K_1 OF CHEVALLEY GROUPS ARE NILPOTENT

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1. Introduction

Let Φ be a reduced irreducible root system and R be a commutative ring. We consider the corresponding simply connected Chevalley group $G = G(\Phi, R)$ and its elementary subgroup $E(\Phi, R)$. When $\mathrm{rk}(\Phi) \geq 2$ it is proven by Suslin and Kopeiko [17], [18], [11] for the classical cases and by Taddei [20] for the exceptional cases, that $E(\Phi, R)$ is normal in $G(\Phi, R)$, so that one can consider the K_1-functor modeled on G:

$$K_1(\Phi, R) = G(\Phi, R)/E(\Phi, R),$$

see references in [15], [25], [13]. Observe that this functor is a generalisation of SK_1 rather than the usual K_1. Namely, $\mathrm{SL}(l+1, R)$ is the Chevalley group of type A_l and $K_1(A_l, R) = \mathrm{SK}_1(l+1, R)$.

It is well known that when R is a field, or, more generally a semi-local ring, the functor $K_1(\Phi, R)$ is trivial, or, in other words, $G(\Phi, R) = E(\Phi, R)$ (see, for example, [2]). In the stable range, i.e. when $\mathrm{rk}(\Phi)$ is large with respect to the dimension of R, the functor $K_1(\Phi, R)$ is abelian. The present paper is an attempt to understand what can be said about $K_1(\Phi, R)$ in the meta-stable range, when dimension of R is large. There are examples due to van der Kallen and Bak [9], [3] which show that non-stable $K_1(\Phi, R)$ can be non-abelian, and the natural question is how non-abelian it can be?

In [3], Bak, developed a beautiful localisation-completion method which allowed him to prove that $\mathrm{SK}_1(n, R)$ is nilpotent, and, more generally, $K_1(n, R)$ is nilpotent-by-abelian when Bass-Serre dimension $\delta(R)$ of the ground ring R is finite. Recall that a group H is called *nilpotent-by-abelian*, if it has a normal subgroup F such that F is nilpotent and H/F is abelian. This clearly implies that H is a solvable group.

In [7] the first author uses the same method to extend this result to non-stable K_1 of general quadratic groups. Classical Chevalley groups fall into this category and it follows from the results of [7] that K_1 are nilpotent for Chevalley groups of types C_l and D_l. In fact, [7] establishes much more general results, namely that certain slightly larger K_1-functors are nilpotent-by-abelian for a huge class of unitary groups over form rings.

Here we show that the same holds for all Chevalley groups. More precisely, the main result of the present work is a construction of a descending central series in the Chevalley group, indexed by the Bass-Serre dimension of the factor-rings of the ground ring. In the case of finite-dimensional rings this leads to the following theorem, which we prove in Section 7.

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Theorem. *Let Φ be a reduced irreducible root system of rank ≥ 2 and R be a commutative ring such that its Bass-Serre dimension $\delta(R)$ is finite. Then for any Chevalley group $G(\Phi, R)$ of type Φ over R the quotient $G(\Phi, R)/E(\Phi, R)$ is nilpotent-by-abelian. In particular $K_1(\Phi, R)$ is nilpotent of class at most $\delta(R) + 1$.*

A special case of this result pertaining to the case when $R = \mathbb{R}^X$ or $\mathbb{C}^X$ is the ring of all continuous real or complex-valued functions on a finite-dimensional topological space X was stated by Vaserstein in [24], Theorem 7. This theorem is accompanied by the following proof, which we reproduce *verbatim*: "PROOF, using the Bruhat decomposition is the same as for $\mathrm{GL}_n(A)$."

Our principal tool is the localisation-completion method of [3] and [7], and we refer the reader to these papers and [4] for more background information and details. Since this method is not as popular as some other techniques and the body of our paper consists of calculations, we take our breath for the moment and explain what really goes on here and how this method stands to other major methods which are used to attack similar problems. To avoid some additional technical complications and present ideas in their simplest form, assume for the time being that $G(\Phi, R)$ is simply connected.

Informally the theorem above may be viewed as an *extremely* strong form of normality of $E(\Phi, R)$ in $G(\Phi, R)$. In fact, normality asserts that for any elementary generator $x_\alpha(a)$ and any $g \in G$ one has $[x_\alpha(a), g] \in E(\Phi, R)$. On the other hand our theorem asserts that for finite-dimensional rings something terribly much stronger occurs, namely $[\dots[[g_1, g_2], g_3], \dots, g_m] \in E(\Phi, R)$ for *any* sufficiently long sequence $g_1, g_2, \dots, g_m \in G$. Of course, in view of the fact that $E(\Phi, R)$ is perfect this implies normality of $E(\Phi, R)$.

Now for arbitrary[1] commutative[2] rings we are aware of five major noticeably different ways to prove such results:

- Suslin's direct factorisation method [17], [18], [11], [8];
- Suslin's factorisation and patching method [22], [10], [5];
- Quillen–Suslin–Vaserstein's localisation and patching method, [17], [23], [20], [19];
- Bak's localisation-completion method [3], [7], [4];
- Stepanov–Vavilov–Plotkin's decomposition of unipotents [25], [27], [16], [26].

Suslin's first method and decomposition of unipotents are based on reduction to groups of smaller rank over the same ring. On the other hand, localisation and patching and localisation-completion are based on reduction to groups of the same type over rings of smaller dimension. Of course, here

[1]There are, of course, many further methods, which only work for some classes of rings, most notably, methods using stability conditions, as developed by Bass, Bak, Dennis, van der Kallen, Stein, Suslin, Vaserstein, and others. There are some further methods, which use topological or metric properties of R, or other similar structures. We do not try to survey such methods here.

[2]There are numerous generalisations and ramifications of these methods for non-commutative rings, including the very powerful Golubchik-Mikhalev non-commutative localisation methods, see references in [25], [5], [16], which we do not discuss here, since we are only interested in commutative rings.

too one has to invoke reduction to a smaller rank at some stage, but the only such reduction there is, occurs at the level of zero-dimensional rings, is classically known and remains invisible to the reader. For example, the only reduction to groups of smaller rank which is ever used in the present paper, appears under the disguise of Gauß decomposition over semi-local rings. Suslin's second method combines reduction in dimension *and* rank. Sometimes these methods are simultaneously used in the same proof, as in [14], which brings into action the combined force of localisation-completion and decomposition of unipotents to obtain length bounds which would be beyond reach of either of these methods individually.

Decomposition of unipotents is a generalisation of Suslin's first method. It is very powerful and extremely straightforward at the same time. When it can be applied, it usually gives by far the best results algorithmically. What you should expect to get in our problem, would be an *explicit* polynomial formula, expressing $[\dots[[g_1, g_2], g_3], \dots, g_m]$ as a product of elementary root unipotents $x_\alpha(a)$ with parameters a depending polynomially on the matrix entries of g_i's in a faithful representation of G. Now everybody, who has seen, how such a formula looks like for the commutator $[x_\alpha(a), g]$ with *one* general matrix $g \in G$ for the classical groups in vector representations, [11], [12], [25], [16], or for the groups of types E_6 or E_7 in micro-weight representations, [25], [27], [26], would immediately recognise, that writing a similar formula for our problem was not an option.

The other three methods are very similar in spirit, they are all based on localisations and partitions of 1 in the ground ring. The real difference is in how they address zero divisors. The relation between Suslin's second method and localisation and patching is exactly the same as the relation between Suslin's and Quillen's solutions of Serre's problem. Both methods are well documented in the existing literature. This is especially true for localisation and patching which was used in dozens of papers published by Suslin, Kopeiko, Tulenbaev, Abe, Vorst, Vaserstein, Taddei, Li Fuan, and many others. The key feature of localisation and patching is throwing in independent variables to fight zero divisors and then applying Quillen's theorem [17].

