A Generalized Poincaré Theorem for Dual Lie Transformation Groups

Tuong Ton-That

Abstract. Let k and n be integers such that $k > 2n > 0$. Let M be the complex analytic manifold defined by $M = \{x \in \mathbb{C}^{n\times k} : xx^t = 0,\ \text{rank } (x) = n\}$. Let $G = \mathrm{SO}(k,\mathbb{C})$ and $G' = \mathrm{GL}(n,\mathbb{C})$, then Witt's theorem on quadratic forms implies that G is a maximal connected Lie group acting transitively on M by right multiplication. Also, G' is a maximal connected Lie group acting freely on M by left multiplication. If $f \in C^\infty(M)$, $x \in M$, $g \in G$, and $g' \in G'$ define $R(g)f$ (resp. $L(g')f$) by

$$(R(g)f)(x) = f(xg) \quad \text{and} \quad (L(g)f)(x) = f(g^{-1}x).$$

If $\mathcal{D}^\omega(M)$ denotes the algebra of all analytic differential operators on M then an element $D \in \mathcal{D}^\omega(M)$ is called right (resp. left)-invariant if $DR(g) = R(g)D$, $\forall\, g \in G$ (resp. $DL(g') = L(g')D$, $\forall\, g' \in G'$).

THEOREM: Let $\mathcal{D}_l^\omega(M)$ (resp. $\mathcal{D}_r^\omega(M)$) denote the subalgebra of $\mathcal{D}^\omega(M)$ of all left (resp. right)-invariant analytic differential operators on M. Let $\widetilde{\mathcal{U}}(\mathfrak{g})$ (resp. $\widetilde{\mathcal{U}}(\mathfrak{g}')$) denote the universal enveloping algebra generated by the infinitesimal action of $R(g)$ (resp. $L(g')$). Then we have

$$\mathcal{D}_l^\omega(M) = \widetilde{\mathcal{U}}(\mathfrak{g}) \text{ and } \mathcal{D}_r^\omega(M) = \widetilde{\mathcal{U}}(\mathfrak{g}').$$

Moreover, the commutant of $\mathcal{D}_l^\omega(M)$ in $\mathcal{D}^\omega(M)$ is $\mathcal{D}_r^\omega(M)$, and vice-versa.

This theorem also holds for other types of dual Lie transformation groups acting on analytic manifolds.

2000 Mathematics Subject Classification: Primary 15A63, 16S32; Secondary 16S30, 14L35.

1 Introduction

In 1900 H. Poincaré established the existence of the universal enveloping algebra of a Lie algebra and proved one of the most fundamental results in the theory of Lie groups and Lie algebras. This theorem which is valid for a Lie algebra over an arbitrary field is usually called the Poincaré-Birkhoff-Witt theorem; however for the case of a real or complex Lie algebra it is entirely due to Poincaré as shown in [TT-T].

THEOREM 1.1 (*Poincaré*). *Let G be a real or complex Lie group with Lie algebra $\mathfrak{g}$. Let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. If $\{X_i : 1 \leq i \leq n\}$ is a basis of $\mathfrak{g}$ then the ordered monomials 1 and $X_{i_1} \cdots X_{i_s} (s \geq 1, i_1 \leq \cdots \leq i_s)$ form a basis for $\mathcal{U}(\mathfrak{g})$.*

Assume that G is a real or complex *connected* Lie group. For each $g \in G$, the translations $l_g, r_g : G \to G$ defined by $l_g(x) = gx$ and $r_g(x) = xg$, $x \in G$, are analytic diffeomorphisms of G onto itself. Let $\mathcal{D}^\omega(G)$ denote the algebra of all analytic differential operators on G. A differential operator D of $\mathcal{D}^\omega(G)$ is said to be *left* (resp. *right*)-*invariant* if it is invariant under all left (resp. right) translations. Let $[\cdot, \cdot]$ denote the commutator product of $\mathcal{D}^\omega(G)$. If $\mathcal{A}$ is a subalgebra of $\mathcal{D}^\omega(G)$ then the *centralizer* (or *commutant*) of $\mathcal{A}$ in $\mathcal{D}^\omega(G)$ is defined as the set $\{D' \in \mathcal{D}^\omega(G) : [D', D] = 0, \forall\, D \in \mathcal{A}\}$, and the *centre* of $\mathcal{A}$ is defined as the set $\{D' \in \mathcal{A} : [D', D] = 0, \forall\, D \in \mathcal{A}\}$. Then the following can be easily deduced from the Poincaré theorem.

