BRAUER EQUIVALENCE IN A HOMOGENEOUS SPACE WITH CONNECTED STABILIZER

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Abstract. Let G be a simply connected algebraic group over a field k of characteristic 0, H a connected k-subgroup of G, $X = H\backslash G$. When k is a local field or a number field, we compute the set of Brauer equivalence classes in $X(k)$.

0. Introduction

In this note we investigate the Brauer equivalence in a homogeneous space $X = H\backslash G$, where G is a simply connected algebraic group over a local field or a number field, and H is a connected subgroup of G.

In more detail, let k be a field of characteristic 0, and let $\bar{k}$ be a fixed algebraic closure of k. For a smooth algebraic variety Y over k, set $\overline{Y} = Y_{\bar{k}} = Y \times_k \bar{k}$. Let $\operatorname{Br} Y$ denote the cohomological Brauer group of Y, $\operatorname{Br} Y = H^2_{\text{ét}}(Y, \mathbb{G}_m)$. Set $\operatorname{Br}_1 Y = \ker[\operatorname{Br} Y \to \operatorname{Br} \overline{Y}]$. There is a canonical pairing

$$Y(k) \times \operatorname{Br}_1 Y \to \operatorname{Br} k, \qquad (y, b) \mapsto b(y) \tag{0.1}$$

called the Manin pairing. We define the Brauer equivalence on $Y(k)$ as follows: $y_1 \sim y_2$ if $(y_1, b) = (y_2, b)$ for all $b \in \operatorname{Br}_1 Y$. We denote the set of classes of Brauer equivalence in $Y(k)$ by $Y(k)/\operatorname{Br}$. Note that we define the Brauer equivalence in terms of $\operatorname{Br}_1 Y$, not in terms of $\operatorname{Br}_1 Y^c$ or $\operatorname{Br} Y^c$, where Y^c is a smooth compactification of Y.

The notion of B-equivalence for a subgroup B of the Brauer group $\operatorname{Br} Y$ was introduced by Manin [Ma1], [Ma2]. Colliot-Thélène and Sansuc [CT/Sa1] investigated the Brauer equivalence in algebraic tori (they defined the Brauer equivalence in terms of the Brauer group of a smooth

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compactification). The Brauer equivalence in reductive groups was studied in [Th].

Let G be a simply connected semisimple algebraic group over k. Let H be a connected subgroup of G. We denote by H^{tor} the biggest toric quotient group of H. We are interested in the Brauer equivalence in the set $X(k)$ where $X = H\backslash G$.

We compute $X(k)/\mathrm{Br}$ when k is a local field. Namely, we prove that there is a bijection

$$X(k)/\mathrm{Br} \xrightarrow{\sim} \mathrm{im}\,[\ker[H^1(k,H) \to H^1(k,G)] \to H^1(k,H^{\mathrm{tor}})]$$

(Theorem 2.1). Moreover, when k is a nonarchimedean local field, we prove that there is a bijection $X(k)/\mathrm{Br} \to H^1(k,H^{\mathrm{tor}})$ (Theorem 2.2).

We also compute $X(k)/\mathrm{Br}$ when k is a number field. We prove that there is a bijection

$$X(k)/\mathrm{Br} \xrightarrow{\sim} \mathrm{im}\left[\ker[H^1(k,H) \to H^1(k,G)] \to \bigoplus_v H^1(k_v,H^{\mathrm{tor}})\right]$$

(Theorem 3.1), where v runs over the set of places of k. Moreover when k is a totally imaginary number field, we prove that there is a bijection

$$X(k)/\mathrm{Br} \xrightarrow{\sim} H^1(k,H^{\mathrm{tor}})/\text{Ш}^1(k,H^{\mathrm{tor}})$$

(Theorem 3.6), where Ш^1 denotes the Shafarevich–Tate kernel.

In Example 3.11 we compute $X(k)/\mathrm{Br}$ when X is a symmetric space of a simply connected almost simple group over a totally imaginary number field.

Note that all our results remain true when G is any simply connected algebraic group (an extension of a simply connected semisimple group by a unipotent group), not necessarily a simply connected semisimple group.

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1. Generalities over an arbitrary field

1.1. We introduce some notation. For a smooth algebraic variety Y over a field k of characteristic 0 let $U(Y) = k[Y]^\times/k^\times$. Let $\mathrm{Pic}\,Y$ denote the Picard group of Y. Let $\mathrm{Br}\,Y$ and $\mathrm{Br}_1\,Y$ be as in the Introduction.