In [3] Bak proposed yet another version of localisation, which does not require passage to a polynomial ring, but operates in R itself. The characteristic feature of this method is reduction to Noetherian rings, where one can easily control the behaviour of zero divisors. The main idea of the method, once you get it, is exceedingly simple. Unfortunately, [3] doesn't constitute an easy reading, since it throws in all possible technical complications simultaneously: non-commutativity, explicit bounds for lengths, second localisation, completion, and more. The crux of the method is buried somewhere in the proof of Lemma 4.11 in the depth of Section 4. And as the final blow, which was especially frustrating for the second author, the notation in [3] fails to clearly distinguish between an element of $g \in G$ and its images under localisations – and at some point in the proof one has to look at the images of g in four different localisations.

We believe that the method introduced in [3] is so natural and important, that it deserves a much better publicity, and one of our broader intentions in writing this paper was to give it the credit it truly deserves. To explain the essence of the method, below we reproduce what we believe is the shortest existing proof for the normality of $E(\Phi, R)$ in $G(\Phi, R)$. In what follows we denote by $F_M : R \longrightarrow R_M$ the localisation homomorphism modulo a maximal ideal and by $F_s : R \longrightarrow R_s$ the localisation homomorphism with respect to $s \in R$.

We wish to prove that for any $g \in G$, any $\alpha \in \Phi$ and any $a \in R$ one has $x = gx_\alpha(a)g^{-1} \in E(\Phi, R)$. As typical for localisation proofs, we use partitions of 1. In other words, we have to pick up $b_1, \dots, b_r \in R$ such that $1 = b_1 + \dots + b_r$ and each of $gx_\alpha(b_i a)g^{-1}$ already lies in $E(\Phi, R)$. Of course, the difference between various localisation methods is in how one chooses such a partition. The following paragraph is a friendly takeover of a theme of Bak, [3], Lemma 4.11.

Since the functors $G(\Phi, -)$ and $E(\Phi, -)$ commute with direct limits and R is a direct limit of its finitely generated subrings, we can from the very start reduce to the case when R is Noetherian. Fix a maximal ideal $M \in \mathrm{Max}(R)$. Since for local rings E coincides with G, one has $F_M(g) \in E(\Phi, R_M)$. Since R_M is the direct limit of R_t, $t \in R \setminus M$, there exists such an $s \in R \setminus M$, that $F_s(g) \in E(\Phi, R_s)$. We will search for a b_i of the form s^l for a sufficiently large exponent l. Set $y = gx_\alpha(s^l a)g^{-1}$. The ring R_s being Noetherian, for a large power of s, say for s^n, the restriction of F_s to the principal congruence subgroup $G(\Phi, R, s^n R)$ is injective. Since $F_s(g) \in E(\Phi, R_s)$, by the Chevalley commutator formula there exists a higher power of s, say, s^l, $l \geq n$, such that $F_s(y) = F_s(g)F_t(x_\alpha(s^l a))F_s(g)^{-1}$ can be expressed as a product $\prod x_{\beta_i}(F_s(s^n c_i))$, $i = 1, \dots, m$. Take the product $z = \prod x_{\beta_i}(s^n c_i)$, $i = 1, \dots, m$. By the very definition $z \in E(\Phi, R)$ and $F_s(y) = F_s(z)$. On the other hand, since $G(\Phi, R, s^n R)$ is normal in $G(\Phi, R)$, one has $y, z \in G(\Phi, R, s^n R)$. Thus $y = z \in E(\Phi, R)$. Since $s^l \notin M$ and the same works for all maximal ideals, we get the desired partition.

There was no completion so far, only localisation, but this is not the end of the story. Bak's method was developed to prove much stronger results, than normality, and that's how completion enters the stage. For suppose we have to prove that $x = [h, g]$ lies in $E(\Phi, R)$ for *two* general matrices h and g. Actually, this is *exactly* what we have to verify in the proof of the above Theorem, not over R itself though, but over its factor-rings. This cannot be easily done by a single localisation. The main idea in the proof of [3] Theorem 4.1, embodied in Theorem 4.16, is to use localisation in an element t, to prove that there exists an element s such that $F_s(h) \in E(\Phi, R_s)$, whereas $g \in E(\Phi, R)G(\Phi, R, s^m R)$ for an arbitrarily large power m. In other words, the element h becomes elementary after localisation in s, whereas g becomes elementary after s-completion, which explains the name *localisation-completion*. Then we can argue exactly as above, by the *second* localisation in the element s. Namely, $F_s(h)$ is elementary, and, taking a sufficiently large power s^m in the congruence for g, we can guarantee that even with all the denominators in $F_s(h)$, enough s's survive for $F_s(x)$ to be in the image of $E(\Phi, R, s^n R)$ for an n such that restriction of F_s to $G(\Phi, R, s^n R)$ is injective.

Of course our actual proof is technically somewhat more demanding. In general there is an extra toral factor to take care of[3] and one has to fiddle with the Chevalley commutator formula a bit to convince herself that she still has large powers of both s and t in the numerator, after all conjugations. But these are details, all the ideas are already there. The paper [7] uses essentially the same ideas, but there are some further technical moments, like non-commutativity, non-triviality of the involution and the form parameter which make calculations in [7] much harder that the ones of the present paper.

The details of our calculations look slightly differently for the non-symplectic and the symplectic case, i.e. when $\Phi \neq C_l$ or $\Phi = C_l$ respectively (recall that $B_2 = C_2$ is symplectic!) and for symplectic case the analysis of the simply-connected group is somewhat easier than the analysis of the adjoint group. By skipping symplectic case altogether we could both spare a page or two of calculations and obtain somewhat better bounds in some of the auxiliary results. Unfortunately this couldn't have been done, if we wish to have our theorem for all groups. In fact, the paper [7] by the first author supplies all the details for the more general case of unitary groups over a form ring. However it does so only in vector representations. Symplectic groups $G_{\mathrm{sc}}(C_l, R) = \mathrm{Sp}(2l, R)$ are obtained as a special case from the general quadratic setting of [7] when the involution is trivial, $\lambda = -1$ and $\Lambda = R$. However the adjoint symplectic case $G_{\mathrm{ad}}(C_l, R) = \mathrm{PGSp}(2l, R)$ requires some extra care.

The rest of the paper is organised as follows. In § 2 we introduce some notation and in §§ 3,4 prove several easy lemmas on commutators. In § 5 we prove a patching result, which in particular provides a shorter proof of Taddei's theorem. In § 6 we introduce the last important ingredient of the proof, completion. Finally, main results are established in § 7.

The main stimulus to undertake this work was a discussion with Benson Farb, who mentioned that solvability of the K_1 functor would have very important geometric consequences. We would like to thank Alexei Stepanov for careful reading of our original manuscript and some very illuminating comments. The second author gratefully acknowledges support of the RFFI projects 98-01-00825 and 00–01–00441.

2. Preliminaries

2.1. Let us fix some notation. Let R be a commutative ring with 1, S be a multiplicative system in R and $S^{-1}R$ be the corresponding localisation. We will mostly use localisation with respect to the two following types of multiplicative systems. If $s \in R$ and the multiplicative system S coincides with $\langle s \rangle = \{1, s, s^2, \dots\}$ we usually write $\langle s \rangle^{-1}R = R_s$. If $M \in \mathrm{Max}(R)$ is a maximal ideal in R, and $S = R \setminus M$, we usually write $(R \setminus M)^{-1}R = R_M$. We denote by $F_S : R \longrightarrow S^{-1}R$ the canonical ring homomorphism called the localisation homomorphism. For the two special cases mentioned above, we write $F_s : R \longrightarrow R_s$ and $F_M : R \longrightarrow R_M$, respectively. When we write an element as a fraction, like a/s or $\dfrac{a}{s}$ we *always* think of it as an element

[3]Actually, there is exactly one case, when the toral factor plays a role: long roots in *adjoint* symplectic groups.

of some localisation $S^{-1}R$, where $s \in S$. If s were actually invertible in R, we would have written as^{-1} instead.

