ON THE GENERIC SPLITTING OF QUADRATIC FORMS IN CHARACTERISTIC 2

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ABSTRACT. In [8] and [9] Knebusch established the basic facts of generic splitting theory of quadratic forms over a field of characteristic not 2. In [10], he generalized this theory to a field of characteristic 2. This note is related to [10]. More precisely, we begin with a complete characterization of quadratic forms of height 1 (in this note we don't exclude anisotropic quadratic forms with radical of dimension at least 1). This allows us to extend the notion of degree in characteristic 2. We prove some results on excellent forms and generic splitting tower of a quadratic form. Some results on quadratic forms of height 2 and degree 1 or 2 are given.

Let F be a field of characteristic not 2. We associate to an anisotropic quadratic form φ of dimension ≥ 3 over F, the function field $F(\varphi)$ of the projective quadric defined by the equation $\varphi = 0$. When φ is an anisotropic quadratic form of dimension 2 (resp. of dimension 1 or isotropic of dimension 2) we set $F(\varphi) = F(\sqrt{-\det\varphi})$ where $\det\varphi$ is the discriminant of φ (resp. $F(\varphi) = F$). An anisotropic quadratic form of dimension at least 2 becomes clearly isotropic over its function field. In [8] Knebusch associated to a non-split quadratic form φ a sequence of extensions and quadratic forms, *the so-called generic splitting tower of* φ, as follows: $\varphi_0 = \varphi_{an}$ (the anisotropic part of φ), $F_0 = F$ and inductively for $n \geq 1$, $F_n = F_{n-1}(\varphi_{n-1})$ and $\varphi_n = ((\varphi_{n-1})_{F_n})_{an}$. The height of φ is the smallest integer $h = \mathrm{h}(\varphi)$ such that $\dim\varphi_h \leq 1$. Clearly, for an anisotropic quadratic form φ of dimension at least 2, we have $\mathrm{h}(\varphi) = \mathrm{h}(\varphi_{F(\varphi)}) + 1$. In particular, an anisotropic quadratic form φ of dimension ≥ 2 is of height 1 if and only if $\dim(\varphi_{F(\varphi)})_{an} \leq 1$. For any details on generic splitting theory of quadratic forms in characteristic not 2, we refer to Knebusch's papers [8], [9].

A quadratic form φ is called a Pfister neighbor if there exists an n-fold Pfister form π and a scalar $a \in F^*$ such that $\dim\varphi > 2^{n-1}$ and $a\pi \cong \varphi \perp \psi$ for some quadratic form ψ, where $\cong$ and $\perp$ denote respectively the isometry and orthogonal sum of quadratic forms. The forms π and ψ are unique up to isometry. We call ψ the complementary form of φ and $\dim\pi - \dim\varphi$ the codimension of φ.

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In general, for a given integer h it is very difficult to describe quadratic forms of height h. In characteristic not 2, Knebusch [8] and Wadsworth [16] have given independently a complete characterization of anisotropic quadratic forms of height 1. Such a quadratic form is a Pfister neighbor of codimension 0 or 1.

In [10] Knebusch generalized the generic splitting theory of quadratic forms to a field of characteristic 2. The height and the generic splitting are defined in the same manner as in characteristic not 2.

From now on, we assume that F is a field of characteristic 2. We will investigate the generic splitting of quadratic forms over F. Along this note, we don't exclude quadratic forms with radical of dimension at least 1. We begin with a complete characterization of anisotropic quadratic forms of height 1. We use that characterization to extend the notion of degree in charactersitic 2, and we prove some results on anisotropic quadratic forms of height 2 and degree 1 or 2 like those obtained in characteristic not 2 [4], [9], [6]. We extend the notion of excellent form and we give some results related to those forms. Some general results on the generic splitting tower of a quadratic form are given.

All basic facts and details on quadratic forms in characteristic 2 can be found in Baeza's book [2]. Let $[a, b]$ (resp. $[a]$) denote the quadratic form $aX^2 + XY + bY^2$ (resp. the quadratic form aX^2).

Every quadratic form φ of dimension at least 1 can be written up to isometry:

$$\varphi = [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp [c_1] \perp \cdots \perp [c_s] \perp [0] \perp \cdots \perp [0] \qquad (1)$$

with $[c_1] \perp \cdots \perp [c_s]$ anisotropic.

Definition 1. *Let φ be as in equation (1).*
(1) If $\dim \varphi = 2r + s$, then φ is called a regular form or a form of type (r, s).
(2) A quadratic form of type $(r, 0)$ is called a nonsingular quadratic form.
(3) A quadratic form of type (r, s) with $s \geq 1$ is called a singular quadratic form.
(4) A quadratic form of type $(0, s)$ is called a totally singular quadratic form.
(5) The quadratic form $[c_1] \perp \cdots \perp [c_s] \perp [0] \perp \cdots \perp [0]$ is called the quasilinear part of φ.

Let $W_q(F)$ denote the Witt group of nonsingular quadratic forms, and let $W(F)$ denote the Witt ring of nonsingular bilinear symmetric forms. It is well known that $W_q(F)$ is a $W(F)$-module.

We denote by $\langle a_1, \cdots, a_n \rangle$ the bilinear form $\sum_{i=1}^{n} a_i X_i Y_i$. For an integer r and a quadratic form φ, we denote by $r \times \varphi$ the quadratic form $\underbrace{\varphi \perp \cdots \perp \varphi}_{r \text{ times}}$. A quadratic form φ is called split if $\varphi \cong r \times \mathbb{H} \perp s \times [0]$ where $\mathbb{H} = [0, 0]$ is the hyperbolic plane. If φ and ψ are quadratic forms, then $\varphi \sim \psi$ means that $\varphi \perp r \times \mathbb{H} \cong \psi \perp s \times \mathbb{H}$. We say that φ and ψ are similar if $\varphi \cong a\psi$ for some

$a \in F^*$. A nonsingular quadratic form φ is hyperbolic if $\varphi \cong r \times \mathbb{H}$. For a field extension K/F and a quadratic form φ over F, the quadratic form $\varphi \otimes K$ is denoted by φ_K.

An n-fold Pfister form is a quadratic form $\langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle \otimes [1, a_{n+1}]$ that we denote by $\langle\langle a_1, \cdots, a_{n+1}]]$. Let P_nF (resp. GP_nF) denote the set of n-fold Pfister forms (resp. the set $\{\alpha\pi \mid \alpha \in F^*, \pi \in P_nF\}$). An n-fold Pfister form π is either anisotropic or hyperbolic [2, Chapter 4, Corollary 3.2]. The $W(F)$-submodule of $W_q(F)$ generated by n-fold Pfister forms is denoted by $I^nW_q(F)$. Let $\mathrm{Br}(F)$ denote the Brauer group of F.