Set $\operatorname{Br}_a Y = \operatorname{coker}[\operatorname{Br} k \to \operatorname{Br}_1 Y]$. Assume that Y has a k-rational point y, and define

$$\operatorname{Br}_y Y = \ker[\operatorname{Br}_1 Y \xrightarrow{y^*} \operatorname{Br} k]$$

where y^* is the specialization map.

We prove that $\operatorname{Br}_y Y \simeq \operatorname{Br}_a Y$. Consider the composed map

$$\operatorname{Br} k \to \operatorname{Br}_1 Y \xrightarrow{y^*} \operatorname{Br} k\,,$$

it is the identity of $\operatorname{Br} k$. It follows that the exact sequence

$$0 \to \operatorname{Br}_y Y \to \operatorname{Br}_1 Y \xrightarrow{y^*} \operatorname{Br} k \to 0$$

splits, and we obtain an isomorphism $\operatorname{Br}_y Y \oplus \operatorname{Br} k \simeq \operatorname{Br}_1 Y$. Thus we obtain an isomorphism $\operatorname{Br}_y Y \to \operatorname{Br}_a Y$ and a splitting $\operatorname{Br}_a Y \to \operatorname{Br}_1 Y$ of the exact sequence

$$0 \to \operatorname{Br} k \to \operatorname{Br}_1 Y \to \operatorname{Br}_a Y \to 0.$$

1.2. We wish to investigate the Brauer equivalence in homogeneous spaces. Let G be a simply connected semisimple algebraic group over a field k of characteristic 0. Let $H \subset G$ be a connected k-subgroup. Set $X = H\backslash G$; then X is a right homogeneous space of G with connected stabilizer. The variety X has a distinguished k-rational point x_0, the image in $X(k)$ of the unit element $e \in G(k)$.

We construct a map $X(k) \to H^1(k, H^{\mathrm{tor}})$ taking x_0 to 1. From an exact sequence in Galois cohomology of nonabelian groups (cf. [Se], I-5.4, Prop. 36) we obtain a canonical bijection

$$X(k)/G(k) \xrightarrow{\sim} \ker[H^1(k,H) \to H^1(k,G)]\,.$$

Thus we obtain a map $X(k) \to H^1(k,H)$. Composing this map with the canonical map $H^1(k,H) \to H^1(k,H^{\mathrm{tor}})$ induced by the homomorphism $H \to H^{\mathrm{tor}}$, we obtain a map

$$\alpha\colon X(k) \to H^1(k, H^{\mathrm{tor}})\,. \tag{1.1}$$

By construction this map is constant on the orbits of $G(k)$ in $X(k)$.

Let $\mathbf{X}(H)$ denote the group of k-characters of H, i.e. $\mathbf{X}(H) = \operatorname{Hom}_k(H, \mathbb{G}_m)$. We have $\mathbf{X}(H) = \mathbf{X}(H^{\mathrm{tor}})$.

Proposition 1.3. *There is a canonical isomorphism* $\beta\colon \mathbf{X}(H) \xrightarrow{\sim} \operatorname{Pic} X$.

Proof. Let $\chi \in \mathbf{X}(H)$. Set $L(\chi) = H\backslash(\mathbb{G}_a \times G)$, where H acts on $\mathbb{G}_a \times G$ by $h * (u,g) = (\chi(h)u, hg)$ (here $h \in H,\ u \in \mathbb{G}_a,\ g \in G$). There is a canonical map $L(\chi) \to H\backslash G = X$, and one can easily see that $L(\chi)$ is a linear bundle over X. We denote its class in $\operatorname{Pic} X$ by $\beta(\chi)$. One can easily see that $\beta\colon \mathbf{X}(H) \to \operatorname{Pic} X$ is a homomorphism.

We wish to prove that β is an isomorphism. We need the following fact: $U(G)=1$. Indeed, by Rosenlicht's theorem [Ro] $U(G)=\mathbf{X}(G)$ and clearly $\mathbf{X}(G)=1$ because G is semisimple. We need also the fact that $\operatorname{Pic} G=0$, cf. [Sa], 6.9(iv).

