DISCRIMINANT AND CLIFFORD ALGEBRAS

ANNE QUÉGUINER-MATHIEU AND JEAN-PIERRE TIGNOL

Abstract. The centralizer of a square-central skew-symmetric unit in a central simple algebra with orthogonal involution carries a unitary involution. The discriminant algebra of this unitary involution is shown to be an orthogonal summand in one of the components of the Clifford algebra of the orthogonal involution. As an application, structure theorems for orthogonal involutions on central simple algebras of degree 8 are obtained.

Throughout this paper, F denotes a field of characteristic different from 2. Let A be a central simple F-algebra of degree $n = 4m$, for some integer m, endowed with an involution σ of orthogonal type. In the first two sections, we assume that the algebra A contains an element θ such that $\sigma(\theta) = -\theta$ and $\theta^2 = a \in F^\star$. We denote by $\tilde{A}$ the centralizer of θ in A. Since θ is skew-symmetric, σ induces an involution $\tilde{\sigma}$ on $\tilde{A}$, and $(\tilde{A}, \tilde{\sigma})$ is a central simple algebra with unitary involution of degree $2m$ over the étale quadratic extension $F(\theta)$ of F.

With this data, we may associate two different central simple algebras with involution, namely the Clifford algebra of (A, σ), and the discriminant algebra of $(\tilde{A}, \tilde{\sigma})$, both endowed with their canonical involution (see [14, §8, §10]). Our aim in this paper is to relate these two algebras with involution. Reversing the viewpoint, we show in section 3 that every central simple algebra of even degree $2m$ with unitary involution and exponent 2 over a quadratic extension of F can be embedded in a central simple F-algebra of degree $4m$ with orthogonal involution as the centralizer of some skew-symmetric, square-central element. Therefore, our results yield information on discriminant algebras of any central simple algebra of even degree with unitary involution and exponent 2.

The algebra with involution (A, σ) is said to be decomposable if A contains a non trivial σ-stable central simple subalgebra A_1. Indeed, if this holds, then A is the tensor product of A_1 and its centralizer A_2, and σ is the tensor product of its restrictions to A_1 and A_2. If A has degree 4, criteria of decomposability have been proven for the different types of involutions. Namely, (A, σ) is always decomposable if σ is of symplectic type; it is decomposable if and only if the discriminant of σ is trivial (resp. the discriminant algebra of (A, σ) is split) if σ is of orthogonal type (resp. of unitary type) (see [14, (16.16)], [12, (5.2)] and [13]). For higher degrees, it is still true that any decomposable orthogonal involution on a 2-power degree algebra has trivial discriminant, but this condition is not sufficient anymore (see [6], [17] and [18] for examples of indecomposable involutions with trivial discriminant). The main result of section 4 is Theorem 4.3, which gives a necessary and sufficient

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condition for a central simple algebra of degree 8 with orthogonal involution to be decomposable: one of the components of the Clifford algebra must contain a square-central symmetric unit whose centralizer is split. In particular, we recover the fact that one of the components of the Clifford algebra of a degree 8 decomposable algebra with orthogonal involution is of index at most 2 proven in [17, 5.17]. We also give in Proposition 4.12 a necessary and sufficient condition for a degree 4 algebra B with a K/F-unitary involution τ to decompose as $(B,\tau)=(B_0,\tau_0)\otimes_F(K,{}^-)$ where (B_0,τ_0) is a central simple algebra with involution of the first kind and ${}^-$ denotes the canonical involution of the quadratic extension K/F (compare with [14, (2.22)]).

Returning to the general situation described at the beginning, we now sketch the relations between the Clifford algebra $\mathcal{C}(A,\sigma)$ and the discriminant algebra $\mathcal{D}(\tilde{A},\tilde{\sigma})$ proven in sections 1 and 2. The discriminant of the orthogonal involution σ is easily determined: since $\sigma(\theta)=-\theta$, we have

$$\operatorname{disc}(\sigma)=\operatorname{Nrd}_A(\theta)=N_{F(\theta)/F}\bigl(\operatorname{Nrd}_{\tilde{A}}(\theta)\bigr)=N_{F(\theta)/F}(\theta^{2m})=a^{2m}\in F^{\star 2}. \tag{1}$$

Therefore, by [14, (8.10)], the center $Z(A,\sigma)$ of the Clifford algebra $\mathcal{C}(A,\sigma)$ is isomorphic to $F\times F$, and $\mathcal{C}(A,\sigma)$ is a product of two central simple F-algebras of degree 2^{2m-1}, which we denote by C_1 and C_2. Jacobson proved that $C_1\otimes_F C_2$ is Brauer-equivalent to A (see [14, §9.C]), and it can be derived from Proposition 3.2.3 in Bayer-Fluckiger–Parimala [2] that one of C_1, C_2 is Brauer-equivalent to the discriminant algebra $\mathcal{D}(\tilde{A},\tilde{\sigma})$. However, we aim to obtain more precise information, taking into account the canonical involutions on the Clifford and discriminant algebra.

To give precise statements, we note that it is possible to use θ to discriminate between the components C_1, C_2 of $\mathcal{C}(A,\sigma)$, as follows: recall from [14, §8.D] that there is a canonical homogeneous polynomial map of degree $2m$ defined on the space $\operatorname{Skew}(A,\sigma)$ of skew-symmetric elements

$$\pi\colon\ \operatorname{Skew}(A,\sigma)\to Z(A,\sigma)$$

such that $T_{Z(A,\sigma)/F}\bigl(\pi(s)\bigr)=0$ and $\pi(s)^2=\operatorname{Nrd}_A(s)$ for all $s\in\operatorname{Skew}(A,\sigma)$. (In the split case, π is essentially the pfaffian map, see [14, (8.26)].) From (1), it follows that $\pi(\theta)^2=a^{2m}$; therefore, the elements

$$z^+=\tfrac12\bigl(1+a^{-m}\pi(\theta)\bigr)\qquad\text{and}\qquad z^-=\tfrac12\bigl(1-a^{-m}\pi(\theta)\bigr)$$

are orthogonal idempotents in $Z(A,\sigma)$, and we have $\mathcal{C}(A,\sigma)=\mathcal{C}^+(A,\sigma)\times\mathcal{C}^-(A,\sigma)$ with

$$\mathcal{C}^+(A,\sigma)=\mathcal{C}(A,\sigma)z^+\qquad\text{and}\qquad\mathcal{C}^-(A,\sigma)=\mathcal{C}(A,\sigma)z^-.$$

Note that, although this is not apparent from the notation, the determination of which component of $\mathcal{C}(A,\sigma)$ is $\mathcal{C}^+(A,\sigma)$ and which is $\mathcal{C}^-(A,\sigma)$ really depends on the choice of θ: a different choice of θ may interchange the components $\mathcal{C}^+(A,\sigma)$ and $\mathcal{C}^-(A,\sigma)$, see Proposition 4.13 and Corollary 4.14.

The canonical involution $\underline{\sigma}$ on $\mathcal{C}(A,\sigma)$ restricts to involutions of the first kind on $\mathcal{C}^+(A,\sigma)$ and $\mathcal{C}^-(A,\sigma)$, which we denote again by $\underline{\sigma}$. These involutions are both orthogonal if m is even and both symplectic if m is odd. In section 2, we consider the case where $a\notin F^{\star 2}$. We construct a quaternion F-algebra Q (whose definition depends on θ) and canonical representations

$$\Theta^+\colon\ \mathcal{C}^+(A,\sigma)\xrightarrow{\sim}\operatorname{End}_Q E^+,\qquad\Theta^-\colon\ \mathcal{C}^-(A,\sigma)\xrightarrow{\sim}\operatorname{End}_{A\otimes Q}E^-$$

(see Propositions 2.3 and 2.8). Moreover, we define $(-1)^{m+1}$-hermitian forms h^+ on E^+ and h^- on E^- (with respect to the conjugation involution $^-$ on Q and the involution $\sigma\otimes{}^-$ on $A\otimes Q$, respectively) whose adjoint involutions correspond to the canonical involutions on $\mathcal{C}^+(A,\sigma)$ and $\mathcal{C}^-(A,\sigma)$ under Θ^+ and Θ^-.

We also construct canonical orthogonal decompositions

$$(E^+,h^+)=\begin{cases}\Big(\bigoplus^{\perp}_{0\le r<m/2}(\hat{E}^+_r,h^+_r)\Big)\overset{\perp}{\oplus}(E^+_{m/2},h^+_{m/2}) & \text{if } m \text{ is even,}\\ \bigoplus^{\perp}_{0\le r\le(m-1)/2}(\hat{E}^+_r,h^+_r) & \text{if } m \text{ is odd,}\end{cases}\tag{2}$$

$$(E^-,h^-)=\begin{cases}\bigoplus^{\perp}_{0\le r<m/2}(\hat{E}^-_r,h^-_r) & \text{if } m \text{ is even,}\\ \Big(\bigoplus^{\perp}_{0\le r<(m-1)/2}(\hat{E}^-_r,h^-_r)\Big)\overset{\perp}{\oplus}(E^-_{(m-1)/2},h^-_{(m-1)/2}) & \text{if } m \text{ is odd,}\end{cases}\tag{3}$$

see Propositions 2.5 and 2.9. The relation with the discriminant algebra $\mathcal{D}(\tilde{A},\tilde{\sigma})$ is the following: there is a canonical isomorphism

$$\Psi_m\colon\ \mathcal{D}(\tilde{A},\tilde{\sigma})\xrightarrow{\sim}\begin{cases}\operatorname{End}_Q E^+_{m/2} & \text{if } m \text{ is even,}\\ \operatorname{End}_{A\otimes Q} E^-_{(m-1)/2} & \text{if } m \text{ is odd,}\end{cases}$$

under which the canonical involution on $\mathcal{D}(\tilde{A},\tilde{\sigma})$ corresponds to the adjoint involution with respect to $h^+_{m/2}$ or $h^-_{(m-1)/2}$ respectively (see Corollary 2.13). Thus, the algebra with involution $\big(\mathcal{D}(\tilde{A},\tilde{\sigma}),\underline{\tilde{\sigma}}\big)$ is an "orthogonal direct summand" (in the sense of [4]) of $\big(\mathcal{C}^+(A,\sigma),\underline{\sigma}\big)$ if m is even, of $\big(\mathcal{C}^-(A,\sigma),\underline{\sigma}\big)$ if m is odd.

The other terms in the decompositions (2) and (3) above can also be related to the centralizer $\tilde{A}$: we show in Propositions 2.5 and 2.10 that $(\hat{E}^+_r,h^+_r)$ for $r<m/2$ and $(\hat{E}^-_r,h^-_r)$ for $r<(m-1)/2$ are in a certain sense "extended" from $(-1)^m$-hermitian spaces (E^+_r,ℓ^+_r) and (E^-_r,ℓ^-_r) over $F(\theta)$ and $A\otimes F(\theta)$ respectively, and that there are canonical isomorphisms

$$\Psi_{2r}\colon\ \lambda^{2r}\tilde{A}\xrightarrow{\sim}\operatorname{End}_{F(\theta)}E^+_r,\qquad \Psi_{2r+1}\colon\ \lambda^{2r+1}\tilde{A}\xrightarrow{\sim}\operatorname{End}_{A\otimes F(\theta)}E^-_r$$

under which the involutions $\tilde{\sigma}^{\wedge 2r}$, $\tilde{\sigma}^{\wedge 2r+1}$ correspond to the adjoint involutions with respect to ℓ^+_r and ℓ^-_r respectively (see Proposition 2.12).

If $a\in F^{\star 2}$, then substituting $\sqrt{a}^{-1}\theta$ for θ we may assume $\theta^2=1$. The element $e=\frac{1}{2}(1+\theta)\in A$ is an idempotent such that $\sigma(e)=1-e$, hence σ is hyperbolic by [3]. In this case, $F(\theta)\simeq F\times F$, so $\tilde{A}\simeq B^{\mathrm{op}}\times B$ for some central simple F-algebra B of degree $2m$ and $\tilde{\sigma}$ is the exchange involution on $B^{\mathrm{op}}\times B$. The results above then take the following simple form: there are canonical representations

$$\Theta^+\colon\ \big(\mathcal{C}^+(A,\sigma),\underline{\sigma}\big)\xrightarrow{\sim}(\operatorname{End}_F E^+,\operatorname{ad}_{f^+}),$$
$$\Theta^-\colon\ \big(\mathcal{C}^-(A,\sigma),\underline{\sigma})\big)\xrightarrow{\sim}(\operatorname{End}_A E^-,\operatorname{ad}_{f^-})$$

for some canonical $(-1)^m$-symmetric form f^+ and some $(-1)^m$-hermitian form f^- with respect to σ (see Propositions 1.1 and 1.5). Moreover, there is a canonical decomposition

$$E^+=\bigoplus_{r=0}^{m}E^+_r$$

such that $f^+(\xi,\eta)=0$ for $\xi\in E_r^+$ and $\eta\in E_s^+$, unless $r+s=m$ (see Proposition 1.1). Therefore, for $r\leq (m-1)/2$ the space $\hat{E}_r=E_r^+\oplus E_{m-r}^+$ is a hyperbolic subspace of E^+, and E^+ is Witt-equivalent to $E_{m/2}^+$ if m is even (and it is hyperbolic if m is odd, but this is clear *a priori* since f^+ is skew-symmetric in this case).

Similarly, there is a decomposition

$$E^-=\bigoplus_{r=0}^{m-1}E_r^-$$

such that $f^-(\xi,\eta)=0$ for $\xi\in E_r^-$ and $\eta\in E_s^-$, unless $r+s=m-1$ (see Proposition 1.5). Therefore, the space $\hat{E}_r^-=E_r^-\oplus E_{m-1-r}^-$ is hyperbolic, and E^- is hyperbolic if m is even, and Witt-equivalent to $E_{(m-1)/2}^-$ if m is odd.

The relation with the centralizer $\tilde{A}=B^{\mathrm{op}}\times B$ is as follows: there are canonical isomorphisms

$$\Psi_{2r}\colon\ \lambda^{2r}B\xrightarrow{\sim}\operatorname{End}_F E_r^+\qquad\text{for } r=0,\,\dots\,,m,$$
$$\Psi_{2r+1}\colon\ \lambda^{2r+1}B\xrightarrow{\sim}\operatorname{End}_A E_r^-\qquad\text{for } r=0,\,\dots\,,m-1,$$

which induce an isomorphism of algebras with involution

$$\Psi_m\colon\ \big(\mathcal{D}(\tilde{A},\tilde{\sigma}),\underline{\tilde{\sigma}}\big)=(\lambda^mB,\gamma)\xrightarrow{\sim}\begin{cases}(\operatorname{End}_F E_{m/2}^+,\operatorname{ad}_{f_{m/2}^+}) & \text{if } m \text{ is even,}\\ (\operatorname{End}_A E_{(m-1)/2}^-,\operatorname{ad}_{f_{(m-1)/2}^-}) & \text{if } m \text{ is odd,}\end{cases}$$

where γ is the canonical involution on λ^mB ([14, §10.B]), see Propositions 1.9 and 1.11.

We thus recover (by substantially different methods) the results obtained by Garibaldi on the Clifford algebras of hyperbolic involutions [7], which provided the main source of inspiration for our investigations. We first discuss this particular case, to which the general case is reduced by scalar extension to $F(\sqrt{a})$.

1. The hyperbolic case

In this section, we consider the case where $a\in F^{\star 2}$. After scaling, we may assume $a=1$. The elements

$$e=\tfrac{1}{2}(1+\theta)\qquad\text{and}\qquad e'=\tfrac{1}{2}(1-\theta)$$

are idempotents and $\sigma(e)=e'=1-e$, hence σ is hyperbolic by [3].

1.1. **Representation of the Clifford algebra.** Let $c\colon A\to\mathcal{C}(A,\sigma)$ be the canonical map (see [14, (8.13)]). Consider

$$\rho(e)=c(eAe')^m\subset\mathcal{C}(A,\sigma)$$

and

$$E^+=\mathcal{C}(A,\sigma)\rho(e)\subset\mathcal{C}(A,\sigma).$$

It is known (see [9, §4] or [14, (8.29)]) that $\rho(e)$ is a 1-dimensional subspace of $\mathcal{C}(A,\sigma)$ and $\dim E^+=2^{2m-1}=\deg\mathcal{C}^{\pm}(A,\sigma)$. Moreover, Lemma 4.2 of [9] shows that $\pi(\theta)u=u$ for all $u\in\rho(e)$, hence $z^+\xi=\xi$ and $z^-\xi=0$ for all $\xi\in E^+$. Therefore, left multiplication induces an F-algebra homomorphism

$$\Theta^+\colon\ \mathcal{C}^+(A,\sigma)\to\operatorname{End}_F E^+.$$

This homomorphism is injective since $\mathcal{C}^+(A,\sigma)$ is simple, hence it is surjective by dimension count.

Consider the map $f^+\colon \mathcal{C}(A,\sigma)\times\mathcal{C}(A,\sigma)\to\mathcal{C}(A,\sigma)$ defined by

$$f^+(\xi,\eta)=\underline{\sigma}(\xi)\eta \qquad \text{for } \xi,\ \eta\in\mathcal{C}(A,\sigma), \tag{4}$$

where $\underline{\sigma}$ is the canonical involution on $\mathcal{C}(A,\sigma)$.

Proposition 1.1. *For all ξ and $\eta\in E^+$, we have $f^+(\xi,\eta)\in\rho(e)$. The map f^+ is a regular bilinear form on E^+. It is symmetric if m is even and skew-symmetric if m is odd. Letting $E_r^+=c(e'Ae)^r\rho(e)$ for $r=0,\ \dots,\ m$, we have $\dim E_r^+=\binom{2m}{2r}$ and*

$$E^+=\bigoplus_{r=0}^{m}E_r^+.$$

Moreover, for $\xi\in E_r^+$ and $\eta\in E_s^+$ we have $f^+(\xi,\eta)=0$ unless $r+s=m$.

