Free Subgroups in Maximal Subgroups of $GL_1(D)$ *

M. Mahdavi-Hezavehi †

Abstract

Let D be a division algebra of finite dimension over its centre F. Given a noncommutative maximal subgroup M of $D^* := GL_1(D)$, it is proved that either M contains a noncyclic free subgroup or there exists a maximal subfield K of D which is Galois over F such that K^* is normal in M and $M/K^* \cong Gal(K/F)$. Using this result, it is shown in particular that if D is a noncrossed product division algebra, then M does not satisfy any group identity.

1 Introduction

Let D be a division algebra of degree m over its centre F. Denote by D' the commutator subgroup of the multiplicative group $D^* = D - \{0\}$. Given a subgroup G of D^*, we shall say that G is *maximal* in D^* if for any subgroup H of D^* with $G \subset H$, one concludes that $H = D^*$. We know, by the Lemma of [9], that $G(D) := D^*/RN(D^*)D'$, where $RN(D^*)$ is the image of D^* under the reduced norm of D to F, is an abelian torsion group of a bounded exponent dividing the degree m of D over F. This group is not trivial in general. For example, if D is the algebra of real quaternions, then $G(D)$ is trivial whereas for rational quaternions $G(D)$ is isomorphic to a direct product of copies of Z_2, as it is easily checked. Assume that $G(D)$ is not trivial, then by Prũfer-Baer Theorem (cf. [11, p. 105]), we conclude that $G(D)$ is isomorphic to a direct

* *Key words:* Free Subgroup, Division ring, maximal subgroup.

† AMS (1991) *Subject classification* : 15A33, 16K

product of Z_{r_i}, where r_i divides m. In this way we may obtain maximal normal subgroups of finite index in D^*. So, if $G(D)$ is not trivial, then D^* contains maximal subgroups. We show later on that even for the case $G(D) = 1$ we may have maximal subgroups in D^*. But the question of whether D^* has a maximal subgroup for any noncommutative division algebra D, is still open. Now, let D be a division ring not necessarily of finite dimension over its centre F. The problem of whether the multiplicative group of D contains a noncyclic free subgroup seems to be posed first by Lichtman in [7]. Stronger versions of this problem which essentially deal with the existence of noncyclic free subgroups in normal or subnormal subgroups of D^* have been investigated in [4] and [5]. It is known so far that these problems have positive answers as long as we work in a division algebra of finite dimension over its centre. Further investigations for the infinite dimensional case are also dealt with in [3] and [4]. The study of maximal subgroups of the multiplicative group of a division ring D begins in [1] in relation with an investigation of the structure of finitely generated normal subgroups of $GL_n(D)$, where D is of finite dimension over its centre F. In [1] and [8] we essentially show that maximal subgroups arise naturally in $GL_n(D)$, $n \geq 1$ and finitely generated subnormal subgroups of $GL_n(D)$, $n \geq 1$ are central. This result is used to prove that a maximal subgroup of $GL_n(D)$ can not be finitely generated for $n \geq 1$. Therefore, we are not able to apply directly Tits' result, that any finitely generated linear group either is soluble-by-finite or contains a noncyclic free group (cf. [17]), to a maximal subgroup M of D^* to explore the structure of M. In [1], it is also shown that there is a similarity between the behaviour of normal or subnormal subgroups of D^* and the maximal ones. So, it is natural to ask if there exists a noncyclic free group in a maximal subgroup of D^*. In this direction, we actually show that if D is a noncrossed product division algebra, then any noncommutative maximal subgroup of D^* contains a noncyclic free group. To deal with the general case, it seems that one must re-examine the technique that Suprunenko used in [14] and [15] to investigate primitive soluble linear groups and maximal soluble irreducible linear groups. Here we shall try to modify Suprunenko's results for irreducible maximal subgroups of D^* containing F^*. We then apply these results to present a version of Tits' Theorem for maximal subgroups of D^*. To be more precise, let D be a division algebra of finite dimension over its centre F. Given a noncommutative maximal subgroup M of D^*, it is proved that either M contains a noncyclic free subgroup or there exists a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$, where $Gal(K/F)$ denotes the Galois group of K over F. Consequently, the Platonov's result on a linear group with a group identity (cf. [19, p. 149]) may be restated for maximal subgroups M of D^*, namely, a noncommutative maximal subgroup M satisfies a group identity if and only if there exists a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$.