2.2. Let as above Φ be a reduced irreducible root system, P, $Q(\Phi) \leq P \leq P(\Phi)$ be a lattice between the root lattice and the weight lattice. We denote by $G = G_P(\Phi, R)$ the Chevalley group of type (Φ, P) over R, by $T = T_P(\Phi, R)$ – a split maximal torus and by $E = E_P(\Phi, R)$ the corresponding (absolute) elementary subgroup. Usually P does not play role in our calculations and we suppress it in the notation. The elementary group $E = E(\Phi, R)$ is generated by all root unipotents $x_\alpha(a)$, $\alpha \in \Phi$, $a \in R$, elementary with respect to T. The subgroup E being normal in G means exactly that E does not depend on the choice of T.

For any Φ, P assignments $R \longrightarrow X(\Phi, R)$, where $X = G, T, E$, define functors from commutative rings to groups, i.e. to a ring homomorphism $\phi : R_1 \longrightarrow R_2$ there corresponds a natural group homomorphism $X(\phi) : X(\Phi, R_1) \longrightarrow X(\Phi, R_2)$, which we usually denote by the same letter ϕ, rather than by their official names $G(\phi)$, $T(\phi)$ and $E(\phi)$. For E this is obvious, it is enough to define ϕ on elementary generators by $\phi(x_\alpha(a)) = x_\alpha(\phi(a))$, whereas G and T are *by the very construction* affine group schemes, i.e. *representable* functors from rings to groups. In fact E is a subfunctor of G in the sense that the restriction of $G(\phi)$ to $E(\Phi, R)$ coincides with $E(\phi)$. In particular, if S is a multiplicative system in R, the localisation homomorphism $F_S : G(\Phi, R) \longrightarrow G(\Phi, S^{-1}R)$ maps $E(\Phi, R)$ inside $E(\Phi, S^{-1}R)$.

2.3. The property of these functors which will be crucial for what follows is that they *commute with direct limits*. In other words, if $R = \varinjlim R_i$, where $\{R_i\}_{i \in I}$ is an inductive system of rings, then $X(\Phi, \varinjlim R_i) = \varinjlim X(\Phi, R_i)$. We will use this property in the following two situations. First, let R_i be the inductive system of all finitely generated subrings of R with respect to the embeddings. Then $X(\Phi, R) = \varinjlim X(\Phi, R_i)$, which reduces most of the proofs to the case of Noetherian rings. Second, let S be a multiplicative system in R and R_s, $s \in S$, the inductive system with respect to the localisation homomorphisms: $F_t : R_s \longrightarrow R_{st}$. Then $X(\Phi, S^{-1}R) = \varinjlim X(\Phi, R_s)$, which allows to reduce localisation in any multiplicative system to localisation in one element.

2.4. Let $\mathfrak{a}$ be an additive subgroup of R. Then $E(\Phi, \mathfrak{a})$ denotes the subgroup of $E(\Phi, R)$ generated by all elementary root unipotents $x_\alpha(t)$ where $\alpha \in \Phi$ and $t \in \mathfrak{a}$. Further, let L denote a nonnegative integer and let $E^L(\Phi, \mathfrak{a})$ denote the *subset* of $E(\Phi, \mathfrak{a})$ consisting of all products of L or fewer elementary root unipotents $x_\alpha(t)$, where $\alpha \in \Phi$ and $t \in \mathfrak{a}$. Thus $E^1(\Phi, \mathfrak{a})$ is the set of all $x_\alpha(t)$, $\alpha \in \Phi$, $t \in \mathfrak{a}$. When $\mathfrak{a} \trianglelefteq R$ is a proper ideal in R, the group $E(\Phi, \mathfrak{a})$ shouldn't be confused with the (relative) elementary group $E(\Phi, R, \mathfrak{a})$ of level $\mathfrak{a}$. By definition $E(\Phi, R, \mathfrak{a})$ is the normal closure of $E(\Phi, \mathfrak{a})$ in $E(\Phi, R)$. In general $E(\Phi, R, \mathfrak{a})$ is *not* generated by elementary transvections of level $\mathfrak{a}$. We use the following easy fact on the interrelation of $E(\Phi, \mathfrak{a})$ with the relative elementary groups, see, for example, [21], Proposition 2.

Lemma 2.5. *Suppose $\mathfrak{a} \trianglelefteq R$ is an ideal in R. In the case $\Phi \neq C_l$ one has $E(\Phi, \mathfrak{a}) \geq E(\Phi, R, \mathfrak{a}^2)$. In the case $\Phi = C_l$ one has $E(\Phi, \mathfrak{a}) \geq E(\Phi, R, ((2)+\mathfrak{a})\mathfrak{a}^2)$.*

2.6. If $\mathfrak{a} \trianglelefteq R$ is an ideal in R, then we denote by $G(\Phi, R, \mathfrak{a})$ the principal congruence subgroup of level $\mathfrak{a}$ in $G(\Phi, R)$, i.e. the kernel of the reduction homomorphism modulo $\mathfrak{a}$: $G(\Phi, R) \longrightarrow G(\Phi, R/\mathfrak{a})$. Clearly, $E(\Phi, \mathfrak{a}) \leq G(\Phi, R, \mathfrak{a})$. Further, set $T(\Phi, R, \mathfrak{a}) = T(\Phi, R) \cap G(\Phi, R, \mathfrak{a})$. Fix an ordering on Φ, let Φ^+ and Φ^- be the corresponding sets of positive and negative roots, respectively. As usual, we set

$$U(\Phi, \mathfrak{a}) = \big\langle x_\alpha(a), a \in \mathfrak{a}, \alpha \in \Phi^+ \big\rangle$$
$$U^-(\Phi, \mathfrak{a}) = \big\langle x_\alpha(a), a \in \mathfrak{a}, \alpha \in \Phi^- \big\rangle$$

Obviously, $U(\Phi, \mathfrak{a}), U^-(\Phi, \mathfrak{a}) \leq E(\Phi, \mathfrak{a})$.

Our reduction to groups of smaller rank is based on the following version of Gauß decomposition, see [1], Corollary 3.3 and [2], Proposition 2.3.

Lemma 2.7. *If $\mathfrak{a}$ is an ideal of R contained in the Jacobson radical, then we have*

$$G(\Phi, R, \mathfrak{a}) = U(\Phi, \mathfrak{a}) T(\Phi, R, \mathfrak{a}) U^-(\Phi, \mathfrak{a}).$$

2.8. Let R^* be the multiplicative group of the ring R. For $\alpha \in \Phi$ and $a \in R^*$ one sets $w_\alpha(a) = x_\alpha(a) x_{-\alpha}(a^{-1}) x_\alpha(a)$ and $h_\alpha(a) = w_\alpha(a) w_\alpha(-1)$. Let $H(\Phi, R)$ be the subgroup of $T(\Phi, R)$ generated by all $h_\alpha(a)$:

$$H(\Phi, R) = \big\langle h_\alpha(a), a \in R^*, \alpha \in \Phi \big\rangle$$

The following formula (see [1], Section 2.2)

$$h_\alpha(a) = x_{-\alpha}(a^{-1} - 1) x_\alpha(1) x_{-\alpha}(a - 1) x_\alpha(-1) x_\alpha(1 - a^{-1})$$

shows that $h_\alpha(a) \in E(\Phi, R, \mathfrak{a})$ if $a \equiv 1 \pmod{\mathfrak{a}}$. In particular, $H(\Phi, R) = T(\Phi, R) \cap E(\Phi, R)$.