COROLLARY 1.2 (*To Poincaré Theorem*). *If $\mathcal{D}_l^\omega(G)$ (resp. $\mathcal{D}_r^\omega(G)$) denotes the subalgebra of $\mathcal{D}^\omega(G)$ of all left (resp. right)-invariant analytic differential operators on G then $\mathcal{D}_l^\omega(G)$ (resp. $\mathcal{D}_r^\omega(G)$) is isomorphic to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Moreover, the centralizer of $\mathcal{D}_l^\omega(G)$ in $\mathcal{D}^\omega(G)$ is $\mathcal{D}_r^\omega(G)$, and vice-versa. Finally, the centres of $\mathcal{D}_l^\omega(G)$ and $\mathcal{D}_r^\omega(G)$ coincide with $\mathcal{D}_l^\omega(G) \cap \mathcal{D}_r^\omega(G)$.*

In the context of Lie transformation groups on analytic manifolds the corollary above can be phrased as follows: Consider G as a G-transformation group acting on the analytic manifold $M = G$ to the right and as a G'-transformation acting on M to the left; then the subalgebras of all left (resp. right)-invariant analytic differential operators on the analytic manifold M are mutual commutants in $\mathcal{D}^\omega(M)$. We shall generalize this result to dual transformation groups acting on analytic manifolds. The simplest case with $G = \mathrm{GL}(k, \mathbb{C})$, $G' = \mathrm{GL}(n, \mathbb{C})$, $M = \{x \in \mathbb{C}^{n \times k} : x \text{ of maximum rank}\}$ was considered in [TT5]. In this article three more cases are considered. They are more intricate and *Witt's theorems on quadratic forms and skew-symmetric bilinear forms* play a crucial role in their resolution. The general case will be considered in a future publication.

2 A Duality Theorem for Commutants in $\mathcal{D}^\omega(M)$

Let $E = \mathbb{C}^{n\times k}$, $G = \mathrm{SO}(k,\mathbb{C})$, $G' = \mathrm{GL}(n,\mathbb{C})$. Then it is clear that G' (resp. $\mathrm{GL}(k,\mathbb{C})$) is the maximum linear group acting on E by left (resp. right) multiplication. As a subgroup of $\mathrm{GL}(k,\mathbb{C})$, G acts on E by right multiplication and leaves the nondegenerate symmetric bilinear form $(x,y) \to \mathrm{tr}(xy^t)$, $x,y \in \mathbb{C}^{n\times k}$, invariant. If $S(E^*)$ is the symmetric algebra of all polynomial functions on E then the action of G on E induces an action of G on $S(E^*)$, denoted by $g\cdot p$, for $g \in G$, $p \in S(E^*)$. We say that $p \in S(E^*)$ is *G-invariant* if $g\cdot p = p$, for all $g \in G$. The G-invariant polynomial functions form a subalgebra $J(E^*)$ of $S(E^*)$. If $J_+(E^*)$ is the subset of all elements in $J(E^*)$ without constant term we let $J_+(E^*)S(E^*)$ denote the ideal in $S(E^*)$ generated by $J_+(E^*)$. Recall ([We, Theorem 2.9A]) that $J_+(E^*)S(E^*)$ is generated by the $n(n+1)/2$ algebraically independent polynomials

$$p_{ij}(x) = \sum_{s=1}^{k} x_{is}x_{js}, \qquad 1 \le i \le j \le n, \quad x \in E, \tag{2.1}$$

together with the $(k\times k)$ minors of the matrix x (which are 0 when $k > n$). If P is the null cone of the common zeros of polynomial functions in $J_+(E^*)S(E^*)$ then by the Hilbert Nullstellensatz the ideal in $S(E^*)$ of all polynomial functions which vanish on P is $\sqrt{J_+(E^*)S(E^*)}$. By [D-TT, Theorem 2.1] the ideal $J_+(E^*)S(E^*)$ is prime if and only if $k > 2n$, and the scheme P which is then equal to the set $\{x \in E : xx^t = 0\}$ is a complete intersection, with one open dense orbit.