2 Notations and conventions

Let D be a division ring with centre F. Given a subgroup G of D^*, we denote by $F[G]$ the F-algebra generated by elements of G over F, and by $F(G)$ the division ring generated by F and G. We shall say that G is *irreducible* if $D = F(G)$. For any group G we denote its centre by $Z(G)$. Given a subgroup H of G, $N_G(H)$ means the *normalizer* of H in G, $[G : H]$ denotes the *index* of H in G, and $< H, K >$ the group generated by H and K, where K is a subgroup of G. Let S be a subset of D, then the *centralizer* of S in D is denoted by $C_D(S)$. Some notations and conventions for linear groups and skew linear groups from [13] and [16] are frequently used throughout.

3 Free groups in maximal subgroups

Given a division ring D with centre F, let M be a maximal subgroup of D^*. This section essentially deals with irreducible maximal subgroups of D^* and how they sit in D^* with respect to the multiplicative groups of maximal subfields of D. Firstly, given a noncommutative maximal subgroup M of D^* containing F^*, let K^* be a maximal abelian normal subgroup of M. Then, it is shown that K^* is the multiplicative group of a subfield K of D. Furthermore, if M is irreducible, then the factor group M/L, where $L = C_M(K^*)$, is isomorphic to a subgroup G of the group of automorphisms of K/F, and the elements of

K that remain fixed by elements of G are contained in F. We then show that $K^* = L$ if and only if $K = C_D(K)$, i.e., K is a maximal subfield of D. Thus, if $K^* = L$, then $M/K^* \cong Gal(K/F)$. To prove our main result, we need to put conditions on M which imply either the commutativity of M or that of D. In fact it is shown that given a maximal subgroup M of D^* containing F^*, if M/F^* is torsion, then M is commutative. We then use these results to prove our main theorem that given a noncommutative maximal subgroup M of D^*, then either M contains a noncyclic free subgroup or there is a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$. Therefore, M satisfies a group identity if and only if there exists a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$. We begin the material of this note with the following lemmas which establish a connection between maximal subgroups of D^* and multiplicative groups of subfields of D that are contained in M. One may compare these results with the ones obtained in [14] for primitive soluble linear groups.

Lemma 1. *Let D be a division ring not necessarily of finite dimension over its centre F. Assume that M is a noncommutative maximal subgroup of D^* containing F^*. Let K^* be a maximal abelian normal subgroup of M. Then we have*

(i) *K^* is the multiplicative group of a subfield K of D.*

(ii) *If M is irreducible, then the factor group M/L, where $L = C_M(K^*)$, is isomorphic to a subgroup G of the group of automorphisms of K/F, and the elements of K that remain fixed by elements of G are contained in F.*

(iii) *If M is irreducible and $[K : F] < \infty$, then K is normal and separable over F and we have $Gal_F(K) \cong M/L$.*

Proof. (i) By maximality of K^*, we conclude that $F^* \subset K^*$. Consider the F-algebras $F[K^*]$ and $F(K^*)$. Since for any $x \in M$ we have $xK^*x^{-1} = K^*$, we obtain $xF[K^*]x^{-1} = F[K^*]$ and consequently, $xF(K^*)x^{-1} = F(K^*)$. Thus, $< F(K^*)^*, M > \subset N_{D*}(F(K^*)^*)$. If $F(K^*)^* \not\subset M$, then, by Cartan-Brauer-Hua Theorem (cf. [12, p. 427]), either $F(K^*) = D$ or $F(K^*)^* \subset F^*$. The first case contradicts the noncommutativity of M and the second case says that $F(K^*)^* \subset F^* \subset M$ which is also a contradiction. Thus, $F(K^*)^* \subset M$. Now, by maximality of K^*, we obtain $F(K^*)^* = K^*$, i. e., $K = K^* \cup \{0\}$ is a subfield of D.

