is the quotient of the tensor algebra $T(A)$ of the underlying vector space of A by the ideal $I_1 + I_2$, where I_1 is the ideal generated by elements of the type $s - \operatorname{Trd}_A(sa)$ for $s \in \operatorname{Sym}(A, *)$, and I_2 is the ideal generated by elements of the type $c - \operatorname{Sand}(c, a)$ for $c \in A \otimes A$ such that $\operatorname{Sand}(c, u) = \operatorname{Sand}(c, u^*)$ for all $u \in A$. On the other hand, the algebra $C(A, \sigma, f)$ is defined in [10, §8B] as the quotient of $T(A)$ by the ideal $J_1 + J_2$, where J_1 is the ideal generated by elements of the type $s - \operatorname{Trd}_A(g^{-1}as)$ for $s \in \operatorname{Sym}(A, \sigma)$ and J_2 is the ideal generated by elements of the type $c - \operatorname{Sand}(c, g^{-1}a)$ for $c \in A \otimes A$ such that $\operatorname{Sand}(c, u) = \operatorname{Sand}\big(c, \sigma(u)\big)$ for all $u \in A$. Multiplication on the right by g is a linear endomorphism of A which maps $\operatorname{Sym}(A, *)$ to $\operatorname{Sym}(A, \sigma)$. The induced automorphism of $T(A)$ maps I_1 to J_1 and I_2 to J_2, hence it induces an isomorphism $\operatorname{Cl}(q) \xrightarrow{\sim} C(A, \sigma, f)$. ∎

Let now q and q' be nonsingular quadratic forms on a D-vector space V. Assume $\operatorname{Arf}(q) = \operatorname{Arf}(q')$. Since the Arf invariant is additive, the direct sum $q \perp \langle\lambda\rangle q'$ has trivial Arf invariant for all $\lambda \in F^\times$, hence

$$\operatorname{Cl}(q \perp \langle\lambda\rangle q') \simeq C_\lambda^+ \times C_\lambda^-$$

for some central simple F-algebras C_λ^+, C_λ^-.

2.2. Proposition. 1. *Let $Z = F[t]/(t^2 - t - \delta)$, where $\delta \in F$ is a representative of $\operatorname{Arf}(q) = \operatorname{Arf}(q') \in F/\wp(F)$, and identify Z to the center of $\operatorname{Cl}(q)$ and to the center of $\operatorname{Cl}(q')$. Then*

$$\{C_\lambda^+ \otimes_F Z, C_\lambda^- \otimes_F Z\} = \{\operatorname{Cl}(q) \otimes_Z \operatorname{Cl}(q'), \operatorname{Cl}(q) \otimes_Z {}^\iota\operatorname{Cl}(q')\},$$

where ${}^\iota\operatorname{Cl}(q')$ is the conjugate algebra of $\operatorname{Cl}(q')$ under the nontrivial automorphism ι of Z/F.

2. *For λ, $\lambda' \in F^\times$, the algebras $C_{\lambda\lambda'}^+$ and $C_{\lambda\lambda'}^-$ are Brauer-equivalent to $C_\lambda^+ \otimes [\delta, \lambda')$ and $C_\lambda^- \otimes [\delta, \lambda')$ respectively.*

Proof. The proofs given for the case where $\operatorname{char} F \neq 2$ in [11, Lemma 1] and [7, Proposition 3.8] apply without any substantial change. □

Suppose again q and q' are nonsingular quadratic forms on a D-vector space V with the same Arf invariant. As observed above, we have

$$\operatorname{Cl}(q \perp q') = C^+ \times C^-$$

for some central simple algebras C^+, C^- (denoted by C_1^+ and C_1^- above). Note that $\deg D \dim_D V$ is even, as observed after Corollary 1.2. Therefore, $\deg D \dim_D(V \perp V)$ is divisible by 4, and it follows from [14, Proposition 7] or [10, (9.14)] that $C^+ \otimes_F C^-$ is Brauer-equivalent to D. Since moreover C^+ and C^- have exponent 2 (since they carry a canonical involution, see [10, (9.13)] or [14, Proposition 7]), it follows that the Brauer classes $[C^+]$ and $[C^-]$ have the same image in the following quotient of the Brauer group of F:

$$B_D = \operatorname{Br}(F)/\{1, [D]\}.$$

We may therefore define a relative invariant $c(q, q')$ of q and q' by

$$c(q, q') = \text{image of } [C^+] \text{ or } [C^-] \text{ in } B_D.$$

This is an analogue of the invariant defined in characteristic different from 2 by Bartels [3, §7]. We shall show in section 5 that it holds the key to the classification of quadratic forms over local or global fields in characteristic 2.

If $D = F$, we have $B_D = \operatorname{Br}(F)$, and $c(q, q')$ is the Brauer class of the *full* Clifford algebra $C(V \oplus V, q \perp q')$, which is the tensor product of the *full* Clifford algebras $C(V, q)$ and $C(V, q')$. (The full Clifford algebras are classically defined, since q and q' are ordinary quadratic forms, see for instance [2, Chapter 2, §2].)

2.3. Proposition. *If q, q', q'' are nonsingular quadratic forms on V with the same Arf invariant, then*

$$c(q, q'') = c(q, q') + c(q', q'').$$

Proof. If $D = F$ the proposition is clear since

$$c(q, q'') = [C(V, q)] + [C(V, q'')], \qquad c(q, q') = [C(V, q)] + [C(V, q')],$$
$$c(q', q'') = [C(V, q')] + [C(V, q'')]$$

and $2[C(V, q')] = 0$. The general case is reduced to the case where $D = F$ by scalar extension to the function field L of the Severi–Brauer variety of D. Since the kernel of the scalar extension map $\mathrm{Br}(F) \to \mathrm{Br}(L)$ is $\{0, [D]\}$, the induced map $B_D \to \mathrm{Br}(L)$ is injective, and the proposition follows. □

To conclude this section, we show to which extent the Clifford algebra of a nonsingular quadratic form is an invariant of its Witt class.