It is shown in [2] that Lemma 2.7 immediately implies

Lemma 2.9. *Let R be semi-local. Then $G(\Phi, R) = E(\Phi, R) T(\Phi, R)$. In particular,*

$$G(\Phi, R)/E(\Phi, R) = T(\Phi, R)/H(\Phi, R)$$

is abelian and if $G(\Phi, R)$ is simply-connected, $G(\Phi, R) = E(\Phi, R)$.

Lemma 2.10. *If $\mathfrak{a}$ is an ideal of local ring R then*

$$G(\Phi, R, \mathfrak{a}) = T(\Phi, R, \mathfrak{a}) E(\Phi, \mathfrak{a}).$$

Proof. If $\mathfrak{a} = R$, the conclusion follows from Lemma 2.9. If $\mathfrak{a}$ is a proper ideal, it is contained in the Jacobson radical and we can apply Lemma 2.7. Since $T(\Phi, R, \mathfrak{a})$ normalises $U(\Phi, \mathfrak{a})$ and both $U(\Phi, \mathfrak{a})$ and $U^-(\Phi, \mathfrak{a})$ are contained in $E(\Phi, \mathfrak{a})$, the left hand side is contained in the right hand side. The inverse inclusion is obvious. ∎

2.11. If a and b are elements of a group, we write ${}^a b = aba^{-1}$ and $[a,b] = aba^{-1}b^{-1}$. In the sequel we make heavy use of the following commutator formulae: $[a,bc] = [a,b]^b[a,c]$ and $[ab,c] = {}^a[b,c][a,c]$. Most of the calculations in the present paper are based on the Chevalley commutator formula

$$[x_\alpha(s), x_\beta(t)] = \prod_{i\alpha+j\beta\in\Phi} x_{i\alpha+j\beta}(N_{\alpha\beta ij}s^i t^j),$$

where $N_{\alpha\beta ij}$ are the structure constants which do not depend on s and t (but for $\Phi = G_2$ may depend on the order of the roots in the product on the right hand side). The following observation was made by Chevalley himself: let $\alpha - p\beta, \dots, \alpha-\beta, \alpha, \alpha+\beta, \dots, \alpha+q\beta$ be the α-series of roots through β, then $N_{\alpha\beta 11} = \pm(p+1)$ and $N_{\alpha\beta 12} = \pm(p+1)(p+2)/2$.

Let i_Φ be the largest integer which may appear as i in a root $i\alpha + j\beta \in \Phi$ for all $\alpha, \beta \in \Phi$. Obviously $i_\Phi = 1, 2$ or 3, depending on whether Φ is simply laced, doubly laced or triply laced. The following result makes the proof for $\Phi \neq C_l$ slightly easier than for the symplectic case. Recall that $A_1 = C_1$ and $B_2 = C_2$ so that root systems of types A_1 and B_2 *are* symplectic. All roots of A_1 are long.

Lemma 2.12. *Let $\beta \in \Phi$ and either $\Phi \neq C_l$ or β is short. Then there exist two roots $\gamma, \delta \in \Phi$ such that $\beta = \gamma + \delta$ and $N_{\gamma\delta 11} = 1$. If $\Phi = C_l$, $l \geq 2$, and β is long, then there exist two roots $\gamma, \delta \in \Phi$ such that either $\beta = \gamma + 2\delta$ and $N_{\gamma\delta 12} = 1$, or $\beta = 2\gamma + \delta$ and $N_{\gamma\delta 21} = 1$.*

Proof. If β is long and $\Phi \neq C_l$, then β can be embedded into a root system of type A_2 consisting of long roots. Take any two roots $\gamma, \delta \in \Phi$ such that $\gamma + \delta = \beta$. Now let β be short. Then β can be embedded into a root system of type B_2 and G_2. Let γ be a short root and δ be a long root such that $\gamma + \delta = \beta$. Finally if β is long and $\Phi = C_l$, let γ be a long root and δ be a short root such that $\gamma + 2\delta = \beta$. In all cases $\gamma - \delta$ is not a root and thus $N_{\gamma\delta 1i} = \pm 1$, where $i = 1$ in the generic case and $i = 2$ in the exceptional case. If $N_{\gamma\delta 1i} = -1$ switch γ and δ. □

Throughout the paper the letters $k, l, m, n, p, q, r, K, L$ are used to denote non-negative integers, a, b, c, d, s, t denote elements of the ground ring R, $\alpha, \beta, \gamma, \delta$ denote roots in Φ and g, h, x, y, z, u, v denote elements of the Chevalley group $G(\Phi, R)$.

3. First localisation

In this and next section we prove some technical results on conjugation calculas of Chevalley groups. If $t \in R$, let $\dfrac{t}{s^k}R$ denote the additive subgroup of R_s consisting of all quotients $\dfrac{ta}{s^k}$, where $a \in R$. All calculations in the present section take place in $E(\Phi, R_s)$. Thus, when we write something like $E(\Phi, s^p t^q R)$, or $x_\alpha(s^p a)$, what we *really* mean, is $E(\Phi, F_s(s^p t^q R))$, or $x_\alpha(F_s(s^p a))$, respectively, but we suppress F_s in our notation. However this shouldn't lead to a confusion since in this section we *never* refer to elements or subgroups of $G(\Phi, R)$. Starting from Section 5, where elements of $G(\Phi, R)$ and several of its localisations may appear in the same formula, we always explicitly cite the corresponding localisation homomorphisms.

Lemma 3.1. *If p and k are given, there is a q such that*

$$E^1\big(\Phi,\frac{1}{s^k}R\big)E(\Phi,s^qt^3R)\subseteq E(\Phi,s^ptR).$$

Proof. Since by definition $E(\Phi,s^qt^3R)$ is generated by $x_\beta(s^qt^3b)$, $b\in R$, it suffices to show that there is a q such that

$$^{x_\alpha\big(\frac{a}{s^k}\big)}x_\beta(s^qt^3b)\in E(\Phi,s^ptR).$$

for any $x_\alpha\big(\frac{a}{s^k}\big)\in E^1\big(\Phi,\frac{1}{s^k}R\big)$ and any $x_\beta(s^qt^3b)\in E(\Phi,s^qt^3R)$.

Case 1. Let $\alpha\neq-\beta$ and set $q\geq i_\Phi k+p$. By the Chevalley commutator formula,

$$x_\alpha\big(\frac{a}{s^k}\big)x_\beta(s^qtb)x_\alpha\big(-\frac{a}{s^k}\big)=x_\beta(s^qtb)\prod_{i\alpha+j\beta\in\Phi}x_{i\alpha+j\beta}\big(N_{\alpha\beta ij}\big(\frac{a}{s^k}\big)^i(s^qtb)^j\big)$$

and a quick inspection shows that the right hand side of the above equality is in $E(\Phi,s^ptR)$.

Case 2. Let $\alpha=-\beta$ and one of the following holds: β is short or $\Phi\neq C_l$. By Lemma 2.1 there exist roots γ and δ such that $\gamma+\delta=\beta$ and $N_{\gamma\delta 11}=1$. We set $q=2(i_\Phi k+p)$ and decompose $x_\beta(s^qt^2b)$ as follows:

$$x_\beta(s^qt^2b)=[x_\gamma(s^{q/2}t),x_\delta(s^{q/2}tb)]\prod x_{i\gamma+j\delta}(-N_{\gamma\delta ij}(s^{q/2}t)^i(s^{q/2}tb)^j),$$

where the product on the right hand side is taken over all roots $i\gamma+j\delta\neq\beta$. Conjugating this expression by $x_\alpha\big(\frac{a}{s^k}\big)$ we get

$$\begin{aligned}^{x_\alpha\big(\frac{a}{s^k}\big)}x_\beta(s^qt^2b) &= \Big[^{x_\alpha\big(\frac{a}{s^k}\big)}x_\gamma(s^{q/2}t),\,^{x_\alpha\big(\frac{a}{s^k}\big)}x_\delta(s^{q/2}tb)\Big]\times\\ &\quad\prod {}^{x_\alpha\big(\frac{a}{s^k}\big)}x_{i\gamma+j\delta}\Big(-N_{\gamma\delta ij}(s^{q/2}t)^i(s^{q/2}tb)^j\Big).\end{aligned}$$

Obviously γ,δ and all the roots $i\gamma+j\delta\neq\beta$ are distinct from $-\alpha$ and now Case 1 shows that each term is in $E(\Phi,s^ptR)$.