(ii) We know that for every $a \in M$ we have $aKa^{-1} = K$. Thus, the mapping $\phi_a : K \to K$ given by $\phi_a(x) = axa^{-1}$ is an automorphism of K that leaves every element of F fixed. We claim that only the elements of F remain fixed under all automorphisms of the above form. This follows from the fact that M is irreducible, i. e., $D = F(M)$, thus the centralizer of M in D^* is exactly F^*. It is clear that all automorphisms ϕ_a, $a \in M$ form a group G, say. Now, consider the mapping $f : M \to G$ given by $f(a) = \phi_a$. By definition, f is an epimorphism and we have $\ker f = \{a \in M \mid \phi_a = 1\} = \{a \in M \mid axa^{-1} = x\} = C_M(K^*) = L$, and this completes the proof of (ii).

(iii) If $[K : F] < \infty$, the fixed field of G is F and $G \subset Aut(K)$, then it is basic Galois theory that K/F is Galois, G is finite, and G is the Galois group.

The next result essentially provides a necessary and sufficient condition under which the multiplicative group of a maximal subfield of D is contained in a maximal subgroup of D^*.

LEMMA 2. *Let D be a division algebra of finite dimension over its centre F. Assume that M is an irreducible maximal subgroup of D^* containing F^*. Let K^* be a maximal abelian normal subgroup of M with $L = C_M(K^*)$. Then we have*

(i) $[F[M] : F[L]] = [M : L]$.

(ii) *$K^* = L$ if and only if $K = C_D(K)$, i.e., K is a maximal subfield of D. Therefore, if $K^* = L$, then $M/K^* \cong Gal(K/F)$.*

PROOF. (i) since $[D : F] < \infty$, using Lemma 1, we obtain $[M : L] < \infty$. Let $m_1, \ldots, m_r$ be distinct representatives of the cosets of L in M, i.e., $M = L_{m1} \cup \ldots \cup Lm_r$. Therefore, each element of $F[M]$ may be represented in the form $\sum_1^r l_i m_i$ with $l_i \in F[L]$. We claim that $\{m_i\}_1^r$ are linearly independent over $F[L]$. To see this, assume that $l_1m_1 + \ldots + l_sm_s = 0$ is a nontrivial relation containing the smallest number of nonzero terms. Since $L = C_M(K^*)$ and m_1, m_2 belong to distinct cosets of L, there exists an element $u \in K^*$ such that $u_1 = m_1 u m_1^{-1} \neq u_2 = m_2 u m_2^{-1}$. Thus, from our minimal relation we conclude that $(l_1 m_1 + \ldots + l_s m_s)u - u_1(l_1 m_1 + \ldots + \ldots l_s m_s) = (u_2 - u_1)l_2 m_2 + \ldots (u_s - u_1)l_s m_s = 0$ with $u_s = m_s u m_s^{-1}$. But this contradicts the choice of s and so the result follows.

(ii) Assume that K is a maximal subfield of D. Then $C_{D*}(K^*) = K^*$ and therefore we have $L = C_M(K^*) = K^*$. On the other hand, assume that $L = K^*$. Since M is irreducible we have $D = F[M]$ and so from (i) we conclude that $[D : K] = [K : F]$, and therefore $[D : F] = [K : F]^2$ which implies that K is a maximal subfield of D. This completes the proof of the lemma.

Before proving our next result we need also the following lemma which will be used frequently throughout.