We construct a map which is inverse to β. We follow an idea of Mumford [Mu], Ch. I, §2. Let π denote the canonical map $G\to X=H\backslash G$. Let $L\to X$ be a line bundle over X. Consider the pullback π^*L, it is a line bundle over G. Since $\operatorname{Pic} G=0$, the line bundle π^*L is trivial. This means that there exists an isomorphism

$$\mu\colon \pi^*L\to\mathbb{G}_a\times G\,.$$

There is a canonical action of H on π^*L, and using μ we obtain an action of H on $\mathbb{G}_a\times G$ extending the standard action on G, $h*g=hg$. The action of H on $\mathbb{G}_a\times G$ can be written as

$$h*(u,g)=(\chi_g(h)u,hg)\quad\text{where } h\in H, g\in G, u\in\mathbb{G}_a,$$

and $\chi\colon G\times H\to\mathbb{G}_m$ is a regular map. If we fix h, then $\chi_g(h)\colon G\to\mathbb{G}_m$ is a regular map, in other words it is a nowhere zero regular function on G. Since $U(G)=1$, we conclude that $\chi_g(h)$ is constant in g. We may therefore write $\chi(h)$ instead of $\chi_g(h)$.

Since $h_1*(h_2*(u,g))=(h_1h_2)*(u,g)$, we obtain that $\chi(h_1)\chi(h_2)=\chi(h_1h_2)$. Hence $\chi\colon H\to\mathbb{G}_m$ is a regular homomorphism, $\chi\in\mathbf{X}(H)$.

If we start from another isomorphism

$$\mu'\colon \pi^*L\to\mathbb{G}_a\times G\,,$$

then μ' differs from μ by a nowhere zero regular function f on G, and this f must be a constant because $U(G)=1$. Then the homomorphism χ is replaced by χ' where $\chi'_g(h)=\chi_g(h)f(hg)f(g)^{-1}$, i.e. $\chi'=\chi$. Thus χ is defined uniquely by L. One can easily check that we have constructed a homomorphism $\delta\colon \operatorname{Pic} X\to\mathbf{X}(H)$ and that this homomorphism is inverse to β. Thus β is an isomorphism. □

1.3.1. *Remark.* We will use Proposition 1.3 only in the case when k is algebraically closed. In this case Proposition 1.3 was proved in [Po], Cor. to Thm. 4.

1.4. We have seen in the proof of Proposition 1.3 that $U(G)=1$. Hence $U(\overline{G})=1$. It follows that $U(\overline{X})=1$.

Since $X(k)\neq\varnothing$ and $U(\overline{X})=1$, we have by [Sa], 6.3(iii)

$$\operatorname{Br}_a X=H^1(k,\operatorname{Pic}\overline{X})\,.$$

We have $\operatorname{Br}_{x_0} X\simeq\operatorname{Br}_a X$. By Proposition 1.3, $\operatorname{Pic}\overline{X}=\mathbf{X}(\overline{H})$. We obtain

$$\operatorname{Br}_{x_0} X=H^1(k,\mathbf{X}(\overline{H}))=H^1(k,\mathbf{X}(\overline{H}^{\mathrm{tor}}))\,. \tag{1.2}$$

There is a canonical cup product pairing

$$H^1(k, H^{\mathrm{tor}}) \times H^1(k, \mathbf{X}(\overline{H}^{\mathrm{tor}})) \to \operatorname{Br} k\,. \tag{1.3}$$

The pairing (1.3) together with the map (1.1) $X(k) \to H^1(k, H^{\mathrm{tor}})$ and the isomorphism (1.2) defines a pairing

$$X(k) \times \operatorname{Br}_{x_0} X \to \operatorname{Br} k\,. \tag{1.4}$$

Theorem 1.5. *The pairing (1.4) up to sign coincides with the restriction of the Manin pairing (0.1) to $X(k) \times Br_{x_0}X \subset X(k) \times \operatorname{Br}_1 X$.*

This theorem will be proved in the Appendix.

