On the Existence of Normal Maximal Subgroups in Division Rings *

S. Akbari, M. Mahdavi-Hezavehi †

Abstract

Let D be a division ring with centre F. Denote by D^* the multiplicative group of D. The relation between valuations on D and maximal subgroups of D^* is investigated. In the finite dimensional case, it is shown that F^* has a maximal subgroup if $Br(F)$ is nontrivial provided that the characteristic of F is zero. It is also proved that if F is a local or an algebraic number field, then D^* contains a maximal subgroup that is normal in D^*. It should be observed that every maximal subgroup of D^* contains either D' or F^*, and normal maximal subgroups of D^* contain D', whereas maximal subgroups of D^* do not necessarily contain F^*. It is then conjectured that the multiplicative group of any noncommutative division ring has a maximal subgroup.

Let D be a division algebra of finite dimension over its centre F. Denote by D' the commutator subgroup of the multiplicative group $D^* = D - \{0\}$. For any field F, we use the notation $Br(F)$ for the Brauer group of F. In [1], [2] and [5], the structure of maximal subgroups and finitely generated subnormal subgroups of D^* is investigated and it is shown how these subgroups sit in D^* with respect to F^* and D'. The aim of this note is to show that the existence of maximal subgroups of F^* is essential to study those of D^*. In fact, it is shown that if F^* has no maximal subgroups, then $Br(F)$ is trivial. In this connection, we observe that the multiplicative group of an algebraically closed field has no maximal subgroups whereas there exist fields that have no maximal subgroups

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but are not algebraically closed. As another example, $\mathbf{R}^*$, the multiplicative group of real numbers of $\mathbf{R}$, has only one maximal subgroup which is associated to the absolute value of $\mathbf{R}$ and there is only one non-commutative division algebra of finite dimension over $\mathbf{R}$. The situation is much better for the field of rational numbers $\mathbf{Q}$. It is shown later that $\mathbf{Q}^*$ has infinitely many maximal subgroups which are associated to the valuations of $\mathbf{Q}$ and it is well known that $Br(\mathbf{Q})$ is infinite. We shall try to establish a connection between valuations on a field F and maximal subgroups of F^* and D^*, where D is a division algebra with centre F. To be more precise, we characterize all maximal subgroups of the field of rational number $\mathbf{Q}^*$ with respect to set of all valuations on $\mathbf{Q}$. As for maximal subgroups of D^*, it is proved that if M is a maximal subgroup of D^* not containing F^*, then $Z(M)$ is a maximal subgroup of F^*. Furthermore, assume that D is of finite dimension over F and m is a maximal subgroup of F^* containing $Z(D')$. It is shown that D^* contains a maximal subgroup M containing m that is normal in D^*. Using these results, we prove if F is a field with a Krull valuation whose value group contains a maximal subgroup, and D is a division algebra of finite dimension over its centre F, then D^* contains a maximal subgroup M which is normal in D^*. We then apply these results to division algebras over algebraic number fields and local fields to show that in these cases D^* contains maximal subgroups which are normal in D^*. In contrast, we shall show that the multiplicative group of the real quaternions contains no normal maximal subgroups. It is generally believed that for any division ring D, D^* has a maximal subgroup. In this connection, it is also proved that if D is finite dimensional over its centre F and F admits a discrete valuation, then D^* contains a maximal subgroup. Finally, it is proved that, under certain mild conditions, each non-zero element of a division algebra D is contained in a maximal subgroup of D^*. We begin the material of this note with the determination of maximal subgroups of multiplicative groups of usual number systems in the following

LEMMA 1. *Let* $\mathbf{Q}$, $\mathbf{R}$, *and* $\mathbf{C}$ *denote the field of rational, real, and complex numbers, respectively. Then we have*
(a) *For any natural number* $r \geq 3$, *consider the epimorphism* f_r *from* Q^* *onto*