Lemma 3. *Let D be a division algebra of finite dimension over its centre F. Then every soluble subgroup of D^* has an abelian normal subgroup of finite index.*

Proof. Let S be a soluble subgroup of D^*. Since $[D : F] < \infty$, S is a linear group. Now, by Kochlin-Maltsev's Theorem (cf. [19, p. 146]), we conclude that S contains a subgroup T of finite index such that T' is unipotent. Since the only unipotent element in a division ring is the identity, we obtain $T' = \{1\}$. Thus S contains an abelian group of finite index and consequently S contains an abelian normal subgroup A of finite index and thus the lemma follows.

Using above results, we are now able to prove a modified version of a theorem of Suprunenko (cf. [14]) for maximal subgroups of D^* which are soluble.

Corollary 4. *Let D be a finite dimensional division algebra with centre F. Assume that M is a noncommutative maximal subgroup of D^*. Then M is soluble if and only if there is a maximal subfield K of D such that K^* is normal in M with $M/K^* \cong Gal(K/F)$, and $Gal(K/F)$ is soluble.*

PROOF. One way is clear. To prove the other way, assume that M is soluble. We have either $F(M)^* = M$ or $F(M) = D$. The first case can not occur, by Hua's Theorem (cf. [6, p. 223]). The same result also implies that D' is not contained in M. Thus M is an irreducible soluble maximal subgroup of D^* containing F^*. Thus, by Lemma 3, M contains an abelian normal subgroup A of finite index. If $A \subset F^*$, then M/F^* is finite. By Corollary 4 of [1], we conclude that M is commutative which is contradiction. Therefore, F^* is properly contained in A. Take a maximal abelian normal subgroup K^*, say, in M which contains A. By part (iii) of Lemma 1, we conclude that $K = K^* \cup \{0\}$ is a normal separable extension field of F and we have $Gal(K/F) \cong M/L$, where $L = C_M(K^*)$. We now claim that K is a maximal subfield of D. To see this, assume that $C_{D^*}(K^*)$ is not contained in M. Then $< C_{D^*}(K^*), M >\subset N_{D^*}(K^*)$ and thus $D^* = N_{D^*}(K^*)$ which implies that $K \subset F$ which is impossible since F^* is contained properly in K^*. Thus, we must have $C_{D^*}(K^*) \subset M$ and since M is soluble we obtain $C_{D^*}(K^*)$ is soluble. Now, by Hua's Theorem, we conclude that $C_D(K)$ is commutative. This implies that K is maximal in D and the claim is established. Now, by part (ii) of Lemma 2, we obtain $L = K^*$ and since M is soluble the result follows.

EXAMPLE. Let D be the real quaternion algebra. It is known that D' consists of elements $a+bi+cj+dk$ with $a^2+b^2+c^2+d^2 = 1$. One may easily check that $D^* = F^*D'$ as well as $G(D) = 1$, where $F = R$ is the field of real numbers. Here we show that the subgroup $M := C^* \cup C^*j$, where C is the field of complex numbers, is a maximal subgroup of D^*. It is shown in [15] that M is soluble and so this maximal subgroup satisfies the conclusion of the above corollary. Here we also observe that $M = N_{D^*}(C^*)$, and thus the normalizer of the multiplicative group of a maximal subfield of D may be a maximal subgroup of D^*. To show that M is maximal in D^*, it is enough to prove that $M \cap D'$ is maximal in D' since $D^* = F^*D'$. To see this, we first identify the quaternion $a+bi+cj+dk$ with the complex 2×2 matrix $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$. Therefore, D' is the group of unitary matrices of determinant 1. Now, one should specify the group $M \cap D'$ in terms of matrices and show that for any

