INVARIANTS OF QUASI-TRIVIAL TORI AND THE ROST INVARIANT

A.S. MERKURJEV, R. PARIMALA, AND J.-P. TIGNOL

Abstract. For any absolutely simple, simply connected linear algebraic group G over a field F, Rost has defined invariants for torsors under G with values in the Galois cohomology group $H^3\big(F, \mathbb{Q}/\mathbb{Z}(2)\big)$. The aim of this paper is to give an explicit description of these invariants for torsors induced from the center of G, when G is of type A_n or D_n. As an application, we show that the multipliers of unitary similitudes satisfy a relation involving the discriminant algebra.

For an arbitrary field F, we let Fields_F denote the category of all field extensions of F and we consider algebraic groups over F as functors from Fields_F to the category Groups of groups. Similarly, if G is an algebraic group over F, the Galois cohomology set $H^1(L, G)$ is defined for every field extension L/F, and this construction yields a functor $\underline{H}^1(G)$ from Fields_F to the category Sets^* of pointed sets. If G is commutative, the Galois cohomology set $H^1(L, G)$ is a group, and we obtain a functor $H^1(G)$ from Fields_F to Groups.

The cycle modules over F defined in [10] also give rise to a sequence of functors $\mathit{Fields}_F \to \mathit{Groups}$. (In the applications considered in this paper, all the cycle modules are given by cohomology groups.) Let M be a cycle module over F and J be a functor from Fields_F to Groups or to Sets^*. As in [9], we define invariants of dimension d of J in M as natural transformations of functors $J \to M_d$. The group of these invariants is denoted by

$$\mathrm{Inv}^d(J, M) \quad \text{or} \quad \mathrm{Inv}(J, M_d).$$

In particular, if C is a commutative algebraic group and M is a cycle module, one can consider the groups $\mathrm{Inv}^d(H^1(C), M)$ and $\mathrm{Inv}^d(\underline{H}^1(C), M)$, and the forgetful functor $\mathit{Groups} \to \mathit{Sets}^*$ yields an embeddding

$$\mathrm{Inv}^d(H^1(C), M) \hookrightarrow \mathrm{Inv}^d(\underline{H}^1(C), M).$$

In this paper, we shall be mostly interested in the group of invariants $\mathrm{Inv}^3(\underline{H}^1(G), M)$, where G is an absolutely simple, simply connected linear algebraic group over F and M is the cohomological cycle module $H^*[\mathbb{Q}/\mathbb{Z}(-1)]$ (see section 1.3). Let C be the center of G. The natural transformation $\underline{H}^1(C) \to \underline{H}^1(G)$ induced by the inclusion $i\colon C \hookrightarrow G$ yields a map

$$i^*\colon\ \mathrm{Inv}^3(\underline{H}^1(G), M) \to \mathrm{Inv}^3(\underline{H}^1(C), M).$$

It turns out that the image of this map is actually in the subgroup $\mathrm{Inv}^3(H^1(C), M)$ of *group*-invariants, see Corollary 1.8. Our goal is to determine this image in the case where G is of type A_n or D_n. (Groups of trialitarian type D_4 are not considered.) In these cases, we construct in section 1.2 a quasi-trivial torus T and a natural transformation $\varphi\colon T \to H^1(C)$ such that the induced map

$$\varphi^*\colon\ \mathrm{Inv}^d(H^1(C), M) \to \mathrm{Inv}^d(T, M)$$

The first author is partially supported by the N.S.F. The third author gratefully acknowledges the hospitality of the Tata Institute for Fundamental Research, Mumbai, where work for this paper was done. He is partially supported by the National Fund for Scientific Research (Belgium).

is injective (for all d). The group of invariants $\operatorname{Inv}^d(T,M)$ has an easy explicit description, given in section 1.1. The image $i^*\bigl(\operatorname{Inv}^3(\underline{H}^1(G),M)\bigr)$ is determined in terms of this description in section 1.4. As an application, we obtain in section 1.5 some conditions on the image of the map $\overline{G}(F)\to H^1(F,C)$, where $\overline{G}=G/C$ is the adjoint group corresponding to G.

1. Statement of results

Throughout this section, F denotes an arbitrary field. Restrictions on the characteristic of F will be explicitly mentioned when needed.

1.1. **Invariants of quasi-trivial tori.** Let A be an étale F-algebra and let $R_{A/F}(\mathbf{G}_{\mathrm{m}}) = \mathbf{GL}_1(A)$ be the torus of invertible elements in A. Let also M be a cycle module over F. Every element $u\in M_{d-1}(A)$ defines an invariant $\alpha^A(u)\in\operatorname{Inv}^d(R_{A/F}(\mathbf{G}_{\mathrm{m}}),M)$ as follows: for any field extension L/F and $t\in R_{A/F}(\mathbf{G}_{\mathrm{m}})(L)=(A\otimes_F L)^\times$,

$$\alpha^A(u)\colon\ t\mapsto N_{A\otimes L/L}(t\cdot u_{A\otimes L})\in M_d(L),$$

where N is the norm map, and the product is taken for the module structure on M over the Milnor K-ring.