Set $\wp(x) = x^2 + x$ for $x \in F$ and $\wp(F) = \{\wp(x) \mid x \in F\}$. If φ is nonsingular, then the Clifford algebra $C(\varphi)$ is a central simple algebra over F, and the center $Z(\varphi)$ of the even Clifford algebra $C_0(\varphi)$ is a separable quadratic algebra over F. In this case, $Z(\varphi) = F(\wp^{-1}(\delta))$ for some $\delta \in F$ and the Arf invariant $\triangle(\varphi)$ of φ is defined as the class of δ in $F/\wp(F)$. More precisely, if $\varphi \cong a_1[1, b_1] \perp \cdots \perp a_r[1, b_r]$ then $\triangle(\varphi) = b_1 + \cdots + b_r \in F/\wp(F)$.

Let φ be a quadratic form of dimension $n \geq 1$ which is not isometric to $s \times [0]$, and let P_φ be the homogenous polynomial given by φ. In [13] it is shown that P_φ is reducible if and only if φ is either isometric to $\mathbb{H} \perp [0] \perp \cdots \perp [0]$ or isometric to $[a] \perp [0] \perp \cdots \perp [0]$ for some $a \in F^*$. If P_φ is irreducible, we define the function field $F(\varphi)$ of φ as the field of fractions of

$$\frac{F\left[X_1, \cdots, X_n\right]}{(P_\varphi)}$$

where (P_φ) is the ideal of $F\left[X_1, \cdots, X_n\right]$ generated by P_φ. In particular, $F(\varphi)$ is well defined for an anisotropic quadratic form φ.

Quadratic forms of height 1 are as follows.

Theorem 2. *Let F be a field of characteristic 2, and let φ be an anisotropic quadratic form, possibly singular, of dimension ≥ 1. Then, φ is of height 1 if and only if φ is one of the following types:*
(1) $\dim \varphi = 2$,
(2) $\varphi \in GP_nF$ *for some integer* $n \geq 1$,
(3) $\varphi \cong \psi \perp [c]$ *for some anisotropic* $\psi \in W_q(F)$ *and* $c \in F^*$ *such that* $\psi \perp c\,[1, \triangle(\psi)] \in GP_mF$ *for some integer* $m \geq 1$. ■

We will need frequently a generalization of the subform notion.

Definition 3. *([11]) Let $\psi = [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp [c_1] \perp \cdots \perp [c_s]$ be a quadratic form.*
(1) We say that ψ is dominated by φ and denote $\psi \prec \varphi$ if there exists a form δ such that

$$\varphi \cong [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \xi_1 \perp \cdots \perp \xi_s \perp \delta$$

and for all $i \in \{1, \cdots, s\}$*, we have* $\xi_i = [c_i]$ *or* $\xi_i = [c_i, d_i]$ *for some* $d_i \in F$*.*
(2) We say that ψ *is weakly dominated by* φ *if* $a\psi \prec \varphi$ *for some scalar* $a \in F^*$*.*
(3) We say that ψ *is a subform of* φ *and denote* $\psi < \varphi$ *if* $\varphi \cong \psi \perp \mu$ *for some quadratic form* μ*.* ■

From Definition 3 we deduce the following remarks.

Remarks 4. *(1) If* ψ *is weakly dominated by* φ *and if* $F(\psi)$ *is well defined, then* $\varphi_{F(\psi)}$ *is isotropic.*
(2) If φ *and* ψ *are nonsingular and* ψ *is dominated by* φ*, then* ψ *is a subform of* φ*.*
(3) With the same notations and hypothesis as in Definition 3 (1) and if φ *is nonsingular, then* $\dim \xi_i = 2$ *for all* $i \in \{1, \cdots, s\}$*.*

In [11], an analogue of the Cassels-Pfister subform theorem was proved.

Proposition 5. *(*[11, Proposition 3.4]*) Let* F *be a field of characteristic* 2*,* $\varphi \in W_q(F)$ *anisotropic and* ψ *be an anisotropic quadratic form, possibly singular, such that* $\varphi_{F(\psi)}$ *is hyperbolic. Then,* ψ *is weakly dominated by* φ*. In particular,* $\dim \varphi \geq \dim \psi$*.*

Proposition 5 was obtained by using a result of Baeza concerning the norm theorem for nonsingular quadratic forms [3] and some results of [1].

The following Corollary is an immediate consequence of Proposition 5.

Corollary 6. *Let* F *be a field of characteristic* 2*,* $\varphi \in W_q(F)$ *anisotropic and* π *be an anisotropic Pfister form. If* $\varphi_{F(\pi)}$ *is hyperbolic, then there exists a bilinear form* ρ *such that* $\varphi \cong \rho \otimes \pi$*.*

The following Lemma is well known, we recall it without proof.

Lemma 7. *Let* F *be a field of characteristic* 2*,* $\varphi, \psi \in W_q(F)$ *and* $u_1, \cdots, u_n, v_1, \cdots, v_n \in F$ *such that* $\varphi \perp [u_1] \perp \cdots \perp [u_n] \cong \psi \perp [v_1] \perp \cdots \perp [v_n]$*. Then the sets* $\{u_1, \cdots, u_n\}$*,* $\{v_1, \cdots, v_n\}$ *engender the same vector space over* F^2*.*

We will need a generalization of the Witt cancellation.

Proposition 8. *(*[7, Proposition 1.2]*) Let* F *be a field of characteristic* 2*,* $\eta \in W_q(F)$ *and* φ, ψ *be quadratic forms, possibly singular, such that* $\varphi \perp \eta \cong \psi \perp \eta$*. Then,* $\varphi \cong \psi$*.*

By using Proposition 8 it is clear that a nonsingular quadratic form φ is hyperbolic if and only if $\varphi \sim 0$.

Proof of Theorem 2. To prove this theorem in the nonsingular case, we use the same idea as in characteristic not 2 but we proceed differently by using a function field argument. The singular case needs some observations, the point is that we can't follow directly the proof given in characteristic not 2 because Knebusch used a Witt cancellation argument which is not allowed in characteristic 2.

We recall a lemma.

Lemma 9. *(*[11, Lemma 3.1]*) Let $\varphi = [a,b] \perp [y]$ be a quadratic form. We have:*
(1) If $z \in F$ is represented by φ, then either $[y] \cong [z]$ or there exists $r \in F$ such that $\varphi \cong [z,r] \perp [y]$.
(2) If φ is isotropic of type $(1,1)$, then $\varphi \cong \mathbb{H} \perp [y]$.
(3) If $a = y$, then $\varphi \sim [y]$.