$G_r := \bigoplus_{i=1}^{+\infty} Z_r^{(i)}$ *given by the rule* $f_r(x) = (\overline{\nu_{p_i}(x)})$, *where* $Z_r^{(i)} \cong Z_r$ *the ring of integers modulo* r, *for each* i, p_i *is the* i*-th prime number and* ν_{p_i} *is the* p_i*-adic valuation on* Q *and* $\overline{\nu_{p_i}(x)}$ *is the remainder of* $\nu_{p_i}(x)$ *modulo* r. *Suppose further that* f_2 *is an epimorphism from* Q^* *onto* $G_2 := Z_2 \bigoplus (\bigoplus_{i=1}^{+\infty} Z_2^{(i)})$, *given by the rule* $f_2(x) = (sgn\, x, (\overline{\nu_{p_i}(x)}))$, *where* $sgn\, x$ *denotes the sign of* x. *If* M *is a maximal subgroup of* Q^*, *then there exists a prime number* q *and a maximal subspace* W *of* G_q *(*G_q *is a vector space over* Z_q*) such that* $M = f_q^{-1}(W)$. *Conversely, for any prime* q *and any maximal subspace* W *of* G_q, $f_q^{-1}(W)$ *is a maximal subgroup of* Q^*.

(b) $\mathbf{R}^*$ *has only one maximal subgroup.*

(c) *If* F *is an algebraically closed field, then* F^* *contains no maximal subgroup. In particular,* $\mathbf{C}^*$ *has no maximal subgroup.*

PROOF. (a) It is clearly seen that the map θ from Q^* onto

$$G := Z_2 \bigoplus (\bigoplus_{i=1}^{+\infty} Z^{(i)}),$$

where $Z^{(i)} \cong Z$, the ring of integers, for each i, given by

$$\theta(x) = (sgn\, x, \nu_{p_1}(x), \nu_{p_2}(x), \ldots),$$

is a group isomorphism. Therefore, there is a $1-1$ correspondence between maximal subgroups of Q^* and G. Let M be a maximal subgroup of Q^* and denote the corresponding maximal subgroup of G by G_M. Thus, there is a prime number q such that $G/G_M \cong Z_q$ and we have $qG \subseteq G_M$. Now two cases can be considered.

Case 1. $q \geq 3$. In this case $qG \subseteq G_M$ implies that $Z_2 \bigoplus (\bigoplus_{i=1}^{+\infty} qZ^{(i)}) \subset G_M$. Thus, $G_M/Z_2 \bigoplus (\bigoplus_{i=1}^{+\infty} qZ^{(i)})$ is a maximal subgroup of $G/Z_2 \bigoplus (\bigoplus_{i=1}^{+\infty} qZ^{(i)}) \cong \bigoplus_{i=1}^{+\infty} Z_q^{(i)}$, where $Z_q^{(i)} \cong Z_q$ for each i. This implies that there is an epimorphism from Q^* onto $\bigoplus_{i=1}^{+\infty} Z_q^{(i)}$, given by $f_q(x) = (\overline{\nu_{p_i}(x)})$. Thus, there exists a maximal subspace W of $\bigoplus_{i=1}^{+\infty} Z_q^{(i)}$ such that $M = f_q^{-1}(W)$.

Case 2. $q = 2$. In this case $qG \subseteq G_M$ yields that $\bigoplus_{i=1}^{+\infty} 2Z^{(i)} \subseteq G_M$ and hence

$G_M / \bigoplus_{i=1}^{+\infty} 2Z$ is isomorphic to a maximal subspace of G_2. Therefore, there exists a maximal subgroup of G_2 say W, such that $M = f_2^{-1}(W)$. The other side of the theorem is clear.

We remark that if we put $W =< e_i \mid i \geq 2 >$, where e_i is the vector whose i-th component is 1 and other components are zero, then we find $Q^+ = f_2^{-1}(W)$.

(b) Assume that M is maximal subgroup of $\mathbf{R}^*$. Then, we must have $\mathbf{R}^*/M \cong Z_p$ for some prime number p. Thus, for each $a \in \mathbf{R}^*$ we have $a^p \in M$. Now, take the equation $x^p = b$ over $\mathbf{R}$. We know that if p is odd, this equation has a solution in $\mathbf{R}$. This means that $a = (a^{1/p})^p \in M$, i.e., $\mathbf{R} = M$. So the only choice of p is $p = 2$. Thus, $M = \mathbf{R}^+$ is the only maximal subgroup of $\mathbf{R}^*$.

(c) Let F be algebraically closed and M be a maximal subgroup of F^*. Then, we have $F^*/M \cong Z_p$ for some prime p. Take an element $x \in F^*$. Since $x^{1/p}$ exists in F^* for any prime p, we conclude that $x = (x^{1/p})^p \in M$, i.e., $F^* = M$ which completes the proof.