Theorem 1.1. *The map $\alpha^A\colon\ M_{d-1}(A)\to\operatorname{Inv}^d(R_{A/F}(\mathbf{G}_{\mathrm{m}}),M)$ is an isomorphism.*

The proof is given in section 2 below. We shall actually construct an inverse of α^A. (When A is split, this theorem is already proven in [9, Proposition 2.5].)

1.2. **Roots of unity.** For any integer n which is not divisible by the characteristic of F, we let μ_n denote the algebraic group of roots of unity, i.e., the kernel of the n-th power map $\mathbf{G}_{\mathrm{m}}\to\mathbf{G}_{\mathrm{m}}$. If K/F is a separable quadratic field extension, $R_{K/F}(\mu_n)$ is the corestriction of μ_n, and $\mu_{n[K]}$ is the kernel of the norm map

$$N_{K/F}\colon\ R_{K/F}(\mu_n)\to\mu_n.$$

The centers of absolutely simple, simply connected linear algebraic groups (except for trialitarian groups D_4) are all of type μ_n, $\mu_2\times\mu_2$, $R_{K/F}(\mu_n)$ or $\mu_{n[K]}$ (see section 1.4 below).

In order to describe the invariants of $H^1(C)$, for C of the type above, it is useful to construct a quasi-trivial torus T_C and a natural transformation $\varphi\colon\ T_C\to H^1(C)$ such that $\varphi_L\colon\ T_C(L)\to H^1(L,C)$ is surjective for every field extension L/F. We thus get an explicit description of the cohomology group $H^1(L,C)$ and an injective map

$$\varphi^*\colon\ \operatorname{Inv}^d(H^1(C),M)\to\operatorname{Inv}^d(T_C,M)$$

for every cycle module M over F.

1.2.1. *Suppose first $C=\mu_n$.* The cohomology sequence associated with the Kummer exact sequence

$$1\to\mu_n\to\mathbf{G}_{\mathrm{m}}\xrightarrow{n}\mathbf{G}_{\mathrm{m}}\to 1$$

yields for every field extension L/F a map

$$\varphi_L\colon\ L^\times\to H^1(L,\mu_n).$$

This map is surjective since $H^1(L,\mathbf{G}_{\mathrm{m}})=1$ by Hilbert's Theorem 90. Therefore, we may take $T_C=\mathbf{G}_{\mathrm{m}}$ for $C=\mu_n$, and the collection of maps φ_L for $L\in\mathsf{Fields}_F$ yields a natural transformation

$$\varphi\colon\ \mathbf{G}_{\mathrm{m}}\to H^1(\mu_n). \tag{1}$$

Since the kernel of φ_L is $L^{\times n}$, we obtain the (well-known) description of $H^1(L,\mu_n)$ by the isomorphism

$$L^\times/L^{\times n} \xrightarrow{\sim} H^1(L,\mu_n). \tag{2}$$

The image of $x\in L^\times$ under φ_L is denoted by $(x)_n$.

Corollary 1.2. *Let M be a cycle module over F. For every invariant $\iota\colon H^1(\mu_n)\to M_d$, there is a uniquely determined element $u\in M_{d-1}(F)$ satisfying $nu=0$ such that for every field extension L/F and $x\in L^\times$,*

$$\iota_L\big((x)_n\big)=x\cdot u_L.$$

The proof is given in section 3.1 below.

The identification $H^1(L,\mu_n\times\mu_n)=H^1(L,\mu_n)\times H^1(L,\mu_n)$ for $L\in \mathsf{Fields}_F$ induces a canonical isomorphism

$$\operatorname{Inv}^d(H^1(\mu_n\times\mu_n),M)=\operatorname{Inv}^d(H^1(\mu_n),M)\times\operatorname{Inv}^d(H^1(\mu_n),M).$$

Corollary 1.3. *Let M be a cycle module over F. For every invariant $\iota\colon H^1(\mu_n\times\mu_n)\to M_d$, there are uniquely determined elements u_1, $u_2\in M_{d-1}(F)$ satisfying $nu_1=nu_2=0$ such that for every field extension L/F and x_1, $x_2\in L^\times$,*

$$\iota_L\big((x_1)_n,(x_2)_n\big)=x_1\cdot(u_1)_L+x_2\cdot(u_2)_L.$$

1.2.2. *Suppose* $C=R_{K/F}(\mu_n)$. The transfer of the Kummer sequence yields the following exact sequence:

$$1\to R_{K/F}(\mu_n)\to R_{K/F}(\mathbf{G}_\mathrm{m})\xrightarrow{n} R_{K/F}(\mathbf{G}_\mathrm{m})\to 1. \tag{3}$$

Since $H^1\big(L,R_{K/F}(\mathbf{G}_\mathrm{m})\big)=1$ for every $L\in\mathsf{Fields}_F$, we may set $T_C=R_{K/F}(\mathbf{G}_\mathrm{m})$ and let

$$\varphi\colon\ R_{K/F}(\mathbf{G}_\mathrm{m})\to H^1\big(R_{K/F}(\mu_n)\big) \tag{4}$$

be the natural transformation given by the connecting map in the cohomology sequence associated to (3).