Henceforth we assume that $k > 2n$. Let $M = \{x \in E : xx^t = 0,\ \mathrm{rank}(x) = n\}$ then obviously M is dense in P. Since $(g'x)(g'x)^t = g'(xx^t)(g')^t$ it follows immediately that G' is the maximum linear group acting on M by left multiplication. For $\gamma \in \mathrm{GL}(k,\mathbb{C})$ and $p \in S(E^*)$ define $R(\gamma)p$ by $(R(\gamma)p)(x) = p(x\gamma)$, then clearly γ leaves M, and hence P, invariant if and only if $R(\gamma)p_{ij} \in J_+(E^*)S(E^*)$ for all $1 \le i \le j \le n$. Obviously, $R(\gamma)p_{ij}$ are quadratic polynomials, and since the p_{ij} form a basis for the quadratic polynomials in $J_+(E^*)S(E^*)$ we have

$$R(\gamma)p_{ij} = \sum_{r,s} C^{ij}_{rs}p_{rs}, \qquad 1 \le r \le s \le n, \tag{2.2}$$

where $C^{ij}_{rs} \in \mathbb{C}$ are constants depending on γ. For $1 \le i \le n$, $1 \le t \le k$ let $x(i,t)$ denote the element of E which has the (i,t)-entry equal to 1 and all other entries equal 0. Then an easy computation shows that

$$(R(\gamma)p_{ii})(x(i,t)) = \sum_{l=1}^{k} \gamma_{tl}^2 = \sum_{r,s} C^{ii}_{rs}p_{rs}(x(i,t)) = C^{ii}_{ii}.$$

It follows that $\sum_{l=1}^{k}\gamma_{tl}^2 = C \in \mathbb{C}$ for all t, and i. Choose x of the form

$x(i,t)+x(i,t')$ with $t \neq t'$ then we obtain

$$\begin{aligned}(R(\gamma)p_{ii})(x(i,t)+x(i,t')) &= \sum_{l=1}^{k}(\gamma_{tl}+\gamma_{t'l})^2 \\ &= \sum_{r,s} C_{rs}^{ii} p_{rs}(x(i,t)+x(i,t')) \\ &= 2C_{ii}^{ii} = 2C.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{l=1}^{k}(\gamma_{tl}+\gamma_{t'l})^2 = \sum_{l=1}^{k}\gamma_{tl}^2 + \sum_{l=1}^{k}\gamma_{t'l}^2 + 2\sum_{l=1}^{k}\gamma_{tl}\gamma'_{t'l} = 2C \\ = C + C + 2\sum_{l=1}^{k}\gamma_{tl}\gamma_{t'l}.\end{aligned}$$

It follows that we have the system of equations

$$\sum_{l=1}^{k}\gamma_{tl}^2 = C, \qquad \sum_{l=1}^{k}\gamma_{tl}\gamma_{t'l} = 0 \qquad \text{for all } t,t' = 1,\dots,k,\ t \neq t', \tag{2.3}$$

or equivalently, $\gamma^t\gamma = CI_k$.
Since $(\det(\gamma))^2 = C^k$ and γ is invertible it follows that $C \neq 0$. Let λ be a fixed square root of C and set $g = \frac{1}{\lambda}\gamma$, then $g^t g = I_k$, or $g \in \mathrm{O}(k,\mathbb{C})$. It follows that the largest group acting on M by right multiplication is $\mathbb{C}^*\mathrm{O}(k,\mathbb{C}) = \{\lambda g : \lambda \in \mathbb{C}, g \in \mathrm{O}(k,\mathbb{C})\}$, and G is a maximal connected linear group acting on M by right multiplication.
By Witt's theorem on symmetric bilinear forms (see, e.g., [Ar] and [TT1, Lemma 2.8]) G acts *analytically* and *transitively* on M. More precisely, if $x_0 \in M$ then M is the G-orbit of x_0, and if G_{x_0} is the stability subgroup at x_0, then it is easy to verify that G_{x_0} is isomorphic to $\mathrm{SO}(k-n,\mathbb{C})$. Moreover, the map $G_{x_0}g \to x_0 g$ is an analytic diffeomorphism of $G_{x_0}\backslash G$ onto M (see, e.g., [Va, Theorem 2.9.4]). Thus M is an analytic manifold of complex dimension $nk - n(n+1)/2$ (this also follows from [TT1, Lemma 2.9] and the implicit function theorem for analytic functions [Hö, Theorem 2.1.2]).
Now let us show that G' acts *freely* on M, i.e., the stability subgroup G'_x is $\{1_{G'}\}$ at every $x \in M$. Indeed, if $x \in M$ then by the assumption $\mathrm{rank}(x) = n$ there exist n columns $x_{i_1}\cdots x_{i_n}, i_1 < \cdots < i_n$, of x such that the $n \times n$ matrix x_n formed by them is invertible. So $g'x = x$ implies that $g'x_n = x_n$ or $g' = x_n x_n^{-1} = 1_{G'}$. Now let us recall the definition of differential operators on a complex manifold M of dimension m (see, e.g., [He, Chapter 10]).
If (φ, U) is a local chart on M with $\varphi(p) = (x_1(p),\dots,x_m(p)) \in \mathbb{C}^m$, $p \in U$, and $f \in C^\infty(M)$, set $f^* = f \circ \varphi^{-1}$: $\varphi(U) \subset \mathbb{C}^m \to \mathbb{C}$. Set $\partial_i = \partial/\partial x_i$ $(1 \le i \le m)$ and if $\alpha = (\alpha_1,\dots,\alpha_m)$ is an m-tuple of indices $\alpha_i \ge 0$ we put