2. Brauer equivalence over a local field

Theorem 2.1. *Let G, H, X be as in 1.2. Assume that k is a local field (archimedean or not). Then the map (1.1) $\alpha\colon X(k) \to H^1(k, H^{\mathrm{tor}})$ induces a bijection*

$$X(k)/\mathrm{Br} \xrightarrow{\sim} \operatorname{im}[\ker[H^1(k,H) \to H^1(k,G)] \to H^1(k, H^{\mathrm{tor}})]\,.$$

Proof. It follows from Theorem 1.5 that two points $x_1, x_2 \in X(k)$ are Brauer-equivalent if and only if $(\alpha(x_1), \eta) = (\alpha(x_2), \eta)$ for every $\eta \in H^1(k, \mathbf{X}(\overline{H}^{\mathrm{tor}}))$. Since k is a local field, the cup product pairing (1.3) is perfect (Tate–Nakayama duality, cf. [Mi], Cor. I.2.4), and it follows that x_1 and x_2 are Brauer-equivalent if and only if $\alpha(x_1) = \alpha(x_2)$. Thus the set of classes of Brauer equivalence is in a bijective correspondence with $\operatorname{im}\alpha$. We see that we must only describe the image of $X(k)$ in $H^1(k, H^{\mathrm{tor}})$. But the image of $X(k)$ in $H^1(k,H)$ is the same as the image of $X(k)/G(k)$ and it equals $\ker[H^1(k,H) \to H^1(k,G)]$. Hence the image of $X(k)$ in $H^1(k, H^{\mathrm{tor}})$ is

$$\operatorname{im}[\ker[H^1(k,H) \to H^1(k,G)] \to H^1(k, H^{\mathrm{tor}})]\,,$$

and the assertion of the theorem follows. □

Theorem 2.2. *Let G, H, X be as in 1.2, and assume that k is a non-archimedean local field. Then the map (1.1) α induces a bijection*

$$X(k)/\mathrm{Br} \to H^1(k, H^{\mathrm{tor}}).$$

Proof. Since G is a simply connected group, by Kneser's theorem (see [Pl/Ra], 6.1, Thm. 4) $H^1(k,G) = 1$. We see now from Theorem 2.1 that $X(k)/\mathrm{Br}$ is in a bijective correspondence with $\operatorname{im}[H^1(k,H) \to H^1(k, H^{\mathrm{tor}})]$. Let H^{ssu} denote $\ker[H \to H^{\mathrm{tor}}]$, it is an extension of a semisimple group by a unipotent group. Since k is local nonarchimedean and $(H^{\mathrm{ssu}})^{\mathrm{tor}} = 1$, the map $H^1(k,H) \to H^1(k, H^{\mathrm{tor}})$ is surjective, cf. [Bo], Cor. 6.4. This proves the theorem. □

3. Brauer equivalence over a number field

Theorem 3.1. *Let k be a number field, and let G, H, X be as in 1.2. Then the map*

$$X(k) \to X(k)/G(k) \xrightarrow{\sim} \ker[H^1(k,H) \to H^1(k,G)] \to \bigoplus_v H^1(k_v, H^{\mathrm{tor}})$$

induces a bijection

$$X(k)/\mathrm{Br} \xrightarrow{\sim} \mathrm{im}\left[\ker[H^1(k,H) \to H^1(k,G)] \to \bigoplus_v H^1(k_v, H^{\mathrm{tor}})\right]$$

where v runs over the set of places of k.

3.2. Before proving Theorem 3.1 we note that $\mathrm{Br}_1 G = \mathrm{Br}\, k$ (cf. [Sa], 6.9(iv)), hence every orbit of $G(k)$ in $X(k)$ is contained in one class of Brauer equivalence. It follows that the map $X(k) \to X(k)/\mathrm{Br}$ factors through $X(k)/G(k)$:

$$X(k) \to X(k)/G(k) \to X(k)/\mathrm{Br}$$

and these maps are surjective.

For a place v of k consider the map

$$X(k) \to X(k)/\mathrm{Br} \to X(k_v)/\mathrm{Br} \to H^1(k_v, H^{\mathrm{tor}}) \tag{3.1}$$

where the last arrow is defined by Theorem 2.1.

Lemma 3.3. *The map (3.1) $X(k) \to H^1(k_v, H^{\mathrm{tor}})$ factors*

$$X(k) \to X(k)/G(k) \xrightarrow{\sim} \ker[H^1(k,H) \to H^1(k,G)] \to H^1(k_v, H^{\mathrm{tor}})$$

with the obvious arrows.