Let φ be an anisotropic quadratic form of type (1), (2) or (3) as in Theorem 2. It is clear that $\mathrm{h}(\varphi) = 1$ when $\dim\varphi = 2$ or $\varphi \in GP_nF$ for some integer $n \geq 1$. Let's assume now that there exists a nonsingular quadratic form ξ and a scalar $c \in F^*$ such that $\varphi \cong \xi \perp [c]$ and $\pi := \xi \perp c\,[1, \triangle(\xi)] \in GP_mF$ for some integer $m \geq 1$. Since φ is weakly dominated by π, it follows that $\pi_{F(\varphi)} \sim 0$ and $\xi_{F(\varphi)} \sim (c\,[1, \triangle(\xi)])_{F(\varphi)}$. In particular, $\varphi_{F(\varphi)} \sim (c\,[1, \triangle(\xi)] \perp [c])_{F(\varphi)}$. By Lemma 9 and Proposition 8, we deduce that $(\varphi_{F(\varphi)})_{an} \cong ([c])_{F(\varphi)}$.

For the converse, we may assume that $\dim\varphi \geq 3$ and $\mathrm{h}(\varphi) = 1$. Set $\varphi = [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp [c_1] \perp \cdots \perp [c_s]$.

(1) If φ is nonsingular: Let k be the maximal integer such that there exists a k-fold Pfister form ψ as a subform of φ. We have to prove $\dim\varphi = 2^{k+1}$. Assume $\dim\psi < \dim\varphi$, and let ψ' be a quadratic form such that $\varphi \cong \psi \perp \psi'$. Since ψ is anisotropic and φ is nonsingular of dimension ≥ 4, it follows that $F(\psi)(\varphi)$ is well defined. However, $\varphi_{F(\psi)(\varphi)} \sim \psi'_{F(\psi)(\varphi)} \sim 0$ and $\dim\psi' < \dim\varphi$. Proposition 5 implies that $\psi'_{F(\psi)} \sim 0$, and thus ψ is weakly dominated by ψ'. Consequently, there exists a scalar $a \in F^*$ and a quadratic form η such that $\psi' \cong a\psi \perp \eta$. The form $\psi \perp a\psi$ is a subform of φ and is similar to a $(k+1)$-fold Pfister form, contradicting the maximality of k. Hence, φ is similar to a Pfister form.

(2) If φ is singular: In this case we need some results on the isotropy problem, and the relation between the height and the type of a singular form.

Proposition 10. *(*[11, Proposition 1.1]*) Let F be a field of characteristic 2, and let φ, ψ be anisotropic quadratic forms.*
(1) If $\dim\varphi = 2$ and $\dim\psi \geq 3$, then $\varphi_{F(\psi)}$ is anisotropic.
(2) If φ is nonsingular of dimension 2 and ψ is totally singular, then $\varphi_{F(\psi)}$ is anisotropic.

Proposition 11. *Let φ be as above. Then, $s = 1$.*

Proof. Assume $s \geq 2$. Since $\mathrm{h}(\varphi) \leq 1$, it follows that

$$\varphi_{F(\varphi)} \cong r \times \mathbb{H} \perp (s-1) \times [0] \perp [\alpha]$$

for some $\alpha \in F(\varphi)$. Lemma 7 implies that the $F(\varphi)^2$-vector space spanned by $c_1, \cdots, c_s$ is of dimension ≤ 1. In particular, the $F(\varphi)^2$-vector space spanned by c_1, c_2 is of dimension ≤ 1, which means that the form $[c_1] \perp [c_2]$ becomes isotropic over $F(\varphi)$, a contradiction with Proposition 10. ∎

We consider the following quadratic forms

$$\mu := [a_1, b_1] \perp \cdots \perp [a_r, b_r],$$

and

$$\nu := [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp c_1 \, [1, \triangle(\mu)] \, .$$

Lemma 12. *Let μ and ν be as above. Then, ν is anisotropic and $\nu_{F(\varphi)}$ is hyperbolic.*

Proof. Since $\varphi_{F(\varphi)} \prec \nu_{F(\varphi)}$ and $\varphi_{F(\varphi)}$ contains r hyperbolic planes, it follows that $\nu_{F(\varphi)} \cong r \times \mathbb{H} \perp \xi$ for some 2-dimensional quadratic form ξ. By comparing Arf invariant we get $\xi \cong \mathbb{H}$. Therefore, $\nu_{F(\varphi)}$ is hyperbolic. The quadratic form ν is not hyperbolic because φ is anisotropic. Assume that ν is isotropic. Thus, $2 \leq \dim \nu_{an} \leq \dim \nu - 2 < \dim \varphi$ and $(\nu_{an})_{F(\varphi)} \sim 0$, a contradiction with Proposition 5. ∎

To complete the proof, let k be the maximal integer such that there exists a k-fold Pfister form ψ as a subform of ν. We have to prove $\dim \nu = 2^{k+1}$. Assume $\dim \nu > 2^{k+1}$, and let ψ' be a quadratic form such that $\nu \cong \psi \perp \psi'$. Notice that $\dim \psi' \leq \dim \nu - 2 < \dim \varphi$. Since ψ is anisotropic, φ is not totally singular and $c \neq 0$, it follows that $F(\psi)(\varphi)$ is well defined. However, $\nu_{F(\psi)(\varphi)} \sim \psi'_{F(\psi)(\varphi)} \sim 0$. By Proposition 5, we have $\psi'_{F(\psi)} \sim 0$, and thus ψ is weakly dominated by ψ'. Hence, there exists $a \in F^*$ and a quadratic form η such that $\psi' \cong a\psi \perp \eta$. The form $\psi \perp a\psi$ is a subform of ν and is similar to a $(k+1)$-fold Pfister form, contradicting the maximality of k. Hence, ν is similar to a Pfister form. ∎

We recall a general result on the isotropy problem.