The left multiplication isomorphism Θ^+ induces an isomorphism of algebras with involution

$$\Theta^+\colon\ (\mathcal{C}^+(A,\sigma),\underline{\sigma})\xrightarrow{\sim}(\operatorname{End}_F E^+,\operatorname{ad}_{f^+})$$

where ad_{f^+} is the involution adjoint to f^+.

Proof. It suffices to check the assertions over a scalar extension of F. Extending scalars to a splitting field of A, we are reduced to the case where A is split. Assume thus $A=\operatorname{End}_F V$ and $\sigma=\operatorname{ad}_q$, for some $4m$-dimensional hyperbolic quadratic space (V,q) over F. The subspaces $U=\operatorname{im}e$ and $W=\ker e$ are supplementary totally isotropic subspaces of V. Let $(u_1,\dots,u_{2m})$ be a basis of U, and $(w_1,\dots,w_{2m})$ a basis of W such that $q(u_i+w_j)=\delta_{ij}$. Under the canonical identification $\mathcal{C}(A,\sigma)\simeq C_0(V,q)$ ([14, (8.8)]), the vector space $c(eAe')$ (resp. $c(e'Ae)$) is the span of the products u_iu_j (resp. w_iw_j). Therefore, $\rho(e)=u_1\dots u_{2m}F$ and E_r^+ is the span of the products $w_{i_1}\dots w_{i_{2r}}u_1\dots u_{2m}$. It is canonically isomorphic to $\bigwedge^{2r}W$ under the map which carries $x_1\wedge\dots\wedge x_{2r}$ to $x_1\dots x_{2r}u_1\dots u_{2m}$, for $x_1,\ \dots,\ x_{2r}\in W$, hence $\dim E_r^+=\binom{2m}{2r}$. The equality $E^+=\bigoplus_{r=0}^m E_r^+$ is easily verified.

If $\xi=w_{i_1}\dots w_{i_{2r}}u_1\dots u_{2m}$ and $\eta=w_{j_1}\dots w_{j_{2s}}u_1\dots u_{2m}$, computation shows that $f^+(\xi,\eta)\neq 0$ if and only if each w_i, $1\le i\le 2m$ appears exactly once among w_{i_1}, $\dots,\ w_{i_{2r}},\ w_{j_1},\ \dots,\ w_{j_{2s}}$. If this condition holds, then $r+s=m$ and $f^+(\xi,\eta)=\pm u_1\dots u_{2m}$. This shows that f^+ is regular and that $f^+(\xi,\eta)\in\rho(e)$ for all ξ, $\eta\in E^+$.

For $\xi,\ \eta,\ \zeta\in\mathcal{C}(A,\sigma)$, we have

$$f^+(\xi\eta,\zeta)=\underline{\sigma}(\eta)\underline{\sigma}(\xi)\zeta=f^+\bigl(\eta,\underline{\sigma}(\xi)\zeta\bigr),$$

hence $\underline{\sigma}$ corresponds under Θ^+ to the adjoint involution with respect to f^+. □

Remark. The idea to consider the spaces E_r^+ is inspired by Garibaldi [7, §4].

Corollary 1.2. *Let $\hat{E}_r^+=E_r^+\oplus E_{m-r}^+$ for $r<m/2$. The restriction of f^+ to $\hat{E}_r^+$ is a hyperbolic form. If m is even, the restriction $f_{m/2}^+$ of f^+ to $E_{m/2}^+$ is a regular symmetric bilinear form, and (E^+,f^+) is Witt-equivalent to $(E_{m/2}^+,f_{m/2}^+)$.*

Proof. The corollary readily follows from Proposition 1.1. □

In order to obtain a representation of $\mathcal{C}^-(A,\sigma)$ similar to the representation Θ^+ of $\mathcal{C}^+(A,\sigma)$, we use the Clifford bimodule $\mathcal{B}(A,\sigma)$ defined in [14, §9]. Recall from [14, (9.7)] that $\mathcal{B}(A,\sigma)$ is a left A-module as well as a $\mathcal{C}(A,\sigma)$-bimodule, and that for $A = \operatorname{End}_F V$, $\sigma = \operatorname{ad}_q$, we have a canonical isomorphism

$$\mathcal{B}(A,\sigma) = V \otimes C_1(V,q)$$

where A acts on V and $\mathcal{C}(A,\sigma) = C_0(V,q)$ acts by multiplication. For $a \in \operatorname{End}_F V$, $c_0 \in C_0(V,q)$, $v \in V$ and $c_1 \in C_1(V,q)$ we thus set

$$a \cdot (v \otimes c_1) = a(v) \otimes c_1, \quad c_0 * (v \otimes c_1) = v \otimes (c_0c_1), \quad (v \otimes c_1) \cdot c_0 = v \otimes (c_1c_0).$$

There is a canonical A-module homomorphism $b\colon A \to \mathcal{B}(A,\sigma)$ which is given in the case where $A = \operatorname{End}_F V$ by the canonical map

$$\operatorname{End}_F V \xrightarrow{\sim} V \otimes V \hookrightarrow V \otimes C_1(V,q).$$

Using the right action of $\mathcal{C}(A,\sigma)$ on $\mathcal{B}(A,\sigma)$, define

$$E^- = \mathcal{B}(A,\sigma) \cdot \rho(e) \subset \mathcal{B}(A,\sigma).$$

The left actions of A and $\mathcal{C}(A,\sigma)$ on $\mathcal{B}(A,\sigma)$ endow E^- with a structure of left A- and $\mathcal{C}(A,\sigma)$-module. For $r = 0, \dots, m-1$, define

$$E_r^- = (Ae)^b \cdot E_r^+ \subset E^-$$

(where $(Ae)^b$ is the image of Ae under b). This is an A-submodule of E^-.

Lemma 1.3. *The dimensions of the F-vector spaces defined above are*

$$\dim E^- = 4m2^{2m-1} = \deg A \deg \mathcal{C}^{\pm}(A,\sigma), \qquad \dim E_r^- = 4m\binom{2m}{2r+1},$$

and

$$E^- = \bigoplus_{r=0}^{m-1} E_r^-.$$

Moreover, $z^+ * \xi = 0$ and $z^- * \xi = \xi$ for all $\xi \in E^-$.

Proof. As in Proposition 1.1, it suffices to consider the case where A is split. Using the same notation as in the proof of that proposition, we have

$$E^- = V \otimes C_1(V,q)u_1 \dots u_{2m}$$

and E_r^- is spanned by the products $v \otimes (w_{i_1} \dots w_{i_{2r+1}} u_1 \dots u_{2m})$. Therefore, there is a canonical isomorphism of F-vector spaces $E_r^- = V \otimes \bigwedge^{2r+1} W$ which maps $v \otimes (x_1 \wedge \dots \wedge x_{2r+1})$ to $v \otimes (x_1 \dots x_{2r+1} u_1 \dots u_{2m})$ for $v \in V$ and $x_1, \dots, x_{2r+1} \in W$, and we have $E^- = \bigoplus_{r=0}^{m-1} E_r^-$. The dimensions of E^- and E_r^- are then easily computed.

Since $\pi(\theta)$ is a trace zero element in the center of $C_0(V,q)$, it anticommutes with every element in $C_1(V,q)$, hence

$$\pi(\theta) * (v \otimes cu_1 \dots u_{2m}) = -v \otimes c\pi(\theta)u_1 \dots u_{2m} = -v \otimes cu_1 \dots u_{2m}$$

for $v \in V$ and $c \in C_1(V,q)$. Therefore, $z^+ * \xi = 0$ and $z^- * \xi = \xi$ for all $\xi \in E^-$. $\square$

From the lemma, it follows that the left action of $\mathcal{C}(A,\sigma)$ on $\mathcal{B}(A,\sigma)$ induces an F-algebra homomorphism

$$\Theta^-\colon\ \mathcal{C}^-(A,\sigma)\to \operatorname{End}_A E^-.$$

This homomorphism is injective since $\mathcal{C}^-(A,\sigma)$ is simple, hence it is surjective by dimension count.

To describe a hermitian form on E^- whose adjoint involution corresponds to $\underline{\sigma}$ under Θ^-, we use the following result:

Lemma 1.4. *There exists a $\mathcal{C}(A,\sigma)$-bimodule isomorphism*

$$\mu\colon\ \mathcal{B}(A,\sigma)\otimes_{\mathcal{C}(A,\sigma)}\mathcal{B}(A,\sigma)\to A\otimes_F\mathcal{C}(A,\sigma)$$

which, in the case where $A=\operatorname{End}_F V$ and $\sigma=\operatorname{ad}_q$, is given by

$$\mu\big((v_1\otimes c_1)\otimes(v_2\otimes c_2)\big)=(v_1\otimes v_2)\otimes c_1c_2$$

under the canonical isomorphisms $\mathcal{B}(A,\sigma)=V\otimes C_1(V,q)$, $\mathcal{C}(A,\sigma)=C_0(V,q)$ and $A=V\otimes V$. This map satisfies

$$\mu\big((a\xi)\otimes\eta\big)=a\otimes 1\cdot\mu(\xi\otimes\eta)\qquad\textit{and}\qquad \mu\big(\xi\otimes(a\eta)\big)=\mu(\xi\otimes\eta)\cdot\sigma(a)\otimes 1$$

for $a\in A$ and ξ, $\eta\in\mathcal{B}(A,\sigma)$.

Proof. Let $T(\underline{A})$ be the tensor algebra on the underlying vector space $\underline{A}$ of A, and let $T_+(\underline{A})$ be the sum of the components of strictly positive degree in $T(\underline{A})$. Recall from [14, (9.4)] the canonical representation $\rho_r\colon\ \mathfrak{S}_{2r}\to\mathrm{GL}(\underline{A}^{\otimes r})$ of the symmetric group $\mathfrak{S}_{2r}$ which for $\underline{A}=V\otimes V$ is given by

$$\rho_r(p)(v_1\otimes\cdots\otimes v_{2r})=v_{p^{-1}(1)}\otimes\cdots\otimes v_{p^{-1}(2r)}.$$

For all $r\geq 1$, let γ_r be the image of the cycle $(1,2,\dots,2r)^{-1}\in\mathfrak{S}_{2r}$ under ρ_r. We define a map $\tilde{\mu}\colon\ T_+(\underline{A})\times T_+(\underline{A})\to\underline{A}\otimes T(\underline{A})$ by

$$\tilde{\mu}(u,v)=\rho_{i+j}\big((1,2)\big)\circ\gamma_{i+j}^{-1}\big(u\otimes\gamma_j(v)\big)\qquad\text{for } u\in\underline{A}^{\otimes i}\text{ and } v\in\underline{A}^{\otimes j}.$$

Straightforward verifications in the split case show that the map $\tilde{\mu}$ induces an isomorphism μ with the required properties under the canonical epimorphisms $T(\underline{A})\to\mathcal{C}(A,\sigma)$ and $T_+(\underline{A})\to\mathcal{B}(A,\sigma)$. □

Remark. This lemma is Exercise 6 (a) in [14, Chapter II].

Recall from [14, (9.10)] the linear involution $\omega\colon\ \mathcal{B}(A,\sigma)\to\mathcal{B}(A,\sigma)$ which in the split case is given by

$$\big(x\otimes(v_1\dots v_{2r+1})\big)^{\omega}=x\otimes(v_{2r+1}\dots v_1)\qquad\text{for } x,\ v_1,\ \dots,\ v_{2r+1}\in V.$$

For ξ, $\eta\in\mathcal{B}(A,\sigma)$ we set

$$f^-(\xi,\eta)=\mu(\xi^{\omega}\otimes\eta)\in A\otimes\mathcal{C}(A,\sigma).$$

Proposition 1.5. *For all ξ, $\eta\in E^-$ we have $f^-(\xi,\eta)\in A\otimes\rho(e)$. The map f^- is a regular $(-1)^m$-hermitian form on the left A-module E^- with respect to σ, and $f^-(\xi,\eta)=0$ for $\xi\in E^-_r$ and $\eta\in E^-_s$, unless $r+s=m-1$.*

The homomorphism Θ^- is an isomorphism of algebras with involution

$$\Theta^-\colon\ \big(\mathcal{C}^-(A,\sigma),\underline{\sigma}\big)\xrightarrow{\sim}(\operatorname{End}_A E^-,\operatorname{ad}_{f^-}).$$

Proof. As in Proposition 1.1, it suffices to consider the split case. Using the same notation as in the proof of that proposition, we let $A = \operatorname{End}_F V$ and $\sigma = \operatorname{ad}_q$. Let $v_1, v_2 \in V$. For $\xi = v_1 \otimes (w_{i_1} \dots w_{i_{2r+1}} u_1 \dots u_{2m})$ and $\eta = v_2 \otimes (w_{j_1} \dots w_{j_{2s+1}} u_1 \dots u_{2m})$, we have

$$f^-(\xi, \eta) = (v_1 \otimes v_2) \otimes u_{2m} \dots u_1 w_{i_{2r+1}} \dots w_{i_1} w_{j_1} \dots w_{j_{2s+1}} u_1 \dots u_{2m}.$$

Computation shows that $f^-(\xi,\eta) \neq 0$ if and only if each w_i for $i = 1, \dots, 2m$ appears exactly once among $w_{i_1}, \dots, w_{i_{2r+1}}, w_{j_1}, \dots, w_{j_{2s+1}}$. If this condition holds, then $r + s = m - 1$ and

$$f^-(\xi, \eta) = \pm (v_1 \otimes v_2) \otimes u_1 \dots u_{2m}.$$

It follows that f^- is regular. The following equations follow from straightforward computations: for $\xi, \eta \in E^-$ and $a \in A$,

$$f^-(\eta, \xi) = (-1)^m \sigma \otimes \operatorname{Id}\bigl(f^-(\xi, \eta)\bigr),$$

$$f^-(a\xi, \eta) = a f^-(\xi, \eta) \qquad \text{and} \qquad f^-(\xi, a\eta) = f^-(\xi, \eta)\sigma(a).$$

Therefore, f^- is a $(-1)^m$-hermitian form on the left A-module E^-.

Finally, for $\xi, \eta \in \mathcal{B}(A, \sigma)$ and $c \in \mathcal{C}(A, \sigma)$ we have

$$(c * \xi)^\omega \otimes \eta = \bigl(\xi^\omega \underline{\sigma}(c)\bigr) \otimes \eta = \xi^\omega \otimes \bigl(\underline{\sigma}(c) * \eta\bigr),$$

hence $f^-(c*\xi, \eta) = f^-\bigl(\xi, \underline{\sigma}(c)*\eta\bigr)$. Therefore, $\underline{\sigma}$ corresponds to ad_{f^-} under Θ^-. □

Corollary 1.6. *Let $\hat{E}_r^- = E_r^- \oplus E_{m-1-r}^-$ for $r < (m-1)/2$. The restriction of f^- to $\hat{E}_r^-$ is a hyperbolic form. If m is even, the hermitian space (E^-, f^-) is hyperbolic. If m is odd, (E^-, f^-) is Witt-equivalent to $(E^-_{(m-1)/2}, f^-_{(m-1)/2})$, where $f^-_{(m-1)/2}$ is the restriction of f^- to $E^-_{(m-1)/2}$.*

Remark. This result was first proved by Garibaldi [7, Proposition 4.8].

1.2. **Representation of the discriminant algebra of the centralizer.** Our next goal is to relate the constructions of the preceding section to the discriminant algebra $\mathcal{D}(\tilde{A}, \tilde{\sigma})$. We shall establish isomorphisms $\mathcal{D}(\tilde{A}, \tilde{\sigma}) \simeq \operatorname{End}_F E^+_{m/2}$ if m is even, $\mathcal{D}(\tilde{A}, \tilde{\sigma}) \simeq \operatorname{End}_A E^-_{(m-1)/2}$ if m is odd, and show that the canonical involution on the discriminant algebra corresponds to the restriction of the adjoint involutions ad_{f^+} and ad_{f^-} respectively.

Lemma 1.7. *$\tilde{A} = eAe \oplus e'Ae'$.*

Proof. Since $e + e' = 1$, every element $x \in A$ decomposes as

$$x = exe + exe' + e'xe + e'xe'.$$

We have $e\theta = \theta e = e$ and $e'\theta = \theta e' = -e'$, hence x commutes with θ if and only if $exe' + e'xe = 0$. Therefore,

$$x = exe + e'xe' \in eAe \oplus e'Ae'.$$

This proves $\tilde{A} \subset eAe \oplus e'Ae'$, and the reverse inclusion is clear. □

Let $B = e'Ae'$, a central simple F-algebra of degree $2m$ which is Brauer-equivalent to A. For $x \in B$ we have $\sigma(x) \in eAe$, hence the map $(x_1^{\mathrm{op}}, x_2) \mapsto \sigma(x_1) + x_2$ defines an F-algebra isomorphism

$$B^{\mathrm{op}} \times B \xrightarrow{\sim} eAe \oplus e'Ae' = \tilde{A}.$$

Under this isomorphism, $\tilde{\sigma}$ corresponds to the exchange involution ε on $B^{\mathrm{op}} \times B$. Therefore, there is a canonical isomorphism

$$\big(\mathcal{D}(\tilde{A},\tilde{\sigma}),\underline{\tilde{\sigma}}\big) = (\lambda^m B, \gamma), \tag{5}$$

where γ is the canonical involution on $\lambda^m B$, by [14, (10.31)].

Recall from [14, (10.4)] that for $r = 0, \dots, 2m$, the algebra $\lambda^r B$ is defined as follows: let $s_r \in B^{\otimes r}$ be the element which in the split case $B = \operatorname{End}_F W$ satisfies

$$s_r(w_1 \otimes \cdots \otimes w_r) = \sum_{p\in\mathfrak{S}_r} \operatorname{sgn}(p) w_{p(1)} \otimes \cdots \otimes w_{p(r)} \qquad \text{for } w_1, \dots, w_r \in W,$$

where the sum runs over all the permutations p of $\{1,\dots,r\}$ and $\operatorname{sgn}(p)$ is the signature of p. Let $S_r = B^{\otimes r} s_r$. Then $\lambda^r B = \operatorname{End}_{B^{\otimes m}} S_r$.

