Free Subgroups in Maximal Subgroups of $GL_1(D)$ *

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Abstract

Let D be a division algebra of finite dimension over its centre F. Given a noncommutative maximal subgroup M of $D^* := GL_1(D)$, it is proved that either M contains a noncyclic free subgroup or there exists a maximal subfield K of D which is Galois over F such that K^* is normal in M and $M/K^* \cong Gal(K/F)$. Using this result, it is shown in particular that if D is a noncrossed product division algebra, then M does not satisfy any group identity.

1 Introduction

Let D be a division algebra of degree m over its centre F. Denote by D' the commutator subgroup of the multiplicative group $D^* = D - \{0\}$. Given a subgroup G of D^*, we shall say that G is *maximal* in D^* if for any subgroup H of D^* with $G \subset H$, one concludes that $H = D^*$. We know, by the Lemma of [9], that $G(D) := D^*/RN(D^*)D'$, where $RN(D^*)$ is the image of D^* under the reduced norm of D to F, is an abelian torsion group of a bounded exponent dividing the degree m of D over F. This group is not trivial in general. For example, if D is the algebra of real quaternions, then $G(D)$ is trivial whereas for rational quaternions $G(D)$ is isomorphic to a direct product of copies of Z_2, as it is easily checked. Assume that $G(D)$ is not trivial, then by Prũfer-Baer Theorem (cf. [11, p. 105]), we conclude that $G(D)$ is isomorphic to a direct

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product of Z_{r_i}, where r_i divides m. In this way we may obtain maximal normal subgroups of finite index in D^*. So, if $G(D)$ is not trivial, then D^* contains maximal subgroups. We show later on that even for the case $G(D) = 1$ we may have maximal subgroups in D^*. But the question of whether D^* has a maximal subgroup for any noncommutative division algebra D, is still open. Now, let D be a division ring not necessarily of finite dimension over its centre F. The problem of whether the multiplicative group of D contains a noncyclic free subgroup seems to be posed first by Lichtman in [7]. Stronger versions of this problem which essentially deal with the existence of noncyclic free subgroups in normal or subnormal subgroups of D^* have been investigated in [4] and [5]. It is known so far that these problems have positive answers as long as we work in a division algebra of finite dimension over its centre. Further investigations for the infinite dimensional case are also dealt with in [3] and [4]. The study of maximal subgroups of the multiplicative group of a division ring D begins in [1] in relation with an investigation of the structure of finitely generated normal subgroups of $GL_n(D)$, where D is of finite dimension over its centre F. In [1] and [8] we essentially show that maximal subgroups arise naturally in $GL_n(D)$, $n \geq 1$ and finitely generated subnormal subgroups of $GL_n(D)$, $n \geq 1$ are central. This result is used to prove that a maximal subgroup of $GL_n(D)$ can not be finitely generated for $n \geq 1$. Therefore, we are not able to apply directly Tits' result, that any finitely generated linear group either is soluble-by-finite or contains a noncyclic free group (cf. [17]), to a maximal subgroup M of D^* to explore the structure of M. In [1], it is also shown that there is a similarity between the behaviour of normal or subnormal subgroups of D^* and the maximal ones. So, it is natural to ask if there exists a noncyclic free group in a maximal subgroup of D^*. In this direction, we actually show that if D is a noncrossed product division algebra, then any noncommutative maximal subgroup of D^* contains a noncyclic free group. To deal with the general case, it seems that one must re-examine the technique that Suprunenko used in [14] and [15] to investigate primitive soluble linear groups and maximal soluble irreducible linear groups. Here we shall try to modify Suprunenko's results for irreducible maximal subgroups of D^* containing F^*. We then apply these results to present a version of Tits' Theorem for maximal subgroups of D^*. To be more precise, let D be a division algebra of finite dimension over its centre F. Given a noncommutative maximal subgroup M of D^*, it is proved that either M contains a noncyclic free subgroup or there exists a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$, where $Gal(K/F)$ denotes the Galois group of K over F. Consequently, the Platonov's result on a linear group with a group identity (cf. [19, p. 149]) may be restated for maximal subgroups M of D^*, namely, a noncommutative maximal subgroup M satisfies a group identity if and only if there exists a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$.