Proposition 13. *(*[11, Corollaire 3.3]*) Let F be a field of characteristic 2, and let φ, ψ be anisotropic quadratic forms over F of dimension ≥ 2 such that φ is totally singular and ψ is not totally singular. Then, $\varphi_{F(\psi)}$ is anisotropic.*

As a corollary we have:

Corollary 14. *Let F be a field of characteristic 2, and let φ, ψ be anisotropic quadratic forms. Assume that ψ is not totally singular and φ is totally singular. Let $(F_i, \varphi_i)_{0 \leq i \leq \mathrm{h}(\varphi)}$ be the generic splitting tower of φ. Then, $F_i(\psi)$ is well defined for all $i \in \{0, \cdots, \mathrm{h}(\varphi)\}$.*

Proof. Set $\psi = [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp [c_1] \perp \cdots \perp [c_s]$. By assumption we have $r \geq 1$. Let $i \in \{0, \cdots, \mathrm{h}(\varphi)\}$ and assume that $F_i(\psi)$ is not defined. Hence,

$$\psi_{F_i} \cong \mathbb{H} \perp [0] \perp \cdots \perp [0] \tag{2}$$

In particular, $r = 1$. If $s \geq 1$, then the F_i^2-vector space spanned by $\{c_1, \cdots, c_s\}$ is of dimension ≥ 1, a contradiction with Lemma 7 and equation (2). Hence, $\psi = [a_1, b_1]$ and $\psi_{F_i} \cong \mathbb{H}$. Since, φ_j is totally singular for all $j \in \{0, \cdots, \mathrm{h}(\varphi)\}$, it follows from Proposition 10 (2) that ψ_{F_i} is anisotropic, a contradiction. ∎

In the following theorem, we give some general results on the generic splitting tower of a quadratic form.

Theorem 15. *Let φ be an anisotropic quadratic form of height $h = \mathrm{h}(\varphi)$ and type (r, s), and let $(F_i, \varphi_i)_{0\leq i\leq \mathrm{h}(\varphi)}$ be its generic splitting tower. When $s \geq 1$, we denote by η the quasilinear part of φ. Then, we have the following statements:*
(1) If φ is totally singular and ψ is not a totally singular quadratic form, then $\mathrm{h}(\varphi) = \mathrm{h}(\varphi_{F(\psi)})$.
(2) If $s \geq 2$, then there exists $j \in \{0, \cdots, \mathrm{h}(\varphi) - 1\}$ such that φ_j is of type $(0, s)$. More precisely, we have $\varphi_j \cong \eta_{F_j}$.
(3) If $r, s \geq 1$, then $\mathrm{h}(\eta) + 1 \leq \mathrm{h}(\varphi) \leq \mathrm{h}(\eta) + r$.
(4) Assume that $r, s \geq 1$ and there exists $j \in \{0, \cdots, \mathrm{h}(\varphi) - 1\}$ such that φ_j is of type $(r_j, 1)$. Then $s = 1$.
(5) Let $j \in \{0, \cdots, \mathrm{h}(\varphi)\}$. Assume that φ_j is of type (r_j, ϵ) with $r_j \geq 1$. Then, for all $i \in \{0, \cdots, j\}$ the type of φ_i is (r_i, ϵ) for some integer r_i.
(6) If $s \geq 2$, then $\varphi_{\mathrm{h}(\varphi)-1}$ is totally singular of dimension 2.
(7) If $s = 1$, then for all $i \in \{0, \cdots, \mathrm{h}(\varphi)\}$ the type of φ_i is $(r_i, 1)$ for some r_i.
(8) If $s = 0$, then $\varphi_{\mathrm{h}(\varphi)-1}$ is similar to a Pfister form. ■

Proof. (1) Set $\psi = [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp [c_1] \perp \cdots \perp [c_s]$ and $\mu = [a_1, b_1]$. By Corollary 14 the fields $F_i(\psi)$, $F_i(\mu)$ are well defined.

(i) If $\mu = \psi$, then by Proposition 10 we deduce that μ_{F_i} is anisotropic, and thus by Proposition 13 $(\varphi_i)_{F_i(\mu)}$ is anisotropic.

(ii) If $\dim \psi > \dim \mu$, then $\dim \psi \geq 3$. By the same argument as in the proof of Corollary 14 and by using $\dim \psi \geq 3$, we deduce that $F_i(\mu)(\psi)$ is well defined. The extension $F_i(\mu)(\psi)/F_i(\mu)$ is purely transcendental [13, Lemma 1]. If $(\varphi_i)_{F_i(\psi)}$ is isotropic, then $(\varphi_i)_{F_i(\mu)}$ is also isotropic, a contradiction with Proposition 13.
Since, $(\varphi_i)_{F_i(\psi)}$ is anisotropic for all $i \in \{0, \cdots, \mathrm{h}(\varphi)\}$, it follows that $\mathrm{h}(\varphi) = \mathrm{h}(\varphi_{F(\psi)})$.

(2) Since $\dim \varphi \geq s \geq 2$, it follows that $\mathrm{h}(\varphi) \geq 1$. Let j be the smallest in $\{1, \cdots, \mathrm{h}(\varphi)\}$ such that φ_j is of type (r_j, s_j) with $s_j < s$. By the choice of j, the form $\eta_{F_{j-1}}$ is anisotropic. Lemma 7 implies that η_{F_j} is isotropic. Since $F_j = F_{j-1}(\varphi_{j-1})$, it follows from Proposition 13 that φ_{j-1} is totally singular. Therefore, by the minimality of j we conclude that φ_{j-1} is of type $(0, s)$, and thus by Lemma 7 we obtain $\varphi_{j-1} \cong \eta_{F_{j-1}}$.

(3) The statement is clear for $s = 1$. So we may assume $s \geq 2$. By statement (2), there exists $j \in \{0, \cdots, \mathrm{h}(\varphi) - 1\}$ such that $\varphi_j \cong \eta_{F_j}$. Since φ is not totally singular and φ_j is totally singular, we obtain $j \geq 1$. Since φ_{j-1} is not totally singular, it follows that φ_l is not totally singular for all $l \in \{0, \cdots, j-1\}$. By statement (1) we conclude that $\mathrm{h}(\eta_{F_j}) = \mathrm{h}(\eta_{F_{j-1}}) = \cdots = \mathrm{h}(\eta)$. Since, $\mathrm{h}(\varphi) \geq \mathrm{h}(\varphi_{j-1}) = \mathrm{h}(\varphi_j) + 1$ and $\eta_{F_j} \cong \varphi_j$, we conclude that $\mathrm{h}(\varphi) \geq \mathrm{h}(\eta) + 1$. The second inequality is obvious.

(4) There is nothing to prove in the case $j = 0$. So we may assume $j \geq 1$. Let (r_{j-1}, s_{j-1}) be the type of φ_{j-1}. Since φ_j is of type $(r_j, 1)$ and $j \leq \mathrm{h}(\varphi) - 1$, it follows that $r_j > 0$, otherwise $\mathrm{h}(\varphi_{j-1}) = 1$ and $j = \mathrm{h}(\varphi)$. In particular, $r_{j-1} > 0$ and φ_{j-1} is not totally singular. Set $\varphi_{j-1} = \xi_{j-1} \perp \eta_{j-1}$ with $\xi_{j-1} \in W_q(F_{j-1})$ and η_{j-1} a totally singular form over F_{j-1}. If $s_{j-1} \geq 2$, then η_{j-1} becomes isotropic over $F_{j-1}(\varphi_{j-1})$, a contradiction with Proposition 13. Hence $s_{j-1} = 1$. We finish the proof by a simple iteration argument.