Proof. The statement of the lemma shows how to define a map from $X(k)/G(k)$ to $H^1(k, H^{\mathrm{tor}})$. We can define a map from $X(k_v)/G(k_v)$ to $H^1(k_v, H^{\mathrm{tor}})$ in a similar way. Consider the diagram

$$
\begin{tikzcd}
& X(k)/\mathrm{Br} \arrow[dd] & \\
X(k)/G(k) \arrow[ur] \arrow[rr] \arrow[dd] & & H^1(k, H^{\mathrm{tor}}) \arrow[dd] \\
& X(k_v)/\mathrm{Br} \arrow[dr] & \\
X(k_v)/G(k_v) \arrow[ur] \arrow[rr] & & H^1(k_v, H^{\mathrm{tor}})
\end{tikzcd}
$$

It is commutative (the lower triangle is commutative by Theorem 2.1). It follows that the diagram

$$
\begin{tikzcd}
X(k)/G(k) \arrow[r] \arrow[d] & X(k)/\mathrm{Br} \arrow[r] & X(k_v)/\mathrm{Br} \arrow[d] \\
X(k_v)/G(k_v) \arrow[rr] & & H^1(k_v, H^{\mathrm{tor}})
\end{tikzcd}
$$

is commutative, which proves the lemma. $\square$

Lemma 3.4. (cf. [Ma/Ts], 4.5). *Let Y be a variety over a global field k. Then the map $Y(k)/\mathrm{Br} \to \prod_v Y(k_v)/\mathrm{Br}$ is injective.*

Proof. Let $y_1, y_2 \in Y(k)$, and assume that y_1 and y_2 are Brauer-equivalent in $Y(k_v)$ for all places v of k. This means that for every $b_v \in \mathrm{Br}_1 Y_{k_v}$, $(y_1, b_v) = (y_2, b_v)$. Let now $b \in \mathrm{Br}_1 Y$. We wish to compare (y_1, b) and (y_2, b). Consider $\mathrm{loc}_v(y_i, b) \in \mathrm{Br}\, k_v$, $i = 1, 2$, where loc means localization. We have $\mathrm{loc}_v(y_i, b) = (y_i, \mathrm{loc}_v b)$, where $\mathrm{loc}_v b \in \mathrm{Br}_1 Y_{k_v}$. By assumption we have $(y_1, \mathrm{loc}_v b) = (y_2, \mathrm{loc}_v b)$. We see that $\mathrm{loc}_v(y_1, b) = \mathrm{loc}_v(y_2, b)$ for all v. It follows that $(y_1, b) = (y_2, b)$, because the map $\mathrm{loc}\colon \mathrm{Br}\, k \to \prod_v \mathrm{Br}\, k_v$ is injective. Thus y_1 and y_2 are Brauer-equivalent in $Y(k)$. $\square$

3.5. *Proof of Theorem 3.1.* By Lemma 3.3 the obvious map

$$\ker[H^1(k, H) \to H^1(k, G)] \to \prod_v H^1(k_v, H^{\mathrm{tor}}) \tag{3.2}$$

has the same image as the map

$$X(k)/\mathrm{Br} \to \prod_v X(k_v)/\mathrm{Br} \to \prod_v H^1(k_v, H^{\mathrm{tor}}), \tag{3.3}$$

where the last arrow is described in Theorem 2.1. But the image of map (3.2) is contained in $\bigoplus_v H^1(k_v, H^{\mathrm{tor}})$ (because the image of $H^1(k, H^{\mathrm{tor}})$ in $\prod_v H^1(k_v, H^{\mathrm{tor}})$ is contained in $\bigoplus_v H^1(k_v, H^{\mathrm{tor}})$, cf. e.g. [Vo], 11.3, Cor. 1 of Prop. 1). Thus the image of map (3.3) is contained in $\bigoplus_v H^1(k_v, H^{\mathrm{tor}})$ and is equal to the image of $\ker[H^1(k, H) \to H^1(k, G)]$.

The first arrow in (3.3) is injective by Lemma 3.4, and the second arrow is injective by Theorem 2.1, and so the composition (3.3) is injective. Since (3.2) and (3.3) have the same images, we obtain a bijection

$$X(k)/\mathrm{Br} \xrightarrow{\sim} \mathrm{im}\left[\ker[H^1(k, H) \to H^1(k, G)] \to \bigoplus_v H^1(k_v, H^{\mathrm{tor}})\right].$$

This proves Theorem 3.1. □

Theorem 3.6. *In Theorem 3.1 assume that k is a totally imaginary number field. Then the bijection of Theorem 3.1 induces a bijection*

$$X(k)/Br \to H^1(k, H^{\mathrm{tor}})/Ш^1(k, H^{\mathrm{tor}}).$$

To prove Theorem 3.6 we need a proposition and two corollaries.

