ON MULTICOMMUTATORS FOR SIMPLE ALGEBRAIC GROUPS

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Abstract. There are several examples of groups for which any pair of commutators can be written such that both of them have a common entry, and one can look for a similar property for n-tuples of commutators.
We here answer, for simple algebraic groups over any field, the weaker question, under which condition the set of n-tuples of commutators with one common entry is Zariski dense in the set of all n-tuples of commutators. Surprisingly, there is a uniform bound on n in terms of the so called Coxeter number of G in order to answer the question positively.
An analogoue result is proved for Lie algebras of simple and simply conncected algebraic groups.

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1. Introduction

Let G be an abstract group and let $G^n = G \times \cdots \times G$ (n factors). Further, let

$$\phi_n : G \times G^n \longrightarrow G^n$$

be the map given by the formula

$$\phi_n((g, g_1, \ldots, g_n)) = ([g, g_1], \ldots, [g, g_n]).$$

We say that the group G satisfies the property $\mathcal{C}_n$, if $\operatorname{Im} \phi_n = [G, G]^n$ where $[G, G]$ is the commutator subgroup of G, i.e., for every $\sigma_1, \ldots, \sigma_n \in [G, G]^n$, there exists a sequence $g, g_1, \ldots, g_n$ such that

$$\sigma_1 = [g, g_1], \;\; \sigma_2 = [g, g_2], \ldots, \sigma_n = [g, g_n].$$

(We define $[g, h] = ghg^{-1}h^{-1}$.)

The case $n = 1$ is well known in group theory and has a long history. If G is a finite simple group, the question about the property $\mathcal{C}_1$ is the well-known Ore problem, whether any element in the commutator subgroup of G is a single commutator, which is answered

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positively for (simple) alternating groups (Ore [O], 1951), for sporadic groups (Neubüser, Pahlings and Cleuvers [NPC], 1984) and for finite simple Lie groups over fields with at least 9 elements (see [EG]). Is is also known that in general the answer is negative, even for groups like $\mathrm{SL}_n(K)$ (cf. [THO, Th. 1]), or for finite groups G including perfect finite groups (cf. [I]).

The question for $n > 1$ was posed several years ago by R. Keith Dennis (cf. [C, prob. 14, p. 605]). In [AD], it was shown, among other things, that the Schur multiplier of the group $\mathrm{SL}_1(\mathbb{H})$ of the Hamilton quaternions $\mathbb{H}$ is generated by the image of the Schur multiplier of the commutative subgroup of complex numbers of norm one, a result, which allowed to determine the group $K_2(\mathbb{H})$. In the proof, the property $\mathcal{C}_2$ for the group $\mathbb{H}^*$ played an important role, and it even turned out that this group as well as the multiplicative group of an arbitrary quaternion skew field satisfies $\mathcal{C}_3$. The main idea for this is already contained in [RS, §4, proof of 4.1].

Many years ago, R. Keith Dennis and the second named author verified, by computer and using a special purpose program (written in the programming language C), that the alternating groups A_n with $n = 5, 6, 7, 8, 9, 10$ satisfy $\mathcal{C}_2$. It was also shown that A_5 does not satisfy $\mathcal{C}_3$.

The first named author verified $\mathcal{C}_2$ for the group $\mathrm{SL}_2(\mathbb{C})$ by an explicit computation using the Bruhat decomposition of that group.

Here we look at the property $\mathcal{C}_n$ for a simple *algebraic* group G defined over a field K. It seems to be rather difficult to investigate the property $\mathcal{C}_n$ for the group $G(K)$ of K-rational points even if K is algebraically closed. If K is not algebraically closed, then already the property $\mathcal{C}_1$ is a problem for $G(K)$: In [THO], examples of elements in $\mathrm{SL}_n(\mathbb{R})$ are given which are in the commutator subgroup but no single commutators, and in [KU] there is an example of this type in some non split group. (See also [EG].) We restrict ourselves to a weaker condition. Namely, we say that a simple algebraic group G satisfies the property $\overline{\mathcal{C}}_n$ if the map ϕ_n is dominant, i.e., if the Zariski closure $\overline{\operatorname{Im}\phi_n}$ of $\operatorname{Im}\phi_n$ in G^n coincides with G^n. Thus, the property $\overline{\mathcal{C}}_n$ is the "property $\mathcal{C}_n$ up to Zariski closure".

The main result of this paper is the following theorem.

Theorem 1. *Let G be a simple algebraic group and let $h = h(G)$ be the Coxeter number of the corresponding root system. Then G satisfies the property $\overline{\mathcal{C}}_n$ if and only if $n \leq h+1$.*

Remark 1. *Recall that the Coxeter number h of a root system is the order of special elements of its Weyl group (Coxeter elements) (see [B]). If the root system belongs to a* *simple algebraic group G, we have*

$$h+1 = \frac{\dim G}{\dim T}$$

where T is a maximal torus of G. Thus, G satisfies the property $\overline{\mathcal{C}}_n$ if and only if $n \leq \dim G/\dim T$.

Remark 2 *At the end of this paper we show that ϕ_n is a separable morphism if $n \leq h$ and that ϕ_n is not separable if $n = h+1$ and if the center of the corresponding Lie algebra is not trivial.*

On the basis of Theorem 1 we could propose a conjecture about the property $\mathcal{C}_k$ for groups of points $G(K)$. Say, we may suppose that $\mathcal{C}_1$ implies $\mathcal{C}_{h+1}$ for such groups under the assumption that G is an adjoint group. Since the property $\mathcal{C}_1$ is satisfied by a big massive of quasisplit groups of adjoint type (see [EG]) we may suppose that the property $\mathcal{C}_{h+1}$ is satisfied by such groups (possibly except in the case when K is a small field). At any rate, there is a strong hope that all groups $G(K)$ satisfy the property $\mathcal{C}_{h+1}$, where G is a group of adjoint type and K is algebraically closed field.

Our example below after Remark 5 shows that $\mathcal{C}_3$ is satisfied by the groups GL_1 and PGL_1 over any quaternion skew field: Here, the latter group is an anisotropic adjoint simple algebraic group, and we have $\dim \mathrm{PGL}_1 = 3$ and $\dim T = 1$ for any maximal torus T of PGL_1.

An analogue of Theorem 1 for Lie algebras is the following result.

Theorem 2. *Let L be the Lie algebra of a simple and simply connected algebraic group defined over a field K and corresponding to an irreducible root system R. Let $L^n = L \oplus \cdots \oplus L$ (n summands) and let*

$$\Psi_n : L \oplus L^n \to L^n$$

be the map given by the formula

$$\Psi_n((\ell, \ell_1, \dots, \ell_n)) = ([\ell, \ell_1], \dots, [\ell, \ell_n]).$$

Let $R \neq C_r, r \geq 1$, or char *$K \neq 2$. Then the map Ψ_n is dominant if and only if $n \leq h$. (Note that $A_1 = C_1, B_2 = C_2$.)*

If $R = C_r, r \geq 1$ and $\mathrm{char}\, K = 2$, *then the map Ψ_n is a dominant map onto $[L, L]^n \underset{\neq}{\subset} L^n$ for every n.*

Remark 3. *Theorem 2 can be considered as an analogue of the property $\overline{\mathcal{C}}_n$ for simple algebraic groups. It is interesting that the bound for dominance in the case of Lie algebras is equal to h, while in the case of groups it is $h+1$.*