(5) There is nothing to prove in the case $j = 0$. So we may assume $j \geq 1$. Then, the type (r_{j-1}, s_{j-1}) of φ_{j-1} satisfies $r_{j-1} \geq r_j \geq 1$ and $s_{j-1} \geq \epsilon$. In particular, φ_{j-1} is not totally singular. Set $\varphi_{j-1} = \xi_{j-1} \perp \eta_{j-1}$ with $\xi_{j-1} \in W_q(F_{j-1})$ and η_{j-1} a totally singular form over F_{j-1}. If $s_{j-1} > \epsilon$, then η_{j-1} becomes isotropic over $F_j = F_{j-1}(\varphi_{j-1})$, a contradiction with Proposition 13. Hence, $s_{j-1} = \epsilon$. We finish the proof by a simple iteration argument.

(6) Assume $s \geq 2$. By statement (2) there exists $j \in \{0, \cdots, \mathrm{h}(\varphi) - 1\}$ such that φ_j is of type $(0, s)$. In particular, $\varphi_{\mathrm{h}(\varphi)-1}$ is totally singular. It follows from Theorem 2 that $\varphi_{\mathrm{h}(\varphi)-1}$ is totally singular of dimension 2.

(7) Obvious.

(8) If $s = 0$. Then, all forms φ_i are nonsingular. By Theorem 2 the form φ_{h-1} is similar to a Pfister form. ■

The notion of a Pfister neighbor form was extended in characteristic 2 as follows.

Definition 16. *(*[11, Définition 1.2]*) A quadratic form φ is a Pfister neighbor if there exists an n-fold Pfister form π such that $\dim \varphi > 2^n$ and φ is weakly dominated by π. We call $\dim \pi - \dim \varphi$ the codimension of φ.*

In [11, Proposition 3.1], it is shown that an anisotropic Pfister neighbor can't be totally singular, and if φ is a Pfister neighbor of π then π is unique up to isometry.

Theorem 2 allows us to extend the notion of degree of a quadratic form. So, let φ be a non-split quadratic form of height $h = \mathrm{h}(\varphi)$, and let $(F_i, \varphi_i)_{0 \leq i \leq h}$ be its generic splitting tower. Assume that $\dim \varphi_0 \geq 2$. Then, $\mathrm{h}(\varphi) \geq 1$. The field F_{h-1} is called the leading field of φ [8]. Since φ_{h-1} is of height 1, it follows from Theorem 2 that φ_{h-1} is either a Pfister neighbor of codimension 0 or 1 (in the sense of Definition 16), or a singular form of dimension 2. We have two possibilities:

(1) If φ_0 is nonsingular, then φ_{h-1} is also nonsingular. Hence, $\varphi_{h-1} \cong a\pi$ for some $a \in F_{h-1}^*$ and $\pi \in P_d F_{h-1}$ for some integer d. In this case, we say that φ is of degree $d + 1$. The form π is called the leading form of φ and is uniquely determined by φ (we use the same argument as in [8]).

(2) If φ_0 is singular, then Theorem 15 implies that φ_{h-1} is also singular. In this case, we say that φ is of degree 0. By Theorem 2 φ_{h-1} is of type $(r_{h-1}, 1)$ with $r_{h-1} \geq 1$ or of type $(0, 2)$. When the type is $(r_{h-1}, 1)$, we conclude by Theorem 2 that $\varphi_{h-1} \cong \xi \perp [c]$ with $\xi \in W_q(F_{h-1})$ and $c \in F^*_{h-1}$ such that $c\xi \perp [1, \triangle(\xi)]$ is a Pfister form that we call also the leading form of φ [8].

If φ is a split form (resp. $\dim \varphi_{an} = 1$), we set the degree of φ as ∞ (resp. we set the degree of φ as 0).

We denote by $\deg(\varphi)$ the degree of φ, $J_n(F)$ the set of all quadratic forms of degree at least n. Clearly, $GP_nF \subset J_{n+1}(F)$ and $W_q(F) = J_1(F)$.

As in [8], we have the following proposition.

Proposition 17. *(1) Let $\tau \in P_nF$ anisotropic and $a \in F^*$. Let φ be a quadratic form of degree $\geq n+2$. Then, the quadratic form $a\tau \perp \varphi$ is of degree $n+1$.*
(2) For $n \geq 0$, the set $J_n(F)$ is closed under addition. In particular, $J_n(F)$ is a subgroup of $W_q(F)$ for $n \geq 1$, and $I^nW_q(F) \subset J_{n+1}(F)$.

Proof. We use the same proof as in [8, Theorem 6.3 and 6.4]. ■

The following proposition allows us to define the complementary form of a Pfister neighbor in characteristic 2.

Proposition 18. *Let $\xi \in W_q(F)$ and $c_1, \cdots, c_s \in F$ such that*

$$\varphi := \xi \perp [c_1] \perp \cdots \perp [c_s]$$

is a Pfister neighbor. Let $\delta \in W_q(F)$ and $d_1, \cdots, d_s \in F$ such that

$$\xi \perp [c_1, d_1] \perp \cdots \perp [c_s, d_s] \perp \delta$$

is similar to a Pfister form.
(1) If $F(\varphi)$ is well defined, then $\varphi_{F(\varphi)} \sim ([c_1] \perp \cdots \perp [c_s] \perp \delta)_{F(\varphi)}$.
(2) The quadratic form $[c_1] \perp \cdots \perp [c_s] \perp \delta$ is unique up to isometry.

Proof. Assume that $1 \in D_F(\varphi)$.
(1) The form $\pi := \xi \perp [c_1, d_1] \perp \cdots \perp [c_s, d_s] \perp \delta$ is a Pfister form. Since $\pi_{F(\varphi)}$ is isotropic, it follows that

$$\xi_{F(\varphi)} \sim ([c_1, d_1] \perp \cdots \perp [c_s, d_s] \perp \delta)_{F(\varphi)}.$$

In particular,

$$\varphi_{F(\varphi)} \sim ([c_1] \perp \cdots \perp [c_s] \perp [c_1, d_1] \perp \cdots \perp [c_s, d_s] \perp \delta)_{F(\varphi)}.$$

By Lemma 9 we have $[c_i, d_i] \perp [c_i] \sim [c_i]$. Hence,

$$\varphi_{F(\varphi)} \sim ([c_1] \perp \cdots \perp [c_s] \perp \delta)_{F(\varphi)}.$$

(2) Let $d'_1, \cdots, d'_s \in F$ and $\delta' \in W_q(F)$ such that $\pi' := \xi \perp [c_1, d'_1] \perp \cdots [c_s, d'_s] \perp \delta'$ is another Pfister form. However, φ is a Pfister neighbor of π and π'. By [11, Proposition 3.1] $\pi \cong \pi'$. In particular,

$$\pi \perp [c_1] \perp \cdots \perp [c_s] \cong \pi' \perp [c_1] \perp \cdots \perp [c_s] .$$

By Lemma 9 and the Witt cancellation (Proposition 8) we get the desired conclusion ■

Definition 19. *With the same notations and hypothesis as in Proposition 18, the quadratic form* $[c_1] \perp \cdots \perp [c_s] \perp \delta$ *is called the partial complementary form of* φ.