To obtain an alternative description of $\lambda^r B$, consider the idealizer $\hat{S}_r$ and the annihilator S_r^0, defined by

$$\hat{S}_r = \{x \in B^{\otimes r} \mid S_r x \subset S_r\} \qquad \text{and} \qquad S_r^0 = \{x \in B^{\otimes r} \mid S_r x = 0\}.$$

As explained in [14, p.9], right multiplication induces an isomorphism

$$\hat{S}_r / S_r^0 \xrightarrow{\sim} \operatorname{End}_{B^{\otimes r}} S_r = \lambda^r B.$$

It turns out that the even powers $\lambda^{2r} B$ can be represented as $\operatorname{End}_F E_r^+$, and the odd powers $\lambda^{2r+1} B$ as $\operatorname{End}_A E_r^-$, as we now show.

First, we consider the case of even powers. Let $\sigma_\star \colon B \otimes B \to \operatorname{End}_F(e'Ae)$ be the algebra homomorphism defined by $\sigma_\star(x \otimes y)(z) = xz\sigma(y)$. It is injective since B is simple, hence it is an isomorphism by dimension count. Therefore, we get an algebra isomorphism

$$\sigma_\star^{\otimes r} \colon\ B^{\otimes 2r} \xrightarrow{\sim} \operatorname{End}_F\big((e'Ae)^{\otimes r}\big).$$

On the other hand, pick a nonzero element $\varkappa \in \rho(e)$ and denote by Φ_r^+ the epimorphism

$$\Phi_r^+ \colon\ (e'Ae)^{\otimes r} \to c(e'Ae)^r \rho(e) = E_r^+$$

which maps $z_1 \otimes \cdots \otimes z_r$ to $c(z_1)\cdots c(z_r)\varkappa$, for $z_1, \dots, z_r \in e'Ae$.

Lemma 1.8. *For any $x \in \hat{S}_{2r}$, $\sigma_\star^{\otimes r}(x)(\ker \Phi_r^+) \subset \ker \Phi_r^+$. Moreover, if $x \in S_{2r}^0$, then $\operatorname{im} \sigma_\star^{\otimes r}(x) \subset \ker \Phi_r^+$.*

Let us assume this lemma for the moment. For any $x \in \hat{S}_{2r}$, $\sigma_\star^{\otimes r}(x)$ induces an endomorphism of E_r^+, which is trivial as soon as $x \in S_{2r}^0$. Hence, we get an algebra homomorphism

$$\Psi_{2r} \colon\ \lambda^{2r} B = \hat{S}_{2r}/S_{2r}^0 \to \operatorname{End}_F E_r^+.$$

This homomorphism does not depend on the choice of $\varkappa \in \rho(e)$.

Proposition 1.9. *The homomorphism Ψ_{2r} is an isomorphism. Moreover, if m is even, Ψ_m is an isomorphism of algebras with involution*

$$\Psi_m \colon\ (\lambda^m B, \gamma) \xrightarrow{\sim} (\operatorname{End}_F E_{m/2}^+, \operatorname{ad}_{f_{m/2}^+})$$

where $f_{m/2}^+$ is the restriction of f^+ to $E_{m/2}^+$.

Proofs of Lemma 1.8 and Proposition 1.9. As in Proposition 1.1, we may assume that A is split. Using the same notation as in the proof of Proposition 1.1, we have $B = \operatorname{End}_F W$,

$$e'Ae = \operatorname{Hom}(U, W) = W \otimes U^* \simeq W \otimes W$$

(where the last isomorphism is induced by the polar form of q), and $\sigma_\star^{\otimes r}$ is the natural isomorphism $(\operatorname{End}_F W)^{\otimes 2r} \simeq \operatorname{End}_F((W \otimes W)^{\otimes r})$. As observed in the proof of Proposition 1.1, $E_r^+ \simeq \bigwedge^{2r} W$, and Φ_r^+ is the canonical epimorphism $\Phi_r^+ \colon W^{\otimes 2r} \to \bigwedge^{2r} W$, whose kernel is $\ker s_{2r}$ (see [14, proof of (10.3)]). Lemma 1.8 then follows by explicit calculations. The homomorphism Ψ_{2r} is thus well-defined. It is injective since $\lambda^{2r}B$ is simple, hence it is an isomorphism by dimension count.

Assume now that m is even. The canonical involution γ of $\lambda^m B \simeq \operatorname{End}_F(\bigwedge^m W)$ is adjoint to the exterior power $\wedge\colon \bigwedge^m W \times \bigwedge^m W \to \bigwedge^{2m} W \simeq F$ (see [14, (10.11)(a)]), and one may check that it coincides with $f^+_{m/2}$ if $m/2$ is even and with $-f^+_{m/2}$ if $m/2$ is odd. Hence, in both cases, we get that γ corresponds to $\operatorname{ad}_{f^+_{m/2}}$, and this finishes the proof of Proposition 1.9. □

Now, we consider the odd powers $\lambda^{2r+1}B$. Letting A act on $Ae \otimes (e'Ae)^{\otimes r}$ by left multiplication on the factor Ae, we get an algebra isomorphism

$$\sigma_{2r+1} \colon B^{\otimes 2r+1} \to \operatorname{End}_A\bigl(Ae \otimes_F (e'Ae)^{\otimes r}\bigr)$$

such that

$$\sigma_{2r+1}(x_1 \otimes \dots \otimes x_{2r+1})(y \otimes z_1 \otimes \dots \otimes z_r) = \\ y\sigma(x_1) \otimes x_2 z_1 \sigma(x_3) \otimes \dots \otimes x_{2r} z_r \sigma(x_{2r+1})$$

for $x_1, \dots, x_{2r+1} \in B$, $y \in Ae$, and $z_1, \dots, z_r \in e'Ae$. Let $\varkappa \in \rho(e)$, $\varkappa \neq 0$, and consider the A-module homomorphism

$$\Phi_r^- \colon Ae \otimes_F (e'Ae)^{\otimes r} \to (Ae)^b \cdot E_r^+ = (Ae)^b \cdot c(e'Ae)^r \cdot \rho(e)$$

which maps $y \otimes z_1 \otimes \ldots \otimes z_r$ to $y^b \cdot c(z_1) \ldots c(z_r)\varkappa$.

Lemma 1.10. *For any $x \in \hat{S}_{2r+1}$, $\sigma_{2r+1}(x)(\ker \Phi_r^-) \subset \ker \Phi_r^-$. Moreover, if $x \in S^0_{2r+1}$, then $\operatorname{im}\bigl(\sigma_{2r+1}(x)\bigr) \subset \ker \Phi_r^-$.*

Assuming the lemma, we may use σ_{2r+1} to define a canonical algebra homomorphism

$$\Psi_{2r+1} \colon \lambda^{2r+1}B = \hat{S}_{2r+1}/S^0_{2r+1} \to \operatorname{End}_A E_r^-.$$

Proposition 1.11. *The homomorphism Ψ_{2r+1} is an isomorphism. Moreover, if m is odd, Ψ_m is an isomorphism of algebras with involution*

$$\Psi_m \colon (\lambda^m B, \gamma) \xrightarrow{\sim} (\operatorname{End}_A E^-_{(m-1)/2}, \operatorname{ad}_{f^-_{(m-1)/2}})$$

where $f^-_{(m-1)/2}$ is the restriction of f^- to $E^-_{(m-1)/2}$.

The proofs of Lemma 1.10 and Proposition 1.11 follow the same lines as those of Lemma 1.8 and Proposition 1.9. We leave the details to the reader.

Recall that central simple algebras with involution are called *Witt-equivalent* if they can be represented as endomorphism algebras of Witt-equivalent hermitian spaces, see [5]. (If (A_1, σ_1) and (A_2, σ_2) are Witt-equivalent and $\deg A_1 \geq \deg A_2$, Garibaldi [7] writes that (A_1, σ_1) is a *hyperbolic extension* of (A_2, σ_2).)

Corollary 1.12. *The algebra* $\big(\mathcal{D}(\tilde{A},\tilde{\sigma}),\underline{\tilde{\sigma}}\big)$ *is Witt-equivalent to* $\big(\mathcal{C}^+(A,\sigma),\underline{\sigma}\big)$ *if* m *is even, to* $\big(\mathcal{C}^-(A,\sigma),\underline{\sigma}\big)$ *if* m *is odd.*

Proof. Combine the isomorphism (5) with Propositions 1.1 and 1.9 if m is even, with Propositions 1.5 and 1.11 if m is odd. □

Remark. For even m, this result was proved by Garibaldi [7, Main Theorem 0.1]. For odd m, Garibaldi proves a different result (see [7, Proposition 3.4]), which turns out to be equivalent to the above by [8, Theorem 1.1].

2. The general case

In this section, we consider the general case described in the introduction: (A,σ) is a central simple F-algebra of degree $4m$ with orthogonal involution, containing an element θ such that $\sigma(\theta)=-\theta$ and $\theta^2=a\in F^\star$. We assume $a\notin F^{\star 2}$ and let $K=F(\sqrt{a})$. We denote $(A_K,\sigma_K)=(A\otimes_F K,\sigma\otimes\mathrm{Id}_K)$. Now, $\theta\in A_K$ satisfies $\sigma_K(\theta)=-\theta$ and $\theta^2\in K^{\star 2}$, hence (A_K,σ_K) is hyperbolic and we may apply the results of the preceding section.

2.1. **Representation of the Clifford algebra.** Let $\alpha\in K$ satisfy $\alpha^2=a$. Let also

$$e=\tfrac12(1+\theta\otimes\alpha^{-1})\in A_K,\qquad e'=\tfrac12(1-\theta\otimes\alpha^{-1})\in A_K,$$

$$\rho(e)=c(eA_Ke')^m\subset\mathcal{C}(A_K,\sigma_K)\qquad\text{and}\qquad E^+=\mathcal{C}(A_K,\sigma_K)\rho(e)\subset\mathcal{C}(A_K,\sigma_K).$$

2.1.1. *Representation of* $\mathcal{C}^+(A,\sigma)$. By Proposition 1.1, left multiplication defines an isomorphism

$$\Theta^+\colon\ \big(\mathcal{C}^+(A_K,\sigma_K),\underline{\sigma_K}\big)\to(\operatorname{End}_K E^+,\operatorname{ad}_{f^+}).\tag{6}$$

Let $^-$ denote the non trivial automorphism of K/F. We also denote by $^-$ the natural action of this automorphism on $\mathcal{C}(A_K,\sigma_K)=\mathcal{C}(A,\sigma)\otimes K$. Our first goal is to determine the corresponding action on $\operatorname{End}_K E^+$ under Θ^+, as in [9, §4]. As we proceed to show, it is given by $f\mapsto\psi\circ f\circ\psi^{-1}$, where ψ is defined in the next lemma. This will enable us to define on E^+ a module structure over a certain quaternion F-algebra Q. This algebra will be defined as the subalgebra of $\operatorname{End}_F E^+$ generated by K and a map ψ as in the following lemma.

Lemma 2.1. *There is an invertible* K*-semilinear map* $\psi\colon E^+\to E^+$ *such that*

$$\psi(\xi\eta)=\overline{\xi}\psi(\eta)\qquad\textit{for all }\xi\in\mathcal{C}(A_K,\sigma_K),\ \eta\in E^+.\tag{7}$$

This map is unique up to a factor in $K^\star$*. It satisfies*

$$\psi^2=\lambda\operatorname{Id}_{E^+}$$

for some $\lambda\in F^\star$.

Proof. The Skolem–Noether theorem (see for instance [16, Chapter 8, Theorem 4.2]) yields an element $t\in A^\star$ such that

$$t\theta t^{-1}=-\theta,$$

hence $et=te'$ and $\sigma(t)e'=e\sigma(t)$. Let

$$g=ete'+e'\sigma(t)^{-1}e.$$

Straightforward computations show that $geg^{-1} = e' = \overline{e}$ and $\sigma_K(g)g = 1$, hence g is in the orthogonal group $\mathcal{O}(A_K, \sigma_K)$. We claim that g is in the *special* orthogonal group $\mathcal{O}^+(A_K, \sigma_K)$. To prove this, observe that $g\theta g^{-1} = -\theta$, hence

$$\pi(g\theta g^{-1}) = (-1)^{2m}\pi(\theta) = \pi(\theta).$$

Since the automorphism of $\mathcal{C}(A_K, \sigma_K)$ induced by the isometry g maps $\pi(\theta)$ to $\pi(g\theta g^{-1})$, it follows that this automorphism restricts to the identity on the center of $\mathcal{C}(A_K, \sigma_K)$. This proves the claim.

Consider then the Clifford group $\Gamma(A_K, \sigma_K)$. Since the vector representation $\chi\colon \Gamma(A_K, \sigma_K) \to \mathcal{O}^+(A_K, \sigma_K)$ is onto (see [14, §13.B]), there exists $\upsilon \in \Gamma(A_K, \sigma_K)$ such that $\chi(\upsilon) = g$. Lemma 4.5 of [9] shows that

$$\upsilon\rho(e)\upsilon^{-1} = \rho(geg^{-1}) = \rho(\overline{e}).$$

For $\xi \in E^+$, we have $\overline{\xi} \in \overline{E^+} = \mathcal{C}(A_K, \sigma_K)\rho(\overline{e})$, hence the preceding equation shows that $\overline{\xi}\upsilon \in E^+$. Define $\psi\colon E^+ \to E^+$ by

$$\psi(\xi) = \overline{\xi}\upsilon \qquad \text{for } \xi \in E^+.$$

The map ψ is clearly invertible and satisfies (7). If $\psi'\colon E^+ \to E^+$ is another such map, then $\psi' \circ \psi^{-1} \in \operatorname{End}_K E^+$ commutes with left multiplication by elements in $\mathcal{C}(A_K, \sigma_K)$. It is therefore central, since the homomorphism Θ^+ of (6) is onto, hence $\psi' \circ \psi^{-1} \in K^\star$. Similarly, $\psi^2 \in \operatorname{End}_K E^+$ is central, hence $\psi^2 = \lambda \operatorname{Id}_{E^+}$ for some $\lambda \in K^\star$. Since ψ^2 commutes with ψ, we must have $\overline{\lambda} = \lambda$, hence $\lambda \in F^\star$. □

Since E^+ is a K-vector space, there is a natural embedding $K \hookrightarrow \operatorname{End}_F E^+$. Since the map ψ of Lemma 2.1 is uniquely determined up to a factor in K, the subalgebra $Q \subset \operatorname{End}_F E^+$ generated by K and ψ is a uniquely determined quaternion F-algebra. The centralizer of Q consists of the endomorphisms in $\operatorname{End}_K E^+$ which commute with ψ. In view of Lemma 2.1, left multiplication by $\xi \in \mathcal{C}(A_K, \sigma_K)$ commutes with ψ if and only if $\overline{\xi} = \xi$; therefore, the centralizer of Q is the image of $\mathcal{C}^+(A, \sigma)$ under Θ.

Since $Q \subset \operatorname{End}_F E^+$, we may consider E^+ as a (left) Q-module. Thus, the isomorphism (6) restricts to

$$\Theta^+\colon \mathcal{C}^+(A, \sigma) \xrightarrow{\sim} \operatorname{End}_Q E^+.$$

To complete the description of $\big(\mathcal{C}^+(A, \sigma), \underline{\sigma}\big)$, we next determine a sesquilinear form on E^+ with respect to the quaternion conjugation on Q whose adjoint involution corresponds to the canonical involution $\underline{\sigma}$ under Θ^+.