Now, let $0 \neq [A] \in Br(F)$ be cyclic. It is known that there is a cyclic extension L/F, an automorphism $\sigma \in Gal(L/F)$, and $a \in F$ such that $A \cong (a, L/F, \sigma)$. Now, the map $\theta : F^* \to Br(L/F) \subset Br(F)$, given by the rule $\theta(c) = [c, L/F, \sigma]$, is a nontrivial group homomorphism (cf. Chapter 10 of [3]). This homomorphism is used in the next result to show that F^* has a maximal subgroup.

Lemma 2. *Let F be a field and $0 \neq [A] \in Br(F)$. If A is cyclic, then F^* has a maximal subgroup.*

Proof. Since A is cyclic, there exists a maximal subfield E, say, of A such that E/F is a finite cyclic extension. Now, consider the homomorphism $r_{E/F} : Br(F) \longrightarrow Br(E)$ given by $r_{E/F}(X) = X \otimes_F E$. We have $r_{E/F}(A) = A \otimes_F E$ and since E is a maximal subfield of A we find $r_{E/F}(A) = 0$, that is $A \in Br(E/F)$. Now since E/F is a finite cyclic extension we have $0 \neq Br(E/F) \simeq F^*/N_{E/F}(E^*)$ and if $[E : F] = n$, then we obtain $F^{*n} \subset N_{E/F}(E^*)$ and this implies that $F^* \neq (F^*)^n$. Now the group $F^*/(F^*)^n$ is a nontrivial abelian group of finite exponent and thus, by Baer-Prufer Theorem (cf. [10]), F^* has a maximal subgroup.

It is known that if F is local or global, then every F-central simple algebra is cyclic (cf. [3] or [11]), and so F^* has a maximal subgroup. We also observe that this result also follows easily from the fact that if F has a discrete valuation, then F^* has a maximal subgroup. Therefore, we record this fact as a corollary.

COROLLARY 3. *If F is a local or global field, then F^* has a maximal subgroup.*

COROLLARY 4. *If $Br(F)$ is non-trivial, then either F^* has a maximal subgroup or there exists a primitive p-th root of unity (p is a prime) ω,say, such that $F(\omega)/F$ is a finite cyclic extension and $(F(\omega))^*$ has a maximal subgroup.*

PROOF. Assume that p is a prime and $0 \neq [A] \in Br(F)$ such that $p[A] = 0$. If Char $F = p > 0$, then, by Albert Main Theorem (cf. [3, p. 110]), A is a cyclic F-algebra. Now, by Lemma 2, F^* has maximal subgroup. So we may assume that $Char F \neq p$. If F contains a primitive p-th root of unity, by Merkurjev-Suslin Theorem (cf. [12, p. 236]), A is similar to a tensor product of cyclic F-algebras. Since $[A] \neq 0$ we conclude that at least one of the components in the tensor product is non-trivial. Now, use Lemma 2 to complete the proof of this case. Thus one may assume that F^* does not contain a primitive p-th root of unity. Set $A' = A \otimes_F L$, where $L = F(\omega)$ and w is a primitive p-th root of unity. If $0 = [A'] \in Br(L)$, then $0 \neq Br(L/F) = F^*/N_{L/F}(L^*)$ and so, by Baer-Prufer Theorem, F^* has a maximal subgroup. So suppose that $0 \neq [A'] \in Br(L)$. Then the order of $[A']$ is clearly p and so by Merkurjev-Suslin Theorem there exists a non-trivial cyclic L-algebra. Now, apply Lemma 2 to complete the proof.

To prove our main theorem in this connection, we shall need the following

LEMMA 5. *Suppose $Br(F)$ is nontrivial. Then $Br_p(F)$ is nontrivial for some prime p. Furthermore, there is a cyclic F-algebra under either of the following situations:*
(i) F has characteristic p, or
(ii) F has characteristic $\neq p$ with enough roots of 1.

Proof. Suppose that $0 \neq [A] \in Br_p(F)$. Now, we have $p[A] = 0$. If $Char F = p$, by Albert's Theorem, A is cyclic and the result follows. Now, assume that F has characteristic $\neq p$ with enough roots of 1. By Merkurjev-Suslin Theorem, A is similar to a tensor product of cyclic algebras. Since $0 \neq [A] \in Br_p(F)$ we obtain a non-trivial cyclic algebra as desired.