The Witt equivalence of quadratic forms over D can be defined as in the classical case: every nonsingular form q decomposes into an orthogonal direct sum of a hyperbolic form and an anisotropic form, uniquely determined up to isometry and called the *anisotropic kernel* of q; two nonsingular quadratic forms are called *Witt-equivalent* if their anisotropic kernels are isometric.

Corresponding notions can be defined for central simple algebras with quadratic pairs (see [8] for the case where $\operatorname{char} F \neq 2$). Suppose (σ, f) is a quadratic pair on a central simple F-algebra A. If $e \in A$ is a symmetric idempotent, then σ restricts to an involution on each of the F-algebras $A_1 = eAe$ and $A_2 = (1-e)A(1-e)$, and the restrictions of f to the symmetric elements in A_1 and A_2 yield quadratic pairs (σ_1, f_1) on A_1 and (σ_2, f_2) on A_2. The quadratic pair (σ, f) is said to be an *orthogonal sum* of (σ_1, f_1) and (σ_2, f_2). This terminology is motivated by the observation that if $A = \operatorname{End}_D V$ and (σ, f) is adjoint to a quadratic form (ψ, q) on V, then every symmetric idempotent in A is the orthogonal projection on a nonsingular subspace $U \subset V$, and (σ_1, f_1), (σ_2, f_2) are the orthogonal pairs adjoint to the restrictions of (ψ, q) to U and its orthogonal complement $U^\perp$.

If (σ_2, f_2) is hyperbolic, the algebra with quadratic pair (A, σ, f) is called a *hyperbolic extension* of (A_1, σ_1, f_1). Using the Witt decomposition of quadratic forms, we see that every algebra with quadratic pair (A, σ, f) is a hyperbolic extension of an algebra with anisotropic quadratic pair, uniquely determined up to conjugation, called an *anisotropic kernel* of (A, σ, f). Two algebras with quadratic pairs are said to be *Witt-equivalent* if they have isomorphic anisotropic kernels. Note that the algebras themselves are then Brauer-equivalent, since for every idempotent $e \in A$, the algebras A and eAe are Brauer-equivalent.

2.4. Theorem. *Let q_0 be the anisotropic kernel of a nonsingular quadratic form q. The Clifford algebra $\mathrm{Cl}(q)$ is isomorphic as an F-algebra to an algebra which is Morita-equivalent to $\mathrm{Cl}(q_0)$. Similarly, if (A_0, σ_0, f_0) is the anisotropic kernel of an algebra with quadratic pair (A, σ, f), then the Clifford algebra $C(A, \sigma, f)$ is isomorphic as an F-algebra to an algebra which is Morita-equivalent to $C(A_0, \sigma_0, f_0)$.*

Note that $\mathrm{Cl}(q)$ and $\mathrm{Cl}(q_0)$ cannot be considered as Morita-equivalent since their centers are not canonically isomorphic. However, if an identification of the centers of $\mathrm{Cl}(q)$ and $\mathrm{Cl}(q_0)$ is chosen, then the theorem asserts that $\mathrm{Cl}(q)$ is Morita-equivalent to $\mathrm{Cl}(q_0)$ or to its conjugate algebra ${}^\iota\,\mathrm{Cl}(q_0)$.

Proof. The same arguments as in the characteristic not 2 case apply, see [8, Proposition 3]. □

3. The Witt exact sequence of a quadratic extension

In this section, we consider the Witt group W_qF of quadratic forms over a field F of characteristic 2. Our goal is to relate this group to the Witt groups of quadratic and hermitian forms over a separable quadratic extension K/F.

Throughout the section, we fix a separable quadratic field extension K/F. We write $K = F(\alpha)$ where $\alpha^2 - \alpha = a \in F$ ($a \notin \wp(F)$).

Let ${}_2\operatorname{Br}(F)$ denote the 2-torsion subgroup of the Brauer group $\operatorname{Br}(F)$. The corestriction (or norm) map $\operatorname{cor}\colon \operatorname{Br}(K) \to \operatorname{Br}(F)$ is described in [13, Ch. 8, §9], [10, (3.12)].

3.1. Lemma. *The kernel of* $\operatorname{cor}\colon {}_2\operatorname{Br}(K) \to {}_2\operatorname{Br}(F)$ *is the image of the scalar extension map* $\operatorname{res}\colon {}_2\operatorname{Br}(F) \to {}_2\operatorname{Br}(K)$.

(When $\operatorname{char} F \neq 2$, the corresponding result follows from the exact sequence of Galois cohomology groups $H^2(F,\mu_2) \overset{\operatorname{res}}{\to} H^2(K,\mu_2) \overset{\operatorname{cor}}{\to} H^2(F,\mu_2)$, see [13, p. 309] or [10, (30.12)].)

Proof. The kernel of the corestriction map consists of the Brauer classes of central simple K-algebras which admit an involution of unitary type, i.e., an involution whose restriction to K is the non-trivial automorphism ι of K/F, see [10, (3.1)]. It is therefore clear that the image of ${}_2\operatorname{Br}(F)$ lies in the kernel of cor. Let B be a central simple K-algebra whose Brauer class is in the kernel of $\operatorname{cor}\colon {}_2\operatorname{Br}(K) \to {}_2\operatorname{Br}(F)$. We may then find on B an involution τ of unitary type and also, since $B \otimes_K B$ is split, an involution σ whose restriction to F is the identity, see [10, (3.1)]. The composition $\tau \circ \sigma \circ \tau$ yields an involution on B which is the identity on K, hence there exists an element $u \in B^\times$ such that $\sigma(u) = u$ and

$$\tau \circ \sigma \circ \tau(x) = u\sigma(x)u^{-1} \qquad \text{for all } x \in B.$$