Obviously if φ is a Pfister neighbor of type (r, s), then the partial complementary form φ' of φ is of type (r', s) for some integer r'. In characteristic not 2, φ' is known as the complementary form of φ.

In characteristic 2, we extend the notion of excellent form as follows.

Definition 20. *Any form of dimension* ≤ 1 *is called excellent. A quadratic form of dimension* ≥ 2 *is called excellent if it is a Pfister neighbor and its partial complementary form is excellent.*

In collaboration with Mammone [12] we extended in characteristic 2 a theorem of Hoffmann on the isotropy of quadratic forms [5]. Here is our result.

Theorem 21. *(*[12]*) Let* F *be a field of characteristic* 2*, and let* φ, ψ *be anisotropic quadratic forms over* F*. Assume that:*
(1) If φ *is nonsingular, then* $\dim\varphi \leq 2^n < \dim\psi$ *for some integer* $n \geq 1$.
(2) If φ *is singular of type* (r, s)*, then* $2r + 2s \leq 2^n < \dim\psi$ *for some* $n \geq 1$.
Then, φ *remains anisotropic over* $F(\psi)$.

As a corollary of Theorem 21, we have:

Corollary 22. *Let* φ *be an anisotropic Pfister neighbor of* $\pi \in P_nF$ *and* ψ *be its partial complementary form. Then, we have* $(\varphi_{F(\varphi)})_{an} \cong \psi_{F(\varphi)}$ *in the following cases:*
(1) φ *is nonsingular.*
(2) φ *is singular of type* (r, s) *such that the type* (r', s) *of* ψ *satisfies* $2r' + 2s \leq 2^n$.
(3) ψ *is a Pfister neighbor.*

Proof. Let (r, s) be the type of φ and (r', s) be the type of ψ. We have $\dim\psi = 2r' + s$ and $\varphi_{F(\varphi)} \sim \psi_{F(\varphi)}$ (Proposition 18). Since $\dim\varphi > 2^n$, it follows that $\dim\psi < 2^n$. The form $\varphi_{F(\pi)}$ is isotropic. Hence, $F(\pi)(\varphi)/F(\pi)$ is a purely transcendental extension [11, Corollaire 3.4].
• Assume that φ satisfies the case (1) or (2). If $\psi_{F(\varphi)}$ is isotropic, then $\psi_{F(\pi)}$ is isotropic, a contradiction with Theorem 21. Hence $\psi_{F(\varphi)}$ is anisotropic.
• Assume that ψ is a Pfister neighbor of a k-fold Pfister form ρ. Since $\dim\psi > 2^k$, we obtain $k < n$, that means, $\dim\rho < \dim\pi$. If $\psi_{F(\varphi)}$ is isotropic, then $\psi_{F(\pi)}$ is isotropic. Consequently, $\rho_{F(\pi)}$ is isotropic and thus hyperbolic, a contradiction with Proposition 5. ■

Definition 23. *For a field extension* K/F*, we say that a quadratic form* φ *over* K *is defined over* F *if there exists a quadratic form* ψ *over* F *such that* $\varphi \cong \psi_K$.

For an excellent quadratic form, we prove a general result which generalizes a known result of Knebusch on such a form in characteristic not 2, and we prove that an anisotropic singular excellent form has a particular type.

Proposition 24. *Let φ be an anisotropic excellent form of dimension ≥ 3, and let $(F_i, \varphi_i)_{0\leq i\leq \mathrm{h}(\varphi)}$ be its generic splitting tower.*
(1) If φ is singular, then φ is of type $(r, 1)$ for some integer $r > 0$. In particular, all quadratic forms φ_i are of type $(r_i, 1)$.
(2) All quadratic forms φ_i are defined over F.

Proof. (1) Let ψ be the partial complementary form of φ. Since φ is a Pfister neighbor, it follows from [11, Proposition 3.1] that φ is not totally singular. We will proceed by induction on $\mathrm{h}(\varphi)$. If $\mathrm{h}(\varphi) = 1$, then the result is an immediate consequence of Theorem 2. Now assume that the result is true for any anisotropic excellent singular form of height $< \mathrm{h}(\varphi)$. By Corollary 22 (3) $(\varphi_{F(\varphi)})_{an} \cong \psi_{F(\varphi)}$. Hence, $\psi_{F(\varphi)}$ is excellent of height $\mathrm{h}(\varphi) - 1$. Since $\psi_{F(\varphi)}$ is singular, we get by the induction hypothesis that $\psi_{F(\varphi)}$ is of type $(r', 1)$ for some integer r'. If $r' = 0$, then $\mathrm{h}(\varphi) = 1$ and the result is true by Theorem 2. If $r' > 0$, then by Theorem 15 (5) φ is of type $(r, 1)$ for some integer $r > 0$.
(2) A consequence of Proposition 18 and Corollary 22 (3). ∎

Now we give some partial results on quadratic form of height 2 and degree 1 or 2.

Theorem 25. *Let F be a field of characteristic 2, and let φ be an anisotropic nonsingular quadratic form over F.*
(1) φ is of height 2 and degree 1 if and only if φ is one of the following types:
• φ is excellent of codimension 2;
• φ is of dimension 4 with non trivial Arf invariant.
(2) φ is excellent of height and degree 2 if and only if $\varphi_{F(\varphi)}$ is not hyperbolic and $(\varphi_{F(\varphi)})_{an} \in GP_1F(\varphi)$ is defined over F.
(3) φ is of height and degree 2 whose leading form is not defined over F if and only if φ is an Albert form, i.e. a nonsingular quadratic form of dimension 6 with trivial Arf invariant.

The characterization of anisotropic quadratic forms of height 2 and degree 1 or 2 is complete in characteristic not 2. Here, we can't settle the case when φ is of height and degree 2 such that $(\varphi_{F(\varphi)})_{an}$ is not defined over F and the leading form of φ is defined over F.