For a simple (abstract) group G, the property $\mathcal{C}_1$ follows from the stronger condition that there exists a conjugacy class $C \subset G$ such that $C^2 = \{c_1c_2 \mid c_1, c_2 \in C\} = G$. Indeed, if $g \in C$ then g is a real element, i.e., g is conjugate to g^{-1} because $1 \in C^2$. Hence every element of G can be written as $gxg^{-1}x^{-1}$ for some $g \in C$ and therefore G has property $\mathcal{C}_1$. The conjecture about the existence of such a conjugacy class C is known as Thompson's conjecture (see [AH]). We can generalize this question in the following way. We say that the group G satisfies the property $\mathcal{T}_n$, if there exists an element $g \in G$ such that, for every sequence $\sigma_1, \dots, \sigma_n \in [G,G]$, there exist elements $x, y_1, \dots, y_n \in G$ with

$$\sigma_1 = xgx^{-1}y_1gy_1^{-1}, \cdots, \sigma_n = xgx^{-1}y_ngy_n^{-1}.$$

We can rewrite this condition in the following way. Let $g \in G$ and let

$$f_{n,g} : G \times G^n \longrightarrow G^n$$

be the map given by the formula

$$f_{n,g}((x, y_1, \dots, y_n)) = (xgx^{-1}y_1gy_1^{-1}, \cdots, xgx^{-1}y_ngy_n^{-1}).$$

Then the group G satisfies the property $\mathcal{T}_n$ if $\operatorname{Im} f_{n,g} = [G,G]^n$ for some $g \in G$. Obviously, $\mathcal{T}_n$ implies $\mathcal{C}_n$.

Now let G be a simple algebraic group defined over a field K and let $g \in G(K)$. Then the map $f_{n,g}$ is a morphism of K-varieties $G \times G^n$ and G^n. We can define a property $\overline{\mathcal{T}}_n$ in the same way as the property $\overline{\mathcal{C}}_n$. Namely, we say that G satisfies the property $\overline{\mathcal{T}}_n$ if $f_{n,g}$ is a dominant map for some $g \in G(K)$. Here we prove that a simple algebraic group G satisfies the property $\overline{\mathcal{T}}_n$ if and only if $n \le h$ where h is the Coxeter number.

Actually, we prove a more general result. Let $\tilde{g} = (g, g_1, \dots, g_n) \in G^{n+1}(K)$ and let

$$f_{n,\tilde{g}} : G^{n+1} \to G^n$$

be the map defined by the formula

$$f_{n,\tilde{g}}((x, y_1, \dots, y_n)) = (xgx^{-1}y_1g_1y_1^{-1}, \dots, xgx^{-1}y_ng_ny_n^{-1}).$$

Theorem 3. *Let G be a simple algebraic group defined over a field K and let $g, g_1, \dots, g_n \in G(K)$ be a sequence of semisimple regular elements. Then the map $f_{n,\tilde{g}}$ is dominant if and only if $n \le h$ where h is the Coxeter number of G.*

Remark 4. *This theorem shows the bound for the property $\overline{\mathcal{T}}_n$. This bound is the Coxeter number h and it is the same as the corresponding bound in Theorem 2 for Lie algebras but it is smaller than the bound $h+1$ for the property $\overline{\mathcal{C}}_n$ in Theorem 1.*

Remark 5. *The property $\overline{\mathcal{T}}_n$ does not imply the property $\overline{\mathcal{C}}_n$, because an element g can be non-real. However, in Theorem 3 we can take a real element g or $g_1 = g_2 = \cdots = g_n = g^{-1}$.*

An Example. *The groups $\mathrm{GL}_{1,D}(K)$ and $\mathrm{PGL}_{1,D}(K)$ associated with a quaternion skew field D over a field K satisfy $\mathcal{C}_3$.*

This fact was observed, for $\mathrm{GL}_{1,D}$, by R. Alperin and R. K. Dennis as well as by the second named author many years ago, some ideas concerning the proof can be found for the case of the real Hamilton quaternions in [AD] and for general fields K in [RS].

Proof. Let D be some quaternion skew field over some field K of any characteristic. The case char $K \neq 2$ is very well known, the case char $K = 2$ is classical as well. For a uniform discussion, we refer to [KMRT, chap. I, §2, 2.C, p. 25 ff.] or [KR, p. 52].

We fix some notations:

Let $x \mapsto \bar{x}$ $(x \in D)$ denote the canonical involution of D. The reduced trace $\mathrm{T} : D \to K$ is a K-linear map and obtained by $\mathrm{T}(x) = x + \bar{x}$. The reduced norm $\mathrm{N} : D \to K$ is a quadratic form on the 4-dimensional K-vector space D and given by $\mathrm{N}(x) = x\bar{x}$. Its associated bilinear form is given by $(x, y) := \mathrm{N}(x+y) - \mathrm{N}(x) - \mathrm{N}(y) = \mathrm{T}(x\bar{y})$. From the explicit formulae in [KMRT, l.c.] it is easily checked that this bilinear form is non-degenerate.

Hence, for any three elements $x_i \in [D^*, D^*],\ i = 1, 2, 3$, the orthogonal complements of the three subspaces $K(1 - \bar{x}_i)$ of D are of dimension 3 and therefore have a non-trivial intersection. If w is a non-zero element of this intersection, this means that $0 = (w, 1 - \bar{x}_i) = \mathrm{T}(w(1 - x_i))$, hence $\mathrm{T}(w) = \mathrm{T}(wx_i)$. But we also have $\mathrm{N}(w) = \mathrm{N}(wx_i)$, since $\mathrm{N}(x_i) = 1$.

It follows that the minimal polynomials of w and of all three elements wx_i coincide, thus, the K-subalgebras $K(w),\ K(wx_i)$ of D are pairwise K-isomorphic, and the theorem of Skolem-Noether [VDW, p. 105] yields the existence of $x_i' \in D^*$ such that $x_i' w {x_i'}^{-1} = wx_i$, and hence $x_i = [w^{-1}, x_i']$ for $i = 1, 2, 3$. This proves the statement for $\mathrm{GL}_{1,D}$.

To handle the case of $\mathrm{PGL}_{1,D}$, we observe that we have an exact sequence of linear algebraic K-groups

$$0 \to \mathbb{G}_m \to \mathrm{GL}_{1,D} \to \mathrm{PGL}_{1,D} \to 0,$$

where $\mathbb{G}_m$ is the multiplicative group over K, which is mapped to the center of $\mathrm{GL}_{1,D}$. This yields, by [S, ch. I, prop. 43, p. I-71], a long exact Galois-cohomology sequence

$$0 \to \mathbb{G}_m(K) \to \mathrm{GL}_{1,D}(K) \to \mathrm{PGL}_{1,D}(K) \to \mathrm{H}^1(K, \mathbb{G}_m) \to \dots.$$

By [S, ch. II, prop. 1, p. II3], we have $\mathrm{H}^1(K, \mathbb{G}_m) = 0$ and hence the map $\mathrm{GL}_{1,D}(K) \to \mathrm{PGL}_{1,D}(K)$ is surjective. Given three commutators $[x_i, y_i], i = 1, 2, 3$ from $\mathrm{PGL}_{1,D}(K)$, we may lift their entries x_i, y_i to preimages $\tilde{x}_i, \tilde{y}_i \in \mathrm{GL}_{1,D}(K)$. By the preceding result we find $\tilde{w}, \tilde{g}_i \in \mathrm{GL}_{1,D}(K)$ such that $[\tilde{w}, \tilde{g}_i] = [\tilde{x}_i, \tilde{y}_i]$, and mapping this down to $\mathrm{PGL}_{1,D}(K)$ we obtain what we want. Let us remark that, by centrality, the commutators $[\tilde{x}_i, \tilde{y}_i]$ in $\mathrm{GL}_{1,D}(K)$ do not depend on the choice of the liftings, but we don't need that here. $\square$

An Application. If $\mathcal{C}_2$ is satisfied by some (abstract) group G, then there is an easy description of its Schur multiplier.