Case 3. Let $\Phi=C_l$ and $\alpha=-\beta$ be a long root. By Lemma 2.1 there exist roots γ and δ such that either $\gamma+2\delta=\beta$ and $N_{\gamma\delta 12}=1$, or $2\gamma+\delta=\beta$ and $N_{\gamma\delta 21}=1$. We look at the first case, the second case is similar (alternatively, if $N_{\gamma\delta 12}=-1$, one could change the sign of $x_\gamma(b)$ in the following formula by $x_\gamma(-b)$). We set $q=3(i_\Phi k+p)$ and decompose $x_\beta(s^qt^3b)$ as follows:

$$x_\beta(s^qt^3b)=[x_\gamma(s^{q/3}tb),x_\delta(s^{q/3}t)]x_{\gamma+\delta}(-N_{\gamma\delta 11}s^{2q/3}t^2b),$$

Again conjugating this expression by $x_\alpha\big(\frac{a}{s^k}\big)$ and applying Case 1, we see that each term on the right hand side is in $E(\Phi,s^ptR)$. This completes the proof. ☐

The following result immediately follows from Lemma 3.1 by an easy induction on K. In its proof we denote by $f^{\circ K}$ the K-th iteration of the function f, namely $f^{\circ 1}=f$ and $f^{\circ n}=f\circ f^{\circ n-1}$ where $n\geq 2$.

Lemma 3.2. *If p, n, k and K are given, there are q and l such that*

$$E^K\big(\Phi, \frac{1}{s^k}R\big)\, E(\Phi, s^q t^l R) \subseteq E(\Phi, s^p t^n R).$$

Proof. Consider the function $f(x) = 3(i_\Phi k + x)$ which appeared in the proof of Lemma 3.1. Clearly $l = 3^K n$ and $q = f^{\circ K}(p)$ satisfy the desired inclusion. □

4. Second localisation

Here we fix two elements $s, t \in R$ and consider localisation $R_{st} \equiv (R_s)_t \equiv (R_t)_s$.

Lemma 4.1. *Let p, q, k, m are given. Then there are l and n such that*

$$\Big[E^1\big(\Phi, \frac{t^l}{s^k}R\big), E^1\big(\Phi, \frac{s^n}{t^m}R\big)\Big] \subseteq E(\Phi, s^p t^q R).$$

Proof. The proof follows the same pattern as in Lemma 3.1. Let $x_\alpha\big(\frac{t^l a}{s^k}\big) \in E^1\big(\Phi, \frac{t^l}{s^k}R\big)$ and $x_\beta\big(\frac{s^n b}{t^m}\big) \in E^1\big(\Phi, \frac{s^n}{t^m}R\big)$.

Case 1. Let $\alpha \neq -\beta$. Then

$$\big[x_\alpha\big(\frac{t^l}{s^k}a\big), x_\beta\big(\frac{s^n}{t^m}b\big)\big] = \prod_{i,j>0} x_{i\alpha+j\beta}\Big(\big(\frac{t^l}{s^k}a\big)^i\big(\frac{s^n}{t^m}b\big)^j\Big).$$

Let $l \geq i_\Phi m + q$ and $n \geq i_\Phi k + p$. Clearly all factors on the right hand side of the above formula are in $E(\Phi, s^p t^q R)$.

Case 2. Let $\alpha = -\beta$ and one of the following holds: β is short or $\Phi \neq C_l$. By Lemma 2.1 there are roots γ and δ such that $\gamma + \delta = \beta$ and $N_{\gamma\delta 11} = 1$. Increasing n, if necessary, we can assume that n is even. Thus we can decompose $x_\beta\big(\frac{s^n}{t^m}b\big)$ as follows

$$x_\beta\big(\frac{s^n b}{t^m}\big) = \big[x_\gamma(s^{n/2}), x_\delta\big(\frac{s^{n/2}b}{t^m}\big)\big] \prod x_{i\gamma+j\delta}\Big(-N_{\gamma\delta ij}(s^{n/2})^i\big(\frac{s^{n/2}}{t^m}b\big)^j\Big),$$

where the product on the right hand side is taken over all roots $i\gamma + j\delta \neq \beta$.

Next, we consider the commutator formula

$$\Big[x, [y,z]\prod_{i=1}^t u_i\Big] = [x,y]\ {}^y[x,z]\ {}^{yz}[x,y^{-1}]\ {}^{yzy^{-1}}[x,z^{-1}] \prod_{i=1}^t {}^{[y,z]\prod_{j=1}^{i-1}u_j}[x,u_i],$$

and plug into this formula $x_\alpha\big(\frac{t^l}{s^k}a\big)$ instead of x, $x_\gamma(s^{n/2})$ instead of y, $x_\delta\big(\frac{s^{n/2}b}{t^m}\big)$ instead of z and the remaining factors on the right hand side of the expression of $x_\beta\big(\frac{s^n b}{t^m}\big)$ instead of u_i. There are not more than 4 factors in the right hand side of the Chevalley commutator formula anyway, one of them is discarded from the very start, and in the conjugation we discard at least one more. This means that the maximum length K of the exponent in the elementary unipotents is at most 6. Now Lemma 3.2 and Case 1 imply that $l \geq f^{\circ 6}(q) + m i_\Phi^2$ and $n \geq 2(k i_\Phi + 3^6 p)$ satisfy the required condition.

Case 3. Let $\Phi = C_l$ and $\alpha = -\beta$ be a long root. By Lemma 2.1 there exist roots γ and δ such that either $\gamma + 2\delta = \beta$ and $N_{\gamma\delta 12} = 1$, or $2\gamma + \delta = \beta$ and $N_{\gamma\delta 21} = 1$. As in the proof of Lemma 3.1 we lose nothing by looking at the first case. Increasing n if necessary we can assume that n is divisible by 3 and decompose $x_\beta(\frac{s^n b}{t^m})$ as follows:

$$x_\beta\big(\frac{s^n b}{t^m}\big) = \big[x_\gamma\big(\frac{s^{n/3} b}{t^m}\big), x_\delta(s^{n/3})\big] x_{\gamma+\delta}\Big(-N_{\gamma\delta 11}\frac{s^{2n/3} b}{t^m}\Big).$$

Repeating the same arguments as in Case 2, and observing that now the maximum length K of the exponent in the elementary unipotents is at most 4, we see that Lemma 3.2 and Case 1 imply that $l \geq f^{\circ 4}(q) + mi_\Phi$ and $n \geq 3(ki_\Phi + 3^4 p)$ satisfy the required condition. Now comparing Case 1, Case 2 and Case 3, it is clear that a bound, for example, $l \geq f^{\circ 6}(q) + mi_\Phi^2$ and $n \geq 3(ki_\Phi + 3^6 p)$ satisfy the lemma. $\square$

Combining Lemma 4.1 and commutator formulae, we get the main result of this section.