Lemma 2.2. *There exists $\varkappa \in \rho(e)$, $\varkappa \neq 0$, such that $\underline{\sigma} \circ \psi\big(\overline{\underline{\sigma} \circ \psi(\varkappa)}\big) = \psi^2(\varkappa)$.*

Proof. Lemma 2.1 (and its proof) shows that we may assume $\psi(\xi) = \overline{\xi}\upsilon$ for some $\upsilon \in \Gamma(A_K, \sigma_K)$ such that $\upsilon\rho(e)\upsilon^{-1} = \rho(\overline{e})$. Then

$$\underline{\sigma} \circ \psi\big(\overline{\underline{\sigma} \circ \psi(\xi)}\big) = \underline{\sigma}(\upsilon)\overline{\xi}\upsilon \qquad \text{for all } \xi \in E^+.$$

Since $\upsilon \in \Gamma(A_K, \sigma_K)$, we have $\underline{\sigma}(\upsilon)\upsilon \in K^\star$, hence $\underline{\sigma}(\upsilon)\overline{\xi}\upsilon \in \upsilon^{-1}\overline{\xi}\upsilon \cdot K^\star$ for all $\xi \in E^+$. It follows that $\underline{\sigma} \circ \psi\big(\overline{\underline{\sigma} \circ \psi(\xi)}\big) \in \rho(e)$ for all $\xi \in \rho(e)$. Pick an arbitrary element $\varkappa_0 \in \rho(e)$, $\varkappa_0 \neq 0$. Since $\dim \rho(e) = 1$, we have

$$\underline{\sigma} \circ \psi\big(\overline{\underline{\sigma} \circ \psi(\varkappa_0)}\big) = \underline{\sigma}(\upsilon)\overline{\varkappa_0}\upsilon = \nu\varkappa_0 \qquad \text{for some } \nu \in K^\star.$$

Applying $^-$ and multiplying on the left by $\underline{\sigma}(\upsilon)$ and on the right by υ, we get

$$\underline{\sigma}(\overline{\upsilon}\upsilon)\varkappa_0\overline{\upsilon}\upsilon = \overline{\nu}\underline{\sigma}(\upsilon)\overline{\varkappa_0}\upsilon = \overline{\nu}\nu\varkappa_0. \tag{8}$$

On the other hand, we have $\psi^2 = \lambda \operatorname{Id}_{E^+}$ for some $\lambda \in F^\star$, hence

$$\psi^2(\xi) = \xi\overline{\upsilon}\upsilon = \xi\lambda \qquad \text{for all } \xi \in E^+.$$

Applying $\underline{\sigma}$, we obtain $\underline{\sigma}(\overline{\upsilon}\upsilon)\underline{\sigma}(\xi) = \lambda\underline{\sigma}(\xi)$ for all $\xi \in E^+$. Since $\underline{\sigma}\big(\rho(e)\big) = \rho(e)$, it follows that $\underline{\sigma}(\overline{\upsilon}\upsilon)\varkappa_0 = \lambda\varkappa_0$, hence

$$\underline{\sigma}(\overline{\upsilon}\upsilon)\varkappa_0\overline{\upsilon}\upsilon = \lambda^2\varkappa_0. \tag{9}$$

Comparing (8) and (9), we find $\lambda^2 = \overline{\nu}\nu$ hence, by Hilbert's Theorem 90 (see for instance [14, (29.3)]), $\nu\lambda^{-1} = \mu\overline{\mu}^{-1}$ for some $\mu \in K^\star$. Setting $\varkappa = \mu\varkappa_0$, we have

$$\underline{\sigma}\circ\psi\big(\overline{\underline{\sigma}\circ\psi(\varkappa)}\big) = \nu\overline{\mu}\varkappa_0 = \lambda\mu\varkappa_0 = \psi^2(\varkappa).$$

□

Fix a map ψ as in Lemma 2.1 and an element $\varkappa \in \rho(e)$ as in Lemma 2.2. Then $\rho(e) = \varkappa K$ and we may modify the bilinear form f^+ of (4) to get a bilinear form with values in K: we let

$$f^+(\xi,\eta) = g^+(\xi,\eta)\varkappa \qquad \text{for } \xi,\, \eta \in E^+.$$

We also define maps $\ell^+\colon E^+ \times E^+ \to K$ and $h^+\colon E^+ \times E^+ \to Q$ by

$$\ell^+(\xi,\eta) = g^+\big(\xi,\psi(\eta)\big) \qquad \text{for } \xi,\, \eta \in E^+$$

and

$$h^+(\xi,\eta) = \alpha\big(\ell^+(\xi,\eta) - g^+(\xi,\eta)\psi\big) \in K \oplus K\psi = Q$$

where $\alpha \in K$ is such that $\alpha^2 = a$.

Proposition 2.3. *For ξ, $\eta \in E^+$,*

$$g^+\big(\xi,\psi(\eta)\big) = \overline{g^+\big(\psi(\xi),\eta\big)}. \tag{10}$$

The map g^+ is a regular bilinear form on the K-vector space E^+. It is symmetric if m is even and skew-symmetric if m is odd. The map ℓ^+ is a regular $(-1)^m$-hermitian[1] form on the K-vector space E^+, and h^+ is a regular $(-1)^{m+1}$-hermitian form on the left Q-module E^+, with respect to the conjugation involution $^-$ on Q. Moreover, the isomorphism Θ^+ of (6) is an isomorphism of algebras with involution

$$\Theta^+\colon\ \big(\mathcal{C}^+(A,\sigma),\underline{\sigma}\big) \xrightarrow{\sim} (\operatorname{End}_Q E^+, \operatorname{ad}_{h^+}).$$

Proof. Let $\lambda \in F^\star$ be such that $\psi^2 = \lambda\operatorname{Id}_{E^+}$ and let ξ, $\zeta \in E^+$. Using $\underline{\sigma}\circ\psi\big(\overline{\underline{\sigma}\circ\psi(\varkappa)}\big) = \lambda\varkappa$, we derive from $g^+(\xi,\zeta)\varkappa = \underline{\sigma}(\xi)\zeta$

$$\lambda\overline{g^+(\xi,\zeta)}\varkappa = \underline{\sigma}\circ\psi\Big(\overline{\underline{\sigma}\circ\psi\big(\underline{\sigma}(\xi)\zeta\big)}\Big).$$

To compute the right side, write $\xi = \xi_0\varkappa$ and $\zeta = \zeta_0\varkappa$ for some ξ_0, $\zeta_0 \in \mathcal{C}(A_K,\sigma_K)$. Using (7), we get

$$\underline{\sigma}\circ\psi\Big(\overline{\underline{\sigma}\circ\psi\big(\underline{\sigma}(\xi)\zeta\big)}\Big) = \underline{\sigma}\circ\psi(\varkappa)\underline{\sigma}(\overline{\xi_0})\overline{\zeta_0}\psi(\varkappa) = \underline{\sigma}\circ\psi(\xi)\psi(\zeta),$$

hence

$$g^+\big(\psi(\xi),\psi(\zeta)\big)\varkappa = \underline{\sigma}\circ\psi(\xi)\psi(\zeta) = \lambda\overline{g^+(\xi,\zeta)}\varkappa.$$

Substituting η for $\psi(\zeta)$, we obtain (10).

---

[1] Note that ℓ^+ is linear in its first slot and semilinear in its second slot.

Proposition 1.1 shows that f^+ is a regular $(-1)^m$-symmetric bilinear form, hence g^+ is also a regular $(-1)^m$-symmetric bilinear form, and ℓ^+ is a regular $(-1)^m$-hermitian form since ψ is semilinear and satisfies (10). The fact that h^+ is a regular $(-1)^{m+1}$-hermitian form follows from straightforward computations.

Finally, since $\mathrm{ad}_{f^+} = \mathrm{ad}_{g^+}$ corresponds to $\underline{\sigma_K}$ under Θ^+, by Proposition 1.1, it is easily checked that

$$h^+(\xi\eta, \zeta) = h^+\big(\eta, \underline{\sigma}(\xi)\zeta\big) \qquad \text{for } \xi \in \mathcal{C}(A,\sigma) \text{ and } \eta,\, \zeta \in E^+,$$

proving the last part. □

To shed some light on the structure of E^+ as $(-1)^{m+1}$-hermitian space over Q, consider as in section 1 the following K-subspaces:

$$E_r^+ = c(e'A_K e)^r \rho(e) \subset E^+ \qquad \text{for } r = 0, \dots, m.$$

In particular, $E_m^+ = \rho(e')\rho(e)$.

Lemma 2.4. *The map ψ satisfies*

$$\psi\big(\rho(e)\big) = \rho(\overline{e})\rho(e) = \rho(e')\rho(e),$$

and $\psi(E_r^+) = E_{m-r}^+$ for $r = 0, \dots, m$.

Proof. It suffices to prove these properties after scalar extension to a splitting field of A. Therefore, we may assume A is split and use the same notation as in Proposition 1.1. We thus let $A = \mathrm{End}_F V$ for some $4m$-dimensional vector space V over F, and $\sigma = \mathrm{ad}_q$ for some quadratic form q on V.

Since $\sigma(\theta) = -\theta$, every $v \in V$ is orthogonal to $\theta(v)$. If V_0 is a maximal nonsingular subspace of V such that V_0 and $\theta(V_0)$ are orthogonal, then $V = V_0 \overset{\perp}{\oplus} \theta(V_0)$. On the other hand, we also have $V_K = U \oplus W$ where $U = e(V_K)$ and $W = e'(V_K)$ are totally isotropic subspaces of V_K, as in the proof of Proposition 1.1. Let v_1, $\dots$, v_{2m} be an orthogonal basis of V_0 and let $\lambda_i = q(v_i)$ for $i = 1, \dots, 2m$. In V_K, consider the vectors

$$u_i = \tfrac{1}{2}\big(v_i + \theta(v_i)\alpha^{-1}\big), \qquad w_i = \tfrac{1}{2}\big(v_i - \theta(v_i)\alpha^{-1}\big)\lambda_i^{-1} \qquad \text{for } i = 1, \dots, 2m.$$

Then $u_1, \dots, u_{2m}$ (resp. $w_1, \dots, w_{2m}$) is a basis of U (resp. W), and these elements satisfy $q(u_i + w_j) = \delta_{ij}$. Moreover, their images in the Clifford algebra satisfy

$$u_i v_i = u_i w_i \lambda_i = v_i w_i \lambda_i \qquad \text{for } i = 1, \dots, 2m \tag{11}$$

hence also, since $\overline{u_i} = \lambda_i w_i$,

$$v_i u_i = \lambda_i w_i u_i = \lambda_i w_i v_i \qquad \text{for } i = 1, \dots, 2m. \tag{12}$$

As observed in Proposition 1.1, we have

$$\rho(e) = u_1 \dots u_{2m} K \subset \mathcal{C}_0(V_K, q) = \mathcal{C}(A_K, \sigma_K).$$

Using (11) and (12), it is easily seen that for $v = v_1 \dots v_{2m}$, the equation $\psi_0(\xi) = \overline{\xi}v$ defines a K-semilinear map $\psi_0\colon E^+ \to E^+$ satisfying condition (7) of Lemma 2.1, hence $\psi = \mu\psi_0$ for some $\mu \in K^\star$. Moreover, we have

$$\overline{u_1 \dots u_{2m}}v = vu_1 \dots u_{2m} = \lambda_1 \dots \lambda_{2m} w_1 \dots w_{2m} u_1 \dots u_{2m}.$$

Since $\rho(e')\rho(e) = w_1 \dots w_{2m} u_1 \dots u_{2m} K$, it follows that $\psi\big(\rho(e)\big) = \psi_0\big(\rho(e)\big) = \rho(e')\rho(e)$, proving the first part of the lemma.

For the second part, observe that $c(e'A_K e)$ is spanned by the products $w_i w_j$, hence E_r^+ is the span of the products $w_{i_1} \dots w_{i_{2r}} u_1 \dots u_{2m}$, where we may assume

$1 \le i_1 < \cdots < i_{2r} \le 2m$. It follows that $\psi(E_r^+) = \psi_0(E_r^+)$ is spanned by the elements of the form $\overline{w_{i_1} \dots w_{i_{2r}} u_1 \dots u_{2m}} v_1 \dots v_{2m}$, hence also, since $\overline{w_i} = \lambda_i^{-1} u_i$ and $\overline{u_i} = \lambda_i w_i$, by the products $u_{i_1} \dots u_{i_{2r}} w_1 \dots w_{2m} v_1 \dots v_{2m}$. By (11) and (12), we have for $1 \le i_1 < \cdots < i_{2r} \le 2m$,

$$\begin{aligned} u_{i_1} \dots u_{i_{2r}} w_1 \dots w_{2m} v_1 \dots v_{2m} &= u_{i_1} \dots u_{i_{2r}} w_1 \dots w_{2m} u_1 \dots u_{2m} \\ &= \pm w_{j_1} \dots w_{j_{2s}} u_1 \dots u_{2m}, \end{aligned}$$

where the indices $j_1, \dots, j_{2s}$ form the complementary subset of $\{i_1, \dots, i_{2r}\}$ in $\{1, \dots, 2m\}$. Therefore, $\psi(E_r^+) = E_{m-r}^+$, and the proof is complete. □

It follows from the lemma that for $r < m/2$ the sum

$$\hat{E}_r^+ = E_r^+ \oplus E_{m-r}^+$$

is a Q-submodule of E^+, and that $E_{m/2}^+$ is a Q-submodule of E^+ if m is even. Their dimensions over Q are respectively equal to $\binom{2m}{2r}$ and $\frac{1}{2}\binom{2m}{m}$.

For $\varepsilon = \pm 1$, let $W^\varepsilon(K, {}^-)$ (resp. $W^\varepsilon(Q, {}^-)$) denote the Witt group of ε-hermitian spaces over K (resp. Q) with respect to ${}^-$. For any ε-hermitian space (E, ℓ) over K, a $(-\varepsilon)$-hermitian space $R(E, \ell)$ over Q is defined by $R(E, \ell) = (E \oplus \psi E, h)$ where

$$h(x_1 + \psi x_2, y_1 + \psi y_2) = \alpha\big(\ell(x_1, y_1) + \overline{\ell(x_2, y_2)}\lambda\big) - \alpha\big(\ell(x_1, y_2) + \overline{\ell(x_2, y_1)}\big)\psi$$

for $x_1, x_2, y_1, y_2 \in E$. As observed in [16, p. 359][2] the map R induces a group homomorphism

$$R\colon W^\varepsilon(K, {}^-) \to W^{-\varepsilon}(Q, {}^-).$$

Proposition 2.5. *If $\xi \in E_r^+$ and $\eta \in E_s^+$, then*

$$g^+(\xi, \eta) = 0 \quad \textit{unless } r + s = m \quad \textit{and} \quad \ell^+(\xi, \eta) = 0 \quad \textit{unless } r = s.$$

Therefore, letting h_r^+ (resp. $h_{m/2}^+$) denote the restriction of h^+ to $\hat{E}_r^+$ (resp. $E_{m/2}^+$, if m is even), we have

$$(E^+, h^+) = \begin{cases} \bigoplus^{\perp}_{0 \le r \le (m-1)/2} (\hat{E}_r^+, h_r^+) & \textit{if } m \textit{ is odd,} \\ \Big(\bigoplus^{\perp}_{0 \le r < m/2} (\hat{E}_r^+, h_r^+)\Big) \overset{\perp}{\oplus} (E_{m/2}^+, h_{m/2}^+) & \textit{if } m \textit{ is even.} \end{cases}$$

Moreover, for $r < m/2$ we have $(\hat{E}_r^+, h_r^+) = R(E_r^+, \ell_r^+)$, where ℓ_r^+ is the restriction of ℓ^+ to E_r^+.

Proof. The first part readily follows from Proposition 1.1 and Lemma 2.4. The equality $(\hat{E}_r^+, h_r^+) = R(E_r^+, \ell_r^+)$ for $r < m/2$ follows from a straightforward computation. □

Recall from [16, p. 359] the "extension of scalars" map $S\colon W^{-1}(Q, {}^-) \to WK$ (denoted by π_2 in [16]) which fits in an exact sequence

$$W^1(K, {}^-) \xrightarrow{R} W^{-1}(Q, {}^-) \xrightarrow{S} WK. \tag{13}$$

Corollary 2.6. *If m is even, $S(\hat{E}_r^+, h_r^+) = 0$ for all $r < m/2$.*

The result holds trivially if m is odd, since the corresponding map S carries hermitian forms over Q to skew-symmetric spaces over K.

[2] Scharlau denotes by ρ the map R. When comparing with [16], one has to keep in mind that Scharlau considers sesquilinear forms which are semilinear in the first slot and linear in the second slot, whereas the opposite convention is used here. Moreover, there is a misprint in Scharlau's definition of the form h, on p. 359 of [16]: it should involve the term $f(x_2, y_2)^*$ instead of $f(x_2, y_2)$.

2.1.2. *Representation of* $\mathcal{C}^-(A,\sigma)$. In order to obtain an analogue of Proposition 2.3 for $\mathcal{C}^-(A,\sigma)$, we consider

$$E^- = \mathcal{B}(A_K,\sigma_K)\cdot\rho(e) \subset \mathcal{B}(A_K,\sigma_K)$$

and the left multiplication isomorphism

$$\Theta^-\colon \big(\mathcal{C}^-(A_K,\sigma_K),\underline{\sigma_K}\big) \xrightarrow{\sim} (\operatorname{End}_{A_K} E^-,\operatorname{ad}_{f^-})$$

of Proposition 1.5.

Lemma 2.7. *There is a unique action of* $\psi\in Q$ *on* E^- *such that*

$$\psi\cdot(\xi\cdot\eta) = \overline{\xi}\cdot\psi(\eta) \qquad \text{for } \xi\in\mathcal{B}(A_K,\sigma_K) \text{ and } \eta\in E^+.$$

This action is A*-linear.*

Proof. Let $\varkappa_0\in\rho(e)$, $\varkappa_0\neq 0$. The proof of Lemma 2.4 shows that $\psi(\varkappa_0)=\overline{\varkappa_0}\upsilon$ for some $\upsilon\in\Gamma(A_K,\sigma_K)$. Since

$$E^- = \mathcal{B}(A_K,\sigma_K)\cdot E^+ = \mathcal{B}(A_K,\sigma_K)\cdot\varkappa_0,$$

it follows that the action, if it exists, is necessarily defined by

$$\psi(x) = \overline{x}\upsilon. \tag{14}$$

This proves uniqueness of the action. On the other hand, one may easily check that equation (14) defines an action of ψ on E^- having the required properties. □

Since K acts naturally on E^-, the lemma yields a left $A\otimes_F Q$-module structure on E^-, and Θ^- restricts to an F-algebra isomorphism

$$\Theta^-\colon \mathcal{C}^-(A,\sigma) \xrightarrow{\sim} \operatorname{End}_{A\otimes Q} E^-. \tag{15}$$

To define a sesquilinear form on E^- whose adjoint involution corresponds to $\underline{\sigma}$ under this isomorphism, fix an element $\varkappa\in\rho(e)$ as in Lemma 2.2. Then $A_K\otimes_K\rho(e) = A_K\otimes\varkappa$ and we may define a map $g^-\colon E^-\times E^-\to A_K$ by the equation

$$f^-(\xi,\eta) = g^-(\xi,\eta)\otimes\varkappa \qquad \text{for } \xi,\ \eta\in E^-.$$

We also define maps $\ell^-\colon E^-\times E^-\to A_K$ and $h^-\colon E^-\times E^-\to A\otimes_F Q$ by

$$\ell^-(\xi,\eta) = g^-(\xi,\psi\cdot\eta)$$

and

$$h^-(\xi,\eta) = \alpha\big(\ell^-(\xi,\eta) - g^-(\xi,\eta)\psi\big) \in A_K\oplus A_K\psi = A\otimes_F Q.$$

Proposition 2.8. *For* ξ, $\eta\in E^-$,

$$g^-(\xi,\psi\cdot\eta) = \overline{g^-(\psi\cdot\xi,\eta)}. \tag{16}$$

The map g^- *is a regular* $(-1)^m$*-hermitian form on the* A_K*-module* E^- *with respect to the involution* σ_K*, the map* ℓ^- *is a regular* $(-1)^m$*-hermitian form on the* A_K*-module* E^- *with respect to the involution* $\sigma\otimes{}^-$*, and* h^- *is a regular* $(-1)^{m+1}$*-hermitian form on the* $A\otimes Q$*-module* E^- *with respect to the involution* $\sigma\otimes{}^-$*. Moreover, the isomorphism* Θ^- *of* (15) *is an isomorphism of algebras with involution*

$$\Theta^-\colon \big(\mathcal{C}^-(A,\sigma),\underline{\sigma}\big) \xrightarrow{\sim} (\operatorname{End}_{A\otimes Q} E^-,\operatorname{ad}_{h^-}).$$

Proof. Let ξ, $\zeta \in E^-$. We compute in two different ways

$$\tau \otimes (\underline{\sigma_K} \circ \psi)\Bigl(\overline{\tau \otimes (\underline{\sigma_K} \circ \psi)\bigl(f^-(\xi,\zeta)\bigr)}\Bigr) \tag{17}$$

where $\tau\colon A_K \to A_K$ is the involution defined by $\tau(x) = \sigma_K(\overline{x})$ for $x \in A_K$.