$D^{(\alpha)} = \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}$. Then a linear transformation D: $C_c^\infty(M) \to C_c^\infty(M)$ is called a *differential operator* on M if the following condition is satisfied:
For each $p \in M$ and each chart (φ, U), $p \in U$, there exists a locally finite set of functions $h_{(\alpha)} \in C^\infty(U)$ such that for each $f \in C_c^\infty(M)$ with support contained in U,

$$\begin{cases} [Df](p) = \sum\limits_{(\alpha)} h_\alpha \left[D^{(\alpha)} f^*\right](\varphi(p)) & \text{if } p \in U, \\ [Df](p) = 0, & \text{if } p \notin U. \end{cases} \tag{2.4}$$

If M is a complex analytic manifold then a differential operator D is called a *holomorphic* or *complex analytic differential operator* if the functions $h_{(\alpha)}$ in Eq. (2.4) are holomorphic (or complex analytic).
By Hilbert's fifth problem G (resp. G') can be equipped with an analytic structure (see, e.g., [M-Z]) so that they act analytically on M. Let $D^\omega(M)$ denote the algebra of (complex) analytic differential operators on M.
Now consider a *global G-transformation group* on an analytic manifold M (see, e.g., [Pa] or [Va, 2.16]). Let φ: $G \times M \to M$ $((g,x) \to g \cdot x,\ g \in G,\ x \in M)$ denote the global action of G on M. For $x \in M$, $f \in C^\infty(M)$ we define $(\Phi(g)f)(x) = f(g^{-1} \cdot x)$. Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of G. Then for $X \in \mathfrak{g}$ and $x \in M$ we define

$$d\Phi(X)_x(f) := \left(\frac{d}{dt} f(\exp(-tX) \cdot x)\right)_{t=0} \tag{2.5}$$

for all f defined and C^∞ in a neighborhood of x. The map $X \to d\phi(X)$ is a homomorphism of $\mathfrak{g}$ into the Lie algebra of analytic vector fields on M. Therefore it extends to a homomorphism $a \to \widetilde{d\phi}(a)$, $a \in \mathcal{U}(\mathfrak{g})$, of $\mathcal{U}(\mathfrak{g})$ into the algebra $\mathcal{D}^\omega(M)$ of analytic differential operators on M (see [Va, Lemma 2.16.1]), where if $a = X_1 \cdots X_r$ $(X_i \in \mathfrak{g})$ then

$$\widetilde{d\phi}(a)_x(f) = \left(\frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t} f(\exp(-t_r X_r) \cdots \exp(-t_1 X_1) \cdot x)\right)\Bigg|_0, \tag{2.6}$$

where the suffix 0 indicates that the derivatives are taken when $t_1 = \cdots = t_r = 0$. For our problem we consider the cases when $\Phi(g') = L(g')$ and $\Phi(g) = R(g)$, where $(L(g')f)(x) = f\left((g')^{-1}x\right)$ and $(R(g)f)(x) = f(xg)$ for $g' \in G'$, $g \in G$, and $x \in M$. Let $\widetilde{\mathcal{U}}(\mathfrak{g}')$ and $\widetilde{\mathcal{U}}(\mathfrak{g})$ denote the images of $\mathcal{U}(\mathfrak{g}')$ and $\mathcal{U}(\mathfrak{g})$ under the maps $\widetilde{dL}$ and $\widetilde{dR}$, respectively.