Theorem 4.2. *Let p, q, k, m and K, L are given. Then there are l and n such that*

$$\Big[E^K\big(\Phi, \frac{t^l}{s^k}R\big), E^L\big(\Phi, \frac{s^n}{t^m}R\big)\Big] \subseteq E(\Phi, s^p t^q R).$$

Proof. The proof follows from Lemma 4.1 by an easy induction. $\square$

5. Patching

Fix an element $s \in R$, $s \neq 0$. In general if R has zero divisors, the group homomorphism $F_s : G(\Phi, R) \longrightarrow G(\Phi, R_s)$ induced by the localisation homomorphism $R \longrightarrow R_s$ is not injective. There are several methods to circumvent this difficulty. Our approach is based on the following observation, [3], Lemma 4.10.

Lemma 5.1. *Suppose R is Noetherian and $s \in R$. Then there exists a natural number k such that the homomorphism $F_s : G(\Phi, R, s^k R) \longrightarrow G(\Phi, R_s)$ is injective.*

Proof. The homomorphism $F_s : G(\Phi, R, s^k R) \longrightarrow G(\Phi, R_s)$ is injective whenever $F_s : s^k R \longrightarrow R_s$ is injective. For $i \geq 0$, let $\mathfrak{a}_i = \mathrm{Ann}_R(s^i)$ be the annihilator of s^i in R. Since R is Noetherian, there exists a k such that $\mathfrak{a}_k = \mathfrak{a}_{k+1} = \dots$. If $s^k a$ vanishes in R_s, then $s^i s^k a = 0$ for some i. But since $\mathfrak{a}_{k+i} = \mathfrak{a}_k$, already $s^k a = 0$ and thus $s^k R$ injects in R_s. $\square$

Now we are all set to start a localisation and patching procedure. As in Section 3 a fraction of the form $\dfrac{a}{s^k}$ is considered as an element of the localisation R_s, unless specified otherwise.

Lemma 5.2. *Fix an element $s \in R$, $s \neq 0$. Then for any k and q, there exists an r such that for any $a \in R$, $g \in G(\Phi, R, s^r R)$ and any maximal ideal M of R, there exist an element $t \in R \setminus M$, and an l such that*

$$\big[x_\alpha\big(\frac{t^l a}{s^k}\big), F_s(g)\big] \in E(\Phi, F_s(s^q R)) \subseteq G(\Phi, R_s).$$

Proof. By 2.3 one has $G(\Phi, R) = \varinjlim G(\Phi, R_i)$, where the limit is taken over all finitely generated subrings of R. Thus without loss of generality one may assume that R is Noetherian (replace R by the ring generated by a, s and the matrix entries of g in a faithful polynomial representation).

Let M be a maximal ideal of R. Then R_M is a local ring and thus by Lemma 2.10 $F_M(g) \in G(\Phi, R_M)$ can be decomposed as $F_M(g) = uh$ where h is an element of $T(\Phi, R_M, s^r R_M)$, and $u \in E(\Phi, s^r R_M) \leq G(\Phi, R_M)$. But since $G(\Phi, R_M) = \varinjlim G(\Phi, R_t)$, over all $t \in R \setminus M$, and the same holds for $E(\Phi, s^r R_M)$, $T(\Phi, R_M, s^r R_M)$, etc., we can find an element $t \in R \setminus M$ such that already $F_t(g)$ can be factored as $F_t(g) = uh$ where $h \in T(\Phi, R_t, s^r R_t)$ and $u \in E(\Phi, R_t, s^r R_t)$.

Now since R is assumed to be Noetherian, R_s is also Noetherian and by Lemma 5.1 there exists an n such that the canonical homomorphism

$$F_t : G(\Phi, R_s, t^n R_s) \longrightarrow G(\Phi, R_{st})$$

is injective. Let $l > n$. Since $x_\alpha(\frac{t^l a}{s^k}) \in G(\Phi, R_s, t^n R_s)$, and $G(\Phi, R_s, t^n R_s)$ is normal in $G(\Phi, R_s)$, we have

$$x = \big[x_\alpha(\frac{t^l a}{s^k}), F_s(g)\big] \in G(\Phi, R_s, t^n R_s).$$

Consider the image $F_t(x) \in G(\Phi, R_{st})$ of x under localisation with respect to t. Since F_t is a homomorphism, one has $F_t(x) = [F_t\Big(x_\alpha(\frac{t^l a}{s^k})\Big), F_{st}(g)]$. Now $F_{st}(g)$ can be factored as $F_{st}(g) = F_s(u)F_s(h) \in G(\Phi, R_{st})$. It follows that

$$\begin{aligned} F_t(x) \quad &= \quad \big[F_t\Big(x_\alpha(\frac{t^l a}{s^k})\Big), F_s(u)F_s(h)\big] = \\ &\big[F_t\Big(x_\alpha(\frac{t^l a}{s^k})\Big), F_s(u)\big] \ {}^{F_s(u)}\big[F_s\Big(x_\alpha(\frac{t^l a}{s^k})\Big), F_s(h)\big]. \end{aligned}$$

For all cases apart from the case of a long root α in the *adjoint* symplectic group of type C_l one could choose a decomposition $F_t(g) = uh$ such that h commutes with $x_\alpha(*)$. Therefore

$$F_t(x) = \big[F_t\Big(x_\alpha(\frac{t^l a}{s^k})\Big), F_s(u)\big]$$

Now by Theorem 4.2, we can choose a suitable r and l such that $F_t(x) \in E(\Phi, F_{st}(s^q t^n R)) \leq G(\Phi, R_{st})$.

We are left with the case of adjoint symplectic...

This means that $F_t(x)$ can be presented as a product of elementary transvections of the form

$$F_t(x) = x_{\alpha_1}(F_{st}(s^q t^n a_1)) \dots x_{\alpha_m}(F_{st}(s^q t^n a_m))$$

for some $a_1, \dots, a_m \in R$. Form the product of elementary root unipotents

$$y = x_{\alpha_1}(F_s(s^q t^n a_1)) \dots x_{\alpha_m}(F_s(s^q t^n a_m)) \in E(\Phi, F_s(s^q R)) \cap G(\Phi, R_s, t^n R_s).$$

Clearly $F_t(y) = F_t(x)$ and since both x and y belong to $G(\Phi, R_s, t^n R_s)$ and by the very choice of n the restriction of F_t to $G(\Phi, R_s, t^n R_s)$ is injective, it follows that $x = y \in E(\Phi, s^q R)$. This completes the proof. $\square$

Remark. For all cases apart from the case of a long root α in the *adjoint* symplectic group of type C_l one could choose a decomposition $F_t(g) = hu$ such that h commutes with $x_\alpha(*)$.

The following result is a broad generalisation of the normality of the elementary subgroup.

Theorem 5.3. *Fix an element $s \in R$, $s \neq 0$. Then for any p , K and k there exists an r such that*

$$\big[E^K\big(\Phi, \frac{1}{s^k}R\big), F_s(G(\Phi, R, s^r R))\big] \subseteq E(\Phi, F_s(s^p R)) \leq G(\Phi, R_s).$$

Proof. We shall show that there is an r such that

$$\big[E^1\big(\Phi, \frac{1}{s^k}R\big), F_s(G(\Phi, R, s^r R))\big] \subseteq E(\Phi, F_s(s^q R)),$$

where $q = f^{\circ K-1}(p)$ and $f(x) = 3(i_\Phi k + x)$ as in Lemma 3.2. Then by the commutator formulae in 2.11,

$$\big[E^K\big(\Phi, \frac{1}{s^k}R\big), F_s(G(\Phi, R, s^r R))\big] \subseteq {}^{E^{K-1}\big(\Phi, \frac{1}{s^k}R\big)}E(\Phi, F_s(s^q R)),$$

and by Lemma 3.2

$$^{E^{K-1}\big(\Phi, \frac{1}{s^k}R\big)}E(\Phi, F_s(s^q R)) \subseteq E(\Phi, F_s(s^p R)),$$

which proves the theorem.