By a result of C. Miller [MI], the Schur multiplier $\mathrm{H}_2(G, \mathbb{Z})$ of G can be described as follows. Let $U(G)$ denote the group generated by symbols $c(x, y),\ x, y \in G$ subject to just the "formal commutator relations" which are generated by

- $c(x, x) = 1$, $c(x, y)c(y, x) = 1$,
- $c({}^xy, {}^xz)c(x, z) = c(xy, z)$,
- $c({}^xy, {}^xz)c(z, x) = c(x, [y, z])$.

Here ${}^xy = xyx^{-1}$. Then the map induced by $c(x, y) \mapsto [x, y]$ is a central extension of $[G, G]$, and its kernel is canonically isomorphic to $\mathrm{H}_2(G, \mathbb{Z})$, so that we have an exact sequence

$$0 \to \mathrm{H}_2(G, \mathbb{Z}) \to U(G) \to G \to \mathrm{H}_1(G, \mathbb{Z}) \to 0.$$

It is known that, in case G is perfect, $U(G)$ is the universal central extension of $G = [G, G]$. The fact we want to mention here is the following:

If G satisfies $\mathcal{C}_2$, then the Schur multiplier is generated by all elements $c(x, y)c(x', y')$ such that $[x, y] = [y', x']$.

That is, the Schur multiplier is generated by relations induced from Abelian subgroups of G (these are the length 1 relators) and from relators of length 2. In many cases, for example for the group of invertible elements of quaternions [AD,RS], but also for $\mathrm{SL}_n(K)$ and other almost simple split linear groups, it is even true that the relators of length 1 are sufficient to generate the Schur multiplier, so the length 2 relations are not necessary. This follows directly from Matsumoto's theorem on the presentation of $K_2(K)$ by symbols [M]. Analogous results hold also for $\mathrm{SL}_n(D),\ n \geq 2$, for any skew field D over K [R1, R2], and for Kac-Moody groups [MR].

Proof of the fact mentioned above: Let $x_i, y_i,\ i = 1, \dots, n$ be elements in G such that $[x_1, y_1] \cdot \dots \cdot [x_n, y_n] = 1$. By $\mathcal{C}_2$, we may find elements $u, v, z \in G$ such that $[x_{n-1}, y_{n-1}] = [u, z]$ and $[x_n, y_n] = [z, v]$.

From the relations above we obtain $[u, z][z, v] = [uv^{-1}, vzv^{-1}]$.

That is, any element $c(x_1, y_1) \cdot \dots \cdot c(x_n, y_n) \in \mathrm{H}_2(G, \mathbb{Z})$ can be replaced, modulo products of length at most 2, in $\mathrm{H}_2(G, \mathbb{Z})$ by a product of length $n-1$. An induction now gives the result.

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2. Notation and Terminology

2.1. R denotes an irreducible root system generated by a simple root system $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$,

$W = W(R)$ is the Weyl group of R;

$w_\alpha \in W$ is the reflection corresponding to $\alpha \in R$;

$w_c = w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_n}$ is a fixed Coxeter element of W;

$\Gamma = \langle w_c \rangle$; $h = |\Gamma|$ is the Coxeter number of R.

2.2. G denotes a simple algebraic group defined over the field K corresponding to the system R;

$T \leq G$ is a maximal torus (also defined over K); we identify the set R with a subset of characters of T, and for our purposes we may assume that T and hence G is split over K, that is, all characters are defined over K.

N is the normalizer of T in G; thus $N/T \cong W$; by $\dot{w}$ we denote an element of N with the image $w \in W$;

B is a Borel subgroup of G (below we assume $T \leq B$);

ϕ_n, $f_{n,g}$, $f_{n.\tilde{g}}$ are the functions defined in the Introduction.

An element $g \in G$ is called regular if $\dim C_G(g) = \dim T$.

2.3. L denotes the Lie algebra of a simple and simply connected algebraic group G defined over a field K and corresponding to the root system R.

If K is an algebraically closed field then $L = H + U$ is a Cartan decomposition where H is a Cartan subalgebra and

$$U = \sum_{\alpha \in R} U_\alpha$$

where U_α is the one-dimensional subspace of L corresponding to a root $\alpha \in R$. Since G is simply connected one can chose a Chevalley basis $h_{\alpha_1}, \cdots, h_{\alpha_r}, \{u_\alpha\}$ of L (see [St1]) where $\{h_{\alpha_i}\}$ is a basis of H and $\{u_\alpha\}$ is a basis of U. Note that, for every $k = 1, \cdots, r$, we have $h_{\alpha_k} = [u_{\alpha_k}, u_{-\alpha_k}]$.([St1, Lemma 2]). Further, for every root $\alpha \in R$ one can define

$$h_\alpha = [u_\alpha, u_{-\alpha}].$$

The adjoint action of $G(K)$ on L will be denoted by $g(\ell)$ where $g \in G(K),\ \ell \in L$. We assume that $L^T = H$, where L^T denotes the invariant elements of L under the (adjoint) T-action.

A semisimple element $l \in L$ is called regular if $\dim C_L(l) = \dim C_L(H)$.

2.4. Let $f : X \to Y$ be a morphism of irreducible affine k-varieties. We say that f is dominant if the comorphism $f^* : K[Y] \to K[X]$ is an injection. Obviously, this is equivalent to the condition $\overline{f(X(\bar{K}))} = Y(\bar{K})$ where $\bar{K}$ is the algebraic closure of K and $\overline{f(X(\bar{K}))}$ is the Zariski closure of $f(X(\bar{K}))$ in $Y(\bar{K})$.

Thus we may assume in the proofs of theorems 1, 2, 3 that K is an algebraically closed field. Also, it is enough to prove theorems 1 and 3 for the cases where G is a simply connected group.

Below, we suppose that K is algebraically closed and G is simply connected. From the context, it will be clear, that some of the statements hold under weaker assumptions, e.g., in the case that G is split over K, or in the case that K is sufficiently large.

3. Some technical results

3.1. Let

$$R = S_1 \cup S_2 \cdots \cup S_r$$

be the decomposition of R into the union of Γ-orbits. Then $r = \operatorname{rank} R$ and there exists a sequence of representatives

$$\theta_1 \in S_1, \theta_2 \in S_2, \ldots, \theta_r \in S_r$$

which is a basis of the group $Q(R)$ ([B, IV, 21, Proposition 33, p. 170]). (Recall that $Q(R)$ is the lattice generated by the roots.)