2 Notations and conventions

Let D be a division ring with centre F. Given a subgroup G of D^*, we denote by $F[G]$ the F-algebra generated by elements of G over F, and by $F(G)$ the division ring generated by F and G. We shall say that G is *irreducible* if $D = F(G)$. For any group G we denote its centre by $Z(G)$. Given a subgroup H of G, $N_G(H)$ means the *normalizer* of H in G, $[G : H]$ denotes the *index* of H in G, and $< H, K >$ the group generated by H and K, where K is a subgroup of G. Let S be a subset of D, then the *centralizer* of S in D is denoted by $C_D(S)$. Some notations and conventions for linear groups and skew linear groups from [13] and [16] are frequently used throughout.

3 Free groups in maximal subgroups

Given a division ring D with centre F, let M be a maximal subgroup of D^*. This section essentially deals with irreducible maximal subgroups of D^* and how they sit in D^* with respect to the multiplicative groups of maximal subfields of D. Firstly, given a noncommutative maximal subgroup M of D^* containing F^*, let K^* be a maximal abelian normal subgroup of M. Then, it is shown that K^* is the multiplicative group of a subfield K of D. Furthermore, if M is irreducible, then the factor group M/L, where $L = C_M(K^*)$, is isomorphic to a subgroup G of the group of automorphisms of K/F, and the elements of

K that remain fixed by elements of G are contained in F. We then show that $K^* = L$ if and only if $K = C_D(K)$, i.e., K is a maximal subfield of D. Thus, if $K^* = L$, then $M/K^* \cong Gal(K/F)$. To prove our main result, we need to put conditions on M which imply either the commutativity of M or that of D. In fact it is shown that given a maximal subgroup M of D^* containing F^*, if M/F^* is torsion, then M is commutative. We then use these results to prove our main theorem that given a noncommutative maximal subgroup M of D^*, then either M contains a noncyclic free subgroup or there is a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$. Therefore, M satisfies a group identity if and only if there exists a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$. We begin the material of this note with the following lemmas which establish a connection between maximal subgroups of D^* and multiplicative groups of subfields of D that are contained in M. One may compare these results with the ones obtained in [14] for primitive soluble linear groups.

LEMMA 1. *Let D be a division ring not necessarily of finite dimension over its centre F. Assume that M is a noncommutative maximal subgroup of D^* containing F^*. Let K^* be a maximal abelian normal subgroup of M. Then we have*

(i) *K^* is the multiplicative group of a subfield K of D.*

(ii) *If M is irreducible, then the factor group M/L, where $L = C_M(K^*)$, is isomorphic to a subgroup G of the group of automorphisms of K/F, and the elements of K that remain fixed by elements of G are contained in F.*

(iii) *If M is irreducible and $[K : F] < \infty$, then K is normal and separable over F and we have $Gal_F(K) \cong M/L$.*

PROOF. (i) By maximality of K^*, we conclude that $F^* \subset K^*$. Consider the F-algebras $F[K^*]$ and $F(K^*)$. Since for any $x \in M$ we have $xK^*x^{-1} = K^*$, we obtain $xF[K^*]x^{-1} = F[K^*]$ and consequently, $xF(K^*)x^{-1} = F(K^*)$. Thus, $< F(K^*)^*, M > \subset N_{D*}(F(K^*)^*)$. If $F(K^*)^* \not\subset M$, then, by Cartan-Brauer-Hua Theorem (cf. [12, p. 427]), either $F(K^*) = D$ or $F(K^*)^* \subset F^*$. The first case contradicts the noncommutativity of M and the second case says that $F(K^*)^* \subset F^* \subset M$ which is also a contradiction. Thus, $F(K^*)^* \subset M$. Now, by maximality of K^*, we obtain $F(K^*)^* = K^*$, i. e., $K = K^* \cup \{0\}$ is a subfield of D.