Therefore let $x_\alpha\big(\frac{a}{s^k}\big) \in E^1\big(\Phi, \frac{1}{s^k}R\big)$ and $F_s(g) \in F_s(G(\Phi, R, s^r R))$. By Lemma 5.2, for k and $3(i_\Phi k + q)$, there is a r such that for every maximal ideal $M \in \mathrm{Max}(R)$ there exists an element $t_M \in A\backslash M$ and a natural number l_M such that

$$\big[x_\alpha\big(\frac{t_M^{l_M} a}{s^k}\big), F_s(g)\big] \in E(\Phi, F_s(s^{3(i_\Phi k+q)}R)).(5.3.1)$$

Since the set $\{t_M^{l_M} | M \in \mathrm{Max}(R)\}$ is not contained in any maximal ideal of R, there exists its finite set $\{t_1^{l_1}, \cdots, t_r^{l_r}\}$ which generates R as an ideal. Choose $x_1, \ldots, x_r \in R$ such that $x_1 t_1^{l_1} + \ldots + x_r t_r^{l_r} = 1$. Then

$$\big[x_\alpha\big(\frac{a}{s^k}\big), F_s(g)\big] = \big[x_\alpha\big(\frac{x_1 t_1^{l_1} a}{s^k}\big) \ldots x_\alpha\big(\frac{x_r t_r^{l_r} a}{s^k}\big), F_s(g)\big].$$

Using the commutator formula in 2.11, Lemma 3.1 and (5.3.1) we see that

$$\big[x_\alpha\big(\frac{a}{s^k}\big), F_s(g)\big] \in E(\Phi, F_s(s^q R)),$$

which concludes the proof. □

In particular the contents of the present section gives a slightly shorter proof of Taddei's result [20]. In fact only a small fraction of our arguments would be necessary to prove this result:

Corollary 5.4. *Assume* $\mathrm{rk}(\Phi) \geq 2$. *Then $E(\Phi, R)$ is a normal subgroup of $G(\Phi, R)$.*

Proof. Set $s = 1$ in the above theorem. □

6. Completion

In the present section we describe the last important ingredient of the proof of the main theorem. Let $s \in R$. Recall that the s-completion $\widehat{R}_s$ of the ring R is usually defined as the following inverse limit:

$$\widehat{R}_s = \varprojlim R/s^n R, \quad n \in \mathbb{N}.$$

However this definition is not quite compatible with our purposes. Namely, as always, to control zero divisors, we have to reduce to Noetherian rings first. However if $R = \varinjlim R_i$ is a direct limit of Noetherian rings, the canonical homomorphism $\varinjlim(\widehat{R}_i)_s \longrightarrow \widehat{R}_s$ is in general neither surjective, nor injective. This forces us to modify the definition of completion as follows:

$$\widetilde{R}_s = \varinjlim(\widehat{R}_i)_s,$$

where the limit is taken over all finitely generated subgrings R_i of R which contain s. Let us denote by $\widetilde{F}_s$ the canonical homomorphism $R \longrightarrow \widetilde{R}_s$. For the case, when R is Noetherian $\widetilde{F}_s = \widehat{F}_s$ coincides with the inverse limit of reduction homomorphisms $\pi_{s^n} : R \longrightarrow R/s^n R$

Theorem 6.1. *Let R be a commutative ring, Φ an irreducible root system of rank ≥ 2. Then*

$$\left[{F_s}^{-1}(E(\Phi, R_s)), \widetilde{F}_s^{-1}(E(\Phi, \widetilde{R}_s)\right] \subseteq E(\Phi, R).$$

Proof. Let R_i be the inductive system of all finitely generated subrings of R, containing s. By 2.3 one has

$$\begin{aligned} {F_s}^{-1}(E(\Phi, R_s)) &= \varinjlim {F_s}^{-1}(E(\Phi, (R_i)_s)), \\ \widetilde{F}_s^{-1}(E(\Phi, \widetilde{R}_s)) &= \varinjlim \widehat{F}_s^{-1}(E(\Phi, (\widehat{R}_i)_s)), \end{aligned}$$

and the proof reduces to the case when R is Noetherian, as in Lemma 5.2.

Let $x \in {F_s}^{-1}(E(\Phi, R_s))$ and $y \in \widehat{F}_s^{-1}(E(\Phi, (\widehat{R}_i)_s))$. By definition the condition on x means that $F_s(x) \in E^K(\Phi, 1/s^k A)$ for some k and K. On the other hand the condition on y means that $\pi_{s^n}(y) \in E(\Phi, R/s^n R)$ for all n, or, what is the same, $y \in E(\Phi, R)G(\Phi, R, s^n R)$. In other words, for *any* n we can present y as a product $y = uz$, $u \in E(\Phi, R)$ and $z \in G(\Phi, R, s^n R)$.

As in the proof of Lemma 5.2 we can choose p such that the restriction of the localisation homomorphism F_s to the principal congruence subgroup $G(\Phi, R, s^p R)$ is injective. Now for k, K and p choose q as in Theorem 5.3. Now $[x, y] = [x, u]^u [x, z]$. The first commutator belongs to $E(\Phi, R)$ together with u since $E(\Phi, R)$ is normal. Thus it remains only to prove that $[x, z] \in E(\Phi, R)$. By Theorem 5.3 the $F_s([x, z]) \in E(\Phi, F_s(s^q))$. On the other hand, since $G(\Phi, R, s^q R)$ is normal, $[x, z] \in G(\Phi, R, s^q R)$ exactly as the proof of Lemma 5.2 we can conclude that $[x, z] \in E(\Phi, s^q R)$. ☐

7. Main theorem

To introduce the main new concept of the paper, we have to recall the notion of Bass-Serre dimension of a ring. Let X be a topological space. The *dimension* of X is the length n of the longest chain $X_0 \subsetneqq X_1 \subsetneqq \ldots \subsetneqq X_n$ of nonempty closed *irreducible* subsets X_i of X, ([6], § III). Define $\delta(X)$ to be the smallest nonnegative integer d such that X is a finite union of irreducible *Noetherian* subspaces of dimension $\leq d$. If there is no such d, then by definition $\delta(X) = \infty$. Let R be a commutative ring. Let $\mathrm{Max}(R)$ be the maximal spectrum of the ring R, endowed with Zariski topology. Then the *Bass-Serre dimension* of R is $\delta(R) = \delta(\mathrm{Max}(R))$. It is easy to see that $\delta(R) = 0$ if and only if R is a semi-local ring.

The following 'induction lemma' (see [3], Lemma 4.17) is the main instrument to conduct induction on dimension.

Lemma 7.1. *Suppose $\delta(R)$ is finite and $\mathrm{Max}(R) = X_1 \cup \ldots \cup X_r$ be a decomposition into a union of irreducible Noetherian subspaces of dimension $\leq \delta(R)$. If $s \in R$ is such that for each k, $1 \leq k \leq r$, the element s is not contained in some member of X_k, then $\delta(\widetilde{R}_s) < \delta(R)$.*

Now we are all set to state and prove our principal result.

Definition 7.2. *Let R be a commutative ring, Φ an irreducible root system of rank ≥ 2. Define*

$$S^d G(\Phi, R) = \bigcap_{\substack{R \to A \\ \delta(A) \leq d}} \mathrm{Ker}\Big(G(\Phi, R) \longrightarrow G(\Phi, A)/E(\Phi, A)\Big).$$

Theorem 7.3. *Let R be a commutative ring, Φ an irreducible root system of rank ≥ 2 and $G(\Phi, R)$ the Chevalley group of Φ with coefficients in R. Then $G(\Phi, R)/S^0G(\Phi, R)$ is abelian, the sequence*

$$S^0G(\Phi, R) \geq S^1G(\Phi, R) \geq S^2G(\Phi, R) \geq \cdots$$

is a descending central series in $S^0G(\Phi, R)$ and $S^dG(\Phi, R) = E(\Phi, R)$ whenever $\delta(R) = d$.