First, since $f^-(\xi,\zeta) = g^-(\xi,\zeta) \otimes \varkappa$ and $\varkappa$ satisfies the condition in Lemma 2.2, the expression in (17) is equal to $\overline{g^-(\xi,\zeta)} \otimes \psi^2(\varkappa)$.

On the other hand, letting $\xi = \xi_0 \cdot \varkappa$ and $\zeta = \zeta_0 \cdot \varkappa$ for some ξ_0, $\zeta_0 \in \mathcal{B}(A_K, \sigma_K)$, we have $\xi^\omega = \underline{\sigma_K}(\varkappa) * \xi_0^\omega$, hence

$$f^-(\xi,\zeta) = \mu\bigl(\underline{\sigma_K}(\varkappa) * \xi_0^\omega \otimes \zeta_0 \cdot \varkappa\bigr) = 1 \otimes \underline{\sigma_K}(\varkappa) \cdot \mu(\xi_0^\omega \otimes \zeta_0) \cdot 1 \otimes \varkappa.$$

Since for $x \in A_K$ and $c \in \mathcal{C}(A_K, \sigma_K)$,

$$\tau \otimes (\underline{\sigma_K} \circ \psi)(x \otimes c \cdot 1 \otimes \varkappa) = 1 \otimes (\underline{\sigma_K} \circ \psi)(\varkappa) \cdot \sigma_K \otimes \underline{\sigma_K}(\overline{x} \otimes \overline{c}),$$

it follows that

$$\tau \otimes (\underline{\sigma_K} \circ \psi)\bigl(f^-(\xi,\zeta)\bigr) = 1 \otimes (\underline{\sigma_K} \circ \psi)(\varkappa) \cdot \sigma_K \otimes \underline{\sigma_K}\bigl(\mu(\overline{\xi_0}^\omega \otimes \overline{\zeta_0})\bigr) \cdot 1 \otimes \overline{\varkappa}$$

hence the expression in (17) is equal to

$$1 \otimes (\underline{\sigma_K} \circ \psi)(\varkappa) \cdot \mu(\overline{\xi_0}^\omega \otimes \overline{\zeta_0}) \cdot 1 \otimes \psi(\varkappa) = \mu\bigl((\psi \cdot \xi)^\omega \otimes (\psi \cdot \zeta)\bigr).$$

Comparing the results, we obtain

$$\lambda \overline{g^-(\xi,\zeta)} = g^-(\psi \cdot \xi, \psi \cdot \zeta)$$

where $\lambda \in F^\star$ is such that $\psi^2 = \lambda \operatorname{Id}_{E^+}$. Substituting η for $\psi \cdot \zeta$, we obtain (16).

Since f^- is a regular $(-1)^m$-hermitian form with respect to σ_K, it is clear that g^- is also a regular $(-1)^m$-hermitian form with respect to σ_K. Using (16), we obtain

$$\ell^-(\eta,\xi) = \overline{g^-(\psi \cdot \eta, \xi)} = (-1)^m \overline{\sigma_K\bigl(g^-(\xi, \psi \cdot \eta)\bigr)} = (-1)^m \overline{\sigma_K\bigl(\ell^-(\xi,\eta)\bigr)}$$

for ξ, $\eta \in E^-$. It follows that ℓ^- is a regular $(-1)^m$-hermitian form with respect to the involution $\sigma \otimes {}^-(= \tau)$ on A_K. Straightforward computations show that h^- is a $(-1)^{m+1}$-hermitian form with respect to the involution $\sigma \otimes {}^-$ on $A \otimes_F Q$. Finally, since $\operatorname{ad}_{f^-} = \operatorname{ad}_{g^-}$ corresponds to $\underline{\sigma_K}$ under Θ^-, by Proposition 1.5, it is easily checked that

$$h^-(\xi \cdot \eta, \zeta) = h^-\bigl(\eta, \underline{\sigma}(\xi) \cdot \zeta\bigr) \qquad \text{for } \xi \in \mathcal{C}(A,\sigma) \text{ and } \eta,\ \zeta \in E^-,$$

completing the proof. □

As in section 1, we define A_K-submodules of E^- by

$$E_r^- = (A_K e)^b \cdot E_r^+ \subset E^-,$$

for $r = 0, \ldots, m-1$. We also set

$$\hat{E}_r^- = E_r^- \oplus E_{m-r-1}^- \qquad \text{for } r < (m-1)/2.$$

Proposition 2.9. *For $r = 0, \ldots, m-1$, we have $\psi \cdot E_r^- = E_{m-r-1}^-$, hence $\hat{E}_r^-$ is an $A \otimes Q$-submodule of E^- for $r < m/2$, and if m is odd $E_{(m-1)/2}^-$ is an $A \otimes Q$-submodule of E^-.*

If $\xi \in E_r^-$ and $\eta \in E_s^-$, then

$$g^-(\xi,\eta) = 0 \text{ unless } r + s = m - 1 \quad \text{and} \quad \ell^-(\xi,\eta) = 0 \text{ unless } r = s.$$

Therefore, letting h_r^- (resp. $h_{(m-1)/2}^-$) denote the restriction of h^- to $\hat{E}_r^-$ (resp. $E_{(m-1)/2}^-$, if m is odd), we have

$$(E^-,h^-)=\begin{cases}\Bigl(\bigoplus^{\perp}_{0\leq r<(m-1)/2}(\hat{E}_r^-,h_r^-)\Bigr)\overset{\perp}{\oplus}(E^-_{(m-1)/2},h^-_{(m-1)/2}) & \text{if } m \text{ is odd,}\\ \bigoplus^{\perp}_{0\leq r<m/2}(\hat{E}_r^-,h_r^-) & \text{if } m \text{ is even.}\end{cases}$$

Proof. Since $E_r^-=(A_Ke)^b\cdot E_r^+$, we have by Lemmas 2.4 and 2.7

$$\psi\cdot E_r^-=\overline{(A_Ke)^b}\cdot\psi(E_r^+)=(A_Ke')^b\cdot E_{m-r}^+.$$

Extending scalars to a splitting field of A_K, we may verify that

$$(A_Ke')^b\cdot E_{m-r}^+=(A_Ke)^b\cdot E_{m-r-1}^+,$$

hence $\psi\cdot E_r^-=E_{m-r-1}^-$. The other assertions easily follow, by Proposition 1.5. □

Parimala, Sridharan and Suresh [2, Appendix 2, Theorem 2] have defined an exact sequence

$$W^\varepsilon(A\otimes K,\sigma\otimes{}^-)\xrightarrow{R}W^{-\varepsilon}(A\otimes Q,\sigma\otimes{}^-)\xrightarrow{S}W^\varepsilon(A\otimes K,\sigma\otimes\mathrm{Id}_K)$$

which is an analogue of (13).

Proposition 2.10. *For $r<(m-1)/2$, we have $R(E_r^-,\ell_r^-)=(\hat{E}_r^-,h_r^-)$, hence $S(\hat{E}_r^-,h_r^-)=0$. If m is even, $S(E^-,h^-)=0$.*

Proof. Straightforward computation. □

2.2. **Representation of the discriminant algebra of the centralizer.** Our next goal is to establish an isomorphism

$$\bigl(\mathcal{D}(\tilde{A},\tilde{\sigma}),\underline{\tilde{\sigma}}\bigr)\xrightarrow{\sim}\begin{cases}(\mathrm{End}_Q\,E^+_{m/2},\mathrm{ad}_{h^+_{m/2}}) & \text{if } m \text{ is even,}\\ (\mathrm{End}_{A\otimes Q}\,E^-_{(m-1)/2},\mathrm{ad}_{h^-_{(m-1)/2}}) & \text{if } m \text{ is odd.}\end{cases}$$

We first define a canonical isomorphism $\tilde{A}=e'A_Ke'$, then use the isomorphism Ψ_m of Proposition 1.9 or 1.11.

For simplicity of notation, we let $B=e'A_Ke'$. Since $\overline{e'}=e=\sigma_K(e')$, we have $\overline{\sigma_K(x)}\in B$ for $x\in B$. Clearly, the map $\tau\colon B\to B$ defined by

$$\tau(x)=\overline{\sigma_K(x)}\qquad\text{for } x\in B$$

is a unitary involution.

Lemma 2.11. *For $x\in B$, we have $x+\overline{x}\in\tilde{A}$. The map $x\mapsto x+\overline{x}$ is an isomorphism of F-algebras*

$$T\colon(B,\tau)\xrightarrow{\sim}(\tilde{A},\tilde{\sigma})$$

such that $T(\alpha)=-\theta$.

Proof. It is clear that $x+\overline{x}\in A$. Since $\theta e'=-\alpha e'=e'\theta$, we have $\theta x=-\alpha x=x\theta$ hence also $\theta\overline{x}=\alpha\overline{x}=\overline{x}\theta$ for all $x\in B$, hence $x+\overline{x}\in\tilde{A}$. Moreover, for x, $x'\in B$ we have $\overline{x}$, $\overline{x'}\in eA_Ke$, hence $x\overline{x'}=\overline{x}x'=0$ and therefore

$$(x+\overline{x})(x'+\overline{x'})=xx'+\overline{xx'}.$$

It follows that T is an F-algebra homomorphism. It is injective since $x + \overline{x} = 0$ implies $x \in (e'A_Ke') \cap (eA_Ke)$, hence also surjective by dimension count. To see that τ corresponds to $\tilde{\sigma}$ under T, observe that for $x \in B$

$$\tilde{\sigma}(x + \overline{x}) = \sigma_K(x) + \sigma_K(\overline{x}) = T\big(\tau(x)\big).$$

□

The isomorphism T induces a canonical isomorphism

$$\big(\mathcal{D}(\tilde{A}, \tilde{\sigma}), \underline{\tilde{\sigma}}\big) \simeq \big(\mathcal{D}(B, \tau), \underline{\tau}\big). \tag{18}$$

We may (and shall) therefore work with (B, τ) instead of $(\tilde{A}, \tilde{\sigma})$. Recall from Propositions 1.9 and 1.11 that there are canonical isomorphisms

$$\Psi_{2r}\colon\ \lambda^{2r}B \xrightarrow{\sim} \operatorname{End}_K E_r^+ \qquad \text{and} \qquad \Psi_{2r+1}\colon\ \lambda^{2r+1}B \xrightarrow{\sim} \operatorname{End}_{A_K} E_r^-.$$

Proposition 2.12. *The involution $\tau^{\wedge 2r}$ on $\lambda^{2r}B$ induced by τ corresponds under Ψ_{2r} to the adjoint involution with respect to ℓ_r^+, the restriction of ℓ^+ to E_r^+. Similarly, the involution $\tau^{\wedge(2r+1)}$ corresponds under Ψ_{2r+1} to the adjoint involution with respect to ℓ_r^-, the restriction of ℓ^- to E_r^-.*

Proof. Extending scalars to a splitting field, we are reduced to proving the proposition in the case where A is split. We may thus assume $A = \operatorname{End}_F V$ for some $4m$-dimensional vector space V, and $\sigma = \operatorname{ad}_q$ for some quadratic form q on V, and use the notation of Lemma 2.4. In particular, we define $\psi\colon\ E^+ \to E^+$ by $\psi(\xi) = \overline{\xi}\upsilon$, where $\upsilon = v_1 \dots v_{2m}$. Computation shows that the element $\varkappa = u_1 \dots u_{2m}$ satisfies the condition of Lemma 2.2, and

$$\psi(\varkappa) = \upsilon\varkappa = \lambda w_1 \dots w_{2m}\varkappa.$$

A basis of E_r^+ is given by the products $(w_{i_1} \dots w_{i_{2r}}\varkappa)_{1 \le i_1 < \dots < i_{2r} \le 2m}$. To compute the Gram matrix of ℓ_r^+ with respect to this basis, observe that

$$\begin{aligned} f^+\big(w_{i_1} \dots w_{i_{2r}}\varkappa, \psi(w_{j_1} \dots w_{j_{2r}}\varkappa)\big) = \\ \underline{\sigma_K}(\varkappa) w_{i_{2r}} \dots w_{i_1} \overline{w_{j_1}} \dots \overline{w_{j_{2r}}}\psi(\varkappa) = \\ \lambda\lambda_{j_1}^{-1} \dots \lambda_{j_{2r}}^{-1} \underline{\sigma_K}(\varkappa) w_{i_{2r}} \dots w_{i_1} u_{j_1} \dots u_{j_{2r}} w_1 \dots w_{2m}\varkappa. \end{aligned} \tag{19}$$

If $\{i_1, \dots, i_{2r}\} \neq \{j_1, \dots, j_{2r}\}$, say $i_1 \notin \{j_1, \dots, j_{2r}\}$, then w_{i_1} commutes with $u_{j_1} \dots u_{j_{2r}}$, and (19) vanishes since $w_{i_1} w_1 \dots w_{2m} = 0$. Assume now $\{i_1, \dots, i_{2r}\} = \{j_1, \dots, j_{2r}\}$. Since $q(u_i + w_i) = 1$, we have $u_i w_i = 1 - w_i u_i$ for all $i = 1, \dots, 2m$. Substituting $1 - u_{i_1} w_{i_1}$ for $w_{i_1} u_{i_1}$ in the rightmost side of (19), the term containing $u_{i_1} w_{i_1}$ vanishes by the same argument as above. Therefore, we obtain

$$\begin{aligned} f^+\big(w_{i_1} \dots w_{i_{2r}}\varkappa, \psi(w_{i_1} \dots w_{i_{2r}})\varkappa\big) = \\ \lambda\lambda_{i_1}^{-1} \dots \lambda_{i_{2r}}^{-1} \underline{\sigma_K}(\varkappa) w_1 \dots w_{2m}\varkappa = \lambda\lambda_{i_1}^{-1} \dots \lambda_{i_{2r}}^{-1}\varkappa. \end{aligned}$$

It follows that $(w_{i_1} \dots w_{i_{2r}}\varkappa)_{1 \le i_1 < \dots < i_{2r} \le 2m}$ is an orthogonal basis for ℓ_r^+, and

$$\ell_r^+(w_{i_1} \dots w_{i_{2r}}\varkappa, w_{i_1} \dots w_{i_{2r}}\varkappa) = \lambda\lambda_{i_1}^{-1} \dots \lambda_{i_{2r}}^{-1}. \tag{20}$$

On the other hand, as in the proof of Lemma 1.8, we have $B = \operatorname{End}_K W$, and it is easily checked that the involution τ is adjoint to the hermitian form h_q defined by

$$h_q(x, y) = b_q(\overline{x}, y) = q(\overline{x} + y) - q(\overline{x}) - q(y) \qquad \text{for } x,\ y \in W,$$

where b_q is the polar form of q. Thus,

$$h_q(w_i, w_j) = b_q(\lambda_i^{-1} u_i, w_j) = \lambda_i^{-1}\delta_{ij} \qquad \text{for } i,\ j = 1, \dots, 2m.$$

The involution $\tau^{\wedge 2r}$ on $\lambda^{2r}B = \operatorname{End}_K(\bigwedge^{2r} W)$ is then adjoint to the hermitian form $h_q^{\wedge 2r}$ such that

$$h_q^{\wedge 2r}(x_1 \wedge \dots \wedge x_{2r}, y_1 \wedge \dots \wedge y_{2r}) = \det\bigl(h_q(x_i, y_j)\bigr)_{1\le i,j\le 2r}.$$

For $1 \le i_1 < \dots < i_{2r} \le 2m$ and $1 \le j_1 < \dots < j_{2r} \le 2m$, we have

$$h_q^{\wedge 2r}(w_{i_1} \wedge \dots \wedge w_{i_{2r}}, w_{j_1} \wedge \dots \wedge w_{j_{2r}}) = \begin{cases} 0 & \text{if } \{i_1, \dots, i_{2r}\} \ne \{j_1, \dots, j_{2r}\},\\ \lambda_{i_1}^{-1} \dots \lambda_{i_{2r}}^{-1} & \text{if } \{i_1, \dots, i_{2r}\} = \{j_1, \dots, j_{2r}\}.\end{cases}$$

Comparing with (20), we see that the map $\bigwedge^{2r} W \to E_r^+$ which carries $x_1 \wedge \dots \wedge x_{2r}$ to $x_1 \dots x_{2r}\varkappa$ is a similitude of $(\bigwedge^{2r} W, h_q^{\wedge 2r})$ with (E_r^+, ℓ_r^+). Therefore, the adjoint involutions $\tau^{\wedge 2r}$ and $\operatorname{ad}_{\ell_r^+}$ correspond to each other.