(ii) We know that for every $a \in M$ we have $aKa^{-1} = K$. Thus, the mapping $\phi_a : K \to K$ given by $\phi_a(x) = axa^{-1}$ is an automorphism of K that leaves every element of F fixed. We claim that only the elements of F remain fixed under all automorphisms of the above form. This follows from the fact that M is irreducible, i. e., $D = F(M)$, thus the centralizer of M in D^* is exactly F^*. It is clear that all automorphisms ϕ_a, $a \in M$ form a group G, say. Now, consider the mapping $f : M \to G$ given by $f(a) = \phi_a$. By definition, f is an epimorphism and we have $\ker f = \{a \in M \mid \phi_a = 1\} = \{a \in M \mid axa^{-1} = x\} = C_M(K^*) = L$, and this completes the proof of (ii).

(iii) If $[K : F] < \infty$, the fixed field of G is F and $G \subset Aut(K)$, then it is basic Galois theory that K/F is Galois, G is finite, and G is the Galois group.

The next result essentially provides a necessary and sufficient condition under which the multiplicative group of a maximal subfield of D is contained in a maximal subgroup of D^*.

LEMMA 2. *Let D be a division algebra of finite dimension over its centre F. Assume that M is an irreducible maximal subgroup of D^* containing F^*. Let K^* be a maximal abelian normal subgroup of M with $L = C_M(K^*)$. Then we have*

(i) $[F[M] : F[L]] = [M : L]$.

(ii) *$K^* = L$ if and only if $K = C_D(K)$, i.e., K is a maximal subfield of D. Therefore, if $K^* = L$, then $M/K^* \cong Gal(K/F)$.*

PROOF. (i) since $[D : F] < \infty$, using Lemma 1, we obtain $[M : L] < \infty$. Let $m_1, \ldots, m_r$ be distinct representatives of the cosets of L in M, i.e., $M = L_{m1} \cup \ldots \cup Lm_r$. Therefore, each element of $F[M]$ may be represented in the form $\sum_1^r l_i m_i$ with $l_i \in F[L]$. We claim that $\{m_i\}_1^r$ are linearly independent over $F[L]$. To see this, assume that $l_1 m_1 + \ldots + l_s m_s = 0$ is a nontrivial relation containing the smallest number of nonzero terms. Since $L = C_M(K^*)$ and m_1, m_2 belong to distinct cosets of L, there exists an element $u \in K^*$ such that $u_1 = m_1 u m_1^{-1} \neq u_2 = m_2 u m_2^{-1}$. Thus, from our minimal relation we conclude that $(l_1 m_1 + \ldots + l_s m_s)u - u_1(l_1 m_1 + \ldots + \ldots l_s m_s) = (u_2 - u_1) l_2 m_2 + \ldots (u_s - u_1) l_s m_s = 0$ with $u_s = m_s u m_s^{-1}$. But this contradicts the choice of s and so the result follows.

(ii) Assume that K is a maximal subfield of D. Then $C_{D*}(K^*) = K^*$ and therefore we have $L = C_M(K^*) = K^*$. On the other hand, assume that $L = K^*$. Since M is irreducible we have $D = F[M]$ and so from (i) we conclude that $[D : K] = [K : F]$, and therefore $[D : F] = [K : F]^2$ which implies that K is a maximal subfield of D. This completes the proof of the lemma.

Before proving our next result we need also the following lemma which will be used frequently throughout.

Lemma 3. *Let D be a division algebra of finite dimension over its centre F. Then every soluble subgroup of D^* has an abelian normal subgroup of finite index.*

Proof. Let S be a soluble subgroup of D^*. Since $[D : F] < \infty$, S is a linear group. Now, by Kochlin-Maltsev's Theorem (cf. [19, p. 146]), we conclude that S contains a subgroup T of finite index such that T' is unipotent. Since the only unipotent element in a division ring is the identity, we obtain $T' = \{1\}$. Thus S contains an abelian group of finite index and consequently S contains an abelian normal subgroup A of finite index and thus the lemma follows.