It follows that for all $x \in B$,

$$\tau \circ \sigma(u) \cdot \tau \circ \sigma(x) \cdot \tau \circ \sigma(u)^{-1} = (\tau \circ \sigma)^3(x) = u \cdot \tau \circ \sigma(x) \cdot u^{-1},$$

hence $\tau \circ \sigma(u) = \tau(u) = u\lambda$ for some $\lambda \in K^\times$. Since τ restricts to the nontrivial automorphism ι of K/F, we have $N_{K/F}(\lambda) = 1$, and Hilbert's Theorem 90 yields $\lambda_0 \in K^\times$ such that $\lambda = \lambda_0 \iota(\lambda_0)^{-1}$. Substituting $u\lambda_0$ for u, we may thus assume that $\tau(u) = u$. Define then an F-algebra $A = B \oplus Bz$ by the following multiplication rules:

$$z^2 = u, \qquad zb = \tau \circ \sigma(b)z \quad \text{for } b \in B.$$

It is easily verified that A is a central simple F-algebra, and that τ extends to an involution on A such that $\tau(z) = z$. (Compare [1, Chapter 11, Theorem 10], [6, Exercice 4, p. 59].) Therefore, the Brauer class $[A]$ lies in ${}_2\operatorname{Br}(F)$. Moreover, the centralizer of K in A is B, hence $\operatorname{res}[A] = [B]$. □

We now turn to quadratic forms. We let W_qF denote the Witt group of (nonsingular) quadratic forms over F and, for $n \geq 1$, let $I^nW_qF = I^nF \cdot W_qF$, where I^nF is the n-th power of the fundamental ideal of the Witt ring WF of symmetric bilinear forms over F. Let $[1,a]$ be the quadratic form $x^2 + xy + ay^2$, which is the norm form of K/F. Abusing notations, we also let $[1,a]$ denote the Witt class of this form in W_qF. The inclusion $i\colon F \hookrightarrow K$ and the trace map $\operatorname{tr}\colon K \to F$ induce Witt group maps $i_*\colon W_qF \to W_qK$ and $\operatorname{tr}_*\colon W_qK \to W_qF$, see [2, Chapter 1, §2].

3.2. Lemma. *The maps* i_* *and* tr_* *fit into exact sequences*

$$0 \to [1,a]WF \to W_qF \xrightarrow{i_*} W_qK \xrightarrow{\mathrm{tr}_*} W_qF \to 0, \tag{10}$$

$$0 \to [1,a]IF \to IW_qF \xrightarrow{i_*} IW_qK \xrightarrow{\mathrm{tr}_*} IW_qF \to 0, \tag{11}$$

$$0 \to [1,a]I^2F \to I^2W_qF \xrightarrow{i_*} I^2W_qK \xrightarrow{\mathrm{tr}_*} I^2W_qF \to 0. \tag{12}$$

Proof. Exactness of the first sequence is proved in [2, Ch. 5, Corollary (5.9)]. The Arf invariant induces a map $\mathrm{Arf}\colon W_qF \to F/\wp(F)$ whose kernel is IW_qF, by [12, Theorem 2]. Since $\mathrm{Arf}\big([1,a]\cdot\langle b_1,\dots,b_r\rangle\big) = ra + \wp(F)$, it follows that

$$\big([1,a]WF\big) \cap IW_qF = [1,a]IF.$$

We have a commutative diagram with exact rows, where $U = \{0, a+\wp(F)\}/\wp(F) \subset F/\wp(F)$,

$$\begin{tikzcd}
0 \arrow[r] & {[1,a]WF} \arrow[r] \arrow[d, "\mathrm{Arf}"'] & W_qF \arrow[r, "i_*"] \arrow[d, "\mathrm{Arf}"'] & W_qK \arrow[r, "\mathrm{tr}_*"] \arrow[d, "\mathrm{Arf}"] & W_qF \arrow[r] \arrow[d, "\mathrm{Arf}"] & 0 \\
0 \arrow[r] & U \arrow[r] & F/\wp(F) \arrow[r, "i"] & K/\wp(K) \arrow[r, "\mathrm{tr}"] & F/\wp(F) \arrow[r] & 0
\end{tikzcd}$$

and the vertical maps are onto. A chase around this diagram yields the exact sequence (11).

Similarly, the Clifford algebra construction yields a map $w\colon IW_qF \to {}_2\mathrm{Br}(F)$. This map is onto and its kernel is I^2W_qF, by [12, Theorem 2]. The kernel ${}_2\mathrm{Br}(K/F)$ of the scalar extension map $\mathrm{res}\colon {}_2\mathrm{Br}(F) \to {}_2\mathrm{Br}(K)$ consists of the Brauer classes of quaternion algebras $[a,x)$, where $x \in F^\times$, and we have

$$w\big([1,a]\cdot\langle b_1,\dots,b_{2r}\rangle\big) = [a, b_1\dots b_{2r}),$$

hence

$$\big([1,a]IF\big) \cap I^2W_qF = [1,a]I^2F.$$

In view of Lemma 3.1 and the first part of the proof, the rows of the following diagram are exact:

(13)

$$\begin{tikzcd}
0 \arrow[r] & {[1,a]IF} \arrow[r] \arrow[d, "w"'] & IW_qF \arrow[r, "i_*"] \arrow[d, "w"'] & IW_qK \arrow[r, "\mathrm{tr}_*"] \arrow[d, "w"] & IW_qF \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & {{}_2\mathrm{Br}(K/F)} \arrow[r] & {}_2\mathrm{Br}(F) \arrow[r, "\mathrm{res}"] & {}_2\mathrm{Br}(K) \arrow[r, "\mathrm{cor}"] & {}_2\mathrm{Br}(F).
\end{tikzcd}$$

Moreover, the vertical maps are onto, and commutativity of the rightmost square can be checked by explicit computation. To make this computation easier, note that IW_qK is additively generated by forms of the type $[1,u]\cdot\langle 1,v\rangle$ with u, $v \in K$, $v \neq 0$, and even by forms of this type where at least one of u, v lies in F. This is because if $v \notin F$ we may write $u = \xi + \eta v$ for some ξ, $\eta \in F$, and then

$$[1,u]\cdot\langle 1,v\rangle = [1,\xi]\cdot\langle 1,v\rangle + [1,\eta v]\cdot\langle 1,\eta\rangle \qquad \text{in } W_qF.$$

Commutativity of the other squares in diagram (13) is easily seen; a chase around this diagram then yields the exact sequence (12). $\square$

Note that (12) also shows that the corestriction map $\mathrm{cor}\colon {}_2\mathrm{Br}(K) \to {}_2\mathrm{Br}(F)$ is onto, since the map w is surjective.