The roots $\theta_1, \dots, \theta_r$ are defined in the following way

$$\theta_i = w_{\alpha_r} w_{\alpha_{r-1}} \cdots w_{\alpha_{i+1}}(\alpha_i) \tag{1}$$

(see [B, l.c.]), and we have

$$\theta_i > 0$$

for every $i = 1, \dots, r$. Moreover, in [St2, Lemma 7.2.c, p. 298] it is proved that $\theta_i + \theta_j \notin R$ for every i, j. The following lemma is proved by the same arguments (which was essentially don in [St2] as well):

Lemma 1. *Let $k_1, \dots, k_r$ be non-negative integers such that $k_1 + \dots + k_r > 1$. Then*

$$\theta = k_1\theta_1 + \dots + k_r\theta_r \notin R.$$

Proof. Let $i = \max\{m \,|\, k_m \neq 0\}$. From (1) we have

$$\theta' = w_{\alpha_i} w_{\alpha_{i+1}} \dots w_{\alpha_r}(\theta) = -k_i\alpha_i + k_{i-1}\alpha_{i-1} + \sum_{j<i-1} k_j w_{\alpha_{i-1}} w_{\alpha_{i-2}} \dots w_{\alpha_{j+1}}(\alpha_j)$$

Applying formulas (1) to the irreducible root system generated by roots $\{\alpha_1, \dots, \alpha_{i-1}\}$ we get

$$w_{\alpha_{i-1}} w_{\alpha_{i-2}} \dots w_{\alpha_{j+1}}(\alpha_j) = \sum_{s \le i-1} \ell_s \alpha_s$$

where $\ell_s \geq 0$ for every $s \leq i-1$ (indeed, all θ_i in (1) are positive). Now we have

$$\theta' = -k_i\alpha_i + \sum_{s \le i-1} m_s \alpha_s$$

where $m_s \geq 0$ for every $s \leq i-1$. Since $\{\alpha_1, \dots, \alpha_i\}$ is a simple root system the vector θ' belongs to R only if $m_1 = m_2 = \dots = m_{i-1} = 0$ and $k_i = 1$. But in this case $k_j = 0$ for every $j < i$ and, therefore, $k_1 + \dots + k_r = 1$. This contradicts to the condition of the lemma. $\square$

3.2.

Lemma 2. *$\{h_{\theta_1}, \dots, h_{\theta_r}\}$ is a basis of H.*

Proof. If $r = 1$ then $\theta_1 = \alpha_1$ and $\{h_{\alpha_1}\}$ is a basis of H. Suppose our assertion holds for root systems of rank $< r$. Let $\varepsilon_1 = w_{\alpha_r}(\theta_1), \dots, \varepsilon_{r-1} = w_{\alpha_r}(\theta_{r-1})$. It follows from (1) that $\varepsilon_1, \dots, \varepsilon_{r-1}$ belong to the root system generated by $\{\alpha_1, \dots, \alpha_{r-1}\}$. Moreover, the elements $\varepsilon_1, \dots, \varepsilon_{r-1}$ are defined in the same way as $\theta_1, \dots, \theta_r$ for the root system R. Therefore the assumption of the induction implies that $\{h_{\varepsilon_1}, \dots, h_{\varepsilon_{r-1}}\}$ is a basis of subspace $H' \subset H$ generated by $\{h_{\alpha_1}, \dots, h_{\alpha_{r-1}}\}$. Further, $\theta_r = \alpha_r$ and $\varepsilon_r = w_{\alpha_r}(\theta_r) = w_{\alpha_r}(\alpha_r) = -\alpha_r$. Hence $h_{\varepsilon_r} = -h_{\alpha_r}$. Thus, $\{h_{\varepsilon_1}, \dots, h_{\varepsilon_{r-1}}, h_{\varepsilon_r}\}$ is a basis of H. But $h_{\varepsilon_i} = w_{\alpha_r}(h_{\theta_i})$ and therefore $\{h_{\theta_1}, \dots, h_{\theta_r}\}$ is also a basis of H. □

3.3. For every reflection $w_\alpha \in W$ we can find a preimage $\dot{w}_\alpha \in N$ such that $\dot{w}_\alpha(u_\beta) = \pm u_{w_\alpha(\beta)}$ for every root $\beta \in R$ ([St1, Lemma 19, (a)]). Now we fix such preimages $\dot{w}_\alpha$ and put

$$\dot{w}_c = \dot{w}_{\alpha_1}\dot{w}_{\alpha_2}\cdots\dot{w}_{\alpha_r}.$$

Let $x_\alpha(s) = \exp(su_\alpha) \in G(K)$ be the corresponding root element where $s \in K$. Put

$$\gamma = x_{\theta_1}(1)x_{\theta_2}(1)\dots x_{\theta_r}(1), \quad \gamma_j = \dot{w}_c^{j-1}\gamma\dot{w}_c^{-j+1} \text{ for } i = 1, \dots, h \tag{2}$$

Further, put

$$\theta_{ij} = w_c^{j-1}\theta_i w_c^{-j+1} \text{ for } i = 1, \dots, h. \tag{3}$$

Lemma 3. *$\gamma_m(u_{-\theta_{ij}}) \equiv 0\,(mod U)$ if $m \neq j$ and $\gamma_j(u_{-\theta_{ij}}) \equiv \pm h_{\theta_{ij}}\,(mod U)$.*

Proof. Let $g \in G(K)$. From the definition of $\dot{w}_c$ we have

$$\dot{w}_c(g(u_\alpha)) = \dot{w}_c g \dot{w}_c^{-1}(\pm u_{w_c(\alpha)}) \tag{4}$$

for every $\alpha \in R$. Moreover, $\dot{w}_c(U) = U$ and $\dot{w}_c(0) = 0$. Acting on both sides of the congruences by an appropriate power of $\dot{w}_c$ and using (2), (3), (4) we can get the equivalent congruences

$$\begin{aligned} \gamma_1(u_{-\theta_{ij}}) &\equiv 0\,(\operatorname{mod} U) \text{ for } j \neq 1, \\ \gamma_1(u_{-\theta_{i1}}) &\equiv \pm h_{\theta_{i1}}\,(\operatorname{mod} U). \end{aligned} \tag{5}$$

Thus, it is enough to prove (5).

Let $\varepsilon, \delta \in R$. If $\varepsilon \neq -\delta$ then

$$x_\varepsilon(1)(u_\delta) = u_\delta + \sum_{\delta + i\varepsilon \in R} \ell_i u_{\delta + i\varepsilon} \tag{6}$$

where $\ell_i \in K$ and

$$x_\varepsilon(1)(u_{-\varepsilon}) = u_{-\varepsilon} \pm h_\varepsilon \mp u_\varepsilon. \tag{7}$$

Equations (6) and (7) follow from [St1, Lemma 72, p. 209].

Further, if $k_1, \dots, k_r$ are non-negative integers then for every $i = 1, \dots, r$ and for every $\beta \in R$ the equality

$$-\theta_i = -\beta + k_1\theta_1 + \cdots + k_r\theta_r$$

is possible only for $k_1 = k_2 = \cdots = k_r = 0$. This follows from Lemma 1. For $\beta \in R$ put

$$M_\beta = \{-\beta + k_1\theta_1 + \cdots + k_r\theta_r \in R \mid k_i \geq 0, k_1 + k_2 + \cdots + k_r \geq 1\}.$$

Thus, for every $i = 1, \dots, r$ the set M_β does not contain the root $-\theta_i$.