For every field extension L/F, we obtain a map $\varphi_L\colon\ (K\otimes_F L)^\times\to H^1\big(L,R_{K/F}(\mu_n)\big)$ which induces an isomorphism

$$(K\otimes_F L)^\times/(K\otimes_F L)^{\times n}\xrightarrow{\sim} H^1\big(L,R_{K/F}(\mu_n)\big).$$

The image of an element $x\in(K\otimes_F L)^\times$ under φ_L is again denoted by $(x)_n$.

Corollary 1.4. *Let M be a cycle module over F. For every invariant $\iota\colon\ H^1\big(R_{K/F}(\mu_n)\big)\to M_d$ there is a uniquely determined element $u\in M_{d-1}(K)$ satisfying $nu=0$ such that for every field extension L/F and every $x\in(K\otimes_F L)^\times$,*

$$\iota_L\big((x)_n\big)=N_{K\otimes L/L}(x\cdot u_{K\otimes L}).$$

The proof is given in section 3.2 below.

1.2.3. *Suppose finally* $C=\mu_{n[K]}$. Let $f\colon\ \mathbf{G}_\mathrm{m}\times R_{K/F}(\mathbf{G}_\mathrm{m})\to\mathbf{G}_\mathrm{m}$ be defined as follows: for any F-algebra L/F,

$$f(x,y)=x^nN_{K\otimes L/L}(y)^{-1}$$

for $x\in L^\times$ and $y\in(K\otimes L)^\times$. Let P be the kernel of f. We then have two exact sequences

$$1\to P\to\mathbf{G}_\mathrm{m}\times R_{K/F}(\mathbf{G}_\mathrm{m})\xrightarrow{f}\mathbf{G}_\mathrm{m}\to 1$$

and

$$1 \to \mu_{n[K]} \to R_{K/F}(\mathbf{G}_{\mathrm{m}}) \xrightarrow{g} P \to 1 \tag{5}$$

where $g = (N_{K/F}, n)$ is the cartesian product of the norm map and the n-th power map. Since $H^1\big(L, R_{K/F}(\mathbf{G}_{\mathrm{m}})\big) = 1$ for every field extension L/F, (5) yields an exact sequence

$$(K \otimes_F L)^\times \xrightarrow{g_L} P(L) \xrightarrow{\delta_L} H^1(L, \mu_{n[K]}) \to 1,$$

hence an isomorphism

$$\frac{P(L)}{g_L\big((K\otimes_F L)^\times\big)} = \frac{\{(x,y) \in L^\times \times (K\otimes L)^\times \mid x^n = N_{K/F}(y)\}}{\{(N_{K\otimes L/L}(z), z^n) \mid z \in (K\otimes L)^\times\}} \xrightarrow{\sim} H^1(L, \mu_{n[K]})$$

(see [5, (30.13)]). We let $(x,y)_n$ denote the image of $(x,y) \in P(L)$ under this isomorphism.

For our purposes, it will be convenient to use slightly different descriptions, depending on the parity of n.

If n is odd, $n = 2m+1$, let $g_1\colon\ R_{K/F}(\mathbf{G}_{\mathrm{m}}) \to R_{K/F}(\mathbf{G}_{\mathrm{m}})$ be the map defined as follows: for $L \in \mathsf{Fields}_F$ and $t \in (K \otimes_F L)^\times$,

$$(g_1)_L(t) = t^n N_{K\otimes L/L}(t)^{-m}.$$

If $t^n = N_{K\otimes L/L}(t)^m$, then taking the norm of each side and dividing by $N_{K\otimes L/L}(t)^{2m}$ we obtain $N_{K\otimes L/L}(t) = 1$. Therefore, the kernel of g_1 is $\mu_{n[K]}$ and we have an exact sequence

$$1 \to \mu_{n[K]} \to R_{K/F}(\mathbf{G}_{\mathrm{m}}) \xrightarrow{g_1} R_{K/F}(\mathbf{G}_{\mathrm{m}}) \to 1. \tag{6}$$

Since $H^1\big(L, R_{K/F}(\mathbf{G}_{\mathrm{m}})\big) = 1$ for every $L \in \mathsf{Fields}_F$, we may set $T_C = R_{K/F}(\mathbf{G}_{\mathrm{m}})$ and let

$$\varphi\colon\ R_{K/F}(\mathbf{G}_{\mathrm{m}}) \to H^1(\mu_{n[K]})$$

be the natural transformation given by the connecting map in the cohomology sequence associated to (6). Note that the exact sequences (5) and (6) can be combined in the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mu_{n[K]} \arrow[r] \arrow[d, equal] & R_{K/F}(\mathbf{G}_{\mathrm{m}}) \arrow[r, "g_1"] \arrow[d, equal] & R_{K/F}(\mathbf{G}_{\mathrm{m}}) \arrow[r] \arrow[d, "h_1"] & 1 \\
1 \arrow[r] & \mu_{n[K]} \arrow[r] & R_{K/F}(\mathbf{G}_{\mathrm{m}}) \arrow[r, "g"] & P \arrow[r] & 1
\end{tikzcd}$$

where the map h_1 is defined by

$$(h_1)_L(t) = (N_{K\otimes L/L}(t), tN_{K\otimes L/L}(t)^m)$$

for $L \in \mathsf{Fields}_F$ and $t \in (K\otimes_F L)^\times$. This map is in fact an isomorphism, with inverse defined by

$$(h_1^{-1})_L(x,y) = yx^{-m}$$

for $L \in \mathsf{Fields}_F$ and $x \in L^\times$, $y \in (K\otimes_F L)^\times$ such that $N_{K\otimes L/L}(y) = x^n$. Thus, for $t \in (K\otimes_F L)^\times$,

$$\varphi_L(t) = (N_{K\otimes L/L}(t), tN_{K\otimes L/L}(t)^m)_n \in H^1(L, \mu_{n[K]}). \tag{7}$$