Using above results, we are now able to prove a modified version of a theorem of Suprunenko (cf. [14]) for maximal subgroups of D^* which are soluble.

Corollary 4. *Let D be a finite dimensional division algebra with centre F. Assume that M is a noncommutative maximal subgroup of D^*. Then M is soluble if and only if there is a maximal subfield K of D such that K^* is normal in M with $M/K^* \cong Gal(K/F)$, and $Gal(K/F)$ is soluble.*

PROOF. One way is clear. To prove the other way, assume that M is soluble. We have either $F(M)^* = M$ or $F(M) = D$. The first case can not occur, by Hua's Theorem (cf. [6, p. 223]). The same result also implies that D' is not contained in M. Thus M is an irreducible soluble maximal subgroup of D^* containing F^*. Thus, by Lemma 3, M contains an abelian normal subgroup A of finite index. If $A \subset F^*$, then M/F^* is finite. By Corollary 4 of [1], we conclude that M is commutative which is contradiction. Therefore, F^* is properly contained in A. Take a maximal abelian normal subgroup K^*, say, in M which contains A. By part (iii) of Lemma 1, we conclude that $K = K^* \cup \{0\}$ is a normal separable extension field of F and we have $Gal(K/F) \cong M/L$, where $L = C_M(K^*)$. We now claim that K is a maximal subfield of D. To see this, assume that $C_{D^*}(K^*)$ is not contained in M. Then $< C_{D^*}(K^*), M >\subset N_{D^*}(K^*)$ and thus $D^* = N_{D^*}(K^*)$ which implies that $K \subset F$ which is impossible since F^* is contained properly in K^*. Thus, we must have $C_{D^*}(K^*) \subset M$ and since M is soluble we obtain $C_{D^*}(K^*)$ is soluble. Now, by Hua's Theorem, we conclude that $C_D(K)$ is commutative. This implies that K is maximal in D and the claim is established. Now, by part (ii) of Lemma 2, we obtain $L = K^*$ and since M is soluble the result follows.

EXAMPLE. Let D be the real quaternion algebra. It is known that D' consists of elements $a+bi+cj+dk$ with $a^2+b^2+c^2+d^2 = 1$. One may easily check that $D^* = F^*D'$ as well as $G(D) = 1$, where $F = R$ is the field of real numbers. Here we show that the subgroup $M := C^* \cup C^*j$, where C is the field of complex numbers, is a maximal subgroup of D^*. It is shown in [15] that M is soluble and so this maximal subgroup satisfies the conclusion of the above corollary. Here we also observe that $M = N_{D^*}(C^*)$, and thus the normalizer of the multiplicative group of a maximal subfield of D may be a maximal subgroup of D^*. To show that M is maximal in D^*, it is enough to prove that $M \cap D'$ is maximal in D' since $D^* = F^*D'$. To see this, we first identify the quaternion $a+bi+cj+dk$ with the complex 2×2 matrix $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$. Therefore, D' is the group of unitary matrices of determinant 1. Now, one should specify the group $M \cap D'$ in terms of matrices and show that for any