The same computations as above show that for $v,\ v' \in V$ and $1 \le i_1 < \dots < i_{2r+1} \le 2m$, $1 \le j_1 < \dots < j_{2r+1} \le 2m$,

$$\ell_r^-(v \otimes w_{i_1} \dots w_{i_{2r+1}}\varkappa, v' \otimes w_{j_1} \dots w_{j_{2r+1}}\varkappa) = \begin{cases} 0 & \text{if } \{i_1, \dots, i_{2r+1}\} \ne \{j_1, \dots, j_{2r+1}\},\\ \lambda\lambda_{i_1}^{-1} \dots \lambda_{i_{2r+1}}^{-1} v \otimes v' & \text{if } \{i_1, \dots, i_{2r+1}\} = \{j_1, \dots, j_{2r+1}\}.\end{cases}$$

Therefore, the map $V \otimes \bigwedge^{2r+1} W \to E_r^-$ which carries $v \otimes (x_1 \wedge \dots \wedge x_{2r+1})$ to $v \otimes x_1 \dots x_{2r+1}\varkappa$ is a similitude of the form ℓ_r^- and the form on $V \otimes \bigwedge^{2r+1} W$ with values in $V \otimes V$ which maps $(v \otimes \xi, v' \otimes \xi')$ to $v \otimes v' h_q^{\wedge 2r+1}(\xi, \xi')$ for $v,\ v' \in V$ and $\xi,\ \xi' \in \bigwedge^{2r+1} W$. Therefore, the adjoint involutions $\operatorname{ad}_{l_r^-}$ and $\tau^{\wedge 2r+1}$ correspond to each other. □

Corollary 2.13. *If m is even, the map Ψ_m induces an isomorphism of algebras with involution*

$$\Psi_m\colon \bigl(\mathcal{D}(B,\tau), \underline{\tau}\bigr) \xrightarrow{\sim} (\operatorname{End}_Q E_{m/2}^+, \operatorname{ad}_{h_{m/2}^+}).$$

Similarly, if m is odd, the map Ψ_m induces an isomorphism of algebras with involution

$$\Psi_m\colon \bigl(\mathcal{D}(B,\tau), \underline{\tau}\bigr) \xrightarrow{\sim} (\operatorname{End}_{A\otimes Q} E_{(m-1)/2}^-, \operatorname{ad}_{h_{(m-1)/2}^-}).$$

Proof. Recall from [14, §10.E] that $\mathcal{D}(B,\tau)$ is the F-subalgebra of $\lambda^m B$ fixed under the automorphism $\tau^{\wedge m} \circ \gamma$.

Suppose first that m is even. Proposition 1.9 shows that γ corresponds to $\operatorname{ad}_{f_{m/2}^+} = \operatorname{ad}_{g_{m/2}^+}$ under Ψ_m, and Proposition 2.12 shows that $\tau^{\wedge m}$ corresponds to $\operatorname{ad}_{\ell_{m/2}^+}$. Therefore, Ψ_m maps $\mathcal{D}(B,\tau)$ to the F-subalgebra of $\operatorname{End}_K E_{m/2}^+$ fixed under $\operatorname{ad}_{\ell_{m/2}^+} \circ \operatorname{ad}_{g_{m/2}^+}$. We claim that for $f \in \operatorname{End}_K E_{m/2}^+$,

$$\operatorname{ad}_{\ell_{m/2}^+} \circ \operatorname{ad}_{g_{m/2}^+}(f) = \psi^{-1} \circ f \circ \psi$$

or, equivalently,

$$\operatorname{ad}_{\ell_{m/2}^+}(f) = \psi^{-1} \circ \operatorname{ad}_{g_{m/2}^+}(f) \circ \psi.$$

To check the latter equality, observe that for ξ, $\eta \in E^+_{m/2}$,

$$\ell^+\bigl(\xi, f(\eta)\bigr) = \overline{g^+\bigl(\psi(\xi), f(\eta)\bigr)} = \overline{g^+\bigl(\mathrm{ad}_{g^+}(f)\circ\psi(\xi), \eta\bigr)} = \ell^+\bigl(\psi^{-1}\circ\mathrm{ad}_{g^+}(f)\circ\psi(\xi), \eta\bigr).$$

Therefore, the subalgebra of $\mathrm{End}_K E^+_{m/2}$ fixed under $\mathrm{ad}_{\ell^+_{m/2}} \circ \mathrm{ad}_{g^+_{m/2}}$ is the centralizer of ψ, which is $\mathrm{End}_Q E^+_{m/2}$. Clearly, $\mathrm{ad}_{\ell^+_{m/2}}$ and $\mathrm{ad}_{g^+_{m/2}}$ both restrict to $\mathrm{ad}_{h^+_{m/2}}$ on $\mathrm{End}_Q E^+_{m/2}$.

If m is odd, it follows from Propositions 1.11 and 2.12 that Ψ_m maps $\mathcal{D}(B,\tau)$ to the F-subalgebra of $\mathrm{End}_{A_K} E^-_{(m-1)/2}$ fixed under $\mathrm{ad}_{\ell^-_{(m-1)/2}} \circ \mathrm{ad}_{g^-_{(m-1)/2}}$. The same computation as above (substituting $\ell^-_{(m-1)/2}$ for $\ell^+_{m/2}$ and $g^-_{(m-1)/2}$ for $g^+_{m/2}$) shows that this subalgebra is the centralizer of ψ, which is $\mathrm{End}_{A\otimes Q} E^-_{(m-1)/2}$. $\square$

3. Algebras of exponent 2 with unitary involution

In this section, we depart from the viewpoint laid out in the introduction, and take as starting point a central simple algebra B of degree $2m$ over a quadratic extension K of the base field F. We assume that B has exponent 2 and carries a unitary involution τ which is the identity on F. Our goal is to construct a central simple F-algebra $\hat{B}$ of degree $4m$ which contains B, and an orthogonal involution $\hat{\tau}$ on $\hat{B}$ which extends τ. Then B is the centralizer of K in $\hat{B}$, so the results of section 2 apply to relate the discriminant algebra $\mathcal{D}(B,\tau)$ to the Clifford algebra $\mathcal{C}(\hat{B},\hat{\tau})$.

Since B has exponent 2, we may find on B an orthogonal involution ν, by [14, (3.1), (2.8)].

Lemma 3.1. *There exists $u \in B^\times$ such that $\nu(u) = \tau(u) = u$ and $(\tau\circ\nu)^2(x) = uxu^{-1}$ for all $x \in B$.*

Proof. The composite $\tau\circ\nu\circ\tau$ is an involution on B of the same type as ν, since $\mathrm{Sym}(B, \tau\circ\nu\circ\tau) = \tau\bigl(\mathrm{Sym}(B,\nu)\bigr)$ has the same dimension as $\mathrm{Sym}(B,\nu)$. Therefore, we may find $u_0 \in B^\times$ such that $\nu(u_0) = u_0$ and

$$\tau\circ\nu\circ\tau = \mathrm{Int}(u_0)\circ\nu$$

(where $\mathrm{Int}(u_0)$ is the inner automorphism which maps $x \in B$ to $u_0 x u_0^{-1}$), hence $(\tau\circ\nu)^2 = \mathrm{Int}(u_0)$. Then $\mathrm{Int}(u_0)$ commutes with $\tau\circ\nu$, hence $\tau\circ\nu(u_0) \equiv u_0 \mod K^\times$. Since $\nu(u_0) = u_0$, it follows that $\tau(u_0) = \lambda u_0$ for some $\lambda \in K^\times$. Then $N_{K/F}(\lambda) = 1$, and Hilbert's Theorem 90 (see for instance [14, (29.3)]) yields $\lambda_0 \in K^\times$ such that $\lambda = \lambda_0\overline{\lambda_0}^{-1}$. The element $u = \lambda_0 u_0$ meets the requirements. $\square$

Let $u \in B^\times$ be as in Lemma 3.1. Define an F-algebra

$$\hat{B} = B \oplus Bz$$

by the following multiplication rules:

$$z^2 = u \qquad \text{and} \qquad zb = \tau\circ\nu(b)z \quad \text{for } b \in B.$$

Define also $\hat{\tau}\colon \hat{B} \to \hat{B}$ by

$$\hat{\tau}(b_1 + b_2 z) = \tau(b_1) + z\tau(b_2).$$

Proposition 3.2. *The F-algebra $\hat{B}$ is central simple of degree $4m$, and $\hat{\tau}$ is an orthogonal involution on $\hat{B}$.*

Proof. To prove the first part, observe that the algebra $\hat{B}$ is constructed from B by a “generalized crossed product” process, see Albert [1, Chapter XI, Theorem 10]. It is readily verified that $\hat{\tau}$ is an involution on $\hat{B}$. Since ν is orthogonal and

$$\mathrm{Sym}(\hat{B},\hat{\tau}) = \mathrm{Sym}(B,\tau) \oplus z\,\mathrm{Sym}(B,\nu),$$

it follows that

$$\dim \mathrm{Sym}(\hat{B},\hat{\tau}) = (2m)^2 + 2m(2m+1) = \frac{4m(4m+1)}{2},$$

hence $\hat{\tau}$ is orthogonal. □

Remark. The element u is determined by the conditions in Lemma 3.1 up to a factor in $F^\times$. The additivity property for crossed products (see [11, (1.15)]) shows that substituting $u' = \lambda u$ for u in the definition of $\hat{B}$ yields an algebra $\hat{B}'$ which is Brauer-equivalent to $\hat{B} \otimes_F (a,\lambda)_F$, if $K = F(\sqrt{a})$. On the other hand, suppose C is an arbitrary central simple F-algebra of degree $4m$ containing B. Left multiplication defines an F-algebra homomorphism $C \to \mathrm{End}_B C$, where C is viewed as a right B-module, hence $C \otimes_F K \simeq \mathrm{End}_B C \simeq M_2(B)$. Therefore, $C \otimes_F K \simeq \hat{B} \otimes_F K$, and it follows that C is Brauer-equivalent to $\hat{B} \otimes_F (a,\lambda)_F$ for some $\lambda \in F^\times$. Thus, any central simple F-algebra of degree $4m$ containing B can be obtained by the construction above.

4. Algebras of degree 8

In this section, we state the main results before giving the proofs, which are based on the results of section 2 and on the strong “triality” relationship between a degree 8 central simple algebra with orthogonal involution and its Clifford algebra described in [14, §42].

Let A be a central simple algebra endowed with an orthogonal involution σ. As mentioned in the introduction, if A contains a square-central skew-symmetric unit θ, the discriminant of σ is trivial. From Proposition 2.3, we get moreover that one of the components of $\mathcal{C}(A,\sigma)$, namely $\mathcal{C}^+(A,\sigma)$, has index at most 2. When A has degree 8, these conditions turn out to be sufficient for the existence of such an element θ, as the following theorem shows:

Theorem 4.1. *Let (A,σ) be a central simple algebra of degree 8 with orthogonal involution. The following conditions are equivalent:*

(1) *A contains a square-central skew-symmetric unit;*
(2) *the discriminant of σ is trivial, so $\mathcal{C}(A,\sigma) = C_1 \times C_2$ for some central simple F-algebras C_1, C_2 of degree 8, and at least one of C_1, C_2 has index 1 or 2.*

If these conditions hold, then A contains square-central units $\theta_1, \theta_2, \theta_3 \in \mathrm{Skew}(A,\sigma)$ such that $F(\theta_1,\theta_2,\theta_3)$ is an étale 8-dimensional commutative F-subalgebra of A.

Remark. Using this theorem, one may easily find degree 8 central simple algebras with orthogonal involution which do not contain any square-central skew-symmetric element. For split examples, use for instance [14, (42.11)].

The following result is proven in [14, (42.11)]:

Theorem 4.2 (Knus–Merkurjev–Rost–Tignol)**.** *Let (A,σ) be a central simple algebra of degree 8 with orthogonal involution. The following conditions are equivalent:*

1. *$(A,\sigma) = (A_1,\sigma_1)\otimes(A_2,\sigma_2)\otimes(A_3,\sigma_3)$ for some quaternion algebras with involution (A_i,σ_i), $i=1,2,3$;*
2. *the discriminant of σ is trivial, so $\mathcal{C}(A,\sigma) = C_1\times C_2$ for some central simple F-algebras C_1, C_2 of degree 8, and at least one of C_1, C_2 splits.*

If (A,σ) is decomposable, so that

$$(A,\sigma) = (H,\sigma_H)\otimes_F(A_0,\sigma_0) \tag{21}$$

where H is a σ-stable quaternion subalgebra of A and A_0 is of degree 4, it contains a square-central skew-symmetric unit, and hence satisfies the equivalent conditions of Theorem 4.1. On the other hand, Example 4.5 below shows that (A,σ) is not always isomorphic to a tensor product of three quaternion algebras with involution.

The next result gives a criterion of decomposability for (A,σ).

Theorem 4.3. *Let (A,σ) be a central simple algebra of degree 8 with orthogonal involution. The following conditions are equivalent:*

1. *(A,σ) is decomposable, i.e. $(A,\sigma)=(H,\sigma_H)\otimes_F(A_0,\sigma_0)$ where H is a quaternion algebra and A_0 is of degree 4;*
2. *the discriminant of σ is trivial, so $\mathcal{C}(A,\sigma) = C_1\times C_2$ for some central simple F-algebras C_1, C_2 of degree 8, and at least one of C_1, C_2 contains a square-central symmetric unit whose centralizer is split.*

Remarks 4.4. 1. Let $g\in C_i$ be a square-central symmetric unit as above. Its centralizer is Brauer-equivalent to $C_i\otimes F[g]$. Hence, the splitting hypothesis implies that C_i has index at most 2. If C_i is represented as $(\operatorname{End}_Q V, \operatorname{ad}_h)$ for some quaternion division algebra Q, the existence of such an element g is equivalent the existence of a certain type of diagonalisation for h (see the appendix). Note that if $g\in F^{\times 2}$, then $F[g]$ is isomorphic to $F\times F$ and by the hypothesis that the centralizer splits, we mean that C_i splits over F.

2. In the situation of Theorem 4.3, we may always find a decomposition of (A,σ) as (21) in which the involutions σ_H and σ_0 are orthogonal. Indeed, if they are symplectic, then it follows from the second proof of [14, (16.1)] (see also (16.16) and exercise 2 of chapter II) that there is a decomposition

$$(A_0,\sigma_0) = (H',\sigma')\otimes_F(H'',\sigma'')$$

where H', H'' are quaternion algebras, and σ' (resp. σ'') is a symplectic (resp. orthogonal) involution. Substituting H'' for H, we are reduced to the case where the restriction of σ to H is orthogonal.

The following examples show that the conditions of Theorems 4.1, 4.2 and 4.3 are not equivalent.

Example 4.5. Let $\langle 1,u,v,w\rangle$ be a quadratic form over F with non-trivial discriminant. Consider the algebra $A = M_8(K)$ over the field $K=F(t)$, where t is an indeterminate over F, endowed with the adjoint involution with respect to the quadratic form $q=\langle 1,t\rangle\otimes\langle 1,u,v,w\rangle$. The algebra (A,σ) is decomposable, and to prove it does not decompose as a tensor product of three σ-invariant quaternion subalgebras, it suffices to prove, according to Theorem 4.2, that both components of $C(A,\sigma)$ are non-split. To compute $C(A,\sigma)$, we proceed as follows: consider an orthogonal basis $(e_1, e_2, \dots, e_8)$ of $q = \langle 1, u, v, w, t, tu, tv, tw\rangle$. One may check that the elements e_1e_2, e_1e_3, $e_1e_2e_3e_4$, $e_1e_2e_3e_5$, $e_1e_2e_3e_4e_5e_6$ and $e_1e_2e_3e_4e_5e_7$ generate a subalgebra of $C(A,\sigma) = C_0(q)$ which is the tensor product of three quaternion algebras $B = (u, v) \otimes (uvw, uvt) \otimes (vw, uw)$. Since q has trivial discriminant, the center of $C_0(q)$ is isomorphic to $F \times F$ and both components of $C(A,\sigma)$ are isomorphic to B, which is Brauer equivalent to $(uvw, uvwt)$ and hence is not split.

Since (A,σ) is decomposable, the conditions of Theorem 4.3 hold. A square-central symmetric unit whose centralizer is split may here be explicitly found: the element $e_1e_2e_3e_4$ has the required properties.

Example 4.6. Let A be the central simple algebra of degree 8 and exponent 2 constructed in [10, (5.7.37)]. As explained there, A contains a square-central element θ, but does not contain any quaternion subalgebra. By Theorem (4.14) of [14] we may endow A with an involution σ of orthogonal type which extends the canonical involution of the quadratic field extension $F(\theta)/F$. Hence (A,σ) satisfies the equivalent conditions of Theorem 4.1, but not those of Theorem 4.3 since A itself is not decomposable.

Finally, we give a new version of Theorem 4.2 which, for a given choice of θ, yields the more precise decomposition described in (3), under the hypothesis that the split component of $\mathcal{C}(A,\sigma)$ is $\mathcal{C}(A,\sigma)^+$.

Theorem 4.7. *Let (A,σ) be a central simple F-algebra of degree 8 with orthogonal involution and let $\theta \in A$ be a square-central skew-symmetric unit. The following statements are equivalent:*

(1) *$\mathcal{C}^+(A,\sigma)$ is split;*
(2) *$\mathcal{D}(\tilde{A},\tilde{\sigma})$ is split;*
(3) *$(A,\sigma) = (A_1,\sigma_1) \otimes (A_2,\sigma_2) \otimes (A_3,\sigma_3)$ for some quaternion algebras with involution (A_i,σ_i), $i = 1, 2, 3$, and $\theta \in A_1$.*

Moreover, if these conditions hold, we may find a decomposition of (A,σ) as above such that $\sigma_1 = \operatorname{Int}(\theta) \circ \gamma_1$, $\sigma_2 = \gamma_2$ and $\sigma_3 = \gamma_3$, where γ_i denotes the canonical involution on A_i.

Remark. The choice of θ is essential in determining which of the two components of $\mathcal{C}(A,\sigma)$ is $\mathcal{C}^+(A,\sigma)$ (see Proposition 4.13).