Let $^-$ denote the nontrivial automorphism of K/F. Abusing notation, we also denote by $^-$ the induced automorphism on $M(K)$ for any cycle module M over F.

Corollary 1.5. *Suppose $n = 2m + 1$. Let M be a cycle module over F. For every invariant $\iota\colon H^1(\mu_{n[K]}) \to M_d$, there is a uniquely determined element $u \in M_{d-1}(K)$ satisfying $nu = 0$ and $u + \overline{u} = 0$ such that for every field extension L/F and every $x \in L^\times$, $y \in (K \otimes_F L)^\times$ with $N_{K\otimes L/L}(y) = x^n$,*

$$\iota_L\big((x, y)_n\big) = N_{K\otimes L/L}(y \cdot u_{K\otimes L}) - mx \cdot N_{K/F}(u)_L.$$

The proof is given in section 3.3 below.

Suppose next n is even, $n = 2m$. Define a map $g_2\colon \mathbf{G}_\mathrm{m} \times R_{K/F}(\mathbf{G}_\mathrm{m}) \to \mathbf{G}_\mathrm{m} \times R_{K/F}(\mathbf{G}_\mathrm{m})$ as follows: for $L \in \mathsf{Fields}_F$ and $x \in L^\times$, $y \in (K \otimes_F L)^\times$,

$$g_2(x, y) = (N_{K\otimes L/L}(y), xy^m).$$

It is easily verified that the following sequence is exact (see [1, Lemma 2.9]):

$$1 \to \mu_{n[K]} \to \mathbf{G}_\mathrm{m} \times R_{K/F}(\mathbf{G}_\mathrm{m}) \xrightarrow{g_2} \mathbf{G}_\mathrm{m} \times R_{K/F}(\mathbf{G}_\mathrm{m}) \to 1, \tag{8}$$

where the map originating in $\mu_{n[K]}$ is the product of the m-th power map to $\mathbf{G}_\mathrm{m}$ and the inclusion in $R_{K/F}(\mathbf{G}_\mathrm{m})$. Since $H^1\big(L, \mathbf{G}_\mathrm{m} \times R_{K/F}(\mathbf{G}_\mathrm{m})\big) = 1$ for every $L \in \mathsf{Fields}_F$, we may set $T_C = \mathbf{G}_\mathrm{m} \times R_{K/F}(\mathbf{G}_\mathrm{m})$ and let

$$\varphi\colon\ \mathbf{G}_\mathrm{m} \times R_{K/F}(\mathbf{G}_\mathrm{m}) \to H^1(\mu_{n[K]})$$

be the natural transformation given by the connecting map in the cohomology sequence associated to (8). Note that the exact sequences (5) and (8) can be combined in the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mu_{n[K]} \arrow[r] \arrow[d, equal] & \mathbf{G}_\mathrm{m} \times R_{K/F}(\mathbf{G}_\mathrm{m}) \arrow[r, "g_2"] \arrow[d, "\pi_2"] & \mathbf{G}_\mathrm{m} \times R_{K/F}(\mathbf{G}_\mathrm{m}) \arrow[r] \arrow[d, "h_2"] & 1 \\
1 \arrow[r] & \mu_{n[K]} \arrow[r] & R_{K/F}(\mathbf{G}_\mathrm{m}) \arrow[r, "g"] & P \arrow[r] & 1
\end{tikzcd}$$

where π_2 is the projection on the second factor and h_2 is defined by

$$(h_2)_L(x, z) = (x, x^m z\overline{z}^{-1})$$

for $L \in \mathsf{Fields}_F$ and $x \in L^\times$, $z \in (K \otimes_F L)^\times$. Therefore, for $x \in L^\times$ and $z \in (K \otimes_F L)^\times$,

$$\varphi_L(x, z) = (x, x^m z\overline{z}^{-1})_n \in H^1(L, \mu_{n[K]}).$$

Corollary 1.6. *Suppose $n = 2m$. Let M be a cycle module over F. For every invariant $\iota\colon H^1(\mu_{n[K]}) \to M_d$, there are uniquely determined elements $u \in M_{d-1}(F)$, $v \in M_{d-1}(K)$ satisfying $u_K + mv = 0$ and $N_{K/F}(v) = 0$, such that for every field extension L/F and every $x \in L^\times$, $y \in (K \otimes_F L)^\times$ with $N_{K\otimes L/L}(y) = x^n$,*

$$\iota_L\big((x, y)_n\big) = x \cdot u_L + N_{K\otimes L/L}(z \cdot v_{K\otimes L}),$$

where $z \in (K \otimes_F L)^\times$ is such that $y = x^m z\overline{z}^{-1}$.