We are now in a position to prove

Theorem 6. *Assume that $Br(F)$ is non-trivial. Then $Br_p(F)$ is nontrivial for some prime p. Furthermore, F^* has a maximal subgroup under either of the following situations:*
(i) F has characteristic zero, or
(ii) F has characteristic p, or
(iii) F has characteristic $\neq p$ with enough roots of 1.

Equivalently, if F has the above conditions and F^* is divisible, then $Br(F)$ is trivial.

Proof. Assume that $Char F = 0$. If F^* has no maximal subgroups, then F^* is divisible. Now, by Lemma 3 of [9], F^* contains all roots of unity which contradicts Corollary 4. If $Char F = p > 0$, by Lemma 5, there exists a cyclic F-algebra B such that $0 \neq [B] \in Br_p(F)$. Now, by Lemma 2 the result follows. Finally, assume that F has characteristic $\neq p$ with enough roots of 1. Again, using Lemma 5 and Lemma 2 as above completes the proof of the theorem.

We observe that the converse of Theorem 6 is not true in general. For let F be algebraically closed and consider $F(x)$. Then we know that $F(x)$ has a discrete valuation and so $F(x)^*$ has a maximal subgroup whereas, by Tsen-Lang Theorem (cf. [12, p. 211]), $Br(F(x))$ is trivial.

It is not known that if F^* is divisible, then $Br(F)$ is trivial without any condition on the characteristic of F.

Remark. Here we establish a connection between the existence of a maximal subgroup of F^* and certain cohomology groups and modules. Let L/F be a finite Galois extension and the 0-th cohomology group of L is nonzero, i. e., $H^0(G, L^*) \neq \{0\}$, where $G = Gal(L/F)$. Therefore, we conclude that $F^*/N_{L/F}(L^*)$ is nonzero which implies that F^* has a maximal subgroup. Now, assume that F_s denotes the separable closure of the field F. Let us put

$G_m = F_s^*$. It is known that the cohomology of the module G_m is important, for example we have $H^0(F, G_m) = F^*$, and by Hilbert's Theorem 90 we also have $H^1(F, G_m) = 0$ and $H^2(F, G_m) \cong Br(F)$. Suppose that μ_n is the group of all n-th roots of unity and n is prime to the characteristic of F. Now, we have the exact sequence,

$$1 \to \mu_n \to G_m \xrightarrow{n} G_m \to 1,$$

which is referred to as the Kummer sequence and n denotes the endomorphism $x \to x^n$. The corresponding cohomology sequence is also called the Kummer sequence, which is as follows:

$$1 \to \mu_n(F) \to F^* \xrightarrow{n} F^* \to H^1(F, \mu_n) \to 1$$

$$1 \to H^2(F, \mu_n) \to Br(F) \xrightarrow{n} Br(F).$$

Thus, we find the isomorphisms $H^1(F, \mu_n) \cong \frac{F^*}{F^{*n}}$ and $H^2(F, \mu_n) \cong_n Br(F)$. If $\mu_n \subset F^*$, we obtain $H^2(F, \mu_n \otimes \mu_n) \cong_n Br(F) \otimes \mu_n(F)$. Therefore, if Char $F = 0$ and $H^2(F, \mu_n \otimes \mu_n) \neq 0$, then F^* has a maximal subgroup. Now, let F be a field that has no maximal subgroup. Then F^* is divisible, and thus $F^{*^p} = F^*$, where p is the characteristic of F, and so F is perfect. Since we have $F^* = F^{*q}$ for all $q \neq p$ we obtain $H^1(F, \mu_q) = 1$. Conversely, if F is perfect and we have $H^1(F, \mu_q) = 1$ for all $q \neq p$, then F^* is divisible.