Let us now prove those results. We start with the following:

Lemma 4.8. *Let (A,σ) be a central simple F-algebra with orthogonal involution of even degree, and let $e_1, \dots, e_r \in A$ be symmetric orthogonal idempotents such that $e_1 + \cdots + e_r = 1$. For $i = 1, \dots, r$, let $A_i = e_iAe_i$, and let σ_i be the restriction of σ to A_i. Assume each A_i has even degree. We may then identify $\mathcal{C}(A_i,\sigma_i)$ to an F-subalgebra of $\mathcal{C}(A,\sigma)$. For $s_1 \in \operatorname{Skew}(A_1,\sigma_1), \dots, s_r \in \operatorname{Skew}(A_r,\sigma_r)$, we have $s_1 + \cdots + s_r \in \operatorname{Skew}(A,\sigma)$ and*

$$\pi(s_1 + \cdots + s_r) = \pi_1(s_1)\cdots\pi_r(s_r),$$

where $\pi\colon \operatorname{Skew}(A,\sigma) \to Z\big(\mathcal{C}(A,\sigma)\big)$, $\pi_1\colon \operatorname{Skew}(A_1,\sigma_1) \to Z\big(\mathcal{C}(A_1,\sigma_1)\big)$, $\dots$, $\pi_r\colon \operatorname{Skew}(A_r,\sigma_r) \to Z\big(\mathcal{C}(A_r,\sigma_r)\big)$ are the generalized pfaffian maps to the centers of the Clifford algebras. Moreover, the elements $\pi_1(s_1), \dots, \pi_r(s_r)$ pairwise commute and $F\big(\pi_1(s_1),\dots,\pi_r(s_r)\big)$ is an étale F-algebra of dimension 2^r containing the center of $\mathcal{C}(A,\sigma)$. If the center is isomorphic to $F \times F$, and if f is *an idempotent* $\neq 0$, 1 *in the center of* $\mathcal{C}(A,\sigma)$*, then* $F\big(\pi_1(s_1),\dots,\pi_r(s_r)\big)\cdot f = F\big(\pi_1(s_1),\dots,\pi_{r-1}(s_{r-1})\big)\cdot f$ *is an* F*-algebra of dimension* 2^{r-1}.

Proof. The inclusion $A_i \hookrightarrow A$ induces an algebra homomorphism on the tensor algebras of the underlying vector spaces $T(A_i)\to T(A)$, which induces an injective algebra homomorphism $\mathcal{C}(A_i,\sigma_i)\to\mathcal{C}(A,\sigma)$, see Dejaiffe [4]. (It suffices to check this last part in the split case. If $A=\operatorname{End}_F V$, then $A_i=\operatorname{End}_F V_i$ for some even-dimensional regular subspace $V_i\subset V$, and the map $\mathcal{C}(A_i,\sigma_i)\to\mathcal{C}(A,\sigma)$ reduces to the canonical map $C_0(V_i)\hookrightarrow C_0(V)$.)

For $s_1\in\operatorname{Skew}(A_1,\sigma_1)$, …, $s_r\in\operatorname{Skew}(A_r,\sigma_r)$, it is clear that $s_1+\dots+s_r\in\operatorname{Skew}(A,\sigma)$. To prove that $\pi_1(s_1)$, …, $\pi_r(s_r)$ pairwise commute and that $\pi(s_1+\dots+s_r)=\pi_1(s_1)\cdots\pi_r(s_r)$, we may extend scalars to a splitting field of A. The previous equality implies in particular that $\pi_1(s_1)\cdots\pi_r(s_r)f\in F^\star f$ so that $F\big(\pi_1(s_1),\dots,\pi_r(s_r)\big)\cdot f = F\big(\pi_1(s_1),\dots,\pi_{r-1}(s_{r-1})\big)\cdot f$. □

Proof of Theorem 4.1. We only have to prove (2) ⇒ (1). Suppose C_1 has index 1 or 2, and let σ_1 be the restriction to C_1 of the canonical involution of $\mathcal{C}(A,\sigma)$. We may then represent $C_1=\operatorname{End}_Q V$, $\sigma_1=\operatorname{ad}_h$ for some quaternion F-algebra Q and some skew-hermitian Q-space (V,h) of rank 4, and we have

$$\mathcal{C}(\operatorname{End}_Q V,\operatorname{ad}_h)\simeq A\times C_2$$

by triality [14, (42.3)]. Consider an orthogonal decomposition

$$V=V_1\overset{\perp}{\oplus}V_2\overset{\perp}{\oplus}V_3\overset{\perp}{\oplus}V_4$$

where V_1, …, V_4 are Q-subspaces of rank 1. For $i=1,\dots,4$, let $e_i\in\mathcal{C}_1$ be the orthogonal projection $V\to V_i\subset V$. The elements e_1, …, e_4 are symmetric orthogonal idempotents such that $e_1+\dots+e_4=1$, and $e_iC_1e_i$ is a quaternion F-algebra isomorphic to Q for $i=1,\dots,4$.

Let $s_i\in e_iC_1e_i$ be an invertible element such that $\sigma(s_i)=-s_i$ for $i=1,\dots,4$ and consider the generalized pfaffian

$$\pi_i(s_i)\in Z\big(\mathcal{C}(e_iC_1e_i,\sigma_1|_{e_iC_1e_i})\big)\subset\mathcal{C}(C_1,\sigma_1)=A\times C_2.$$

Let

$$\pi_i(s_i)=(\theta_i,\theta_i')\in A\times C_2.$$

Since $\pi_i(s_i)$ is skew-symmetric for the canonical involution of $\mathcal{C}(C_1,\sigma_1)$, we have $\sigma(\theta_i)=-\theta_i$ for $i=1,\dots,4$. Moreover,

$$\pi_i(s_i)^2=\operatorname{Nrd}_{e_iC_1e_i}(s_i)\in F^\star,$$

hence θ_1^2, …, $\theta_4^2\in F^\star$.

The last statement is a consequence of lemma 4.8. □

Remark 4.9. The idea to use an orthogonal basis of V to produce skew-symmetric units in the Clifford algebra of $\operatorname{End}_Q V$ is borrowed from [15].

Proof of Theorem 4.3: "if" part. Suppose C_1 contains a unit g such that $g^2\in F^\star$, $\sigma_1(g)=g$, where σ_1 is the canonical involution on C_1, and the centralizer of g in C_1 is split.

If C_1 is split (see Remark 4.4), then the existence of a stable decomposition of A into a tensor product of quaternion subalgebras follows from Theorem 4.2. We may thus assume C_1 is not split for the rest of the proof of the "if" part.

Then $F[g]$ is a quadratic field extension of F and the index of C_1 is 2. As in the proof of Theorem 4.1, we may choose a representation $C_1 = \operatorname{End}_Q(V)$, $\sigma_1 = \operatorname{ad}_h$ for some quaternion division F-algebra Q split by $F[g]$ and some 4-dimensional skew-hermitian Q-space (V, h). Let $j \in Q$ be a pure quaternion such that $j^2 = g^2$.

We claim that there exists an anisotropic vector $v \in V$ such that $g(v) = vj$. Assuming this result, we may complete the proof by considering $s \in \operatorname{End}_Q V$ defined by

$$s(x) = vh(v, x) \qquad \text{for } x \in V.$$

For x, $y \in V$ we have

$$h\bigl(s(x), y\bigr) = \overline{h(v, x)}h(v, y) = -h(x, v)h(v, y) = -h\bigl(x, s(y)\bigr),$$

so s is skew-symmetric. Moreover, since $\sigma_1(g) = g$ we have

$$s\bigl(g(x)\bigr) = vh\bigl(v, g(x)\bigr) = -vjh(v, x),$$

whereas

$$g\bigl(s(x)\bigr) = g(v)h(v, x) = vjh(v, x),$$

so $s \circ g = -g \circ s$.

Consider now the Clifford algebra

$$\mathcal{C}(C_1, \sigma_1) = A \times C_2,$$

and let $c(s) = (\theta, \theta')$ be the image of s in $A \times C_2$ under the canonical map $c\colon C_1 \to \mathcal{C}(C_1, \sigma_1)$. If we call e the orthogonal projection on vF, s belongs to eC_1e which is a quaternion algebra. Hence, the restriction of c to eC_1e coincides with the generalized pfaffian $\pi\colon \operatorname{Skew}(eC_1e, \sigma_1|_{eC_1e}) \to Z(\mathcal{C}(eC_1e, \sigma_1|_{eC_1e})$. Hence, as in the proof of Theorem 4.1, the element $\theta \in A$ is a square-central skew-symmetric unit. The inner automorphism $\operatorname{Int}(g)$ commutes with σ_1 since $\sigma_1(g)g \in F^\star$. Therefore, it induces an automorphism $\mathcal{C}\bigl(\operatorname{Int}(g)\bigr)$ of the Clifford algebra $\mathcal{C}(C_1, \sigma_1)$. This automorphism is the identity on the center $F \times F$, since $\operatorname{Nrd}_{C_1}(g) = g^8 = \bigl(\sigma_1(g)g\bigr)^4$. Its restriction to A is therefore an inner automorphism $\operatorname{Int}(\rho)$ for some $\rho \in A$. Since $g^2 \in F^\star$, we have $\operatorname{Int}(g)^2 = \operatorname{Id}_{C_1}$, hence also $\operatorname{Int}(\rho)^2 = \operatorname{Id}_A$. Thus, $\rho^2 \in F^\star$. Moreover, $\mathcal{C}\bigl(\operatorname{Int}(g)\bigr)$ commutes with the canonical involution of $\mathcal{C}(C_1, \sigma_1)$, hence $\operatorname{Int}(\rho)$ commutes with σ, and therefore $\sigma(\rho)\rho \in F^\star$. Finally, since

$$\mathcal{C}\bigl(\operatorname{Int}(g)\bigr)\bigl(c(s)\bigr) = c(gsg^{-1}) = -c(s),$$

we have $\rho\theta = -\theta\rho$. It follows that the subalgebra H of A generated by ρ and θ is a quaternion algebra stable under σ. The proof of the "if" part is thus complete if we prove the claim on the existence of $v \in V$. This claim follows from the next lemma.

Lemma 4.10. *Let (V, h) be a skew-hermitian space over some quaternion division algebra Q. Assume $g \in \operatorname{End}_Q V$ is symmetric for the adjoint involution ad_h and satisfies $g^2 = j^2$ for some (non-zero) pure quaternion $j \in Q$. Then there exists an orthogonal basis $(v_1, \dots, v_n)$ of V such that $g(v_\ell) = v_\ell j$ for $\ell = 1, \dots, n$.*

Proof. Let $L = F(j) \subset Q$ and let $i \in Q$ be a non-zero pure quaternion which anticommutes with j, so that $Q = L \oplus iL$. We may then decompose the skew-hermitian form h by letting

$$h(x, y) = h_0(x, y) + ih_1(x, y)$$

with $h_0(x,y)$, $h_1(x,y) \in L$ for x, $y \in V$. The form $h_0\colon V \times V \to L$ is skew-hermitian, and h_1 is symmetric bilinear (compare [16, p. 359]). Consider now $\psi \in \operatorname{End}_L V$ defined by

$$\psi(x) = \tfrac{1}{2}\bigl(x - g(x)j^{-1}\bigr) \qquad \text{for } x \in V.$$

Since $g^2 = j^2$, we have $\psi^2 = \psi$, hence the L-vector space V decomposes as

$$V = \ker\psi \oplus \operatorname{im}\psi.$$

For $x \in V$, we have

$$\psi(xi) = \tfrac{1}{2}\bigl(x + g(x)j^{-1}\bigr)i = \bigl(x - \psi(x)\bigr)i,$$

hence $(\ker\psi)i = \operatorname{im}\psi$. Therefore,

$$\dim_L \ker\psi = \dim_L \operatorname{im}\psi = \dim_Q V,$$

and every L-basis of $\ker\psi$ is a Q-basis of V. For x, $y \in \ker\psi$, we have $g(x) = xj$, $g(y) = yj$, hence the equation $h\bigl(g(x), y\bigr) = h\bigl(x, g(y)\bigr)$ yields

$$-jh(x,y) = h(x,y)j$$

and it follows that $h_0(x,y) = 0$. Therefore, every h_1-orthogonal basis $(v_1, \ldots, v_n)$ of $\ker\psi$ is an h-orthogonal basis of V such that $g(v_\ell) = v_\ell j$ for $\ell = 1, \ldots, n$. □

Remark 4.11. The same arguments can be used in the case where g is skew-symmetric. Then $h_1(x,y) = 0$ for all x, $y \in \ker\psi$, and every h_0-orthogonal basis $(v_1, \ldots, v_n)$ of $\ker\psi$ is an h-orthogonal basis of V such that $g(v_\ell) = v_\ell j$ for $\ell = 1, \ldots, n$.

To prove the "only if" part of Theorem 4.3, we shall need the following result of independent interest:

Proposition 4.12. *Let B be a central simple algebra of degree 4 over a quadratic field extension K of F, and let τ be a unitary involution on B whose restriction to K is the non-trivial automorphism of K/F. There exists an F-subalgebra $B_0 \subset B$ stable under τ such that $B = B_0 \otimes_F K$ if and only if the discriminant algebra $D(B,\tau)$ contains a square-central unit g of reduced trace zero which is symmetric under the canonical involution $\underline{\tau}$. Moreover, if this condition holds, then $D(B,\tau)$ is Brauer-equivalent to the quaternion algebra $(a,\lambda)_F$, where $K = F(\sqrt{a})$ and $\lambda = g^2$.*

Proof. If it exists, the F-subalgebra B_0 is the algebra of fixed points of a semi-linear automorphism φ of B such that $\varphi \circ \tau = \tau \circ \varphi$ and $\varphi^2 = \operatorname{Id}_B$. By [14, (15.26)](see also (15.24)), we have

$$\operatorname{Aut}_F(B,\tau) = \operatorname{Aut}_F\bigl(D(B,\tau), \underline{\tau}\bigr),$$

and the semi-linear automorphisms of (B,τ) correspond to inner automorphisms of $D(B,\tau)$ induced by improper similitudes. Therefore, an automorphism φ as above exists if and only if there is in $D(B,\tau)$ an element y such that

$$\underline{\tau}(y)y \in F^\star, \quad y^2 \in F^\star \quad \text{and} \quad \operatorname{Nrd}_{D(B,\tau)}(y) = -\bigl(\underline{\tau}(y)y\bigr)^3. \tag{22}$$

It remains to prove that such an element y exists if and only if there is in $D(B,\tau)$ a square-central symmetric unit of reduced trace zero.

Suppose first $g \in D(B,\tau)$ is such that $g^2 \in F^\star$, $\underline{\tau}(g) = g$ and $\operatorname{Trd}_{D(B,\tau)}(g) = 0$. If $g^2 = \lambda$, the reduced characteristic polynomial of g has the form $(X^2 - \lambda)^3$, hence

$$\operatorname{Nrd}_{D(B,\tau)}(g) = -\lambda^3 = -\bigl(\underline{\tau}(g)g\bigr)^3$$

and we may take $y = g$.

Conversely, if there is in $D(B,\tau)$ an element y satisfying (22), let $\underline{\tau}(y)y = \mu$ and $y^2 = \lambda$. If $\lambda \notin F^{\star 2}$, then the reduced characteristic polynomial of y is $(X^2 - \lambda)^3$, hence $\mathrm{Trd}_{D(B,\tau)}(y) = 0$, $\mathrm{Nrd}_{D(B,\tau)}(y) = -\lambda^3$ and (22) implies $\mu^3 = \lambda^3$. For $\omega = \mu\lambda^{-1}$, we thus have $\omega^3 = 1$ and $\underline{\tau}(y)y = \omega y^2$, hence $\underline{\tau}(y) = \omega y$. This last equation implies $\omega^2 = 1$, hence $\omega = 1$. It follows that y is a square-central symmetric unit of reduced trace zero.

If $\lambda \in F^{\star 2}$, let $\lambda = \lambda_0^2$ with $\lambda_0 \in F^\star$. The reduced characteristic polynomial of y then has the form $(X - \lambda_0)^{m_1}(X + \lambda_0)^{m_2}$ with $m_1 + m_2 = 6$, hence $\mathrm{Nrd}_{D(B,\tau)}(y) = (-1)^{m_1}\lambda^3$. Conditions (22) yield $-\mu^3 = (-1)^{m_1}\lambda^3$. Arguing as in the preceding case, we obtain $\underline{\tau}(y) = (-1)^{m_1+1}y$. If m_1 is even, we have $\underline{\tau}(y) = -y$, hence $-y$ has the same reduced characteristic polynomial as y. This is a contradiction since the reduced characteristic polynomial of $-y$ is $(X - \lambda_0)^{m_2}(X + \lambda_0)^{m_1}$ and m_1 even implies $m_1 \neq m_2$.

If $m_1 = m_2 = 3$, then y is a square-central symmetric unit of trace zero. Finally, if $m_1 = 1$ (resp. 5), then the right ideal generated by $y - \lambda_0$ (resp. $y + \lambda_0$) has dimension 6 (note that y is diagonalisable), hence the algebra $D(B,\tau)$ is split, and we may represent $D(B,\tau)$ as the endomorphism algebra of some 6-dimensional vector space, and $\underline{\tau}$ as the adjoint involution with respect to some symmetric bilinear form. The orthogonal reflection with respect to any non-singular 3-dimensional subspace is a square-central symmetric unit of trace zero.