Let $\beta = \theta_{ij}$ where $j > 1$. Since the group $\Gamma = \langle w_c \rangle$ acts free on the Γ-orbits $S_1, \dots, S_r$ we have $\beta \neq \theta_1, \dots, \theta_r$. From the definition $\gamma_1 = \gamma$ and (6) we get

$$\gamma_1(u_{-\beta}) = u_{-\beta} + \sum_{\alpha \in M_\beta} \ell_\alpha u_\alpha \tag{8}$$

where $\ell_\alpha \in K$.

Let $\beta = \theta_{i1} = \theta_i$ and $M_\theta = \{\theta_i, i = 1, \dots, h\}$. From (6) and (7) we obtain

$$\gamma_1(u_{-\beta}) = u_{-\beta} \pm h_\beta + \sum_{\alpha \in M_\beta \cup M_\theta} \ell_\alpha u_\alpha \tag{9}$$

where $\ell_\alpha \in K$. Now (8) and (9) imply (5). □

3.4. We define the subspace $\tilde{U}$ of $L^h = \underbrace{L \oplus \cdots \oplus L}_{h-\text{times}}$ by

$$\tilde{U} := \{(\gamma_1(u) + u_1, \gamma_2(u) + u_2, \dots, \gamma_h(u) + u_h) \,|\, u, u_i \in U\}.$$

Lemma 4. $\tilde{U} = L^h$.

Proof. Obviously, $U^h = \underbrace{U \oplus \cdots \oplus U}_{h-\text{times}} \subset \tilde{U}$. Since $H^h + U^h = L^h$ we have to prove $H^h \subset \tilde{U}$.

Let $u = u_{-\theta_{ij}}$. By Lemma 3 we have $\gamma_m(u) \in U$ if $m \neq j$ and $\gamma_j(u) = \pm h_{\theta_{ij}} + u'$ for some $u' \in U$. Thus, for every i, j, the element

$$(0, 0, \dots, h_{\theta_{ij}}, 0, \dots, 0) \tag{10}$$

belongs to $\tilde{U}$. Since $h_{\theta_1}, \dots, h_{\theta_r}$ is a basis of H (Lemma 2), the sequence $h_{\theta_{1j}}, h_{\theta_{2j}} \dots, h_{\theta_{rj}}$ is also a basis of H and, therefore, the set of $r \times h$ elements of the form (10) is a basis of H^h. □

Now let us fix the sequence of elements $\delta_1 \in T\gamma_1 T, \dots, \delta_h \in T\gamma_h T$ and let us define

$$\tilde{U}' = \{(\delta_1 - 1)(l) + u_1, \dots, (\delta_h - 1)(l) + u_h) \;\;|\;\; l \in L, u_i \in U\}.$$

The same arguments as in the proof of the previous lemma give

Lemma 5. $\tilde{U}' = L^h$

3.5. Now we formulate an analogue for lemma 4 for the action of Lie algebra on itself. Namely, let

$$y = u_{\theta_1} + u_{\theta_2} + \cdots + u_{\theta_r}$$

and

$$y_1 = y, \; y_2 = w_c(y), \ldots, \; y_h = w_c^{h-1}(y).$$

We define, similarly to $\tilde{U}$, the set

$$\hat{U} = \{[l, y_1] + u_1, \cdots, [l, y_h] + u_h \;\mid l \in L, \, u_i \in U\}.$$

It is easy to see that the congruences of Lemma 3 also hold if we use elements y_m, y_j instead of γ_m, γ_j. Thus, the same arguments as in the proof of Lemma 4 give

Lemma 6. $\hat{U} = L^h$.

3.6.

Lemma 7. *1.Assume* $\operatorname{char} K \neq 2$ *if* $R = C_r$, $r \geq 1$. *Then there exists an element* $h \in H$ *such that* $C_L(h) = H$.

2. Let $R = C_r$, $r \geq 1$ *and* $\operatorname{char} K = 2$. *Further, let* R_l *be the set of all long roots of* R. *Then there exists an element* $h \in H$ *such that*

$$C_L(h) = C_L(H) = \{H + \sum_{\alpha \in R_l} U_\alpha\}.$$

Proof. 1. If $\operatorname{char} K \neq 2$ or $R \neq C_r$ then for every root $\alpha \in H$ the corresponding linear function $\alpha : H \longrightarrow K$ is not trivial. Since K is an algebraically closed field it is infinite and therefore the set

$$H \setminus (\cup_{\alpha \in R} \operatorname{Ker} \alpha)$$

is not empty.

2. If $R = C_r$ and $\operatorname{char} K = 2$ a map $\alpha : H \longrightarrow K$, $\alpha \in R$ is trivial if and only if $\alpha \in R_l$. Thus we can get the assertion in the same way as above. □

4. Proof of Theorem 2

Let $\operatorname{char} K \neq 2$ if $R = C_r$, $r \geq 1$.

Obviously, if Ψ_n is dominant, then $\Psi_1, \Psi_2, \ldots, \Psi_{n-1}$ is also dominant. Thus we have to prove that Ψ_h is dominant but Ψ_{h+1} is not. Let $(\ell, \ell_1, \ldots, \ell_n) \in L \oplus L^n$. Assume that

$l \in H$ is a regular element. Since for every regular element $h \in H$ we have $[h, L] = U$ (Lemma 7) then for every regular element $l' \in H$ there exist $l'_1, \cdots, l'_n \in L$ such that

$$[l', l'_1] = [l, l_1], \cdots, [l', l'_n] = [l, l_n]. \tag{11}$$

Further, in (11) we can replace every element l'_i by $l'_i + x_i$ where x_i is an element of the centralizer of l', i.e., an element of H. Thus, the dimension of the fiber $\Psi_n^{-1}(\Psi_n((l, l_1, \cdots, l_n))) \geq (n+1) \dim H$ and therefore

$$\dim \Psi_{h+1}^{-1}(\Psi_{h+1}((\ell, \ell_1, \ldots, \ell_{h+1}))) \geq (h+2) \operatorname{rank} L > \dim L \tag{12}$$

(recall, $\dim L = (h+1) \operatorname{rank} L$). Since the set of regular semisimple elements is dense in L the inequality (12) holds for a "generic fiber" of Ψ_{h+1}. Hence

$$\dim \operatorname{Im} \Psi_{h+1} < \dim L^{h+1}$$

and therefore Ψ_{h+1} cannot be dominant.

Consider now Ψ_h. Let $\tilde{a} = (a_1, \ldots, a_h)$, $b = (a, a_1, \ldots, a_h) \in L \oplus L^h$ be fixed points and let $c = \Psi_h(b) \in L^h$. Consider the differential $d_b\Psi_h$ of Ψ_h at the point b

$$d_b\Psi_h : T_b \to T_c$$

where T_b, T_c are the corresponding tangent spaces. We identify T_b with $L \oplus L^h$ and T_c with L^h. Then we have

$$d_b\Psi_h : L \oplus L^h \to L^h$$

and

$$\operatorname{Im} d_b\Psi_h = [a, L^h] + [L, \tilde{a}]. \tag{13}$$

(Here we used the rules for the differential of the map Ψ_h at the point $b = (a, \tilde{a})$ in the following sense: $d_{(a,a_i)}[x, y](l_1, l_2) = [a, l_2] + [l_1, a_i]$ where $x, y \in L$ are variables and l_1, l_2 are elements in the tangent space T_a, T_{a_i} of L at the points a, a_i, which we identify with L.)