The proof is given in section 3.4 below.

1.3. **The cycle module $H^*[\mathbb{Q}/\mathbb{Z}(-1)]$.** Let F_sep be a separable closure of F and let $\Gamma = \mathrm{Gal}(F_\mathrm{sep}/F)$ be the absolute Galois group. For every integer n prime to the characteristic of F, let

$$r_n\colon\ \Gamma \to \mathrm{Aut}\,\mu_n(F_\mathrm{sep})$$

be the action of Γ on $\mu_n(F_{\mathrm{sep}})$. Let $s_n\colon \Gamma \to \operatorname{Aut}\mu_n(F_{\mathrm{sep}})$ be the homomorphism such that $s_n(\gamma) = r_n(\gamma)^{-1}$ for all $\gamma \in \Gamma$, and let $\mu_n^{\otimes -1}(F_{\mathrm{sep}})$ be the corresponding Galois module. For every integer $i \geq 0$, we let

$$\mathbb{Q}/\mathbb{Z}(i-1) = \varinjlim \mu_n^{\otimes(i-1)}(F_{\mathrm{sep}}),$$

where the limit runs over the integers n prime to the characteristic of F. The cohomological cycle module $H^*[\mathbb{Q}/\mathbb{Z}(-1)]$ is defined by

$$H^d[\mathbb{Q}/\mathbb{Z}(-1)](L) = H^d\big(L, \mathbb{Q}/\mathbb{Z}(d-1)\big)$$

for any field extension L/F (see [10, Remark 1.11]). (This cycle module is denoted by $H^*[\mu^{\otimes -1}]$ in [9].)

As part of the cycle module structure, $\bigoplus_{d\geq 0} H^d[\mathbb{Q}/\mathbb{Z}(-1)](L)$ is a module over the Milnor ring K_*L. For later use, we give an explicit description of the multiplication

$$L^\times \times H^d\big(L, \mathbb{Q}/\mathbb{Z}(d-1)\big) \to H^{d+1}\big(L, \mathbb{Q}/\mathbb{Z}(d)\big).$$

We have already observed the isomorphism $L^\times/L^{\times n} \xrightarrow{\sim} H^1(L, \mu_n)$, see (2) above. If m is a multiple of n, there is a canonical (inclusion) map $j\colon \mu_n \to \mu_m$, and the induced maps in cohomology make the following diagram commute for all $d \geq 0$ and $x \in L^\times$:

$$\begin{array}{ccc}
H^d(L, \mu_n^{\otimes(d-1)}) & \xrightarrow{j_*} & H^d(L, \mu_m^{\otimes(d-1)}) \\
{\scriptstyle (x)_n\cup\bullet}\big\downarrow & & \big\downarrow{\scriptstyle (x)_m\cup\bullet} \\
H^{d+1}(L, \mu_n^{\otimes d}) & \xrightarrow{j_*} & H^{d+1}(L, \mu_m^{\otimes d})
\end{array}$$

where the vertical maps are the cup-products with $(x)_n$ and with $(x)_m$ respectively. (In checking the commutativity of this diagram, one has to keep in mind that for $a \in \mu_m(L)$ and $b \in \mu_n^{\otimes(d-1)}(L)$, $a \otimes j(b) = j(a^{m/n} \otimes b)$.) Therefore, if $\xi \in H^d\big(L, \mathbb{Q}/\mathbb{Z}(d-1)\big) = \varinjlim H^d(L, \mu_n^{\otimes(d-1)})$ is represented by an element $\xi_n \in H^d(L, \mu_n^{\otimes(d-1)})$ for some n, we may define $x \cdot \xi \in H^{d+1}\big(L, \mathbb{Q}/\mathbb{Z}(d)\big)$ as the element represented by $(x)_n \cup \xi_n \in H^{d+1}(L, \mu_n^{\otimes d})$; the result does not depend on the choice of n.

1.4. **Rost invariants.** Let G be an absolutely simple, simply connected group over F. The group

$$\operatorname{Inv}^3(\underline{H}^1(G), H^*[\mathbb{Q}/\mathbb{Z}(-1)]) = \operatorname{Inv}\big(\underline{H}^1(G), H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)\big)$$

has been investigated by Rost, who showed that it is cyclic with a distinguished generator, see Proposition (31.40) in [5]. (Note that this group is denoted $\operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)$ in [5].) The distinguished generator is called the *Rost invariant* of $\underline{H}^1(G)$. It has been explicitly determined in a few cases only. In particular, its explicit description is not known for groups of outer type A_{n-1} with n even, nor for groups of type D_n. It is known however that the Rost invariant of a torsor $X \in H^1(F, G)$ generates the kernel of the scalar extension map $H^3\big(F, \mathbb{Q}/\mathbb{Z}(2)\big) \to H^3\big(F(X), \mathbb{Q}/\mathbb{Z}(2)\big)$, by [2, Theorem B.11]. This property is sufficient to determine the group $\operatorname{Inv}\big(\underline{H}^1(G), H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)\big)$ in certain cases.[1]

Let C be the center of G. The inclusion $i\colon C \hookrightarrow G$ yields a canonical map

$$i^*\colon \operatorname{Inv}^3(\underline{H}^1(G), H^*[\mathbb{Q}/\mathbb{Z}(-1)]) \to \operatorname{Inv}^3(\underline{H}^1(C), H^*[\mathbb{Q}/\mathbb{Z}(-1)]).$$

[1] A case in point is $G = \mathbf{SL}(D)$, where D is a central division algebra. The invariant described in [5, p. 437] generates the group of invariants, but it is not known whether it is the Rost invariant.