To prove the last statement, observe that g is an improper similitude of $D(B,\tau)$ with multiplier λ, hence Theorem (13.38) of [14] shows that $D(B,\tau)$ is Brauer-equivalent to the quaternion algebra $(\operatorname{disc}\underline{\tau}, \lambda)_F$. By the equivalence $A_3 \equiv D_3$ (see [14, (15.24)]), the algebra B is isomorphic to the Clifford algebra of $\bigl(D(B,\tau), \underline{\tau}\bigr)$, hence $\operatorname{disc}\underline{\tau} = a \cdot F^{\star 2}$ and the proof is complete. □

Proof of Theorem 4.3: "only if" part. As observed in Remark 4.4, we may assume

$$(A,\sigma) = (H,\sigma_H) \otimes_F (A_0,\sigma_0)$$

where σ_H is an orthogonal involution on the quaternion algebra H. Let $\theta \in H$ be a skew-symmetric unit. Since H is a quaternion algebra, θ is square-central.

Let $a = \theta^2 \in F^\star$. If $a \in F^{\star 2}$, then the discriminant of σ_H is trivial, hence σ_H is hyperbolic by [14, (7.4)]. It follows that σ is also hyperbolic, hence one of C_1, C_2 is split, by [14, (8.31)]. It is easy to find in the split component a square-central symmetric unit of trace zero (for instance an orthogonal reflection), hence the theorem follows.

For the rest of the proof, we may thus assume $a \notin F^{\star 2}$, and apply the results of section 2. These results yield a quaternion F-algebra Q and a representation

$$\bigl(\mathcal{C}^+(A,\sigma), \underline{\sigma}\bigr) \xrightarrow{\sim} (\operatorname{End}_Q E^+, \operatorname{ad}_{h^+})$$

for some skew-hermitian form h^+ (see Proposition 2.3). Proposition 2.5 yields an orthogonal decomposition

$$E^+ = \hat{E}_0^+ \overset{\perp}{\oplus} E_1^+,$$

where $\dim_Q \hat{E}_0^+ = 1$ and $\dim_Q E_1^+ = 3$, and Corollary 2.13 yields a representation

$$\bigl(D(\tilde{A},\tilde{\sigma}), \underline{\tilde{\sigma}}\bigr) \xrightarrow{\sim} (\operatorname{End}_Q E_1^+, \operatorname{ad}_{h_1^+}),$$

where h_1^+ is the restriction of h^+ to E_1^+. Since A_0 is a $\tilde{\sigma}$-stable F-subalgebra of $\tilde{A}$ such that $\tilde{A} = F(\theta) \otimes_F A_0$, Proposition 4.12 shows that $D(\tilde{A}, \tilde{\sigma})$ contains a square-central symmetric unit of reduced trace zero. We may thus find $g_1 \in \operatorname{End}_Q E_1^+$ such that $\operatorname{ad}_{h_1^+}(g_1) = g_1$ and $g_1^2 \in F^\star$. Let $\lambda = g_1^2$. Proposition 4.12 also shows that $D(\tilde{A}, \tilde{\sigma})$ is Brauer-equivalent to the quaternion algebra $(a, \lambda)_F$, hence $Q = (a, \lambda)_F$.

Since $\operatorname{disc} \sigma = 1$, we also have $\operatorname{disc} \underline{\sigma} = 1$ by triality [14, (42.3)]. On the other hand, $\operatorname{disc} \underline{\tilde{\sigma}} = a \cdot F^{\star 2}$ because $\tilde{A}$ is the Clifford algebra of $\big(D(\tilde{A}, \tilde{\sigma}), \underline{\tilde{\sigma}}\big)$ by [14, (15.24)]. Therefore, the restriction of ad_{h^+} to $\operatorname{End}_Q \hat{E}_0^+$ has discriminant $a \cdot F^{\star 2}$. Since $\dim_Q \hat{E}_0^+ = 1$, we have $\operatorname{End}_Q \hat{E}_0^+ \simeq Q$, and there is an isomorphism under which the restriction of ad_{h^+} to $\operatorname{End}_Q \hat{E}_0^+$ corresponds to the involution of $Q = (a, \lambda)_F$ which maps the elements i, j of the standard quaternion basis to $-i$, j respectively, since this orthogonal involution also has discriminant $a \cdot F^{\star 2}$ (see [14, (7.4)]). The element $g_0 \in \operatorname{End}_Q \hat{E}_0^+$ corresponding to j is such that $\operatorname{ad}_{h^+}(g_0) = g_0$ and $g_0^2 = \lambda$. Now, the endomorphism $g = g_0 \oplus g_1$ of $E^+ = \hat{E}_0^+ \overset{\perp}{\oplus} E_1^+$ satisfies $\operatorname{ad}_{h^+}(g) = g$ and $g^2 = \lambda \in F^\star$, and its centralizer is split. □

Proof of Theorem 4.7. The equivalence of (1) and (2) follows from the Brauer-equivalence of $\mathcal{C}^+(A, \sigma)$ and $\mathcal{D}(\tilde{A}, \tilde{\sigma})$, see Proposition 2.3 and Corollary 2.13.

(3) $\Rightarrow$ (2) If (3) holds, we have $\tilde{A} = F(\theta) \otimes_F Q_2 \otimes_F Q_3$ and $\tilde{\sigma} = {}^- \otimes \sigma_2 \otimes \sigma_3$, hence $\mathcal{D}(\tilde{A}, \tilde{\sigma})$ is split, by [14, (10.33)].

(2) $\Rightarrow$ (3) By Karpenko-Quéguiner [13], (2) implies that

$$(\tilde{A}, \tilde{\sigma}) = \big(F(\theta) \otimes Q_2 \otimes Q_3, {}^- \otimes \gamma_2 \otimes \gamma_3\big)$$

for some quaternion F-algebras Q_2, Q_3 with conjugation involutions γ_2, γ_3. The centralizer of $Q_2 \otimes Q_3$ in A is a quaternion algebra Q_1 containing θ.

Finally, since $\gamma_2 \otimes \gamma_3$ is orthogonal, the restriction σ_1 of σ to Q_1 must be orthogonal, otherwise σ would be symplectic. Since it maps θ to $-\theta$, it coincides with $\operatorname{Int}(\theta) \circ \gamma_1$ □

As explained in the introduction, the choice of which of the two components of $\mathcal{C}(A, \sigma)$ is $\mathcal{C}^+(A, \sigma)$ depends on the choice of θ. The following proposition makes this fact clear and emphasizes the difference between Theorems 4.7 and 4.2.

Proposition 4.13. *Let Q be a quaternion division algebra over an arbitrary field F, and let $A = M_4(Q)$. Let also σ be an orthogonal hyperbolic involution on A and let $\mathcal{C}(A, \sigma) = C_1 \times C_2$ for some central simple F-algebras C_1, C_2. Then one of the components (C_1, say) is split, and the other one is Brauer-equivalent to Q. There exist θ_1, $\theta_2 \in \operatorname{Skew}(A, \sigma)$ such that $\theta_1^2 = \theta_2^2 \in F^\star$ and, letting $a = \theta_1^2 = \theta_2^2$,*

$$\tfrac{1}{2}\big(1 + a^{-2}\pi(\theta_1)\big) \cdot \mathcal{C}(A, \sigma) = C_1, \qquad \tfrac{1}{2}\big(1 + a^{-2}\pi(\theta_2)\big) \cdot \mathcal{C}(A, \sigma) = C_2.$$

Thus, the equivalent conditions of Theorem 4.7 hold for θ_1 but not *for θ_2.*

Proof of Proposition 4.13. Note that since hyperbolic involutions on A are isomorphic, it is enough to prove the result for a particular one.

Let $A_0 = M_2(F) \otimes_F Q$ and let σ_0 be the tensor product of the symplectic involutions on $M_2(F)$ and Q. Since the symplectic involution on $M_2(F)$ is hyperbolic, the involution σ_0 is hyperbolic. Letting $M_2(F)^0$, Q^0 denote the vector spaces of (reduced) trace zero elements in $M_2(F)$ and Q respectively, we have

$$\operatorname{Skew}(A_0, \sigma_0) = \big(M_2(F)^0 \otimes 1\big) \oplus \big(1 \otimes Q^0\big).$$

Moreover, $\mathcal{C}(A_0,\sigma_0) = M_2(F)\times Q$, by [14, (8.19)], and the generalized pfaffian map

$$\pi_0\colon \ \mathrm{Skew}(A_0,\sigma_0) \to Z\big(\mathcal{C}(A_0,\sigma_0)\big) = F\times F$$

satisfies

$$\pi_0(u\otimes 1) = (u^2,-u^2) \qquad \text{for } u\in M_2(F)^0, \tag{23}$$

$$\pi_0(1\otimes q) = (-q^2,q^2) \qquad \text{for } q\in Q^0, \tag{24}$$

see [14, (16.32)]. Let

$$A = M_2(F)\otimes_F A_0 \quad (\simeq M_4(Q))$$

and let $\sigma = t\otimes\sigma_0$, the tensor product of the transpose involution on $M_2(F)$ and σ_0. Since σ_0 is hyperbolic, the involution σ is hyperbolic. The elements

$$e_1 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\otimes 1\in A, \qquad e_2 = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\otimes 1\in A$$

are symmetric orthogonal idempotents such that $e_1+e_2=1$. Using the same notation as in Lemma 4.8, we have canonical isomorphisms $(A_1,\sigma_1) = (A_0,\sigma_0) = (A_2,\sigma_2)$.

Let $q\in Q^0$ be an invertible pure quaternion and let $a = q^2\in F^\star$. We may find a matrix $u\in M_2(F)^0$ such that $u^2=a$. In $A = M_2(F)\otimes M_2(F)\otimes Q$, consider the elements

$$\theta_1 = 1\otimes 1\otimes q$$

and

$$\theta_2 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\otimes 1\otimes q + \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\otimes u\otimes 1.$$

We have θ_1, $\theta_2\in\mathrm{Skew}(A,\sigma)$ and $\theta_1^2=\theta_2^2=a$. Lemma 4.8 yields

$$\pi(\theta_1) = \pi_1(1\otimes q)\pi_2(1\otimes q), \qquad \pi(\theta_2) = \pi_1(1\otimes q)\pi_2(u\otimes 1).$$

By (23) and (24), we have $\pi_2(1\otimes q) = -\pi_2(u\otimes 1)$, hence $\pi(\theta_1) = -\pi(\theta_2)$. Therefore, letting

$$C_1 = \tfrac12\big(1+a^{-2}\pi(\theta_1)\big)\cdot\mathcal{C}(A,\sigma) \quad\text{and}\quad C_2 = \tfrac12\big(1-a^{-2}\pi(\theta_1)\big)\cdot\mathcal{C}(A,\sigma),$$

we have $\mathcal{C}(A,\sigma) = C_1\times C_2$ and $\frac12\big(1+a^{-2}\pi(\theta_2)\big)\cdot\mathcal{C}(A,\sigma) = C_2$.

Since condition (3) of Theorem 4.7 holds for θ_1, it follows that C_1 is split. Since $C_1\otimes_F C_2$ is Brauer-equivalent to A by a theorem of Jacobson (see [14, (9.14)]), C_2 is Brauer-equivalent to A, hence also to Q. □

The index of the algebra A in Proposition 4.13 is 2. This turns out to be a general property for algebras exhibiting the behaviour of Proposition 4.13, as the following corollary shows:

Corollary 4.14. *Let (A,σ) be a central simple algebra of degree 8 with orthogonal involution of trivial discriminant. Suppose the index of A is not 2 and one of the components of $\mathcal{C}(A,\sigma)$ is split. Then for every square-central unit $\theta\in\mathrm{Skew}(A,\sigma)$, there is a decomposition of A into a tensor product of σ-invariant quaternion F-subalgebras $A = Q_1\otimes_F Q_2\otimes_F Q_3$ such that $\theta\in Q_1$ and the restriction of σ to Q_1 is an orthogonal involution.*

Proof. Let $\mathcal{C}(A,\sigma) = C_1 \times C_2$ and suppose C_1 is split. By Jacobson's theorem (see [14, (9.14)]), it follows that C_2 is Brauer-equivalent to A. On the other hand, Proposition 2.3 shows that for each square-central unit $\theta \in \mathrm{Skew}(A,\sigma)$ the index of $\mathcal{C}^+(A,\sigma)$ is at most 2. Therefore, we must have $\mathcal{C}^+(A,\sigma) = C_1$ if the index of A is 4 or 8, and $\mathcal{C}^+(A,\sigma) \simeq C_1 \simeq C_2$ if A is split. Thus in all cases $\mathcal{C}^+(A,\sigma)$ is split and Theorem 4.7 applies. □

Appendix

In this appendix, we show that the existence of a square-central symmetric or skew-symmetric element in the endomorphism algebra of some skew-hermitian space over a quaternion division algebra can be accounted for by a certain type of diagonalisation of the corresponding skew-hermitian form.

Proposition. *Let Q be a division quaternion algebra over F, and (V,h) a skew-hermitian module over Q. The following conditions are equivalent:*

(1) *the algebra with involution $(\mathrm{End}_Q(V), \mathrm{ad}_h)$ contains a square-central symmetric (resp. skew-symmetric) element g whose centralizer splits;*
(2) *the hermitian form h admits a diagonalisation of the form $h = \langle q_1, \dots, q_n\rangle$, where $q_1, \dots, q_n$ are pure quaternions satisfying $q_\ell j = -jq_\ell$ (resp. $h = \langle \lambda_1 j, \dots, \lambda_n j\rangle$, where $\lambda_\ell \in F^\star$), for some pure quaternion j.*

If these conditions hold, the element g and the quaternion j are related by $j^2 = g^2$.

Proof. (1) ⇒ (2) Since the centralizer of g splits, the quaternion algebra Q contains a pure quaternion j satisfying $j^2 = g^2$. If g is symmetric, we apply Lemma 4.10 to find an orthogonal basis $(v_1, \dots, v_n)$ of V such that $g(v_\ell) = v_\ell j$ for $\ell = 1, \dots, n$. The equation $h\big(g(v_\ell), v_\ell\big) = h\big(v_\ell, g(v_\ell)\big)$ then yields $q_\ell j = -jq_\ell$ for $q_\ell = h(v_\ell, v_\ell)$. The same arguments can be used in the case where g is skew-symmetric, see Remark 4.11.

(2) ⇒ (1) Let us consider an orthogonal basis $(v_1, \dots, v_n)$ of V over Q in which the form h has a diagonalisation as in (2). The endomorphism g defined by $g(v_\ell) = v_\ell j$ for $\ell = 1, \dots, n$ has the required properties. □

Note that if g is skew-symmetric and V_0 is the F-vector space spanned by the basis $v_1, \dots, v_n$ constructed in the proof above, then $\mathrm{End}_Q V = Q \otimes_F \mathrm{End}_F V_0$ and Q, $\mathrm{End}_F V_0$ are stable under ad_h.

References

[1] A.A. Albert, *Structure of Algebras*, Amer. Math. Soc. Colloquium Publ. 24, Providence, RI, 1961.

[2] E. Bayer-Fluckiger, R. Parimala, Galois cohomology of the classical groups over fields of cohomological dimension ≤ 2, *Invent. Math.* **122**, 195–229 (1995).

[3] E. Bayer-Fluckiger, D. Shapiro, J.-P. Tignol, Hyperbolic involutions, *Math. Z.* **214**, 461–476 (1993).

[4] I. Dejaiffe, Somme orthogonale d'algèbres à involution et algèbre de Clifford, *Comm. Algebra* **26**, no. 5, 1589–1612 (1998).

[5] I. Dejaiffe, D.W. Lewis, J.-P. Tignol, Witt equivalence of central simple algebras with involution, *Rend. Circ. Mat. Palermo, Ser. 2*, **49**, 325–342 (2000).

[6] H. Dherte, Decomposition of Mal'cev-Neumann division algebras with involution, *Math. Z.* **216**, 629–644 (1994).

[7] R. Garibaldi, Clifford algebras of hyperbolic involutions, *Math. Z.*, to appear.

[8] R. Garibaldi, A. Quéguiner, J.-P. Tignol, Involutions and trace forms on exterior powers of a central simple algebra, in preparation.

[9] R. Garibaldi, J.-P. Tignol, A. Wadsworth, Galois cohomology of special orthogonal groups, *Manuscripta Math.* **93**, 247–266 (1997).

[10] N. Jacobson, *Finite Dimensional Division Algebras over Fields*, Springer, 1996.

[11] W. Jehne, Separable adel algebras and a presentation of Weil groups, *J. reine angew. Math.* **375/376**, 211–237 (1987).

[12] M.-A. Knus, R. Parimala, R. Sridharan, Involutions on rank 16 central simple algebras, *J. Indian Math. Soc.* (N.S.) **57**, 143–151 (1991).

[13] N. Karpenko, A. Quéguiner, A criterion of decomposability for degree 4 algebras with unitary involution, *J. Pure App. Algebra* **147**, 303–309 (2000).

[14] M.-A. Knus, A.S. Merkurjev, M. Rost, J.-P. Tignol, *The Book of Involutions*, Amer. Math. Soc. Colloquium Publ. 44, Providence, RI, 1998.

[15] M. Rost, J-P. Serre, J.-P. Tignol, The trace form of a central simple algebra of degree four, in preparation.

[16] W. Scharlau, *Quadratic and Hermitian Forms*, Springer, Berlin, 1985.

[17] D. Tao, The generalized even Clifford algebra, *J. Algebra* **172**, 184–204 (1994).

[18] V. I. Yanchevskii, Symmetric and skew-symmetric elements of involutions, associated groups and the problem of decomposability of involutions, in *K-theory and algebraic geometry: connections with quadratic forms and division algebras*, (B. Jacob, A. Rosenberg, eds) Proceedings of Symposia in Pure Mathematics, **58** Part 2, A.M.S. (1995), pp. 431–444.

Anne Quéguiner-Mathieu
Laboratoire Analyse, Géométrie
et Applications, UMR 7539
Institut Galilée
Université Paris Nord
F-93430 Villetaneuse
France

e-mail:
queguin@math.univ-paris13.fr

Jean-Pierre Tignol
Institut de Mathématique Pure
et Appliquée
Université catholique de Louvain
B-1348, Louvain-la-Neuve
Belgium

e-mail:
tignol@agel.ucl.ac.be