Proposition 26. *Let F be a field of characteristic 2, and let φ be an anisotropic nonsingular quadratic form of dimension 4 with non trivial Arf invariant, or an Albert form. We have the following statements:*
(1) If $\dim \varphi = 4$, then φ is of height 2, degree 1 and $(\varphi_{F(\varphi)})_{an}$ is not defined over F.
(2) If φ is an Albert form, then φ is of height and degree 2 with leading form not defined over F. In particular, $(\varphi_{F(\varphi)})_{an}$ is not defined over F.

Proof. (1) Since $\dim \varphi = 4$ and $\triangle(\varphi) \neq 0$, we have $\varphi \notin GP_1F$ and $\dim(\varphi_{F(\varphi)})_{an} = 2$. Hence $\mathrm{h}(\varphi) = 2$ and $\deg(\varphi) = 1$. Assume that $(\varphi_{F(\varphi)})_{an}$ is defined over F and let $\alpha \in F^*$ such that $\varphi_{F(\varphi)} \sim (\alpha\,[1, \triangle(\varphi)])_{F(\varphi)}$. Hence, $\mu := \varphi \perp \alpha\,[1, \triangle(\varphi)]$ is hyperbolic over $F(\varphi)$. Let ρ be a 3-dimensional quadratic form

dominated by φ. Clearly, ρ is a Pfister neighbor of some 1-fold Pfister form $\tau \in P_1F$. The forms $\varphi_{F(\rho)}$ and $\rho_{F(\tau)}$ are isotropic, thus the extensions $F(\rho)(\varphi)/F(\rho)$ and $F(\tau)(\rho)/F(\tau)$ are purely transcendentals [11, Corollaire 3.4]. Consequently, $\mu_{F(\tau)} \sim 0$. The form μ is not hyperbolic because φ is anisotropic. By Corollary 6 we deduce that μ is isotropic and μ_{an} is similar to τ. Therefore, $\tau_{F(\varphi)} \sim 0$. By Proposition 5 φ is similar to τ, a contradiction.
(2) Assume that φ is an Albert form. By Theorem 2, we have $\mathrm{h}(\varphi) \geq 2$. Since $\triangle(\varphi) = 0$, we conclude that $(\varphi_{F(\varphi)})_{an} \in GP_1F(\varphi)$. Hence, $\mathrm{h}(\varphi) = 2$. Let $\tau \in P_1F(\varphi)$ be the leading form of φ. Assume that τ is defined over F. Consequently, $\varphi_{F(\tau)(\varphi)} \sim 0$. If $\varphi_{F(\tau)}$ is anisotropic, then $\varphi_{F(\tau)}$ is of height 1, a contradiction with Theorem 2. Hence $\varphi_{F(\tau)}$ is isotropic and thus $F(\tau)(\varphi)/F(\tau)$ is a purely transcendental extension [11, Corollaire 3.4]. We conclude that $\varphi_{F(\tau)} \sim 0$ and by Corollary 6 φ is isotropic, a contradiction. ∎

Proposition 27. *Let F be a field of characteristic 2, and let $\varphi, \eta \in W_q(F)$ be anisotropic forms such that $(\varphi_{F(\varphi)})_{an} \cong \eta_{F(\varphi)}$ and $\dim \eta = 2$ or $\eta \in GP_1F$. Then, φ is an excellent form of partial complementary form η.*

Proof. Set $\triangle = \triangle(\varphi)$ and $\mu = \varphi \perp \eta$. We have $2\dim\varphi > \dim\mu$. The form μ is not hyperbolic because φ is anisotropic, and we have $\mu_{F(\varphi)} \sim 0$. We have to prove that μ is similar to an anisotropic Pfister form.

(1) If $\dim\eta = 2$: In this case $\triangle \neq 0$. Let $\alpha \in F^*$ such that $\eta = \alpha\,[1, \triangle]$.
(1.1) Assume that μ is isotropic. Hence,

$$\varphi \cong e\,[1, f] \perp \varphi'$$

and

$$\alpha\,[1, \triangle] \cong e\,[1, \triangle]$$

for some quadratic form φ' and $e, f \in F^*$. It follows that

$$(\varphi' \perp e\,[1, f + \triangle])_{F(\varphi)} \sim 0 \tag{3}$$

The forms $\varphi' \perp e\,[1, f + \triangle]$ and φ have the same dimension.
• If $\varphi' \perp e\,[1, f + \triangle]$ is isotropic, it follows from Proposition 5 and equation (3) that $\varphi' \perp e\,[1, f + \triangle] \sim 0$. Since φ' is anisotropic, we deduce $\varphi' \cong e\,[1, f + \triangle]$. Hence φ is isotropic, a contradiction.
• If $\varphi' \perp e\,[1, f + \triangle]$ is anisotropic, then Proposition 5 implies that φ and $\varphi' \perp e\,[1, f + \triangle]$ are similar. In particular, $\varphi_{F(\varphi)} \sim 0$, a contradiction.
Hence, μ is anisotropic.
(1.2) Assume that $\mu_{F(\mu)}$ is not hyperbolic and let $\mu_1 = (\mu_{F(\mu)})_{an}$. As in case (1.1) we have $\varphi_{F(\mu)} \cong \beta\mu_1$ for some $\beta \in F(\mu)^*$, and thus $\dim\varphi < \dim\mu$ and $\varphi_{F(\varphi)(\mu)} \sim 0$. Therefore by Proposition 5 $\varphi_{F(\varphi)} \sim 0$, a contradiction.
Hence, $\mu_{F(\mu)}$ is hyperbolic, and thus μ is similar to a Pfister form.

(2) If $\dim\eta = 4$: We may assume $\eta \in P_1F$. Set $\eta = [1, y] \perp z\,[1, y]$.
(2.1) If μ is isotropic, then

$$\varphi \cong r\,[1, x] \perp \varphi''$$

and

$$\eta \cong r\,[1,y] \perp rz\,[1,y]\,,$$

for some quadratic form φ'' and $r \in F^*$. In particular,

$$(\varphi'' \perp r\,[1,x+y] \perp rz\,[1,y])_{F(\varphi)} \sim 0.$$

Since φ is anisotropic, it follows that $\varphi'' \perp r\,[1,x+y] \perp rz\,[1,y]$ is not hyperbolic.

(i) If $\varphi'' \perp r\,[1,x+y] \perp rz\,[1,y]$ is anisotropic, then Proposition 5 implies that

$$\varphi'' \perp r\,[1,x+y] \perp rz\,[1,y] \cong u\varphi \perp \xi$$

for some 2-dimensional form ξ and $u \in F^*$. By comparing Arf invariant we deduce that $\xi \cong \mathbb{H}$, a contradiction.