Our goal is to describe the image $I(G)$ of the map i^* above, i.e.,

$$I(G) = i^*\big(\mathrm{Inv}\big(\underline{H}^1(G), H^3(\mathbb{Q}/\mathbb{Z}(2))\big)\big) \subset \mathrm{Inv}\big(\underline{H}^1(C), H^3(\mathbb{Q}/\mathbb{Z}(2))\big).$$

The first step is to prove that $I(G)$ is in the subgroup of group-invariants

$$\mathrm{Inv}\big(H^1(C), H^3(\mathbb{Q}/\mathbb{Z}(2))\big) \subset \mathrm{Inv}\big(\underline{H}^1(C), H^3(\mathbb{Q}/\mathbb{Z}(2))\big).$$

As a preparation, we show how the Rost invariant behaves under twisting. Let ω be a 1-cocycle of the absolute Galois group Γ in $G(F_{\mathrm{sep}})$, and let G_ω be the group obtained by twisting G by ω (or, more precisely, by the cocycle of inner automorphisms $\mathrm{Int}(\omega)$). Multiplication on the right by ω defines a canonical bijection

$$\theta_\omega\colon\ H^1(F, G_\omega) \to H^1(F, G)$$

which carries the distinguished element of $H^1(F, G_\omega)$ to the class $[\omega]$ of ω, see [5, (23.8)]. Let ρ, ρ_ω be the Rost invariants of $\underline{H}^1(G)$ and $\underline{H}^1(G_\omega)$ respectively.

Proposition 1.7. *For $L \in \mathsf{Fields}_F$, let $t_{\omega,L}\colon\ H^3\big(L, \mathbb{Q}/\mathbb{Z}(2)\big) \to H^3\big(L, \mathbb{Q}/\mathbb{Z}(2)\big)$ be the translation by $\rho_L([\omega_L])$,*

$$t_{\omega,L}(\zeta) = \zeta + \rho_L([\omega_L]) \qquad \textit{for } \zeta \in H^3\big(L, \mathbb{Q}/\mathbb{Z}(2)\big).$$

The following diagram commutes:

$$\begin{array}{ccc} H^1(L, G_\omega) & \xrightarrow{\theta_{\omega,L}} & H^1(L,G) \\ {\scriptstyle \rho_{\omega,L}}\big\downarrow & & \big\downarrow{\scriptstyle \rho_L} \\ H^3\big(L, \mathbb{Q}/\mathbb{Z}(2)\big) & \xrightarrow{t_{\omega,L}} & H^3\big(L, \mathbb{Q}/\mathbb{Z}(2)\big). \end{array}$$

Proof. This is proved by Gille in [4, Lemme 7, p. 76]. We give a different proof, which relies only on the following "additivity" property:

$$\mathrm{Inv}\big(H^1(G\times G), H^3(\mathbb{Q}/\mathbb{Z}(2))\big) = \mathrm{Inv}\big(H^1(G), H^3(\mathbb{Q}/\mathbb{Z}(2))\big) \times \mathrm{Inv}\big(H^1(G), H^3(\mathbb{Q}/\mathbb{Z}(2))\big) \tag{9}$$

(see [5, (31.38)]). We start with the following easy observation: if ω, ω' are cohomologous 1-cocycles, then every $f \in G(F_{\mathrm{sep}})$ such that $\omega'_\gamma = f\omega_\gamma{}^\gamma f^{-1}$ for every $\gamma \in \Gamma$ defines an isomorphism $\alpha\colon\ G_\omega \to G_{\omega'}$. The induced natural transformation $H^1(G_\omega) \to H^1(G_{\omega'})$ makes the following diagram commute for every $L \in \mathsf{Fields}_F$:

$$\begin{array}{ccc} H^1(L, G_\omega) & \xrightarrow{\theta_{\omega,L}} & H^1(L,G) \\ {\scriptstyle \alpha_*}\big\downarrow & & \big\| \\ H^1(L, G_{\omega'}) & \xrightarrow{\theta_{\omega',L}} & H^1(L,G). \end{array}$$

Moreover, since α is an isomorphism, the integer n_α attached to this map as on [5, p. 436] is 1, hence the following diagram commutes:

$$\begin{array}{ccc} H^1(L, G_\omega) & \xrightarrow{\rho_{\omega,L}} & H^3\big(L, \mathbb{Q}/\mathbb{Z}(2)\big) \\ {\scriptstyle \alpha_*}\big\downarrow & & \big\| \\ H^1(L, G_{\omega'}) & \xrightarrow{\rho_{\omega',L}} & H^3\big(L, \mathbb{Q}/\mathbb{Z}(2)\big). \end{array}$$