Finally, we turn to hermitian forms over K, relative to the nontrivial automorphim ι of K/F. The corresponding Witt group is denoted by $W(K,\iota)$. The determinant $\det h$ of a hermitian form h is defined as the image in $F^\times/N_{K/F}(K^\times)$ of the determinant of any Gram matrix of h. We let $I_1(K,\iota)$ denote the subgroup of $W(K,\iota)$ generated by forms of even dimension and $I_2(K,\iota)$ denote the subgroup of $I_1(K,\iota)$ generated by forms of even dimension and trivial determinant.

Every hermitian form h on a K-vector space V yields a quadratic form $q_h\colon V\to F$ defined by $q_h(x)=h(x,x)$ for $x\in V$. The assignment $h\mapsto q_h$ defines a map $\Delta\colon W(K,\iota)\to W_qF$.

3.3. Theorem. *The following sequences are exact:*

$$0\to W(K,\iota)\xrightarrow{\Delta} W_qF\xrightarrow{i_*} W_qK\xrightarrow{\mathrm{tr}_*} W_qF\to 0,$$
$$0\to I_1(K,\iota)\xrightarrow{\Delta} IW_qF\xrightarrow{i_*} IW_qK\xrightarrow{\mathrm{tr}_*} IW_qF\to 0,$$
$$0\to I_2(K,\iota)\xrightarrow{\Delta} I^2W_qF\xrightarrow{i_*} I^2W_qK\xrightarrow{\mathrm{tr}_*} I^2W_qF\to 0.$$

Proof. By Theorem 1.2 of [13, p. 348], the image of $\Delta\colon W(K,\iota)\to W_qF$ is the kernel $[1,a]WF$ of i_*. Moreover, Remark 1.4 in [13, pp. 349–350] shows that

$$\mathrm{Arf}(q_h)=(\dim h)a+\wp(F)\in F/\wp(F)\qquad\text{and}\qquad w(q_h)=[a,\det h)\in{}_2\mathrm{Br}(F),$$

hence

$$[1,a]IF=\big([1,a]WF\big)\cap IF=\Delta\big(I_1(K,\iota)\big)$$

and

$$[1,a]I^2F=\big([1,a]IF\big)\cap I^2F=\Delta\big(I_2(K,\iota)\big).$$

The theorem then follows from Lemma 3.2. □

3.4. Corollary. *If $I^2W_qF=0$, then $I^2W_qK=0$ and $I_2(K,\iota)=0$.*

4. The Witt kernel of a separable quadratic extension

In this section, we let A be a central simple algebra over a field F of characteristic 2 and let K be a separable quadratic field extension $K=F(\alpha)$ with $\alpha^2-\alpha=a\in F$. Our goal is to determine necessary and sufficient conditions for a quadratic pair on A to become hyperbolic under scalar extension to K. In the special case where A is split, Proposition 4.2 below proves the exactness of sequence (10) at W_qF, since a quadratic space (V,q) admits a similitude r with multiplier a such that $r^2-r=a$ if and only if $q=[1,a]\cdot b$ for some symmetric bilinear form b.

We first consider the case of involutions.

4.1. Proposition. *Let σ be an involution on A which is the identity on F. If A contains an element r such that*

$$r^2-r=a\qquad\text{and}\qquad\sigma(r)=1-r,\tag{14}$$

then the involution $\sigma_K=\sigma\otimes\mathrm{Id}_K$ on $A_K=A\otimes_F K$ is hyperbolic. Conversely, if σ is anisotropic (i.e., A does not contain any nonzero isotropic right ideal for σ) and σ_K is hyperbolic, then A contains an element r satisfying (14).

Proof. If A contains an element r satisfying (14), then A_K contains $r+\alpha$ ($= r\otimes 1 + 1\otimes\alpha$), which satisfies $(r+\alpha)^2 = r+\alpha$ and $\sigma_K(r+\alpha) = 1-(r+\alpha)$. Therefore, σ_K is hyperbolic by Corollary 1.9.

For the converse, assume A_K contains an idempotent e such that $\sigma_K(e) = 1-e$. We may write $e = e_1\otimes 1 + e_2\otimes\alpha$ for some e_1, $e_2\in A$. The condition $e^2 = e$ yields

$$e_1^2 + ae_2^2 = e_1 \tag{15}$$

and

$$e_2^2 + e_1e_2 + e_2e_1 = e_2. \tag{16}$$

We shall use the hypothesis that σ is anisotropic to prove that e_2 is invertible. Assuming this fact, we may let $r = e_1e_2^{-1}$ and use (15) and (16) to prove that $r^2 - r = a$. Similarly, the conditions $\sigma(e_1) = 1-e_1$ and $\sigma(e_2) = e_2$, which follow from $\sigma_K(e) = 1-e$, imply that $\sigma(r) = 1-r$.

To complete the proof, we show that e_2 is invertible in A. It clearly suffices to show that the right ideal

$$I = \{x\in A \mid e_2x = 0\}$$

is $\{0\}$ or, in view of the hypothesis on σ, that I is isotropic.

We first show that e_1I is isotropic: for x, $y\in I$ we have

$$\sigma(e_1x)e_1y = \sigma(x)(1-e_1)e_1y = \sigma(x)(e_1-e_1^2)y.$$

By (15) we have $e_1 - e_1^2 = ae_2^2$, hence $\sigma(e_1x)e_1y = 0$ since $e_2y = 0$. Therefore, e_1I is isotropic, hence $e_1I = \{0\}$ since σ is assumed to be anisotropic.

Now, for $x\in I$ we have $e_1x = 0$ hence, applying σ,

$$\sigma(x) = \sigma(x)e_1.$$

It follows that for x, $y\in I$,

$$\sigma(x)y = \sigma(x)e_1y$$

and the right side vanishes since $e_1y = 0$ for $y\in I$. This shows that I is isotropic and completes the proof. $\square$

We next turn to quadratic pairs. If (σ, f) is a quadratic pair on A, we denote by

$$f_K\colon\ \mathrm{Sym}(A_K,\sigma_K) = \mathrm{Sym}(A,\sigma)\otimes_F K\to K$$

the linear map extended from f by linearity. It is clear that (σ, f) is a quadratic pair on A_K.