Let $a \in H$ be a regular element and let $a_1 = y_1, \ldots, a_h = y_h$ be the sequence defined in 3.5. We have $[a, L^h] = U^h$ (Lemma 7) and, therefore,

$$[a, L^h] + [L, \tilde{a}] = \{([\ell, y_1] + u_1, [\ell, y_2] + u_2, \ldots, [\ell, y_h] + u_h) |\, \ell \in L,\ u_i \in U\} = \hat{U}$$

By Lemma 6,

$$[a, L^h] + [L, \tilde{a}] = L^h$$

for these particular a and $\tilde{a}$. Hence there exists a point $b \in L \oplus L^h$ where the rank of differential is equal $\dim L^h$ and therefore it holds for points from some open subset

$X \subset L \oplus L^h$. We can find the point $x \in X$ such that $\Psi_h(x)$ is a regular point of $\overline{\Psi_h(L \oplus L^h)}$. Since the tangent space at the point $\Psi_h(x)$ has rank $= \dim L^h$ we have $\dim \overline{\Psi_h(L \oplus L^h)} = \dim L^h$.

Let $R = C_r,\ r \geq 1$ and $\operatorname{char} K = 2$.

Denote by R_{sh} the set of all short roots in R. It is easy to check

$$[L, L] = H + \sum_{\alpha \in R_{sh}} U_\alpha \ .$$

Further, let

$$l = h + \sum_{\beta \in R_l^+} u_\beta \ ,$$

where $h \in H$ is an element satisfying the condition $C_L(h) = C_L(H)$ (Lemma 7), and u_β are elements of the Chevalley basis. We have $[l, L] = [L, L]$.(One can check this using the definition of h, l and Lemma 7.) This implies our assertion.

Theorem 2 has been proved.

5. Proof of Theorem 3

Let C_g, C_{g_i} be the conjugacy classes of the regular semisimple elements g, g_i. Then $\dim C_g = \dim C_{g_i} = \dim G - \operatorname{rank} G$. We assume $n > 1$ and define the subset of G^n

$$M_{n,\tilde{g}} = \{(gy_1g_1y_1^{-1}, \ldots , gy_ng_ny_n^{-1}) \,|\, y_i \in G\}.$$

Obviously $\dim M_{n,\tilde{g}} = n \dim C_g = n(\dim G - \operatorname{rank} G)$ and $GM_{n,\tilde{g}} = \{(xm_1x^{-1}, \ldots , xm_nx^{-1}) \,|\, (m_1, \ldots , m_n) \in M_{n,\tilde{g}}\} = \operatorname{Im} f_{n,\tilde{g}}$. Since a "generic point" $m \in M_{n,\tilde{g}}$ has a stabilizer which is equal to $Z(G)$ (because $n > 1$) and $\dim(M_{n,\tilde{g}} \cap Gm) \geq \operatorname{rank} G$ we have

$$\begin{aligned} \dim GM_{n,g} &\leq n(\dim G - \operatorname{rank} G) + (\dim G - \operatorname{rank} G) = \\ &= n \dim G + (\dim G - (n + 1)\operatorname{rank} G). \end{aligned} \tag{14}$$

If $\overline{\operatorname{Im} f_{n,\tilde{g}}} = G^n$ then (14) implies $(n + 1)\operatorname{rank} G \leq \dim G$ and therefore $n \leq h$.

Now we prove that inequality $n \leq h$ implies $\overline{\operatorname{Im} f_{n,\tilde{g}}} = G^n$. Obviously, it is enough to prove this for $n = h$.

We may assume $g, g_1, \ldots , g_h \in T$. Also we may assume that $(g\sigma_1g_1\sigma_1^{-1}, \ldots , g\sigma_hg_h\sigma_h^{-1})$ is a regular point of $\overline{\operatorname{Im} f_{n,\tilde{g}}}$ for some $\sigma_1, \ldots , \sigma_h \in G(K)$. Moreover, the set of such sequences $(\sigma_1, \ldots , \sigma_h)$ contains a non-empty open set of G^h. Put

$$s_1 = \sigma_1g_1\sigma_1^{-1}, \ldots , s_h = \sigma_hg_h\sigma_h^{-1}$$

and consider the map

$$\Psi :\, G \times G^h \to G^h$$

given by the formula

$$\Psi(x, y_1, \dots , y_h) = (xgx^{-1}y_1s_1y_1^{-1}s_1^{-1}g^{-1}, \dots , xgx^{-1}y_hs_hy_h^{-1}s_h^{-1}g^{-1}).$$

The definitions of Ψ and $s_1, \dots , s_h$ imply

$$\dim \operatorname{Im} \Psi = \dim \operatorname{Im} f_{h,\tilde{g}}. \tag{15}$$

Moreover, $\Psi((1, \dots , 1))$ is a regular point of $\operatorname{Im} \Psi$. (Indeed, $\operatorname{Im} \Psi = (\operatorname{Im} f_{h,\tilde{g}})s)$ where $s = (s_1^{-1}g^{-1}, \dots , s_h^{-1}g^{-1})$. The differential $d\Psi$ at the point $(1, \dots , 1)$ gives the linear map

$$d\Psi :\, L \times L^h \to L^h$$

which is

$$d\Psi\,((\ell, \ell_1, \dots , \ell_h)) = ((1 - g)\ell + g(1 - s_1)\ell_1, \dots , (1 - g)\ell + g(1 - s_h)\ell_1) \tag{$*$}$$

(this follows from the standard formulas for differentials).

Now we want to prove, that for some sequence $s_1, \dots , s_h$ defined above

$$\operatorname{Im} d\Psi = L^h. \tag{16}$$

Then (16) with (15) give us our statement.

Since $g \in T$ is a regular element, we get

$$(1 - g)L = U. \tag{17}$$

Further,

$$g(1 - s_i)L = g(1 - s_i)g^{-1}gL = (1 - gs_ig^{-1})gL = (1 - gs_ig^{-1})L. \tag{18}$$

Recall that $s_i = \sigma_ig_i\sigma_i^{-1}$ and $g_i \in T$. Since g_i is a regular element of T

$$(1 - g_i)L = U. \tag{19}$$

Put $\delta_i = g\sigma_i$. Then (17), (18), (19) imply

$$g(1 - s_i)L = (1 - \delta_ig_i\delta_i^{-1})L = \delta_i(1 - g_i)\delta_i^{-1}L = \delta_i(1 - g_i)L = \delta_i(U). \tag{20}$$

From $(*)$, (17) and (20) we obtain

$$\operatorname{Im} d\Psi = \{(u + \delta_1(u_1), \dots , u + \delta_h(u_h)) \mid u_i \in U\}.$$

Put $\delta = (\delta_1, \dots , \delta_h)$. Then

$$\delta^{-1}(\operatorname{Im} d\Psi) = \{(\delta_1^{-1}(u) + u_1, \dots , \delta_h^{-1}(u) + u_h \mid u_i \in U\}. \tag{21}$$

Let $\tau = (\tau_1, \dots, \tau_h) \in G^h$ and let

$$\Theta_\tau : U \oplus U^h \to L^h$$

be the map

$$\Theta_\tau\left((u, u_1, \dots, u_h)\right) = (\tau_1(u) + u_1, \dots, \tau_h(u) + u_h).$$

The set of $\tau \in G^h$ such that $\dim \operatorname{Im} \Theta_\tau < \dim L^h$ is closed in G^h. By Lemma 4, $\operatorname{Im} \Theta_\tau = L^h$ where $\tau_0 = (\gamma_1, \dots, \gamma_h)$. Hence the set

$$X = \{\tau \in G^h \mid \operatorname{Im} \Theta_\tau = L^h\}$$

is a non-empty open subset of G^h. Further, the set of sequences $Y = \{\sigma = (\sigma_1, \dots, \sigma_h) \in G^h\}$, such that $(g\sigma_1 g_1 \sigma_1^{-1}, \dots, g\sigma_h g_h \sigma_g^{-1})$ is a regular point of $\overline{\operatorname{Im} f_{h,\tilde{g}}}$, contains a non-empty open subset of G^h. Since $\delta_i^{-1} = \sigma_i^{-1} g^{-1}$ we find an element

$$\delta^{-1} \in X \cap Y^{-1} g^{-1}. \tag{22}$$

Now, from (21) and (22) we get (16).