It follows from this observation that

$$\rho_{\omega,L} \circ \theta_{\omega,L}^{-1} = \rho_{\omega',L} \circ \theta_{\omega',L}^{-1},$$

hence this map depends only on the cohomology class of ω. We may then define an invariant Ψ of $H^1(G \times G) = H^1(G) \times H^1(G)$ as follows: for $L \in \mathit{Fields}_F$ and ξ, $\eta \in H^1(L, G)$, we choose a cocycle ω representing ξ and set

$$\Psi(\xi, \eta) = \rho_{\omega,L} \circ \theta_{\omega,L}^{-1}(\eta).$$

Clearly, this is an invariant of $H^1(G \times G)$. By the additivity property (9), we have

$$\Psi(\xi, \eta) = \Psi(\xi, 1) + \Psi(1, \eta),$$

hence

$$\rho_{\omega,L} \circ \theta_{\omega,L}^{-1}(\eta) = \rho_{\omega,L} \circ \theta_{\omega,L}^{-1}(1) + \rho_L(\eta). \tag{10}$$

If $\xi = \eta$, we have $\theta_{\omega,L}^{-1}(\eta) = 1$, hence (10) yields

$$\rho_{\omega,L} \circ \theta_{\omega,L}^{-1}(1) = -\rho_L(\xi).$$

Substituting in (10), we obtain

$$\rho_{\omega,L} \circ \theta_{\omega,L}^{-1}(\eta) = \rho_L(\eta) - \rho_L(\xi) = t_{\omega,L}^{-1} \circ \rho_L(\eta),$$

proving the proposition. □

Corollary 1.8. $I(G) \subset \mathrm{Inv}\big(H^1(C), H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)\big)$.

Proof. We have to show that for every $\iota \in \mathrm{Inv}^3(\underline{H}^1(G), H^*[\mathbb{Q}/\mathbb{Z}(-1)])$ and every field extension L/F, the composition

$$H^1(L, C) \xrightarrow{i_*} H^1(L, G) \xrightarrow{\iota_L} H^3\big(L, \mathbb{Q}/\mathbb{Z}(2)\big)$$

is a group homomorphism (even though $H^1(L, G)$ is not a group). Clearly, it suffices to prove this for $\iota = \rho$, the Rost invariant. If ω is a 1-cocycle of the absolute Galois group of L in $C(L_{\mathrm{sep}})$, then $(G_L)_{i_*(\omega)} = G_L$, and Proposition 1.7 shows that

$$\rho_L \circ \theta_{i_*(\omega),L} = t_{i_*(\omega),L} \circ \rho_L.$$

For $[\omega'] \in H^1(L, C)$, we have $\theta_{i_*(\omega),L}\big(i_*([\omega'])\big) = i_*([\omega])i_*([\omega'])$, hence the last equation yields

$$\rho_L \circ i_*([\omega][\omega']) = \rho_L \circ i_*([\omega]) + \rho_L \circ i_*([\omega']).$$

□

The description of $I(G)$ will be obtained by a case-by-case analysis, using the explicit determination of dimension 3 invariants of $H^1(C)$ in Corollaries 1.2, 1.3, 1.4, 1.5 and 1.6.

1.4.1. *Inner type A_{n-1}.* Suppose the characteristic of F does not divide n, and let $G = \mathbf{SL}(A)$, where A is a central simple F-algebra of degree n. Then $C = \mu_n$ and the invariants of $H^1(C)$ are described in Corollary 1.2. For every field extension L/F, we denote by A_L the central simple L-algebra $A \otimes_F L$, and we let $[A_L]$ be the Brauer class of A_L, viewed as an element in $H^2\big(L, \mathbb{Q}/\mathbb{Z}(1)\big)$.

Theorem 1.9. *The group $I\big(\mathbf{SL}(A)\big)$ is generated by the invariant $\iota\colon H^1(\mu_n) \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ such that for every field extension L/F and for every $x \in L^\times$,*

$$\iota_L\big((x)_n\big) = x \cdot [A_L].$$

This follows from the description of a generating invariant for $\mathrm{Inv}\big(\underline{H}^1\big(\mathbf{SL}(A)\big), H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)\big)$ in [5, p. 437]; see also [12].

1.4.2. *Outer type A_{n-1}.* Suppose the characteristic of F does not divide n, and let $G = \mathbf{SU}(B,\tau)$, where (B,τ) is a central simple algebra of degree n over a separable quadratic extension K/F and τ is an involution of B whose restriction to K is the nontrivial automorphism $^-$ of K/F. The center of G is $C = \mu_{n[K]}$, and the invariants of $H^1(C)$ are described in Corollary 1.5 if n is odd, in Corollary 1.6 if n is even.

Theorem 1.10. *If n is odd, the group $I\big(\mathbf{SU}(B,\tau)\big)$ is generated by the invariant $\iota\colon H^1(\mu_{n[K]}) \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ such that for every field extension L/F and every $x \in L^\times$, $y \in (K \otimes_F L)^\times$ with $N_{K\otimes L/L}(y) = x^n$,*

$$\iota_L\big((x,y)_n\big) = N_{K\otimes L/L}(y \cdot [B_{K\otimes L}]).$$

This follows from the description of a generator of $\mathrm{Inv}\big(\underline{H}^1\big(\mathbf{SU}(B,\tau)\big), H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)\big)$ in [5, (31.45)].