4.2. Proposition. *Let (σ, f) be a quadratic pair on A. If A contains an element r such that*

$$r^2 - r = a \qquad \textit{and} \qquad f(s) = \mathrm{Trd}_A(rs) \quad \textit{for all } s\in\mathrm{Sym}(A,\sigma), \tag{17}$$

then the quadratic pair (σ_K, f_K) is hyperbolic. Conversely, suppose (σ, f) is anisotropic, i.e., A does not contain any nonzero isotropic right ideal for (σ, f). If (σ_K, f_K) is hyperbolic, then A contains an element r for which (17) holds.

Proof. Suppose A contains an element r satisfying (17). As in the proof of Proposition 4.1, the element $r+\alpha\in A_K$ is an idempotent. Moreover, for $s\in\mathrm{Sym}(A,\sigma)$ we have $\mathrm{Trd}_A(s) = 0$ since σ is symplectic, hence

$$\mathrm{Trd}_{A_K}\big((r+\alpha)s\big) = \mathrm{Trd}_A(rs) = f(s)$$

and it follows by linearity that

$$\operatorname{Trd}_{A_K}\big((r+\alpha)s\big) = f_K(s) \qquad \text{for all } s \in \operatorname{Sym}(A_K, \sigma_K).$$

Therefore, Corollary 1.8 shows that (σ_K, f_K) is hyperbolic.

To prove the converse, suppose (σ_K, f_K) is hyperbolic, and let $e = e_1 \otimes 1 + e_2 \otimes \alpha \in A_K$ be an idempotent such that

$$\operatorname{Trd}_{A_K}(es) = f_K(s) \qquad \text{for all } s \in \operatorname{Sym}(A_K, \sigma_K). \tag{18}$$

As in the proof of Proposition 4.1, the condition $e^2 = e$ leads to (15) and (16), hence, assuming that e_2 is invertible, to $(e_1e_2^{-1})^2 - e_1e_2^{-1} = a$. On the other hand, (18) yields

$$\operatorname{Trd}_A(e_1 s) = f(s) \qquad \text{and} \qquad \operatorname{Trd}_A(e_2 s) = 0 \qquad \text{for all } s \in \operatorname{Sym}(A, \sigma). \tag{19}$$

The last equality shows that $e_2 \in \operatorname{Alt}(A, \sigma)$, by [10, (2.3)], and the first equality implies that $\sigma(e_1) = 1 - e_1$, by [10, (5.7)]. Using (15) and (16), computation yields

$$e_1e_2^{-1} + e_1 = e_1e_2^{-1}\sigma(e_1) + ae_2$$

(assuming again that e_2 is invertible). Letting $e_2 = \ell + \sigma(\ell)$ with $\ell \in A$, we have

$$e_2^{-1} = e_2^{-1}e_2e_2^{-1} = e_2^{-1}\ell + \sigma(e_2^{-1}\ell)$$

hence

$$e_1e_2^{-1}\sigma(e_1) + ae_2 = \big(e_1e_2^{-1}\ell\sigma(e_1) + a\ell\big) + \sigma\big(e_1e_2^{-1}\ell\sigma(e_1) + a\ell\big) \in \operatorname{Alt}(A, \sigma).$$

Therefore, $e_1e_2^{-1} + e_1 \in \operatorname{Alt}(A, \sigma)$ and it follows from (19) that for all $s \in \operatorname{Sym}(A, \sigma)$

$$\operatorname{Trd}_A(e_1e_2^{-1}s) = \operatorname{Trd}_A(e_1 s) = f(s).$$

The element $r = e_1e_2^{-1}$ thus satisfies (17).

To complete the proof, we show, as in the proof of Proposition 4.1, that e_2 is invertible. We again let

$$I = \{x \in A \mid e_2 x = 0\}$$

and aim to prove that I is isotropic for (σ, f). Consider first the right ideal $e_1 I$. The same argument as in the proof of Proposition 4.1 shows that $\sigma(e_1x)e_1y = 0$ for x, $y \in I$. To prove that $e_1 I$ is isotropic for (σ, f), we still have to show that $f(e_1x) = 0$ if $x \in I$ is such that $e_1x \in \operatorname{Sym}(A, \sigma)$.

Multiplying equation (15) by $x \in I$, we get $e_1^2x = e_1x$. Therefore, we have by (19)

$$f(e_1x) = \operatorname{Trd}_A(e_1^2x) = \operatorname{Trd}_A(e_1x),$$

and it follows that $f(e_1x) = 0$ if $e_1x \in \operatorname{Sym}(A, \sigma)$ since the reduced trace of every symmetric element is 0. Thus, $e_1 I$ is isotropic for (σ, f), hence $e_1 I = \{0\}$ since (σ, f) is anisotropic. Now, for $x \in I \cap \operatorname{Sym}(A, \sigma)$ we have by (19)

$$f(x) = \operatorname{Trd}_A(e_1x) = 0$$

since $e_1x = 0$. Moreover, the same argument as in the proof of Proposition 4.1 shows that $\sigma(x)y = 0$ for x, $y \in I$. Therefore, I is isotropic for (σ, f), hence $I = \{0\}$ since (σ, f) is anisotropic. It follows that e_2 is invertible and the proof is complete. ∎

For later use, we conclude this section with a few observations on central simple algebras with quadratic pair (A,σ,f) which contain an element r for which (17) holds.

Let $\deg A = n = 2m$ and let $\tilde{A} = C_A(r)$ be the centralizer of r in A. This is a simple algebra of degree m with center $F(r) \simeq K$. Since $\sigma(r) = 1 - r$, it follows that σ restricts to an involution $\tilde{\sigma}$ of unitary type on $\tilde{A}$.

4.3. Lemma. *If $\tilde{\sigma}$ is hyperbolic, then (σ, f) is hyperbolic.*

Proof. Let $e \in \tilde{A}$ be an idempotent such that $\tilde{\sigma}(e) = 1 - e$. To prove that (σ, f) is hyperbolic, it suffices to show that $f(s) = \operatorname{Trd}_A(es)$ for all $s \in \operatorname{Sym}(A,\sigma)$.