$x \in D'$ not in $M \cap D'$ we have $< x, M \cap D' >= D'$. This involves a lot of calculations and we skip this method. But there is a simpler geometric method that is presented here for which I am indebted to Professor C. Ohn: We let complex matrices act on $C \cup \{\infty\}$ by homographies. We then identify $C \cup \{\infty\}$ with the unit sphere $S^2 \subset R^3$ via the stereographic projection, it is then well known that D' acts by rotations on S^2, and that the corresponding morphism π from D' to $SO(3)$ is surjective with kernel $\{\pm 1\}$. Furthermore, π maps $C \cap D'$ to the rotations around the polar axis and $Cj \cap D'$ to the half-turns around an equatorial axis. Therefore, $H = \pi((C \cup Cj) \cap D') \subset SO(3)$ is the subgroup of rotations that leave the equator invariant, and it is clearly enough to show that H is maximal in $SO(3)$. Suppose $x \in SO(3)$ not in H and we show that $K =< H, x >= SO(3)$. To do this, we shall use a well known result which asserts that if a group G acts on a set X, and if K is a subgroup of G such that K acts transitively on X and K contains the stabilizer G_x of some $x \in X$, then $K = G$. Here, put $G = SO(3)$ and $X = S^2$. The second condition of the mentioned result is satisfied for x the north pole. The H-orbits in S^2 are the sets of the form $P_\alpha \cup P_{-\alpha}$, where P_α $(-\pi/2 \leq \alpha \leq \pi/2)$ is the parallel of latitude α. Since $x \notin H$, x maps the equator E to a great circle $E' \neq E$. Since E' hits all parallels between some extreme latitudes $-\alpha$ and α (α the angle between E and E'), for $-\alpha \leq \beta \leq \alpha$ the whole zone $Z_{[-\alpha,\alpha]} = \cup P_\beta$ between those extreme latitudes will be contained in a unique K-orbit. This argument may be repeated with E replaced by $Z_{[-\alpha,\alpha]}$, showing that $Z_{[-2\alpha,2\alpha]}$ is contained in a unique K-orbit. Repeating again and again, this K-orbit is eventually seen to cover the whole sphere S^2, so the first condition is also satisfied, and the proof is complete.

Corollary 5. *Let D be a division algebra of finite dimension over its centre F. Assume that M is a noncommutative maximal subgroup of D^*. If M is soluble, then D is a crossed product division algebra. Equivalently, the multiplicative group of a noncrossed product division algebra can not have any noncommutative soluble maximal subgroup.*

Given a finite dimensional division algebra D with centre F whose characteristic is different from the degree of D over F, in [1] it is shown that if M is a maximal subgroup of D^* and for each element $x \in M$ there exists a positive integer $n(x)$, depending on x, such that $x^{n(x)} \in F$, then $D = F$. Here, we present a variation of this result which deals only with the commutativity of M as follows:

THEOREM 6. *Let D be a division algebra of finite dimension over its centre F. Suppose M is a maximal subgroup of $D^* \neq F^*$ and $M/(M \cap F^*)$ is torsion, then $F^* \subset M$, $M = K^*$ for K a maximal subfield of D, F has characteristic $p > 0$, K/F is purely inseparable, and D has degree p.*

PROOF. Suppose M is a maximal subgroup of $D^* \neq F^*$ and $M/(M \cap F^*)$ is torsion. We claim that $F^* \subset M$, $M = K^*$ for K a maximal subfield of D, F has characteristic $p > 0$, K/F is purely inseparable. Once the claim is established, then using the result mentioned before the theorem, we conclude that D has degree p. To prove our claim, consider the division algebra $F(M)$ generated by F and M. By maximality of M, we have either $F(M)^* = M$ or $D = F(M)$. If the first case occurs, by a result of Kaplansky (cf. [6, p. 259]), $K := F(M)$ is commutative. Therefore, K is a maximal subfield of D and it is radical over F. Thus, by Kaplansky's Lemma (cf. [6, p. 258]), we conclude F has characteristic $p > 0$ and either K is algebraic over the prime subfield or K is purely inseparable over F. If K is algebraic over the prime subfield, then, by a result of Jacobson (cf. [6, p. 219]), $D = F$ which is impossible. Thus, K is purely inseparable over F. The second case implies that M is irreducible. We assume first that the characteristic of F is $p > 0$. Take an element $x \in D' \cap M$. Since M/F^* is torsion, we know that $x^{n(x)} = a \in F^*$. Thus, we conclude that $1 = RN_{D/F}(x)^{n(x)} = a^m$, where $RN_{D/F}$ is the reduced norm function of D to F. Therefore, $M' \subset M \cap D'$ is torsion and consequently M' is locally finite by Schur's Theorem (cf. [6, p. 154]). If $a, b \in M'$, then the subgroup $< a, b >$ is finite. Since $Char F = p$ we conclude that $< a, b >$ is cyclic (cf. [6]), and thus M' is abelian. Therefore, M is a maximal irreducible subgroup of D^* which is soluble. If M is commutative, we are done. Otherwise, by Corollary 4, we conclude that there exists a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$. Since M/F^* is torsion, this implies that K is radical over F. Now, by Kaplansky's Lemma, we have either K is purely inseparable over F or K is algebraic over its prime subfield. The first case can not happen since K/F is Galois and the second case, by the result of Jacobson again, leads to the commutativity of D which is nonsense.