Definition 2.1 *A differential operator D of $\mathcal{D}^\omega(M)$ is said to be* right (*resp.* left)-invariant *if $D(R(g)f) = R(g)(Df)$ (resp. $D(L(g')f) = L(g')(Df)$) for all $g \in G$ (resp. $g' \in G'$), and for all $f \in C^\infty(M)$.*

Theorem 2.2 *Let $\mathcal{D}_l^\omega(M)$ (resp. $\mathcal{D}_r^\omega(M)$) denote the subalgebra of $\mathcal{D}^\omega(M)$ of all left (resp. right)-invariant analytic differential operators on M. Then*

(i) $\mathcal{D}_l^\omega(M) = \widetilde{\mathcal{U}}(\mathfrak{g})$ *and* $\mathcal{D}_r^\omega(M) = \widetilde{\mathcal{U}}(\mathfrak{g}')$,

(ii) *the commutant of* $\mathcal{D}_l^\omega(M)$ *in* $\mathcal{D}^\omega(M)$ *is* $\mathcal{D}_r^\omega(M)$, *and vice-versa. Moreover, the centres of* $\mathcal{D}_l^\omega(M)$ *and* $\mathcal{D}_r^\omega(M)$ *coincide with the subalgebra* $\mathcal{D}_l^\omega(M) \cap \mathcal{D}_r^\omega(M)$.

PROOF. (i) Let $X \in \mathfrak{g}'$, $g \in G$, $x \in M$ and $f \in C^\infty(M)$. Then

$$\begin{aligned} dL(X)(R(g)f)(x) &= \frac{d}{dt}\left((R(g)f)(\exp(-tX)x)\right)_{t=0} \\ &= \frac{d}{dt}(f(\exp(-tX)xg))_{t=0}, \end{aligned}$$

while

$$\begin{aligned} R(g)(dL(X)f)(x) &= (dL(X)f)(xg) \\ &= \frac{d}{dt}(f(\exp(-tX)xg))_{t=0}. \end{aligned}$$

Thus any vector field $\tilde{X} = dL(X) \in \widetilde{\mathcal{U}}(\mathfrak{g}')$ is right-invariant, and it follows immediately that $\widetilde{\mathcal{U}}(\mathfrak{g}') \subset \mathcal{D}_r^\omega(M)$. Similarly we have $\widetilde{\mathcal{U}}(\mathfrak{g}) \subset \mathcal{D}_l^\omega(M)$. Let us show that $\mathcal{D}_r^\omega(M) \subset \widetilde{\mathcal{U}}(\mathfrak{g}')$ and $\mathcal{D}_l^\omega(M) \subset \widetilde{\mathcal{U}}(\mathfrak{g})$.
Let $\mathcal{L}$ denote the Lie algebra of all right-invariant analytic vector fields on M. Then $\mathcal{L}$ is an *involutive analytic system* (see [Va, p. 25] for the definition), i.e., if U is an open subset of M and X, Y are right-invariant vector fields on M then $[X, Y]$ is (obviously) right-invariant. Then the Global Frobenius Theorem (see, e.g., [Va, Theorem 1.3.6]) implies that: given any point of M, there is one and exactly one maximal *integral manifold* $\mathcal{S}$ *of* $\mathcal{L}$ containing that point, i.e., $\mathcal{S}$ is a connected analytic submanifold of M and for each $y \in \mathcal{S}$, $\mathcal{L}_y$ is the tangent space $T_y(\mathcal{S})$. In fact since $\mathcal{L}$ is an infinitesimal group [Pa, Theorem IV, p. 98] implies that $\mathcal{S}$ is the image of a unique connected Lie transformation group H of M. Since G' is the largest linear group acting on M by left multiplication and $dL(\mathfrak{g}') \subset \mathcal{L}$ it follows that $G' \subset H$, and hence, $G' = H$. It follows that if $\{X_1, \ldots, X_{n^2}\}$ is a basis of $\mathfrak{g}'$ then $\{\tilde{X}_1, \ldots, \tilde{X}_{n^2}\}$, where $\tilde{X}_i = dL(X_i)$, $1 \le i \le n^2$, is a basis for right-invariant analytic vector fields on M. Therefore if $D \in \mathcal{D}_r^\omega(M)$ then we can find a unique set of locally finite functions $h_{(\alpha)} \in C^\omega(M)$ such that

$$D = \sum_{(\alpha)} h_{(\alpha)} \tilde{X}^{(\alpha)}. \tag{2.7}$$

Since the $\tilde{X}^{(\alpha)}$ are right-invariant, we have

$$D = D^{r_g} = \sum_{(\alpha)} h_{(\alpha)}^{r_g} \tilde{X}^{(\alpha)} \qquad (g \in G). \tag{2.8}$$

The relations (2.7) and (2.8) imply that all the $h_{(\alpha)}$ are right-invariant, and since G acts transitively on M, they must be all constant. Thus $D \in \widetilde{\mathcal{U}}(\mathfrak{g}')$ for all $D \in \mathcal{D}_r^\omega(M)$, and hence, $\mathcal{D}_r^\omega(M) \subset \widetilde{\mathcal{U}}(\mathfrak{g}')$. To show that $\mathcal{D}_l^\omega(M) \subset \widetilde{\mathcal{U}}(\mathfrak{g})$ we need the following