Now, we turn to study maximal subgroups of D^* and show how normal maximal subgroups of D^* are related to maximal subgroups of F^* and valuations on F. We recall that if D is a division ring with center F, then D is called of *type* 2 if for any two elements $a, b \in D$, the F-algebra $F[a, b]$ is finite dimensional. To state our next result, we need some more preparations. Denote by $G(D)$ the group D^*/F^*D'. When D is algebraic over its centre F, $G(D)$ is torsion (cf. [6]). Some algebraic properties of $G(D)$ are investigated in [4] and [8]. We continue our study with

THEOREM 7. *Let D be a division ring of type 2 with centre F and v be a discrete valuation on F. If there exists a natural number m such that $G(D)$ has no element of order m, then D^* has a maximal subgroup.*

PROOF. First we show that there exists a non-zero homomorphism of D^* into the additive group of rational numbers $\mathbf{Q}$. By assumption there is a prime

p and natural number r such that $G(D)$ has no element of order p^r. Now define a function $w : D^* \longrightarrow \mathbf{Q}$ by $w(a) = \frac{1}{[F(a):F]}v(N_{F(a)/F}(a))$. We claim that w is a homomorphism. To see this, suppose that S is a subalgebra of finite dimension over F and $a \in S$, we prove that $w(a) = \frac{1}{[S:F]}v(N_{S/F}(a))$. Put $[F(a) : F] = n$ and $[S : F] = s$, it follows that $n|s$ and $N_{S/F}(a) = (N_{F(a)/F}(a))^{s/n}$. This implies that $\frac{1}{s}\nu(N_{S/F}(a)) = \frac{1}{n}N_{F(a)/F}(a) = w(a)$. Thus if $a, b \in D^*$ and $F(a, b) = S$, then we find $[S : F] < \infty$ since D is of type 2. Now since $N_{S/F}(ab) = N_{S/F}(a)N_{S/F}(b)$ we have $w(ab) = w(a) + w(b)$. Thus w is a homomorphism as claimed. Now, since $G(D)$ is torsion for any $x \in D^*$ there exists $n(x) > 0$ such that $x^{n(x)} = tc$, where $t \in F$ and $c \in D'$, and p^r does not divide $n(x)$. So we obtain $w(x) = \frac{1}{n(x)}w(t)$, and this implies that $\frac{1}{p^r}$ does not belong to the image of w, $Im(w)$, i.e., $Im(w) \neq \mathbf{Q}$. We now claim that $Im(w)$ is not divisible. Since otherwise, assume that $h/q \in Q\backslash Im(w)$. It is easily seen that $w|F = v$ and so $Im(w) \neq 0$. Now if $0 \neq u \in Im(w)$, then we have $uh \in Im(w)$ and the equation $qux = uh$ has a solution in $Im(w)$ which is a contradiction. Consequently, $Im(w)$ is not divisible and so $Im(w)$ has a maximal subgroup and therefore D^* has a maximal subgroup.

Corollary 8. *Let D be a finite dimensional division algebra over its centre F. If F has a discrete valuation, then D^* has a maximal subgroup.*

In the next result we deal with maximal subgroups of D^* which do not contain F^*, and the theorem also shows how maximal subgroups of D^* arise from those of F^*.

Theorem 9. *Let D be a division ring with centre $Z(D) = F$. Then we have the following*
(a) *If M is a maximal subgroup of D^* not containing F^*, then $Z(M)$ is a maximal subgroup of F^*.*
(b) *Assume that D is of finite index n over F and m is a maximal subgroup of F^* containing $Z(D')$. Then D^* has a maximal subgroup M containing m that is normal in D^*.*

Proof. (a) We have $D^* = F^*M$ and thus $D' = M'$. This shows that M is normal in D^*. Now, by a result of [8], we have $Z(M) = F^* \cap M$ and so $Z_p \cong D^*/M = F^*M/M \cong F^*/Z(M)$, for some prime number p. This implies that $Z(M)$ is maximal in F^*.

(b) Assume that m is a maximal subgroup of F^* which contains $Z(D')$. Thus $F^*/m \cong Z_p$ for some prime number p. Consider the normal subgroup mD' of D^*. If $mD' = D^*$, then we obtain $m = mZ(D') = F^*$ which is a contradiction. So, take the nontrivial group D^*/mD'. We know that D^*/F^*D' is torsion of a bounded exponent dividing the index of D over F (cf. [7] or [8]). Now, since $F^*/m \cong Z_p$ we conclude that the group D^*/mD' is torsion of a bounded exponent. Therefore, by Baer-Prufer theorem (cf. [10]), D^*/mD' is isomorphic to a direct product of cyclic groups Z_{r_i}, where r_i divides the index n for all i. In this way, we may obtain a maximal subgroup N of D^* containing mD' and thus the result follows.