Theorem 3 has been proved.

6. Proof of Theorem 1

Obviously, the property $\overline{\mathcal{C}}_n$ implies the property $\overline{\mathcal{C}}_{n-1}$. Thus we have to prove $\overline{\mathcal{C}}_{h+1}$ for the group G and we need to show that $\overline{\mathcal{C}}_{h+2}$ does not hold for G. The latter follows from the inequality

$$\dim \phi_{h+2}^{-1}(\phi_{h+2}((g, g_1, \dots, g_{h+2}))) \geq (h+2) \dim (C_G(g)) \geq$$

$$\geq (h+2) \operatorname{rank} G > (h+1) \operatorname{rank} G = \dim G.$$

Now we will prove the property $\overline{\mathcal{C}}_{h+1}$ for the group G. Recall, that we assume that G is simply connected.

Lemma 8. *Let $\gamma_1, \dots, \gamma_h$ be the sequence defined in (2). Then for every regular element $t \in T$ and for every $i = 1, \cdots, h$ there exists an element $t_i \in T$ such that*

$$[t\,,\, t_i \gamma_i t_i^{-1}] = \gamma_i$$

Moreover, for every regular t there is only a finite number of such t_i.

Proof. Obviously it is enough to prove the statement only for $\gamma_1 = \gamma$.

Since t is a regular element $\theta_j(t) \neq 1$ for every $j = 1, \dots, r$. Put

$$v_j = (\theta_j(t) - 1)^{-1} \tag{23}$$

Further, there exists an element $t_1 \in T$ such that

$$\theta_j(t_1) = v_j \tag{24}$$

for every j. Indeed, every element $x \in T$ can be presented in the form $x = h_{\alpha_1}(x_1) \dots h_{\alpha_r}(x_r)$ where $x_i \in K^*$ and $h_{\alpha_i}(x_i)$ is the corresponding semisimple element of the α_i- root subgroup ([St1], Lemma 28). The system of equations $\theta_j(x) = v_j$ can be written in the form

$$\prod_{k=1}^{r} x_k^{n(k,j)} = v_j \tag{25}$$

where $n(k,j) = 2(\alpha_k, \theta_j)/(\alpha_k, \alpha_k)$. Since $\theta_1, \dots, \theta_r$ is a basis of the group $Q(R)$ the matrix $\{(\alpha_k, \theta_j)\}_{1\leq k\leq r, 1\leq j\leq r}$ has rank $= r$. Hence the matrix $\{n(k,j)\}$ also has rank $= r$. Thus we can find the solution of (25) which gives us the element $t_1 \in T$ satisfying (24).

From (24)

$$t_1\gamma_1 t_1^{-1} = x_{\theta_1}(v_1) \dots x_{\theta_r}(v_r). \tag{26}$$

Now $[t,\, t_1\gamma t_1^{-1}] = \gamma = \gamma_1$ follows from (23) ,(26) and the Chevalley commutator formula. (Note that, by Lemma 1, the sum $\theta_i + \theta_j$ is not a root for every i, j and therefore $x_{\theta_i}(a)x_{\theta_j}(b) = x_{\theta_j}(b)x_{\theta_i}(a)$ for every a, b.) On the other hand, the commutator equation for t_1 implies the equation (26) which in turn leads us to (25). Since (25) has only a finite number of solutions (because the rank of the matrix $\{n_{k,j}\}$ is equal to r) we obtain that only finitely many t_1 are possible. $\square$

Let

$$X = \{(t, t_1\gamma_1 t_1^{-1}, \dots, t_h\gamma_h t_h^{-1}) \mid t, t_i \in T, [t, t_i\gamma_i t_i^{-1}] = \gamma_i \text{ for every } i = 1, \dots, h\}.$$

The set X is a constructible subset of G^{h+1}. Indeed, X is the image of a closed subset of T^{h+1} (which is defined by commutator equations) under the morphism

$$(t, t_1, \dots, t_h) \longrightarrow (t, t_1\gamma_1 t_1^{-1}, \dots, t_h\gamma_h t_h^{-1}).$$

Let X_0 be an irreducible component of X such that $\dim X_0 = \dim X$. Lemma 8 implies $\dim X = \dim T = \operatorname{rank} G$. Hence

$$\dim X_0 = \operatorname{rank} G \tag{27}$$

Moreover, Lemma 8 implies that the projection of X_0 on the first coordinate contains an open subset of T.

Further, let $(1, T^h)$ be the subset of T^{h+1} consisting of the elements which have the first coordinate 1. Then we consider $(1, T^h)$ as a subset of G^{h+1} and put

$$Y = X_0(1, T^h) = \{(t, t_1\gamma_1 t_1^{-1} t_1', \dots, t_h\gamma_h t_h^{-1} t_h') \mid t, t_i, t_i' \in T\} \tag{28}$$

(note that the elements t_i in (28) depend on the first coordinate t, while elements t_i' run independently through the set T). Since X_0 is an irreducible locally closed subset of G^h the same is Y and (27),(28) imply

$$\dim Y = (\operatorname{rank} G)^{h+1} = \dim G. \tag{29}$$

Further, the definition of the set X_0 and (28) imply

$$Y \subset \phi_h^{-1}((\gamma_1, \dots, \gamma_h)). \tag{30}$$

Lemma 9. *The Zariski closure $\overline{Y}$ of the set Y coincides with an irreducible component of the pre-image $\phi^{-1}((\gamma_1, \dots, \gamma_h))$.*

Proof. Let $y = (t, d_1, \dots, d_h) \in Y$ where $d_i = t_i\gamma_i t_i^{-1} t_i'$ (see (28)). Consider the map

$$\chi_y : \ G^{h+1} \longrightarrow G^h$$

given by the formula

$$\chi_y((x, x_1, \dots, x_h)) = ([xt, x_1 d_1][d_1, t], \dots, [xt, x_h d_h][d_h, t]).$$

The differential of χ_y at the point $(1, \dots, 1)$ gives the linear map

$$d(\chi_y) : \ L^{h+1} \longrightarrow L^h.$$

This map can be easily calculated using usual differentiation formulas. Namely, writing the first component as

$$(x, x_1) \mapsto xt(x_1 d_1 t^{-1}(x^{-1} d_1^{-1} x_1^{-1} d_1) t d_1^{-1}) t^{-1},$$

we obtain for its differential:

$$(l, l_1) \mapsto (l + t(l_1 + d_1 t^{-1}(-l - d_1^{-1}(l_1)))) = (1 - t d_1 t^{-1})(l) + t(1 - d_1 t^{-1} d_1^{-1})(l_1)$$

and therefore for the whole map:

$$d(\chi_y)((l, l_1, \dots, l_h)) =$$
$$((1 - t d_1 t^{-1})(l) + t(1 - d_1 t^{-1} d_1^{-1})(l_1), \dots, (1 - t d_h t^{-1})(l) + t(1 - d_h t^{-1} d_h^{-1})(l_h))$$