We have $\sigma(e - r) = e - r$, hence

$$e - r = (e - r)r + \sigma\big((e - r)r\big) \in \operatorname{Alt}(A,\sigma).$$

Therefore, $\operatorname{Trd}_A\big((e - r)s\big) = 0$ for all $s \in \operatorname{Sym}(A,\sigma)$, by [10, (2.3)], and it follows that

$$\operatorname{Trd}_A(es) = \operatorname{Trd}_A(rs) = f(s) \qquad \text{for all } s \in \operatorname{Sym}(A,\sigma).$$

□

For the next statement, recall that for every central simple algebra with unitary involution (B,τ), a discriminant algebra $D(B,\tau)$ is defined in [10, §10.E].

4.4. Lemma. *Use the same notation as above. If $\deg A \equiv 0 \mod 4$, then the discriminant of (σ, f) is trivial, hence $C(A,\sigma,f) = C^+ \times C^-$ for some central simple F-algebras C^+, C^-. Moreover, one (at least) of C^+, C^- is Brauer-equivalent to $D(\tilde{A},\tilde{\sigma})$.*

Proof. On a separable closure, the characteristic polynomial of r has the form $(X-\alpha)^{m_1}(X-\alpha-1)^{m_2}$ for some m_1, m_2 such that $m_1+m_2 = n$. Since $\sigma(r) = 1-r$ has the same characteristic polynomial, we must have $m_1 = m_2 = m$, hence the coefficient of X^{n-2} is

$$\operatorname{Srd}(r) = ma + \tfrac{1}{2}m(m-1),$$

and it follows that

$$\operatorname{disc}(\sigma, f) = ma + \wp(F) \in F/\wp(F).$$

This proves the first statement.

To prove the second part, we first consider the case where A is split. We may then represent A as $A = \operatorname{End}_F V$ for some F-vector space V, and consider a quadratic form q on V whose adjoint quadratic pair is (σ, f). Identifying $\operatorname{End}_F V$ with $V \otimes V$ through φ_ψ, where ψ is the polar bilinear form of q, we then have $q(v) = f(v \otimes v)$, hence (17) yields

$$q(v) = \operatorname{Trd}(r(v) \otimes v) = \psi\big(v, r(v)\big). \tag{20}$$

We use $r \in \operatorname{End}_F V$ to define on V a K-vector space structure by

$$v\alpha = r(v) \qquad \text{for } v \in V.$$

The centralizer $\tilde{A}$ of r in A is then $\operatorname{End}_K V$. We also define a map $h\colon V \times V \to K$ by

$$h(v,w) = \psi\big(v, r(w)\big) + \alpha\psi(v,w) \qquad \text{for } v,\ w \in V.$$

Since ψ is symmetric and $\sigma(r) = 1 - r$, it follows that h is a hermitian form on V. For $f \in \operatorname{End}_K V$, we clearly have

$$h\big(v, f(w)\big) = h\big(\sigma(f)(v), w\big) \qquad \text{for } v,\, w \in V,$$

hence $\tilde{\sigma}$ is the adjoint involution of h, and it follows from [10, (10.35)] that

$$[D(\tilde{A}, \tilde{\sigma})] = [a, \det h) \qquad \text{in } \operatorname{Br}(F).$$

To relate $D(\tilde{A}, \tilde{\sigma})$ to q, observe that (20) yields $h(v, v) = q(v)$ for $v \in V$, hence $q = q_h$ in the notation of section 3. Therefore, Remark 1.4 in [13, pp. 349–350] shows that the Clifford algebra $C(V, q)$ is Brauer-equivalent to the quaternion algebra $[a, \det h)$. Since C^+ ($\simeq C^-$) is Brauer-equivalent to $C(V, q)$, it follows that $D(\tilde{A}, \tilde{\sigma})$ is Brauer-equivalent to C^+ (and to C^-), proving the lemma in the split case.

In the general case, we extend scalars to the function field L of the Severi–Brauer variety of A. Since the lemma holds in the split case, we have

$$[D(\tilde{A}, \tilde{\sigma}) \otimes_F L] = [C^+ \otimes_F L] \quad (= [C^- \otimes_F L]) \qquad \text{in } \operatorname{Br}(L),$$

hence $[D(\tilde{A}, \tilde{\sigma})] - [C^+]$ lies in the kernel of the scalar extension map $\operatorname{Br}(F) \to \operatorname{Br}(L)$, which consists of 0 and $[A]$. Thus, $[D(\tilde{A}, \tilde{\sigma})] = [C^+]$ or $[C^+] + [A]$, and the proof is complete since $[C^+] + [A] = [C^-]$ by [10, (9.14)]. $\square$

4.5. Lemma. *With the notation above, suppose* $\deg A \equiv 0 \mod 4$ *and one of* C^+, C^- *is split. If* $\tilde{A}$ *is not split, then* $D(\tilde{A}, \tilde{\sigma})$ *is split. If* $\tilde{A}$ *is split, the same property holds either for* A *or for a hyperbolic extension of* A. *More precisely, if* $D(\tilde{A}, \tilde{\sigma})$ *is not split, then there is a hyperbolic extension* (A', σ', f') *of* (A, σ, f) *and an element* $r' \in A'$ *such that* ${r'}^2 - r' = a$, $\operatorname{Trd}_{A'}(r's) = f'(s)$ *for all* $s \in \operatorname{Sym}(A', \sigma')$, *and letting* $\tilde{A}'$ *be the centralizer of* r' *in* A' *and* $\tilde{\sigma}'$ *be the restriction of* σ' *to* $\tilde{A}'$, *the algebra* $D(\tilde{A}', \tilde{\sigma}')$ *is split.*

Proof. Suppose C^+ is split. Then C^- is Brauer-equivalent to A, by [10, (9.14)], hence

$$[C^- \otimes_F K] = [A \otimes_F K] = [\tilde{A}] \qquad \text{in } \operatorname{Br}(K).$$

On the other hand, we have $[D(\tilde{A}, \tilde{\sigma}) \otimes_F K] = m[\tilde{A}]$ by [10, (10.30)], hence K splits $D(\tilde{A}, \tilde{\sigma})$ since m is even. Therefore, if $\tilde{A}$ is not split $D(\tilde{A}, \tilde{\sigma})$ cannot be Brauer-equivalent to C^-; it follows from Lemma 4.4 that $D(\tilde{A}, \tilde{\sigma})$ is Brauer-equivalent to C^+, hence it is split.