$x \in D'$ not in $M \cap D'$ we have $< x, M \cap D' >= D'$. This involves a lot of calculations and we skip this method. But there is a simpler geometric method that is presented here for which I am indebted to Professor C. Ohn: We let complex matrices act on $C \cup \{\infty\}$ by homographies. We then identify $C \cup \{\infty\}$ with the unit sphere $S^2 \subset R^3$ via the stereographic projection, it is then well known that D' acts by rotations on S^2, and that the corresponding morphism π from D' to $SO(3)$ is surjective with kernel $\{\pm 1\}$. Furthermore, π maps $C \cap D'$ to the rotations around the polar axis and $Cj \cap D'$ to the half-turns around an equatorial axis. Therefore, $H = \pi((C \cup Cj) \cap D') \subset SO(3)$ is the subgroup of rotations that leave the equator invariant, and it is clearly enough to show that H is maximal in $SO(3)$. Suppose $x \in SO(3)$ not in H and we show that $K =< H, x >= SO(3)$. To do this, we shall use a well known result which asserts that if a group G acts on a set X, and if K is a subgroup of G such that K acts transitively on X and K contains the stabilizer G_x of some $x \in X$, then $K = G$. Here, put $G = SO(3)$ and $X = S^2$. The second condition of the mentioned result is satisfied for x the north pole. The H-orbits in S^2 are the sets of the form $P_\alpha \cup P_{-\alpha}$, where P_α $(-\pi/2 \leq \alpha \leq \pi/2)$ is the parallel of latitude α. Since $x \notin H$, x maps the equator E to a great circle $E' \neq E$. Since E' hits all parallels between some extreme latitudes $-\alpha$ and α (α the angle between E and E'), for $-\alpha \leq \beta \leq \alpha$ the whole zone $Z_{[-\alpha,\alpha]} = \cup P_\beta$ between those extreme latitudes will be contained in a unique K-orbit. This argument may be repeated with E replaced by $Z_{[-\alpha,\alpha]}$, showing that $Z_{[-2\alpha,2\alpha]}$ is contained in a unique K-orbit. Repeating again and again, this K-orbit is eventually seen to cover the whole sphere S^2, so the first condition is also satisfied, and the proof is complete.

Corollary 5. *Let D be a division algebra of finite dimension over its centre F. Assume that M is a noncommutative maximal subgroup of D^*. If M is soluble, then D is a crossed product division algebra. Equivalently, the multiplicative group of a noncrossed product division algebra can not have any noncommutative soluble maximal subgroup.*

Given a finite dimensional division algebra D with centre F whose characteristic is different from the degree of D over F, in [1] it is shown that if M is a maximal subgroup of D^* and for each element $x \in M$ there exists a positive integer $n(x)$, depending on x, such that $x^{n(x)} \in F$, then $D = F$. Here, we present a variation of this result which deals only with the commutativity of M as follows:

THEOREM 6. *Let D be a division algebra of finite dimension over its centre F. Suppose M is a maximal subgroup of $D^* \neq F^*$ and $M/(M \cap F^*)$ is torsion, then $F^* \subset M$, $M = K^*$ for K a maximal subfield of D, F has characteristic $p > 0$, K/F is purely inseparable, and D has degree p.*