Finally, consider the zero characteristic case. Since $[D : F] = n$, M is a linear group in $GL_n(F)$. By a theorem of Tits (cf. [17]), either M contains a non-abelian free subgroup or it is soluble-by-finite. The first case can not occur since M/F^* is torsion. Therefore, there is a soluble subgruop S in M with $[M : S] < \infty$. By Lemma 3, we conclude that S contains an abelian normal subgroup of finite index and consequently M contains an abelian normal subgroup A of finite index. Put $K = F(A)$. Then we have $< K^*, M >\subset N_{D^*}(K^*)$. If $K^* \not\subset M$, then $N_{D^*}(K^*) = D^*$ and so, by Cartan-Brauer-Hua Theorem, we conclude that $K = F$, i.e., $K^* = F^* \subset M$ which is nonsense. Otherwise, assume that $K^* \subset M$. Therefore, K is radical over F. Thus, using Kaplansky's Lemma again, we obtain $CharF = p > 0$ which is a contradiction. This completes the proof of the theorem.

We observe that in Theorem 6, in characteristic $p > 0$, if $M/M \cap F^*$ is torsion, it is not known if D is commutative. But we have the following

Corollary 7. *Let D be a division algebra of finite dimension over its centre F and assume that D^* has maximal subgroups. If $M/F^* \cap M$ is torsion for every maximal subgroup M of D^*, then $D = F$.*

Proof. Consider the group $G(D) = D^*/F^*D'$. By Corollary 1 of [9], we know that $G(D)$ is torsion of a bounded exponent dividing the index of D over F . If $G(D)$ is not trivial, then by Bear-Prufer Theorem (cf. [11, p. 105]), we conclude that there is a maximal subgroup M, say, of D^* containing D'. But then since $M/F^* \cap M$ is torsion we obtain that $D'/Z(D')$ is torsion. Now, by Lemma 2 of [10], we conclude that $D = F$. Therefore, we may assume that $D^* = F^*D'$ and none of the maximal subgroups of D^* contains D'. Thus, by Proposition 1 of [1], every maximal subgroup M of D^* contains F^*. By Theorem 6, we conclude that M is commutative. Since $G(D)$ is trivial we obtain $M = F^*(M \cap D')$. Because M is maximal in D^* one can easily conclude that $L := M \cup \{0\}$ is a maximal subfield of D. Now, L is radical over F and thus, by Kaplansky's Lemma, we conclude that $charF = p$ and either L

is algebraic over the prime subfield or L is purely inseparable over F. The first case via a theorem of Jacobson leads to $D = F$ and the second case implies that $L \cap D' = M \cap D'$ contains purely inseparable elements. Now, by Corollary 8 of [10], we know that this is not possible unless $M \cap D' = F^* \cap D' = Z(D')$. Therefore, we obtain $M = F^*(M \cap D') = F^*(F^* \cap D') = F^* Z(D') = F^*$, which is a contradiction and so the result follows.