Proof. By 2.9, if A is a semi-local ring, then $G(\Phi, A)/E(\Phi, A)$ is abelian. Since $\delta(A) = 0$ if and only if A is semi-local, one sees that the following homomorphism is injective

$$G(\Phi, R)/S^0G(\Phi, R) \longrightarrow \prod_{\delta(A)=0} G(\Phi, A)/E(\Phi, A).$$

Thus it follows that $G(\Phi, R)/S^0G(\Phi, R)$ is an abelian group.

For the main part of the theorem, we proceed by induction on $\delta(R)$. The theorem holds for zero dimensional rings. It suffices to show that for any $x \in S^0G(\Phi, R)$ and $y \in S^dG(\Phi, R)$, the commutator $[x, y] \in S^{d+1}G(\Phi, R)$. Since

$$G(\Phi, R)/S^{d+1}G(\Phi, R) \longrightarrow \prod_{\substack{R \to A \\ \delta(A) \leq n+1}} G(\Phi, A)/E(\Phi, A)$$

is a monomorphism, it is enough to prove the theorem for rings of dimension $d+1$. Hence $S^{d+1}G(\Phi, R) = E(\Phi, R)$.

Let $X_1 \cup \ldots \cup X_r$ be a decomposition of $\mathrm{Max}(R)$ into irreducible Noetherian subspaces of dimension $\leq \delta(R)$. For any $1 \leq i \leq r$, let $M_i \in X_i$. Take the multiplicative set $S = R \setminus (M_1 \cup \cdots \cup M_r)$. Since $S^{-1}R$ is a semi-local ring, $\delta(\varinjlim R_s) = \delta(S^{-1}R) = 0$, where the limit is taken over all $s \in S$. Therefore there exists an element $s \in S$ such that the $F_s(x) \in E(\Phi, R_s)$. Thus $x \in {F_s}^{-1}(E(\Phi, R_s))$. On the other hand by Lemma 7.1 for any $s \in S$, $\delta(\widetilde{R}_s) < \delta(R)$. Thus $\widetilde{F}_s(y) \in E(\Phi, \widetilde{R}_s)$. Now by Theorem 6.1 one has

$[x, y] \in E(\Phi, R)$. This shows that $S^0G(\Phi, R) \geq S^1G(\Phi, R) \geq \cdots$ is a descending central series. The fact that $S^dG(\Phi, R) = E(\Phi, R)$ whenever $\delta(R) = d$ is immediate from the definition of $S^dG(\Phi, R)$. □

Corollary 7.4. *Let* $\mathrm{rk}(\Phi) \geq 2$ *and* R *be a finite-dimensional ring. Then the quotient* $G(\Phi, R)/E(\Phi, R)$ *is nilpotent-by-abelian. In particular it is solvable.*

Proof. The corollary is an immediate consequence of Theorem 7.3. □

Corollary 7.5. *Let* $\mathrm{rk}(\Phi) \geq 2$ *and* R *be a finite-dimensional ring. Then* $K_1(\Phi, R)$ *is nilpotent.*

Proof. Since G is simply connected, for any semi-local ring R, $G(\Phi, R) = E(\Phi, R)$ and thus $G(\Phi, R) = S^0G(\Phi, R)$. Now the corollary follows from Theorem 7.3. □

References

[1] E. Abe, Chevalley groups over local rings, *Tôhoku Math. J.* **21** (1969), 474–494.

[2] E. Abe, K. Suzuki, On normal subgroups of Chevalley groups over commutative rings, *Tôhoku Math. J.* **28** (1), (1976), 185–198.

[3] A. Bak, Non-abelian K-Theory: The nilpotent class of K_1 and general stability, *K-Theory* **4** (1991), 363–397.

[4] A. Bak, N. Vavilov, Normality for elementary subgroup functors, *Math. Proc. Cambridge Philos. Soc.* **118** (1), (1995), 35–47.

[5] A. Bak, N. Vavilov, Structure of hyperbolic Unitary groups I: Elementary subgroups, *Algebra Coll.* **7** (1), (2000), 159–196.

[6] H. Bass, *Algebraic K-Theory*, Benjamin, New York, 1968.

[7] R. Hazrat, Dimension theory and non-stable K_1 of quadratic module, To appear in *K-Theory.*

[8] A. J. Hahn, O. T. O'Meara, *The Classical Groups and K-Theory*, Springer-Verlag, Berlin, 1989.

[9] W. van der Kallen, A module structure on certain orbit sets of unimodular rows, *J. Pure Appl. Algebra*, **57** (3) (1989), 281–316.

[10] S. Khlebutin, *Elementary subgroups of linear groups over rings*, Ph. D. Thesis, Moscow State Univ., 1987 (in Russian).

[11] V. Kopeiko, The stabilisation of symplectic groups over a polynomial ring, *Math. USSR Sbornik*, **34** (1978), 655–669.

[12] H. Park, C. Woodburn, An algorithmic proof of Suslin's stability theorem for polynomial rings, *J. Algebra* **178** (1) (1995), 277–298.

[13] E. Plotkin, On the stability of the K_1-functor for Chevalley groups of type E_7, *J. Algebra*, **210** (1), (1998), 67–95.

[14] A. Sivatsky, A. Stepanov, On the word length of commutators in $\mathrm{GL}_n(R)$, *K-Theory*, **17**, (1999), 295–302.

[15] M. R. Stein, Stability theorems for K_1, K_2 and related functors modeled on Chevalley groups, *Japan. J. Math.*, **4** (1), (1978), 77–108.

[16] A. Stepanov, N. Vavilov, Decomposition of transvections: a theme with variations, *K-Theory*, **19**, (2000), 109–153.

[17] A. Suslin, On the structure of the general linear group over polynomial rings, *Soviet Math. Izv.*, **41** (2), (1977), 503–516.

[18] A. Suslin, V. Kopeiko, Quadratic modules and orthogonal groups over polynomial rings, *J. Sov. Math.*, **20** (6), (1985), 2665–2691.

[19] K. Suzuki, Normality of the elementary subgroups of twisted Chevalley groups over commutative rings, *J. Algebra* **175** (2), (1995), 526–536.

[20] G. Taddei, Normalité des groupes élémentaires dans les groupes de Chevalley sur un anneau, *Contemp. Math.*, **55** (II), (1986), 693–710.

[21] J. Tits, Systèmes générateurs de groupes de congruence, *C. R. Acad. Sci. Paris, Sér A*, **283**, (1976), 693–695.

[22] M. Tulenbaev, Schur multiplier of the group of elementary matrices of finite order, *J. Sov. Math.*, **17** (4), (1981), 2062–2067.

[23] L. Vaserstein, On normal subgroups of GL_n over a ring, in *Lecture Notes Math.*, Vol. 854, Springer, 1981, pp.456–465.

[24] L. Vaserstein, Unstable K_1-theory of topological spaces is nilpotent, *Contemp. Math.*, **126** (1992), 193–196.

[25] N. Vavilov, Structure of Chevalley groups over commutative rings, *Non-associative algebras and related topics* (*Hiroshima, 1990*), World Sci. Publishing, London et al., 1991, 219–335.

[26] N. Vavilov, A third look at weight diagrams, *Rend. Sem. Math. Univ. Padova*, **104** (2), (2000), 1–50.

[27] N. Vavilov, E. Plotkin, Chevalley groups over commutative rings I: Elementary calculations, *Acta Applic. Math.*, **45**, (1996), 73–113.

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