PROOF. Suppose M is a maximal subgroup of $D^* \neq F^*$ and $M/(M \cap F^*)$ is torsion. We claim that $F^* \subset M$, $M = K^*$ for K a maximal subfield of D, F has characteristic $p > 0$, K/F is purely inseparable. Once the claim is established, then using the result mentioned before the theorem, we conclude that D has degree p. To prove our claim, consider the division algebra $F(M)$ generated by F and M. By maximality of M, we have either $F(M)^* = M$ or $D = F(M)$. If the first case occurs, by a result of Kaplansky (cf. [6, p. 259]), $K := F(M)$ is commutative. Therefore, K is a maximal subfield of D and it is radical over F. Thus, by Kaplansky's Lemma (cf. [6, p. 258]), we conclude F has characteristic $p > 0$ and either K is algebraic over the prime subfield or K is purely inseparable over F. If K is algebraic over the prime subfield, then, by a result of Jacobson (cf. [6, p. 219]), $D = F$ which is impossible. Thus, K is purely inseparable over F. The second case implies that M is irreducible. We assume first that the characteristic of F is $p > 0$. Take an element $x \in D' \cap M$. Since M/F^* is torsion, we know that $x^{n(x)} = a \in F^*$. Thus, we conclude that $1 = RN_{D/F}(x)^{n(x)} = a^m$, where $RN_{D/F}$ is the reduced norm function of D to F. Therefore, $M' \subset M \cap D'$ is torsion and consequently M' is locally finite by Schur's Theorem (cf. [6, p. 154]). If $a, b \in M'$, then the subgroup $< a, b >$ is finite. Since $Char F = p$ we conclude that $< a, b >$ is cyclic (cf. [6]), and thus M' is abelian. Therefore, M is a maximal irreducible subgroup of D^* which is soluble. If M is commutative, we are done. Otherwise, by Corollary 4, we conclude that there exists a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$. Since M/F^* is torsion, this implies that K is radical over F. Now, by Kaplansky's Lemma, we have either K is purely inseparable over F or K is algebraic over its prime subfield. The first case can not happen since K/F is Galois and the second case, by the result of Jacobson again, leads to the commutativity of D which is nonsense.

Finally, consider the zero characteristic case. Since $[D : F] = n$, M is a linear group in $GL_n(F)$. By a theorem of Tits (cf. [17]), either M contains a non-abelian free subgroup or it is soluble-by-finite. The first case can not occur since M/F^* is torsion. Therefore, there is a soluble subgruop S in M with $[M : S] < \infty$. By Lemma 3, we conclude that S contains an abelian normal subgroup of finite index and consequently M contains an abelian normal subgroup A of finite index. Put $K = F(A)$. Then we have $< K^*, M >\subset N_{D^*}(K^*)$. If $K^* \not\subset M$, then $N_{D^*}(K^*) = D^*$ and so, by Cartan-Brauer-Hua Theorem, we conclude that $K = F$, i.e., $K^* = F^* \subset M$ which is nonsense. Otherwise, assume that $K^* \subset M$. Therefore, K is radical over F. Thus, using Kaplansky's Lemma again, we obtain $Char F = p > 0$ which is a contradiction. This completes the proof of the theorem.

We observe that in Theorem 6, in characteristic $p > 0$, if $M/M \cap F^*$ is torsion, it is not known if D is commutative. But we have the following

Corollary 7. *Let D be a division algebra of finite dimension over its centre F and assume that D^* has maximal subgroups. If $M/F^* \cap M$ is torsion for every maximal subgroup M of D^*, then $D = F$.*

Proof. Consider the group $G(D) = D^*/F^*D'$. By Corollary 1 of [9], we know that $G(D)$ is torsion of a bounded exponent dividing the index of D over F . If $G(D)$ is not trivial, then by Bear-Prufer Theorem (cf. [11, p. 105]), we conclude that there is a maximal subgroup M, say, of D^* containing D'. But then since $M/F^* \cap M$ is torsion we obtain that $D'/Z(D')$ is torsion. Now, by Lemma 2 of [10], we conclude that $D = F$. Therefore, we may assume that $D^* = F^*D'$ and none of the maximal subgroups of D^* contains D'. Thus, by Proposition 1 of [1], every maximal subgroup M of D^* contains F^*. By Theorem 6, we conclude that M is commutative. Since $G(D)$ is trivial we obtain $M = F^*(M \cap D')$. Because M is maximal in D^* one can easily conclude that $L := M \cup \{0\}$ is a maximal subfield of D. Now, L is radical over F and thus, by Kaplansky's Lemma, we conclude that $char F = p$ and either L

is algebraic over the prime subfield or L is purely inseparable over F. The first case via a theorem of Jacobson leads to $D = F$ and the second case implies that $L \cap D' = M \cap D'$ contains purely inseparable elements. Now, by Corollary 8 of [10], we know that this is not possible unless $M \cap D' = F^* \cap D' = Z(D')$. Therefore, we obtain $M = F^*(M \cap D') = F^*(F^* \cap D') = F^* Z(D') = F^*$, which is a contradiction and so the result follows.