The next result shows how valuations on the centre F of a division ring D enable one to obtain maximal subgroups of D^* which are normal.

Corollary 10. *Let F be a field with a Krull valuation whose value group contains a maximal subgroup. Assume that D is a division algebra of finite dimension over its centre F. Then D^* contains a maximal subgroup M which is normal in D^*.*

Proof. Let v be a Krull valuation on F whose value group Γ, say, has a maximal subgroup. Then, we have $F^*/U \cong \Gamma$, where U is the group of units of the valuation. Since Γ contains a maximal subgroup , the isomorphism above induces a maximal subgroup L of F^* containing U. We know that $Z(D')$ is torsion. Thus $Z(D') \subset L$. Now, Theorem 9 completes the proof.

Corollary 11. *Let F be an algebraic number field, and assume that D is an F -central division algebra. Then D^* contains a maximal subgroup M which is normal in D^*.*

Proof. It is known that the p-adic valuation of $\mathbf{Q}$ extends to a discrete valuation on F. Now, using Corollary 10 completes the proof.

Given a local field F, one may easily check that F has a discrete valuation. Now, using Corollary 10 again, we obtain the following

Corollary 12. *Let F be a local field, and assume that D is an F -central division algebra. Then D^* contains a maximal subgroup M which is normal in*

D^*.

In contrast to above results, we may observe that not any multiplicative group of a division algebra contains normal maximal subgroups as the following result shows.

THEOREM 13. *Let D be the real quaternion division algebra. Then D^* contains no normal maximal subgroups.*

PROOF. Let M be a maximal subgroup of D^* which is normal in D^*. Since M is maximal we have $D^*/M \cong Z_p$ for some prime number p. Take an element $x \in D \backslash \mathbf{R}$, then $\mathbf{R}[x] \cong \mathbf{C}$. Therefore, every element in D^* has a p-th root and so $D^* = M$ which is a contradiction and so the result follows.

Generally, in view of the above results, one is tempted to state the following

CONJECTURE. *Let D be a non-commutative division ring. Then D^* contains a maximal subgroup.*

Let F be a field. It is not true in general that each element $a \in F^*$ is contained in a maximal subgroup of F^* even if F^* has maximal subgroups. For example, by Lemma 1, $\mathbf{R}^+$ is the only maximal subgroup of $\mathbf{R}$ which does not contain negative real numbers. But for finite dimensional division algebras the situation is different as the following result shows. To state the theorem, we observe that when D is of finite dimension over its centre F, $G(D)$ is torsion of a bounded exponent dividing the index of D over F (cf. [8]).

THEOREM 14. *Let D be a division algebra of finite dimension over its centre F. Then we have the following*

(a) *If $G(D)$ is not cyclic, then each element $x \in D^*$ is contained in a normal maximal subgroup of D^*.*

(b) *If $G(D)$ is cyclic and non-trivial, and F^* contains a maximal subgroup containing $Z(D')$, then each element $x \in D^*$ is contained in a normal maximal subgroup of D^*.*

PROOF. (a) We know, by the Lemma of [6], that $G(D)$ is torsion of a bounded exponent dividing the index m of D over F. Since $G(D)$ is not trivial, by Baer-Prufer Theorem (cf. [10]), we have $G(D) \cong Z_{r_1} \times Z_{r_2} \times \cdots$, where $r_i|m$ for each i. Therefore, there are maximal subgroups M of D^* which contain F^*D'. Thus, if $x \in F^*D'$, then we obtain $x \in M$ and the result follows. So, we may assume that $x \notin F^*D'$. Put $H = F^*D' < x >$. Since $D' \subset H$ we conclude that H is normal in D^*. If $F^*D' < x >= D^*$, then we have

$$G(D) = \frac{F^*D' < x >}{F^*D'} \cong \frac{< x >}{F^*D' \cap < x >} \cong Z_t,$$