Lemma 2.3 *For each $x \in M$ let $G'x = \{g'x : g' \in G'\}$ denote the orbit of x. Let $\mathcal{X} = M/G'$ be the set of all orbits $G'x$, $x \in M$, and define π: $M \to \mathcal{X}$ by assigning to each $x \in M$ its orbit $G'x$. Then $(M, \mathcal{X}, \pi, G')$ is a principal G'-bundle.*

Proof. Define γ: $M \times G' \to M \times M$ by $\gamma(x, g') = (x, g'x)$ and $\Gamma = \gamma(M \times G') = \{(x, g'x) : x \in M,\ g \in G\}$. Then since G' acts freely on M, γ is injective. Now suppose that $\lim_{n\to\infty} x_n = x$, and $\lim_{n\to\infty} g'_n x_n = y$ for $x_n, x, y \in M$, $g'_n \in G$. The same argument used in the proof that G' acts freely on M implies that there exists a submatrix $s[x]$ of x such that $s[x] \in G'$. If $s[x_n]$ denotes the corresponding submatrix of x_n then clearly $\lim_{n\to\infty} s[x_n] = s[x]$. So for n sufficiently large we may assume that $s[x_n] \in G'$. Then clearly $\lim_{n\to\infty} s^{-1}[x_n] = s^{-1}[x]$ and $\lim_{n\to\infty} g'_n s[x_n] = s[y]$. By the continuity of the action of G' on M we have $s[y] \in G'$. Write $g'_n = (g'_n s[x_n]) s^{-1}[x_n]$ for sufficiently large n then $\lim_{n\to\infty} g'_n = s[y]s^{-1}[x]$. Set $g' = s[y]s^{-1}[x]$, then $g' \in G'$, $\lim_{n\to\infty} g'_n = g'$, and $\lim_{n\to\infty} g'_n x_n = g'x = y$. Thus Γ is closed in $M \times M$, and γ is a homeomorphism of $M \times G$ onto Γ. Now all the hypotheses of [Va, Theorem 2.9.10] are met, and we can conclude that there exists an analytic structure on $\mathcal{X}$ such that π is an analytic *immersion* (i.e., $(d\pi)_x$ is injective for all $x \in M$). Moreover for each $p \in \mathcal{X}$ we can select an open subset $\mathcal{Y}$ of $\mathcal{X}$ containing p and an analytic diffeomorphism $\xi_{\mathcal{Y}} = \xi$ of $G' \times \mathcal{Y}$ onto $\pi^{-1}(\mathcal{Y})$, such that

$$\xi(h'g', y) = h'\xi(g', y) \qquad (g', h' \in G,\ y \in \mathcal{Y}).$$

That is, in other words, $(M, \mathcal{X}, \pi, G')$ is a principal G'-bundle. ∎

Now let us finish the proof of part (i) of the theorem.

Since G acts analytically and transitively on the analytic manifold M [Va, Lemma 2.9.2] implies that for each $x \in M$ the map r: $g \to xg$ ($g \in G$) is an *analytic submersion* of G onto M (i.e., $(dr)_g$ is surjective for all $g \in G$). It follows that if $\{Y_1, \ldots, Y_d\}$ is a basis of $\mathfrak{g}$ then there exists a basis for analytic vector fields of M of the form $\{\tilde{Y}_1, \ldots, \tilde{Y}_m\}$, where $m = \dim(M)$, and each $\tilde{Y}_i = dR(Y_j)$ for some j, $1 \le j \le d$. It follows that every $D \in \mathcal{D}_l^\omega(M)$ can be expressed as

$$D = \sum_{(\alpha)} k_{(\alpha)} \tilde{Y}^{(\alpha)}, \tag{2.9}$$

where $\{k_{(\alpha)}\}$ is a set of locally finite analytic functions. Since the $\tilde{Y}^{(\alpha)}$ are left-invariant, we have

$$D = D^{l_{g'}} = \sum_{(\alpha)} k_{(\alpha)}^{l_{g'}} \tilde{Y}^{(\alpha)} \qquad (g' \in G'). \tag{2.10}$$