We are now in a position to prove our main result as

THEOREM 8. *Let D be a division algebra of finite dimension over its centre F. Assume that M is a noncommutative maximal subgroup of D^*. Then either M contains a noncyclic free subgroup or there exists a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$.*

PROOF. Let M be a noncommutative maximal subgroup of D^*. We know, by proposition 1 of [1], that either $D' \subset M$ or $F^* \subset M$. If $D' \subset M$, then M is normal in D^* and so the result follows by a theorem of [5]. So, we may assume that $F^* \subset M$ but $D' \not\subset M$. Since $[D : F] < \infty$ we may consider M as a linear group. If M does not contain a noncyclic free subgroup, then every finitely generated subgroup of M does not contain a noncyclic free subgroup. By Tit's theorem (cf. [17]), we conclude that every finitely generated subgroup of M contains a soluble normal subgroup of finite index. Therefore, by a result of Wehrfritz (cf. [18]), $M/Solv(M)$ is a torsion linear group, where $Solv(M)$ is the unique maximal soluble normal subgroup obtained by Zassenhaus-Maltsev Theorem (cf. [2]). Put $S = Solv(M)$. If $M = S$, then M is a soluble maximal subgroup of D^*. Therefore, by Corollary 4, the result follows for the case $Solv(M) = S = M$. Thus, we assume that $M \neq S$, i.e., S is a proper maximal soluble normal subgroup of M. If $S = F^*$, then M/F^* is torsion. Thus, by Theorem 6, we conclude that M is commutative which is a contradiction to the fact that M is noncommutative. So, we may assume that $F^* \subset S \subset M$. Now, consider the division subring $E = F(S)$ generated by F and S.

If $E^* = F(S)^* \subset M$, then by a theorem of [5], E^* contains a noncyclic free subgroup and so does M unless $F(S)$ is commutative. Now, since $F(S)$ is commutative $F(S)^*$ is a solube normal subgroup of M. By maximality of S we obtain $F(S)^* = S$. Therefore, S is a maximal abelian normal subgroup of

M properly containing F^*. By Lemma 1, $K = S \cup \{0\}$ is a normal separable field extension of F such that K^* is normal in M and $M/L \cong Gal(K/F)$ with $L = C_M(K^*)$. By Lemma 2, it remains to show that K is a maximal subfield of D. To see this, assume that $C_{D^*}(K^*)$ is not contained in M. Then $< C_{D^*}(K^*), M > \subset N_{D^*}(K^*)$ and thus $D^* = N_{D^*}(K^*)$ which implies that $K \subset F$ which is impossible since F^* is contained properly in K^*. Thus, we must have $C_{D^*}(K^*) \subset M$ and since M/K^* is torsion, by Kaplansky's Theorem, we conclude that $C_D(K)$ is commutative. This implies that K is maximal in D and so the result follows in this case.

If $E^* = F(S)^* \not\subset M$, then by maximality of M in D^* we have $D^* =< E^*, M > \subset N_{D^*}(E^*)$ and so we have $D^* = N_{D^*}(E^*)$. Thus, by Cartan-Brauer-Hua Theorem, we have either $E \subset F$ or $F(S) = E = D$. If $E \subset F$, then $S \subset F^*$ which is not possible. Therefore, $D = F(S)$. Now, S is a soluble linear group. By Lemma 3, we conclude that S contains an abelian normal subgroup A of finite index. If $A \subset F^*$, then M/F^* is torsion and, by Theorem 6, we conclude that M is commutative which is a contradiction. Therefore, there is a maximal abelian normal subgroup K^* of M containing A which is also contained in S, by maximality of S. Since M is irreducible, by Lemma 1, $K = K^* \cup \{0\}$ is a separable normal field extension of F such that K^* is normal in M and $M/L \cong Gal(K/F)$ with $L = C_M(K^*)$. As in the previous case, one can show that K is a maximal subfield of D. Therefore, by Lemma 2, we conclude that $K^* = L$ and so the result follows.