Proposition 3.7. *Let k be a totally imaginary number field, and $L = (\overline{F}, \kappa)$ a k-kernel (k-lien) (see [Sp], [Bo], [F/S/S] for a definition), where $\overline{F}$ is a connected linear $\bar{k}$-group such that $\overline{F}^{\mathrm{tor}} = 1$. Then every element of $H^2(k, L)$ is neutral.*

Proof. The proposition follows from [Bo], Thm. 6.8(iii) and Thm. 6.3(ii). Note that in the case when $\overline{F}$ is semisimple, the proposition was proved by Douai ([Do], Cor. 5.1), see also [Bo], Cor. 6.9. The proposition follows also from Douai's result and [Bo], Prop. 4.1. □

Corollary 3.8. *Let k be a totally imaginary number field and let*

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

be an exact sequence of linear k-groups. If G_1 is connected and $G_1^{\mathrm{tor}} = 1$, then the map $H^1(k, G_2) \to H^1(k, G_3)$ is surjective.

Proof. We argue as in [Bo], the proof of Cor. 6.4. Let $\xi \in H^1(k, G_3)$, and let $\psi \in Z^1(k, G_3)$ be a cocycle from the class ξ. According to Springer ([Sp], 1.20), one can associate to ψ a k-kernel $L_\psi = (G_{1\bar{k}}, \kappa_\psi)$ and a cohomology class $\delta(\psi) \in H^2(k, L_\psi)$ which is the obstruction to lifting ξ to $H^1(k, G_2)$. Since $G_{1\bar{k}}^{\mathrm{tor}} = 1$, by Prop. 3.7 the class $\delta(\psi)$ is neutral, and hence ξ comes from $H^1(k, G_2)$. $\square$

Corollary 3.9. *Let F be a connected linear group over a totally imaginary number field k. Then the map $H^1(k, F) \to H^1(k, F^{\mathrm{tor}})$ is surjective.*

Proof. We have an exact sequence

$$1 \to F^{\mathrm{ssu}} \to F \to F^{\mathrm{tor}} \to 1$$

where $(F^{\mathrm{ssu}})^{\mathrm{tor}} = 1$. Now the corollary follows from Corollary 3.8. $\square$

3.10. *Proof of Theorem 3.6.* Since G is simply connected and k is a totally imaginary number field, $H^1(k, G) = 1$ (Kneser–Harder–Chernousov, see [Pl/Ra] §6.1, Thm. 6). Thus $\ker[H^1(k, H) \to H^1(k, G)] = H^1(k, H)$. By Theorem 3.1 $X(k)/\mathrm{Br}$ is in a bijective correspondence with

$$\mathrm{im}\left[H^1(k, H) \to H^1(k, H^{\mathrm{tor}}) \to \bigoplus_v H^1(k_v, H^{\mathrm{tor}})\right].$$

By Corollary 3.9 the map $H^1(k, H) \to H^1(k, H^{\mathrm{tor}})$ is surjective. We see that $X(k)/\mathrm{Br}$ is in a bijective correspondence with

$$\mathrm{im}\left[H^1(k, H^{\mathrm{tor}}) \to \bigoplus_v H^1(k_v, H^{\mathrm{tor}})\right] = H^1(k, H^{\mathrm{tor}})/\text{Ш}^1(k, H^{\mathrm{tor}}).$$

$\square$

Example 3.11. Let G be a simply connected almost simple group over a number field k, $H \subset G$ a connected k-subgroup, $X = H\backslash G$. Assume that X is a symmetric space, i.e. H is the group of invariants of an involution of G. From the classification of involutions of simple Lie algebras (see e.g. [He], X-5, p. 514) it follows that $\dim H^{\mathrm{tor}} \leq 1$.

If $H^{\mathrm{tor}} = 1$ or H^{tor} is a one-dimensional split torus, then $H^1(k_v, H^{\mathrm{tor}}) = 1$ for all v, and by Theorem 3.1 $X(k)/\mathrm{Br}$ consists of one element.