We are now in a position to prove our main result as

Theorem 8. *Let D be a division algebra of finite dimension over its centre F. Assume that M is a noncommutative maximal subgroup of D^*. Then either M contains a noncyclic free subgroup or there exists a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$.*

Proof. Let M be a noncommutative maximal subgroup of D^*. We know, by proposition 1 of [1], that either $D' \subset M$ or $F^* \subset M$. If $D' \subset M$, then M is normal in D^* and so the result follows by a theorem of [5]. So, we may assume that $F^* \subset M$ but $D' \not\subset M$. Since $[D : F] < \infty$ we may consider M as a linear group. If M does not contain a noncyclic free subgroup, then every finitely generated subgroup of M does not contain a noncyclic free subgroup. By Tit's theorem (cf. [17]), we conclude that every finitely generated subgroup of M contains a soluble normal subgroup of finite index. Therefore, by a result of Wehrfritz (cf. [18]), $M/Solv(M)$ is a torsion linear group, where $Solv(M)$ is the unique maximal soluble normal subgroup obtained by Zassenhaus-Maltsev Theorem (cf. [2]). Put $S = Solv(M)$. If $M = S$, then M is a soluble maximal subgroup of D^*. Therefore, by Corollary 4, the result follows for the case $Solv(M) = S = M$. Thus, we assume that $M \neq S$, i.e., S is a proper maximal soluble normal subgroup of M. If $S = F^*$, then M/F^* is torsion. Thus, by Theorem 6, we conclude that M is commutative which is a contradiction to the fact that M is noncommutative. So, we may assume that $F^* \subset S \subset M$. Now, consider the division subring $E = F(S)$ generated by F and S.

If $E^* = F(S)^* \subset M$, then by a theorem of [5], E^* contains a noncyclic free subgroup and so does M unless $F(S)$ is commutative. Now, since $F(S)$ is commutative $F(S)^*$ is a solube normal subgroup of M. By maximality of S we obtain $F(S)^* = S$. Therefore, S is a maximal abelian normal subgroup of

M properly containing F^*. By Lemma 1, $K = S \cup \{0\}$ is a normal separable field extension of F such that K^* is normal in M and $M/L \cong Gal(K/F)$ with $L = C_M(K^*)$. By Lemma 2, it remains to show that K is a maximal subfield of D. To see this, assume that $C_{D^*}(K^*)$ is not contained in M. Then $< C_{D^*}(K^*), M > \subset N_{D^*}(K^*)$ and thus $D^* = N_{D^*}(K^*)$ which implies that $K \subset F$ which is impossible since F^* is contained properly in K^*. Thus, we must have $C_{D^*}(K^*) \subset M$ and since M/K^* is torsion, by Kaplansky's Theorem, we conclude that $C_D(K)$ is commutative. This implies that K is maximal in D and so the result follows in this case.

If $E^* = F(S)^* \not\subset M$, then by maximality of M in D^* we have $D^* =< E^*, M > \subset N_{D^*}(E^*)$ and so we have $D^* = N_{D^*}(E^*)$. Thus, by Cartan-Brauer-Hua Theorem, we have either $E \subset F$ or $F(S) = E = D$. If $E \subset F$, then $S \subset F^*$ which is not possible. Therefore, $D = F(S)$. Now, S is a soluble linear group. By Lemma 3, we conclude that S contains an abelian normal subgroup A of finite index. If $A \subset F^*$, then M/F^* is torsion and, by Theorem 6, we conclude that M is commutative which is a contradiction. Therefore, there is a maximal abelian normal subgroup K^* of M containing A which is also contained in S, by maximality of S. Since M is irreducible, by Lemma 1, $K = K^* \cup \{0\}$ is a separable normal field extension of F such that K^* is normal in M and $M/L \cong Gal(K/F)$ with $L = C_M(K^*)$. As in the previous case, one can show that K is a maximal subfield of D. Therefore, by Lemma 2, we conclude that $K^* = L$ and so the result follows.

Corollary 9. *Let D be a division algebra of finite dimension over its centre F, and M be a noncommutative maximal subgroup of D^*. Then M satisfies a group identity if and only if there exists a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$.*

Therefore, if D is a noncrossed product division algebra and M is a noncommutative maximal subgroup of D^*, then M does not satisfy any group identity.

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