(ii) If $\varphi'' \perp r\,[1,x+y] \perp rz\,[1,y]$ is isotropic, then $\dim(\varphi'' \perp r\,[1,x+y] \perp rz\,[1,y])_{an} \leq \dim\varphi$. By Proposition 5 we get that φ is similar to $(\varphi'' \perp r\,[1,x+y] \perp rz\,[1,y])_{an}$. In particular, $\varphi_{F(\varphi)} \sim 0$, a contradiction.

Hence, μ is anisotropic.

(2.2) If $\varphi_{F(\mu)}$ is isotropic, then $F(\mu)(\varphi)/F(\mu)$ is a purely transcendental extension [11, Corollaire 3.4]. Hence, $\mu_{F(\mu)} \sim 0$ and thus μ is similar to a Pfister form.

(2.3) If $\mu_{F(\mu)}$ is not hyperbolic, then by case (2.2) $\varphi_{F(\mu)}$ is anisotropic. Let $\mu_1 = (\mu_{F(\mu)})_{an}$. We have $(\mu_1)_{F(\mu)(\varphi)} \sim 0$ and $\dim\mu_1 \leq \dim\varphi + 2$. By the same arguments as in cases (i) and (ii) we obtain that $\varphi_{F(\mu)}$ is similar to μ_1. In particular, $\dim\varphi < \dim\mu$ and $\varphi_{F(\varphi)(\mu)} \sim 0$. It follows from Proposition 5 that $\varphi_{F(\varphi)} \sim 0$, a contradiction.

Hence, $\mu_{F(\mu)}$ is hyperbolic and thus μ is similar a Pfister form. ∎

Proof of Theorem 25. Let φ be an anisotropic nonsingular quadratic form. Set $\triangle = \triangle(\varphi)$.

(1) If φ is excellent of codimension 2, it follows from Corollary 22 (3) that $\mathrm{h}(\varphi) = 2$ and degree 1. By Proposition 26, an anisotropic quadratic form of dimension 4 with non trivial Arf invariant is of height 2 and degree 1. For the converse, assume that φ is of height 2 and degree 1. In particular, $\dim\varphi \geq 4$ and $\dim(\varphi_{F(\varphi)})_{an} = 2$. Let $\alpha \in F(\varphi)^*$ such that $\varphi_{F(\varphi)} \sim \alpha\,[1,\triangle]$.

(1.1) If $\alpha\,[1,\triangle]$ is defined over F, then we get by Proposition 27 that φ is excellent of codimension 2.

(1.2) If $\alpha\,[1,\triangle]$ is not defined over F. We have

$$C(\varphi)_{F(\varphi)} = [\triangle,\alpha) \in \mathrm{Br}(F(\varphi)).$$

Assume that $\dim\varphi \geq 6$. By Proposition 26 (2) φ is not an Albert form. However, by the reduction index theorem [13, Theorem 4] the Schur index of the algebra $C(\varphi)$ is less or equal to 2. Let $\tau = [1,r] \perp s\,[1,r] \in P_1F$ such that $C(\varphi) = C(\tau) \in \mathrm{Br}(F)$. Clearly

$$([1,r] \perp s\,[1,r])_{F(\varphi)} \cong [1,\triangle] \perp \alpha\,[1,\triangle]\,.$$

Thus $([1, r+\triangle] \perp s\,[1,r])_{F(\varphi)} \sim \alpha\,[1,\triangle]$. By [11, Théorème 1.3] the form $[1, r+\triangle] \perp s\,[1,r]$ is isotropic. Consequently, $\alpha\,[1,\triangle]$ is defined over F, a contradiction. Hence, $\dim\varphi = 4$.

(2) A consequence of Proposition 27 and Corollary 22 (3).

(3) The sufficient condition is a consequence of Proposition 26 (2). For the necessary condition, we follow the proof of Kahn in characteristic not 2 [6] by using the reduction index theorem in characteristic 2 [13]. ∎

Let us finish this note with some questions. The first one is related to the case that we don't settle in Theorem 25.

Question 28. *Let φ be an anisotropic quadratic form of height and degree 2 and let $\tau \in P_1F$ anisotropic such that $(\varphi_{F(\varphi)})_{an}$ is similar to τ but not defined over F. Is φ of dimension 8?*

It is well known that Question 28 has a positive answer in characteristic not 2 [4, 1.6].

Question 29. *Let φ be an anisotropic form, and let $(F_i, \varphi_i)_{0\le i\le \mathrm{h}(\varphi)}$ be its generic splitting tower. Assume that φ is of type (r,ϵ) such that $\epsilon \le 1$ and all forms φ_i are defined over F. Is φ an excellent form?*

More precisely we ask the following question.

Question 30. *Let φ be an anisotropic form. Assume that φ is of type (r,ϵ) such that $\epsilon \le 1$ and $(\varphi_{F(\varphi)})_{an}$ is defined over F. Is φ a Pfister neighbor?*

Proposition 27 gives a positive answer to Question 29 when $\dim(\varphi_{F(\varphi)})_{an} = 2$ or $(\varphi_{F(\varphi)})_{an} \in GP_1F(\varphi)$.

To formulate Question 30 when φ is a singular form with radical of dimension ≥ 2, it is necessary to add other hypothesis on the type of φ. In fact, Proposition 31 gives an example of an anisotropic singular quadratic form φ such that $(\varphi_{F(\varphi)})_{an}$ is defined over F but φ is not a Pfister neighbor.

Proposition 31. *Let F be a field of characteristic 2, n be an integer such that $2+2n$ is not a power of 2, and $X, Y, Z_1, \cdots, Z_n$ some variables over F. Let $\varphi = [X,Y] \perp [Z_1] \perp \cdots \perp [Z_n]$ and $K = F(X,Y,Z_1,\cdots,Z_n)$. Then, $(\varphi_{K(\varphi)})_{an}$ is defined over K but φ is not a Pfister neighbor. More precisely, $(\varphi_{K(\varphi)})_{an} = ([Z_1] \perp \cdots \perp [Z_n])_{K(\varphi)}$.*

Proof. Let $\eta = [Z_1] \perp \cdots \perp [Z_n]$. Assume that φ is a Pfister neighbor over K. A simple computation implies that $2+2n$ is a power of 2, a contradiction. It follows from Proposition 13 that $\eta_{K(\varphi)}$ is anisotropic and thus $\varphi_{K(\varphi)} \cong \mathbb{H} \perp \eta_{K(\varphi)}$, as desired. ∎

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