since $G(D)$ is torsion. This contradicts our assumption that $G(D)$ is not cyclic. Therefore, $F^*D' < x > \neq D^*$ and $D^*/F^*D' < x >$ is an abelian torsion group of bounded exponent. Thus, By Baer-Prufer Theorem, we obtain $D^*/F^*D' < x > \cong Z_{t_1} \times Z_{t_2} \times \cdots$, where $t_i|m$ for all i. Now, take a maximal subgroup L, say, of $Z_{t_1} \times Z_{t_2} \times \cdots$ and consider the inverse image M, say, of L under the indicated isomorphism. Then, M is a maximal subgroup of D^* which contains x and so the result follows.
(b) By the argument used in part (a), it is enough to consider $F^*D' < x >= D^*$, where $x \notin F^*D'$. Now, we have $D^*/D' < x > \cong F^*/F^* \cap D' < x >$. If $D^* = D' < x >$, then $D^*/D' \cong < x > /D' \cap < x >$. The cyclic group $< x > /D' \cap < x >$ can not be finite since otherwise D^*/D' would be torsion. This is not possible by Proposition 1 of [7]. Thus, we must have $< x > /D' \cap < x > \cong Z \cong D^*/D'$. But then we obtain $D^{(1)}/D' \cong nZ$ for some $n \in Z$, where $D^{(1)}$ is the group of reduced norm 1 elements. This is not possible either since $D^{(1)}/D'$ is a torsion group. Therefore, $D^* \neq D' < x >$. Now, since $Z(D')(F^* \cap < x >) = (F^* \cap D')(F^* \cap < x >) \subset F^* \cap D' < x >$ and F^* contains a maximal subgroup L, say, containing $Z(D')$, by Theorem 9, we conclude that there is a maximal subgroup M in D^* containing $D' < x >$. So $x \in M$ which completes the proof.

It is believed that the condition in Theorem 14 for $G(D)$ to be trivial is superfluous. In fact, it is a conjecture in [4] that $G(D)$ is rarely trivial and it only happens for the real quaternions. So, Theorem 14 applies to a wide range of division algebras. For examples in which all the conditions of Theorem 14 are satisfied, see [4].

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References

[1] Akbari, S., Mahdavi-Hezavehi, M., *Normal Subgroups of $GL_n(D)$ Are not Finitely Generated*, Proc. Amer. Math. Soc., Vol. 128, No. 6, 1827-1832 (2000).

[2] Akbari, S., Mahdavi-Hezavehi, M., Mahmudi, M. G., *Maximal Subgroups of $GL_1(D)$*, Journal of Algebra, 217, 422-433 (1999).

[3] Draxl P. K., Skew Fields, Cambridge University Press, (1983).

[4] Hazrat, R., Mahdavi-Hezavehi, M., Mirzaii, B., *Reduced K-Theory and the Group $G(D) = D^*/F^*D'$*, Algebraic K-Theory and Its Applications, Proc. of the Workshop and Symposium, World Scientific Pub., 403-409 (1999).

[5] Mahdavi-Hezavehi, M., Mahmudi, M. G., Yasamin, S., *Finitely Generated Subnormal Subgroups of $GL_n(D)$ Are Central*, Journal of Algebra, 225, 517-521 (2000).

[6] Mahdavi-Hezavehi, M., *Extending Valuations to Algebraic Division Algebras*, Comm. Algebra, Vol.22(11), 4373-4378 (1994).

[7] Mahdavi-Hezavehi, M., *Determinant-like Functions for Matrices over Finite Dimensional Division Algebras*, Algebraic K-Theory and Its Applications, Proc. of the Workshop and Symposium, World Scientific Pub., 489-502 (1999).

[8] Mahdavi-Hezavehi, M., *Commutator Subgroups of Finite Dimensional Division Algebras*, Rev. Roumaine Math. Pures Appl., 43(9-10), 853-867 (1998).

[9] May, W., *Multiplicative Groups of Fields*, Proc. London Math. Soc. Vol. 3, No. 21, 295-306 (1972).

[10] Robinson, Derek J. S., *A Course in the Theory of Groups*, GTM, No. 80, Springer-Verlag, (1982).

[11] Rowen L., *Ring Theory*, Volume II, Academic Press, INC, (1988).

S. Akbari, `s_akbari@sharif.edu`
M. Mahdavi-Hezavehi, `mahdavih@sharif.edu`
Department of Mathematical Sciences
Sharif University of Technology
P.O.Box 11365-9415
Tehran, Iran