If H^{tor} is a one-dimensional nonsplit torus, then H^{tor} splits over a quadratic extension K of k. Assume in addition that k is totally imaginary. Then by Theorem 3.6 $X(k)/\mathrm{Br} = H^1(k, H^{\mathrm{tor}})/\text{Ш}^1(k, H^{\mathrm{tor}})$. Since K/k is cyclic, we have $\text{Ш}^1(k, H^{\mathrm{tor}}) = 1$ ([Vo], 11.6, Cor.3), and we see that

$$X(k)/\mathrm{Br} = H^1(k, H^{\mathrm{tor}}) = k^\times/N_{K/k}K^\times$$

where $N_{K/k}$ denotes the norm map.

4. Appendix

In this appendix, we prove Theorem 1.5.

We use the description of the Manin pairing with the help of torsors given in [CT/Sa2], §2. We first recall some generalities.

4.1. *Torsors.* Let X be a smooth geometrically integral k-variety such that $U(\overline{X}) = 1$ and $\operatorname{Pic}\overline{X}$ is a finitely generated $\mathfrak{g}$-module, where $\mathfrak{g} = \operatorname{Gal}(\bar{k}/k)$. Let S be a k-torus. Let $(Y, p_Y \colon Y \to X)$ be a right X-torsor under S.

The set of isomorphism classes of X-torsors under S is the first étale cohomology group $H^1(X, S)$. We have a canonical map

$$\chi\colon H^1(X,S) \to \operatorname{Hom}_{\mathfrak{g}}(\mathbf{X}(\overline{S}), \operatorname{Pic}\overline{X})$$

defined as follows. To any X-torsor Y under S we associate a homomorphism χ_Y sending a character $\lambda \in \mathbf{X}(\overline{S})$, $\lambda\colon \overline{S} \to \mathbb{G}_{m,\bar{k}}$ to the push-forward $\lambda_*(Y)$ which is an X-torsor under $\mathbb{G}_{m,\bar{k}}$; since $\operatorname{Pic}\overline{X} = H^1(\overline{X}, \mathbb{G}_m)$, we identify the isomorphism class of $\lambda_*(Y)$ with an element of $\operatorname{Pic}\overline{X}$. We call χ_Y the type of Y. If S is the Néron–Severi torus of X (i.e. $\mathbf{X}(\overline{S}) = \operatorname{Pic}\overline{X}$) and χ_Y is the identity map, the torsor Y is called universal.

4.2. *Evaluation map.* Let Y be an X-torsor under S. We then have a natural evaluation map $\theta_Y\colon X(k) \to H^1(k, S)$ which associates to every rational point $x \in X(k)$ the isomorphism class of the fibre Y_x of Y at x. The map θ_Y allows us to identify the Manin pairing with a cup-product. We use the following construction. To $x \in X(k)$ we can associate a homomorphism of $\mathfrak{g}$-modules $\sigma\colon \bar{k}(X)^\times \to \bar{k}^\times$, $\sigma(f) = f(x)$ for $f \in \bar{k}(X)^\times$. There is a canonical isomorphism $H^1(k, \operatorname{Pic}\overline{X}) \xrightarrow{\sim} \operatorname{Br}_a X$, cf. [Sa], 6.3(iii). Together with the isomorphism $\operatorname{Br}_a X \xrightarrow{\sim} \operatorname{Br}_x X$ (see 1.1) this yields an embedding $t_x\colon H^1(k, \operatorname{Pic}\overline{X}) \to \operatorname{Br}_1 X$ (in [CT/Sa2], p. 449, this embedding is denoted by t_σ).

Let Y^x denote an X-torsor under S of type $\chi_Y\colon \mathbf{X}(\overline{S}) \to \operatorname{Pic}\overline{X}$ which is trivial at x (such a torsor exists and is unique up to isomorphism, cf. [CT/Sa2], p. 449).

Theorem 4.3. ([CT/Sa2], Prop. 2.7.10) *Let*

$$\rho\colon H^1(k, \mathbf{X}(\overline{S})) \to H^1(k, \operatorname{Pic}\overline{X}) \xrightarrow{t_x} \operatorname{Br}_1 X$$

be the composed map where the left arrow is induced by χ_Y. Then the following diagram is commutative up to sign:

$$
\begin{array}{ccccc}
X(k) & \times & \operatorname{Br}_1 X & \longrightarrow & \operatorname{Br} k \\
{\scriptstyle \theta_{Y^x}}\big\downarrow & & \big\uparrow{\scriptstyle \rho} & & \big\| \\
H^1(k,S) & \times & H^1(k,\mathbf{X}(\overline{S})) & \longrightarrow & \operatorname{Br} k
\end{array}
\tag{4.1}
$$

Here the top row is the Manin pairing and the bottom row is the cup-product.