For the rest of the proof, we assume $\tilde{A}$ is split and $D(\tilde{A}, \tilde{\sigma})$ is not split; it is then Brauer-equivalent to C^-, by Lemma 4.4, hence also to A. Since $\tilde{A}$ is Brauer-equivalent to $A \otimes_F K$, it follows that A is split by K, hence it is equivalent to a quaternion division algebra Q. Let $A = \operatorname{End}_Q V$ for some Q-vector space V, and let q be a quadratic form on V whose adjoint quadratic pair is (σ, f). Let (V_0, q_0) be a hyperbolic plane over Q, let $A_0 = \operatorname{End}_Q(V_0)$, and let (σ_0, f_0) be the quadratic pair on A_0 adjoint to q_0. We consider the orthogonal sum $(V', q') = (V \oplus V_0, q \perp q_0)$ and let

$$A' = \operatorname{End}_Q(V \oplus V_0) = \begin{pmatrix} \operatorname{End}_Q V & \operatorname{Hom}_Q(V_0, V) \\ \operatorname{Hom}_Q(V, V_0) & \operatorname{End}_Q V_0 \end{pmatrix},$$

endowed with the quadratic pair (σ', f') adjoint to $q \perp q_0$. Suppose $r_0 \in A_0$ is such that $r_0^2 - r_0 = a$ and $\operatorname{Trd}_{A_0}(r_0 s) = f_0(s)$ for all $s \in \operatorname{Sym}(A_0, \sigma_0)$. We then let

$r' = \left(\begin{smallmatrix} r & 0 \\ 0 & r_0 \end{smallmatrix}\right) \in A'$ be the endomorphism of $V \oplus V_0$ defined by

$$r'(v + v_0) = r(v) + r_0(v_0).$$

Clearly, ${r'}^2 - r' = a$, and it is easily verified that $f'(s) = \mathrm{Trd}_{A'}(s)$ for all $s \in \mathrm{Sym}(A', \sigma')$. Consider the centralizer $\tilde{A}'$ of r' in A' and the centralizer $\tilde{A}_0$ of r_0 in A_0, with the restrictions $\tilde{\sigma}'$ and $\tilde{\sigma}_0$ of σ' and σ_0 respectively. The orthogonal projections $e_V = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in A'$, $e_{V_0} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in A'$ of $V \oplus V_0$ on V and V_0 respectively are symmetric idempotents in $\tilde{A}'$ such that

$$e_V \tilde{A}' e_V = \tilde{A}, \qquad e_{V_0} \tilde{A}' e_{V_0} = \tilde{A}_0,$$

and the restrictions of $\tilde{\sigma}'$ to $e_V \tilde{A}' e_V$ and $e_{V_0} \tilde{A}' e_{V_0}$ are $\tilde{\sigma}$ and $\tilde{\sigma}_0$. Therefore, $(\tilde{A}', \tilde{\sigma}')$ is an orthogonal sum of $(\tilde{A}, \tilde{\sigma})$ and $(\tilde{A}_0, \tilde{\sigma}_0)$. Since these algebras are split, their discriminant algebras are given by the determinant of the corresponding hermitian forms (see [10, (10.35)]), hence

$$[D(\tilde{A}', \tilde{\sigma}')] = [D(\tilde{A}, \tilde{\sigma})] + [D(\tilde{A}_0, \tilde{\sigma}_0)] = [A] + [D(\tilde{A}_0, \tilde{\sigma}_0)] \qquad \text{in } \mathrm{Br}(F).$$

To complete the proof, it suffices to show that A_0 contains an element r_0 as above such that $D(\tilde{A}_0, \tilde{\sigma}_0)$ is Brauer-equivalent to A.

Since (A_0, σ_0, f_0) is hyperbolic we may identify $A_0 = M_2(F) \otimes_F Q$ in such a way that σ_0 is the tensor product of the symplectic involutions on $M_2(F)$ and Q, and f_0 is the canonical form $f_\otimes$, see [10, (15.14)]. Define

$$r_0 = \begin{pmatrix} 0 & a \\ 1 & 1 \end{pmatrix} \otimes 1 \in M_2(F) \otimes Q.$$

Computation shows that $r_0^2 - r_0 = a$ and $f_0(s) = \mathrm{Trd}_{A_0}(r_0 s)$ for all $s \in \mathrm{Sym}(A_0, \sigma_0)$. The centralizer of r_0 is $\tilde{A}_0 = F(r_0) \otimes_F Q$, and the involution $\tilde{\sigma}_0$ is the tensor product of the non-trivial automorphism of $F(r_1)/F$ and the canonical involution of Q. Therefore, $D(\tilde{A}_0, \tilde{\sigma}_0) \simeq Q$, by [10, p. 129]. □

4.6. Proposition. *Suppose A contains an element r satisfying* (17)*, and* $\deg A \equiv 0 \mod 4$*, hence $C(A, \sigma, f) \simeq C^+ \times C^-$ for some central simple F-algebras C^+, C^-. Assume moreover that $\tilde{A}$ and one (at least) of C^+, C^- are split. If $I^2 W_q F = 0$, then (A, σ, f) is hyperbolic.*

Proof. Lemma 4.5 shows that, up to a hyperbolic extension, we may assume $D(\tilde{A}, \tilde{\sigma})$ is split. Since $\tilde{A}$ is split, we may represent it as $\mathrm{End}_K W$ for some K-vector space W; then $\tilde{\sigma}$ is the adjoint involution of some hermitian form h on W relative to the non-trivial automorphism ι of K/F. The dimension of W is even since $\deg A \equiv 0 \mod 4$; moreover the determinant of h is trivial since $D(\tilde{A}, \tilde{\sigma})$ is split, hence h represents an element of $I_2(K, \iota)$. By Corollary 3.4, h is hyperbolic, hence (A, σ, f) is hyperbolic by Lemma 4.3. □

5. Classification over local and global fields

In this section, we prove classification theorems for quadratic forms and quadratic pairs on central division algebras over fields F of characteristic 2 with the following properties:

(a) every central simple F-algebra of exponent 2 is Brauer-equivalent to a quaternion algebra;
(b) there is no Cayley division algebra over F.