Corollary 9. *Let D be a division algebra of finite dimension over its centre F, and M be a noncommutative maximal subgroup of D^*. Then M satisfies a group identity if and only if there exists a maximal subfield K of D such that K^* is normal in M and $M/K^* \cong Gal(K/F)$.*

Therefore, if D is a noncrossed product division algebra and M is a noncommutative maximal subgroup of D^*, then M does not satisfy any group identity.

The author thanks the referee for his constructive comments. He is also indebted to the Research Council of Sharif University of Technology for support.

References

[1] S. Akbari, M. Mahdavi-Hezavehi, M. G. Mahmudi, *Maximal subgroups of $GL_1(D)$*, Journal of Alg. 217, 422-433 (1999).

[2] D. Dixon, The structure of linear groups, Van Nostrand, London, (1971).

[3] J. Goncalves, M. Shirvani, *On free group algebras in division rings with uncountable center*, Proc. Amer. Math. Soc. Vol **124**, No. **3** (1996), 685-687.

[4] J. Goncalves, and A. Mandel, *Are there free groups in division rings*, Israel J. Math. Vol. 51, No. 1, (1986), 69-80.

[5] J. Goncalves, *Free groups in subnormal subgroups and the residual nilpotence of the group of units of group rings*, Can. Math. Bull. 27(1982), 365-370.

[6] T. Y. Lam, A first course in non-commutative rings, GTM, No. 131, Springer-Verlag, (1991).

[7] A. E. Lichtman, *On subgroups of the multiplicative group of skew fields*, Proc. Amer. Math. Soc. 63(1977), 15-16.

[8] M. Mahdavi-Hezavehi, M. G. Mahmudi, S. Yasamin, *Finitely generated subnormal subgroups of $GL_n(D)$ are central*, Journal of Algebra 225, 517-521 (2000).

[9] M. Mahdavi-Hezavehi, *Commutator subgroups of finite dimensional division algebras*, Rev. Roumaine Math. Pures Appl., 43(1998), 9-10, 853-867.

[10] M. Mahdavi-Hezavehi, *On derived groups of division rings*, Comm. Algebra, Vol.23(3), 913-926, (1995).

[11] Derek J. S. Robinson, A course in the theory of groups, Graduate Text in Mathematics, No. 80, Springer-Verlag, (1982).

[12] W. R. Scott, Group theory, Dover Publication, INC, 1987.

[13] M. Shirvani, B. A. F. Wehrfritz, Skew linear groups, LMS Lecture Note Series, No. 118, (1986).

[14] D. A. Suprunenko, *Soluble and nilpotent linear groups*, Translation of Mathematical Monographs, Vol. 9, (1963).

[15] D. A. Suprunenko, *On solvable subgroups of multiplicative groups of a field*, English Transl., Amer. Math. Soc. Transl., (2)46(1965), p. 153-161.

[16] D. A. Suprunenko, Matrix groups, Translations of Math. Mono., Amer. Math. Soc., Providence, Rhode Island, (1976).

[17] J. Tits, *Free subgroups in linear groups*, J. of Algebra, 20, 250-270 (1972).

[18] B. A. F. Wehrfritz, *2-generator conditions in linear groups*, Archiv. Math. 22, 237-240 (1971).

[19] A. E. Zalesskii, Linear groups, in "Algebra IV," part II (A. I. Kostrikin, I. R. Shafarevich, Eds.), Encyclopedia of Mathematics and Science, Springer-Verlag, Berlin/New York, 1993.

mahdavi@math.sharif.ac.ir
Department of Mathematical Sciences
Sharif University of Technology
P. O. Box 11365-9415
Tehran, Iran