The relations (2.9) and (2.10) imply that all the $k_{(\alpha)}$ are left-invariant. By Lemma 2.3 a basic open set M is diffeomorphic to $G' \times \mathcal{Y}$ where $\mathcal{Y}$ is an open subset of $\mathcal{X}$. A typical point in that basic open set is, for example, of the form $x = (*g'*) \in M$. A function $k_{(\alpha)}$ that is left-invariant will be independent of the n^2 variables in the block containing g', and since we can let g' occupy any block in the matrix x it follows that $k_{(\alpha)}$ must be constant. Hence $D \in \widetilde{\mathcal{U}}(\mathfrak{g})$, and the proof of part (i) is completed.
(ii) The proof of part (ii) depends on the following

LEMMA 2.4 *Let $D \in \mathcal{D}^\omega(M)$ then the following statements hold.*

(i) *$[dL(X), D] = 0$ for all $X \in \mathfrak{g}'$ if and only if $D(L(g') = L(g')D$ for all $g' \in G$.*

(ii) *$[dR(Y), D] = 0$ for all $Y \in \mathfrak{g}$ if and only if $D(R(g)) = R(g)D$ for all $g \in G$.*

PROOF. Since both G' and G are connected the two statements are similar, so we will only prove (i). To prove (i) we first consider $g' = g'(t) = \exp tX$, $X \in \mathfrak{g}'$; then we have

$$L(g')DL((g')^{-1}) = (\exp(dL(tX))D.$$

It follows that

$$L(g')D = DL'(g) \Longleftrightarrow [L(X), D] = 0.$$

Since G' is connected, G' is generated by the image of the exponential map (cf. [Go, Cor. I, p. 6.9]), i.e., an arbitrary element g' of G' can be expressed in the form $g' = \exp(X_1)\exp(X_2)\cdots\exp(X_r)$, $X_i \in \mathfrak{g}'$, it follows from [Na, Prop. 2.10.10] that

$$[dL(X), D] = 0, \forall X \in \mathfrak{g}' \Longleftrightarrow L(g')D = DL(g'), \forall g' \in G'.$$

■

Now part (i) of the theorem and Lemma 2.4 imply that the commutant of $\mathcal{D}_l^\omega(M)$ in $\mathcal{D}^\omega(M)$ is $\mathcal{D}_r^\omega(M)$, and vice-versa. Finally, by definition the centre of $\mathcal{D}_l^\omega(M)$ is the subalgebra of elements of $\mathcal{D}_l^\omega(M)$ which commute with all elements of $\mathcal{D}_l^\omega(M)$. So obviously the centre of $\mathcal{D}_l^\omega(M)$ and similarly the centre of $\mathcal{D}_r^\omega(M)$ coincide with $\mathcal{D}_l^\omega(M) \cap \mathcal{D}_r^\omega(M)$. ■
Now let us consider the Lie transformation group $G' \times G$ acting on the analytic manifold M, where $G' = \mathrm{GL}(n, \mathbb{C})$, $G = \mathrm{Sp}(2k, \mathbb{C})$, $M = \{x \in \mathbb{C}^{n\times 2k} : xs_kx^t = 0, \operatorname{rank}(x) = n\}$, $k \geq n$, and

$$s_k = \begin{bmatrix} 0 & -I_k \\ I_k & 0 \end{bmatrix}$$

with I_k denoting the identity matrix of order k.

Recall that $\mathrm{Sp}(2k,\mathbb{C})$ is the group of all complex $2k \times 2k$ matrices g satisfying $g s_k g^t = s_k$. Then by Witt's theorem on skew-symmetric bilinear form (see, e.g., [Ar] and [TT2, Lemma 1.7]) it follows that G acts analytically and transitively by right multiplication on M. Obviously G' acts freely by left multiplication on M, and G and G' are both connected. Thus we have

THEOREM 2.5 *For $k \geq n$ let $G = \mathrm{Sp}(2k,\mathbb{C})$, $G' = \mathrm{GL}(n,\mathbb{C})$, and $M = \{x \in \mathbb{C}^{n\times 2k} : x s_k x^t = 0,\ \mathrm{rank}(x) = n\}$. Then Theorem 2.2 holds for this pair of Lie transformation groups acting on M.*

Finally, let p, q, and k be positive integers such that $k \geq \max(p,q)$ and consider $G' = \mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$, $G = \{(g, g^{\checkmark}) : g \in \mathrm{GL}(k,\mathbb{C}),\ g^{\checkmark} = (g^{-1})^t\} \approx \mathrm{GL}(k,\mathbb{C})$, and

$$M = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^{(p+q)\times k} : x_1 \in \mathbb{C}^{p\times k},\ x_2 \in \mathbb{C}^{q\times k}, \right.$$
$$\left. x_1 x_2^t = 0,\ \mathrm{rank}(x_1) = p,\ \mathrm{rank}(x_2) = q \right\}.$$