4.4. *Abelianization of torsors.* Let us now suppose that X is as in 1.2, i.e. $X = H\backslash G$. We have a canonical map $G \to X$. Define a right action of H on G by $g * h = h^{-1}g$, where $g \in G$, $h \in H$. With this action the map $G \to H$ is a right (non-abelian) X-torsor under H. Set $S = H^{\text{tor}}$. We consider the natural homomorphism $\psi\colon H \to S$. Let H^{ssu} denote the kernel of ψ. Let $\psi_*\colon H^1(k,H) \to H^1(k,S)$ and $\psi_*^X\colon H^1(X,H) \to H^1(X,S)$ be the induced push-forward maps, they send non-abelian torsors under H to abelian torsors under S. Explicitly, to any right X-torsor Z under H we associate an X-torsor $\psi_*^X(Z) = Z/H^{\text{ssu}}$ under S (abusing notation, we use the same symbol ψ_* for maps of torsors and their cohomology classes). Denote by Y the torsor under S obtained by applying ψ_*^X to $G \to X$, i.e. $Y = H^{\text{ssu}}\backslash G$. Note that by Proposition 1.3, we have an isomorphism $\mathbf{X}(\overline{S}) \xrightarrow{\sim} \operatorname{Pic} \overline{X}$ so that Y is a universal torsor. With the notation of Theorem 4.3, $Y = Y^{x_0}$ where $x_0 \in X(k)$ is the image of the unit element of G.

Lemma 4.5. *The map $\theta_Y\colon X(k) \to H^1(k, H^{\text{tor}})$ coincides with the map α defined by (1.1).*

Proof. Indeed, α is the composition

$$\psi_* \circ \delta\colon X(k) \xrightarrow{\delta} H^1(k,H) \xrightarrow{\psi_*} H^1(k, H^{\text{tor}})$$

where δ is the connecting map. Note that the image of $x \in X(k)$ under δ coincides with the (isomorphism class of the) fibre of $G \to X$ at x (cf. [Se], I.5.4, Prop. 36, [Ha/Sk], 4.1). Furthermore, push-forward commutes with specialization, i.e. for any X-torsor Z under H and any $x \in X(k)$ the fibre $[\psi_*^X(Z)]_x$ coincides with $\psi_*(Z_x)$. Hence $\psi_*(\delta(x))$ coincides with the fibre of the abelian X-torsor Y under H^{tor} at x, which is equal to $\theta_Y(x)$ (by the definition of θ_Y). Thus $\alpha = \theta_Y$. □

4.6. To finish the proof of Theorem 1.5, it only remains to apply diagram (4.1). Indeed, denote by i the isomorphism $\operatorname{Br}_{x_0} X \xrightarrow{\sim} H^1(k, \mathbf{X}(\overline{S}))$. Then the pairing (1.4) $X(k) \times \operatorname{Br}_{x_0} X \to \operatorname{Br} k$ is given by

$$(x,b) \mapsto \alpha(x) \cup i(b), \tag{4.2}$$

whereas the Manin pairing (0.1) $X(k) \times \mathrm{Br}_1 X \to \mathrm{Br}\, k$ restricted to $X(k) \times \mathrm{Br}_{x_0}X$ is given by

$$(x, b) \mapsto b(x). \tag{4.3}$$

Any $b \in \mathrm{Br}_{x_0}X$ can be written as $b = \rho(z)$ with $z \in H^1(k, \mathbf{X}(\overline{S}))$, where in the definition of ρ (in Theorem 4.3) we use x_0 instead of x. On applying diagram (4.1) to the Manin pairing (4.3), we get an equality (up to sign)

$$(x, \rho(z)) = \theta_Y(x) \cup z,$$

where $Y = Y^{x_0}$ in θ_Y. On the other hand, taking into account that $\alpha = \theta_Y$ (Lemma 4.5) and $i(\rho(z)) = z$, formula (4.2) gives the same value $\theta_Y(x) \cup z$ for $(x, \rho(z))$. Thus formulas (4.2) and (4.3) coincide (up to sign), and the Manin pairing restricted to $\mathrm{Br}_{x_0}X$ coincides (up to sign) with pairing (1.4). The theorem is proved.

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