Condition (b) can be rephrased in various ways, as was shown by Sah [12, Theorem 3]; for instance, it is equivalent to the requirement that quadratic forms over F are classified by their dimension, Arf invariant and Clifford algebra, or to the condition that $I^2W_qF = 0$. Local and global fields (of characteristic 2) satisfy (a) and (b).

5.1. Theorem. *Let A be a central simple algebra over a field F satisfying* (a) *and* (b)*, and let (σ, f) be a quadratic pair on A. Suppose* $\deg A \equiv 0 \mod 4$ *and* $\operatorname{disc}(\sigma, f) = 0$*, hence $C(A, \sigma, f) = C^+ \times C^-$ for some central simple F-algebras C^+, C^-. If one (at least) of C^+, C^- is split, then (A, σ, f) is hyperbolic.*

Proof. If A is split, then (σ, f) is adjoint to a quadratic form over F. From the hypotheses, it follows that this quadratic form has trivial Arf invariant and split Clifford algebra, hence it is hyperbolic since $I^2W_qF = 0$. We may thus assume A is not split. Moreover, substituting for (A, σ, f) its anisotropic kernel, we may assume (A, σ, f) is anisotropic. Condition (a) shows that A is split by a quadratic field extension K/F. The extended quadratic pair (σ_K, f_K) on $A_K = A \otimes_F K$ is adjoint to some quadratic form over K with trivial Arf invariant and split Clifford algebra. Since $I^2W_qK = 0$ by Corollary 3.4, it follows that (A_K, σ_K, f_K) is hyperbolic. Therefore, Proposition 4.2 shows that A contains an element r satisfying (17), and it follows from Proposition 4.6 that (A, σ, f) is hyperbolic, since the centralizer $\tilde{A}$ of r in A is Brauer-equivalent to $A \otimes K$, hence split. □

Our first classification result follows. Recall from section 2 the relative invariant $c(q, q')$.

5.2. Theorem. *Let V be a vector space over a central division F-algebra D, and let q, q', q_0 be quadratic forms on V. Suppose* $\operatorname{Arf}(q) = \operatorname{Arf}(q') = \operatorname{Arf}(q_0)$ *and* $c(q_0, q) = c(q_0, q')$*. If F satisfies* (a) *and* (b)*, then q and q' are isometric.*

Proof. Since $c(q_0, q) = c(q_0, q')$, it follows from Proposition 2.3 that $c(q, q') = 0$, which means that one of the direct factors of the Clifford algebra $\operatorname{Cl}(q \perp q')$ is split. Theorem 5.1 shows that the quadratic pair adjoint to $q \perp q'$ on $\operatorname{End}_D(V \oplus V')$ is hyperbolic, hence $q \perp q'$ is hyperbolic and therefore $q \simeq q'$. □

Since local and global fields satisfy (a) and (b), Theorem A in the introduction follows from Theorem 5.2. Note that if F is local and D is a quaternion algebra, the condition $c(q_0, q) = c(q_0, q')$ is automatically satisfied, since the quotient ${}_2\operatorname{Br}(F)/\{0, [D]\}$ is trivial.

The corresponding classification theorem for quadratic pairs is next.

5.3. Theorem. *Let A be a central simple F-algebra and let (σ, f), (σ', f') be quadratic pairs on A. Suppose the Clifford algebras $C(A, \sigma, f)$ and $C(A, \sigma', f')$ are F-isomorphic. If F satisfies* (a) *and* (b)*, then (σ, f) and (σ', f') are conjugate.*

Proof. Choose a representation $A = \operatorname{End}_D V$ and quadratic forms q, q' on V whose adjoint quadratic pairs are (σ, f) and (σ', f') respectively. For any $\lambda \in F^\times$, we have

$$\operatorname{Cl}(q \perp \langle\lambda\rangle q') = C^+_\lambda \times C^-_\lambda$$

for some central simple F-algebras C^+_λ, C^-_λ, since the hypothesis $C(A, \sigma, f) \simeq C(A, \sigma', f')$ implies $\operatorname{disc}(\sigma, f) = \operatorname{disc}(\sigma', f')$, hence $\operatorname{Arf}(q) = \operatorname{Arf}(q')$. Let Z be the center of $C(A, \sigma, f)$, which we identify (non-canonically) to the center of $C(A, \sigma', f')$. If $\deg A \equiv 0 \mod 4$, the canonical involution on $C(A, \sigma, f)$ is the identity on Z, hence $C(A,\sigma,f)\otimes_Z C(A,\sigma,f)$ is split. If $\deg A \equiv 2 \mod 4$, the canonical involution restricts to the non-trivial automorphism ι of Z/F, hence the tensor product $C(A,\sigma,f)\otimes_Z {}^{\iota}C(A,\sigma',f')$ is split. Thus, in all cases Proposition 2.2 shows that Z splits one of C^+_λ, C^-_λ. Say Z splits C^+_λ; then C^+_λ is Brauer-equivalent to a quaternion algebra $[\delta,\mu)$ for some $\mu\in F^\times$, where $\delta\in F$ is such that $Z\simeq F[t]/(t^2-t-\delta)$. It then follows from the second part of Proposition 2.2 that $C^+_{\lambda\mu}$ is split. This means that $c(q,\langle\lambda\mu\rangle q')=0$, hence Theorem 5.2 yields $q\simeq\langle\lambda\mu\rangle q'$, hence (σ,f) and (σ',f') are conjugate. $\square$

Since local and global fields satisfy (a) and (b), Theorem B in the introduction is a particular case of Theorem 5.3. If F is local and A has index 2, it suffices to assume $\operatorname{disc}(\sigma,f)=\operatorname{disc}(\sigma',f')$ to conclude that (σ,f) and (σ',f') are conjugate.

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