Then by Witt's theorem on quadratic forms, [TT3, Lemma 1.1] and [TT4, Theorem 5.1], it follows that G acts analytically and transitively on M via the action

$$\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, g \right) \longrightarrow \begin{bmatrix} x_1 g \\ x_2 g^{\checkmark} \end{bmatrix}.$$

Obviously G' acts on M freely via the action

$$\left((g_1', g_2'), \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \longrightarrow \begin{bmatrix} g_1' x_1 \\ g_2' x_2 \end{bmatrix}, \qquad g_1' \in \mathrm{GL}(p,\mathbb{C}),\ g_2' \in \mathrm{GL}(q,\mathbb{C}).$$

Moreover, both G and G' are connected. Thus we have

THEOREM 2.6 *Theorem 2.2 holds for the pair of Lie transformation groups G, G' acting on the analytic manifold M described above.*

3 CONCLUSION

In [TT5] we used the duality theorem for commutants in $\mathcal{D}^\omega(M)$ with $G = \mathrm{GL}(k,\mathbb{C})$, $G' = \mathrm{GL}(n,\mathbb{C})$, $M = \{x \in \mathbb{C}^{n\times k} : x \text{ of maximum rank}\}$ to find the Casimir invariants of the infinite-dimensional group $\mathrm{GL}(\infty,\mathbb{C})$. In turn, a set of generators of these Casimir invariants determine the irreducible unitary representations of the group $U(\infty)$. We hope that Theorems 2.2, 2.5, and 2.6 will allow us to find the Casimir invariants of some other infinite-dimensional groups.

ACKNOWLEDGEMENTS: The author was supported by a Carver Research Initiative Grant during the preparation of this paper. He also wishes to thank Ms. Cymie Wehr for her patience in preparing this manuscript.

REFERENCES

[Ar] E. Artin, *Geometric Algebra*, Wiley-Interscience Publication, John Wiley & Sons, New York (1988), reprint of the 1957 original.

[D-TT] O. Debarre and T. Ton-That, *Representations of* $\mathrm{SO}(k,\mathbb{C})$ *on harmonic polynomials on a null cone*, Proc. Amer. Math. Soc. 112 (1991), 31–44.

[Go] R. Godement, *Introduction à la Théorie des Groupes de Lie*, 2 vols., Publ. Math. Univ. Paris VII, Université de Paris VII, Paris, 1982.

[He] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962.

[Hö] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, third edition, North-Holland, Amsterdam, 1990.

[M-Z] D. Montgomery and L. Zippin, *Topological Transformation Groups*, Robert E. Krieger, Huntington, NY, 1974, reprint of the 1955 original.

[Na] R. Narasimhan, *Analysis on Real and Complex Manifolds*, North-Holland Mathematical Library, vol. 35, North-Holland, Amsterdam, 1985, reprint of the 1973 second edition.

[Pa] R. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. 22 (1957).

[TT1] T. Ton-That, *Lie group representations and harmonic polynomials of a matrix variable*, Trans. Amer. Math. Soc. 216 (1976), 1–46.

[TT2] T. Ton-That, *Symplectic Stiefel harmonics and holomorphic representations of symplectic groups*, Trans. Amer. Math. Soc. 232 (1977), 265–277.

[TT3] T. Ton-That, *Sur la décomposition des produits tensoriels des représentations irréductibles de* $\mathrm{GL}(k,\mathbf{C})$, J. Math. Pures Appl. (9) 56 (1977), 251–261.

[TT4] T. Ton-That, *Dual representations and invariant theory*, Representation Theory and Harmonic Analysis (T. Ton-That, et al., eds.), Contemp. Math., vol. 191, American Mathematical Society, Providence, 1995, pp. 205–221.

[TT5] T. Ton-That, *Poincaré-Birkhoff-Witt theorems and generalized Casimir invariants for some infinite-dimensional Lie groups: I*, J. Phys. A 32 (1999), 5975–5991.

[TT-T] T. Ton-That and T.D. Tran, *Poincaré's proof of the so-called Birkhoff-Witt theorem*, Rev. Histoire Math. 5 (1999), 249–284.

[Va] V.S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Graduate Texts in Math., vol. 102, Springer-Verlag, New York, 1984, reprint of the 1974 original, Prentice-Hall, Englewood Cliffs, NJ.

[We] H. Weyl, *The Classical Groups: Their Invariants and Representations*, second edition, Princeton University Press, Princeton, NJ, 1946.

Department of Mathematics
The University of Iowa
Iowa City, IA 52242-1419
USA
tonthat@math.uiowa.edu