(recall that, for $g \in G$ and $l \in L$, we write $g(l)$ for $\mathrm{ad}\, g(l)$). Put $u_i = (1 - t^{-1})d_i^{-1}(l_i), l' = t^{-1}(l)$. Then $u_i \in U$. Applying the invertible linear operator $(d_1^{-1}t^{-1}, \dots , d_h^{-1}t^{-1})$ to the image of $d(\chi_y)$ we get the linear space

$$\{(d_1^{-1} - 1)(l') + u_1, \dots , (d_h^{-1} - 1)(l') + u_h) \;\mid\; l' \in L, u_i \in U\}$$

which, according to Lemma 5, coincides with L^h. Thus the differential of the map χ_y at the point $(1, \dots , 1)$ has rank $= \dim L^h$. Note that the map χ_y is the composition of two translations and the map ϕ_h. This implies that the differential of ϕ_h at the point y also is of rank $\dim L^h$. According to Theorem 3 the map ϕ_h is dominant. Hence $\phi_h(y)$ is a regular point of $\overline{\mathrm{Im}\phi_h} = G^h$. Since the rank of the differential of ϕ_h at the point y is equal to $\dim L^h = \dim G^h$, there exists an irreducible component Y' of the pre-image $\phi_h^{-1}(\phi_h(y))$ such that $y \in Y'$ and $\dim Y' = \dim G$. We may assume that a point y belongs only to those irreducible components of $\phi_h^{-1}(\phi_h(y)) = \phi_h^{-1}(\gamma_1, \dots , \gamma_h)$ which contain the whole set Y. Thus we obtain $Y \subset Y'$, and our statement will follow from (29)

□

Lemma 10. *Let $Y' \subset G^{h+1}$ be an irreducible component of $\phi_h^{-1}((\gamma_1, \dots , \gamma_h))$ which is the closure of the set Y. Then the projection of Y' to the first component of G^{h+1} is contained in T.*

Proof. Let $p_1 : G^{h+1} \longrightarrow G$ be the corresponding projection. Since $Y \subset Y'$, the set $p_1(Y')$ contains an open subset of T. This follows from the definition of Y. Further, the set $\overline{p_1(Y')}$ is an irreducible closed subset of the dimension $\mathrm{rank}\, G$. (The latter follows from the definitions Y and Y'.) Hence this set coincides with T. □

Now let $t \in T$ be a regular element and $s = [t, w_c]$ (recall that w_c is a fixed Coxeter element) and let $M_s = \phi_{h+1}^{-1}((\gamma_1, \dots , \gamma_h, s))$. We want to chose the element t satisfying the following conditions:

I. Let Y' be as in Lemma 10. There exists an element $(t, d_1, \dots , d_h) \in Y'$ which does not belong to any other irreducible component of $\phi_h^{-1}((\gamma_1, \dots , \gamma_h))$.

II. The set of all elements $g \in G$ such that $[t, g] = s$ consists only of elements of the form $w_c t'$ where t' runs through T.

Suppose we find an element t satisfying conditions I. and II. Put

$$z = (t, d_1, \dots , d_h, w_c).$$

From the definition of t, s we get the inclusion $z \in M_s$. Let M_{sz} be an irreducible component of M_s containing the element z. Further, let $P = P_{h+1}^{h+2} : G^{h+2} \longrightarrow G^{h+1}$

be the projection to the first $(h+1)$ components. Then $P(M_{sz}) \subset \phi_h^{-1}((\gamma_1, \dots, \gamma_h))$. Since the set M_{sz} is irreducible the set $\overline{P(M_{sz})}$ is also irreducible and I. implies that $P(M_{sz})$ is contained only in the irreducible component Y' of $\phi_h^{-1}((\gamma_1, \dots, \gamma_h))$. Since w_c is a Coxeter element there is only a finite number of $t' \in T$ satisfying the condition $[t', w_c] = s$. Together with Lemma 10 this implies

$$\overline{P(M_{sz})} = \overline{\{(t, a_1, \dots, a_h) \in Y\}} \tag{31}$$

(here t is a fixed element from the torus T but the elements a_i run through the sets of elements of the form $t_i \gamma_i t_i^{-1} t'$; see the definition of Y).

From (31) and the definition of Y we get

$$\dim P(M_{sz}) = (\operatorname{rank} G)^h.$$

Further, II. implies that the dimension of every fiber of the projection $M_{sz} \longrightarrow P(M_{sz})$ has dimension $\operatorname{rank} G$. Hence $\dim M_{sz} = (\operatorname{rank} G)^{h+1} = \dim G$. Thus we find an irreducible component of a pre-image of a point in G^{h+1} with respect to the map ϕ_{h+1} which has dimension $\dim G$. Therefore the dimension of the image of ϕ_{h+1} has the dimension $\dim G^{h+1}$. This gives our assertion.

Now we have to prove the existence of a regular element $t \in T$ satisfying conditions I.-II.

We can choose a point of Y which does not belong to other irreducible components of $\phi_h^{-1}((\gamma_1, \dots, \gamma_h))$ and which has a regular element $t \in T$ as its first coordinate. This follows from the definition of Y and Lemma 10. Thus we have I.

Now we show II. for a chosen t. Let $[t, g] = s$ for some $g \in G$. Then $g \in BwB$ for some $w \in W$. Hence $g = vwt'u$ where $v, u \in U$ (here U is the product of all positive root subgroups (see [St1])) $t' \in T$. We may assume that in u only those factors u_α from root subgroups are non-trivial which have the property $wu_\alpha w^{-1} \in U^-$ (here U^- is the product of all negative root subgroups) ([St1], Theorem 4').

Consider the equality

$$[t, g] = tvwt'ut^{-1}u^{-1}(t')^{-1}w^{-1}v^{-1} = tv(wt'ut^{-1}u^{-1}(t')^{-1}w^{-1})v^{-1} = s$$

The expression in brackets lies in the group $B^- = TU^-$. This follows from the choice of u. The elements on both sides of the brackets lie in the Borel subgroup B. Since $s \in T$, the expression in the bracket is in T. This implies

$$[t, u] = 1. \tag{32}$$

Since t is a regular element (32) implies $u = 1$. The same arguments show $v = 1$. Thus, $g = wt'$. But the equality $[t, w_c] = s = [t, wt']$ implies $w = w_c$ (because we assume that G is simply connected and t is a regular element of T).

Theorem 1 has been proved.

The proof of Lemma 9 shows that dim $\mathrm{Im}\, d_y(\phi_n) = (\dim G)^n$ if $n \leq h$ in the generic point y. Thus, if $n \leq h$ the map ϕ_n is always a separated morphism. Now let $n = h + 1$. We can consider the map $\chi_y : G^{h+2} \longrightarrow G^h$ which is constructed in the same way as the corresponding map in the proof of Lemma 9 changing h to $h + 1$. From the definition we have an equality of ranks of the differentials of χ_y at the point $(1, \cdots, 1)$ and ϕ_{h+1} at the generic point y. The formula (*) in that lemma shows that this rank cannot be $(\dim\ G)^{h+1}$ if the center of the Lie algebra is not trivial.

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