If n is even, a discriminant algebra $D(B,\tau)$ is defined in [5, §10]. It is a central simple F-algebra, hence it defines a Brauer class $[D(B,\tau)] \in H^2\big(F, \mathbb{Q}/\mathbb{Z}(1)\big)$.

Theorem 1.11. *If n is even, $I\big(\mathbf{SU}(B,\tau)\big)$ is generated by the invariant $\iota\colon H^1(\mu_{n[K]}) \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ such that for every field extension L/F and every $x \in L^\times$, $y \in (K \otimes_F L)^\times$ with $N_{K\otimes L/L}(y) = x^n$,*

$$\iota_L\big((x,y)_n\big) = x \cdot [D(B,\tau)_L] + N_{K\otimes L/L}(z \cdot [B_{K\otimes L}]),$$

where $z \in (K \otimes_F L)^\times$ is such that $yx^{-n/2} = z\overline{z}^{-1}$.

The proof is given in section 4.1.

1.4.3. *Type B_n.* Suppose the characteristic of F is not 2, and let $G = \mathbf{Spin}(q)$, where q is a $(2n+1)$-dimensional quadratic form over F (with $n \geq 2$). The center C of G is μ_2.

Theorem 1.12. $I\big(\mathbf{Spin}(q)\big) = 0$.

Indeed, it is shown in [5, p. 437] that the Rost invariant of a class in $H^1\big(F, \mathbf{Spin}(q)\big)$ only depends on its image in $H^1\big(F, \mathbf{O}^+(q)\big)$.

1.4.4. *Type C_n.* Suppose the characteristic of F is not 2, and let $G = \mathbf{Sp}(A,\sigma)$, where A is a central simple F-algebra of degree $2n$ (with $n \geq 2$) and σ is a symplectic involution on A. The center C of G is μ_2.

Theorem 1.13. *If n is even, $I\big(\mathbf{Sp}(A,\sigma)\big) = 0$. If n is odd, $I\big(\mathbf{Sp}(A,\sigma)\big)$ is generated by the invariant $\iota\colon H^1(\mu_2) \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ defined as follows: for every field extension L/F and $x \in L^\times$,*

$$\iota_L\big((x)_2\big) = x \cdot [A_L].$$

This follows from the explicit description of the Rost invariant given in [5, p. 440].

1.4.5. *Inner type D_n.* Suppose the characteristic of F is not 2, and let $G = \mathbf{Spin}(A,\sigma)$, where A is a central simple F-algebra of degree $2n$ (with $n \geq 3$) and σ is an orthogonal involution on A. Assume the discriminant of σ is trivial, hence the center of the Clifford algebra $C(A,\sigma)$ is the split étale algebra $F \times F$. Therefore, $C(A,\sigma)$ decomposes as

$$C(A,\sigma) \simeq C^+(A,\sigma) \times C^-(A,\sigma) \tag{11}$$

for some central simple F-algebras $C^+(A,\sigma)$, $C^-(A,\sigma)$. These algebras satisfy

$$2[C^+(A,\sigma)] = 2[C^-(A,\sigma)] = [A] \text{ and } [C^+(A,\sigma)] + [C^-(A,\sigma)] = 0 \text{ if } n \text{ is odd,}$$
$$2[C^+(A,\sigma)] = 2[C^-(A,\sigma)] = 0 \text{ and } [C^+(A,\sigma)] + [C^-(A,\sigma)] = [A] \text{ if } n \text{ is even.}$$

Let C be the center of G. This group embeds in the center of $C(A,\sigma)$, and we have

$$C \simeq \begin{cases} \mu_4 & \text{if } n \text{ is odd,} \\ \mu_2 \times \mu_2 & \text{if } n \text{ is even.} \end{cases}$$

For the following statement, we fix such an isomorphism. If n is even, this amounts to fixing an isomorphism (11); we then have $H^1(C) = H^1(\mu_2) \times H^1(\mu_2)$. For every field extension L/F, the elements in $H^1(L,C)$ may then be represented as pairs $\big((x^+)_2, (x^-)_2\big)$, where x^+, $x^- \in L^\times$.

Theorem 1.14. *If n is odd, the group $I\big(\mathbf{Spin}(A,\sigma)\big)$ is generated by the invariant $\iota\colon H^1(\mu_4) \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ such that for every field extension L/F and every $x \in L^\times$,*

$$\iota_L\big((x)_4\big) = x \cdot [C^+(A,\sigma)_L].$$

If n is even, the group $I\big(\mathbf{Spin}(A,\sigma)\big)$ is generated by the invariant $\iota\colon H^1(\mu_2) \times H^1(\mu_2) \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ such that for every field extension L/F and every x^+, $x^- \in L^\times$,

$$\iota_L\big((x^+)_2, (x^-)_2\big) = \begin{cases} x^+ \cdot [C^+(A,\sigma)_L] + x^- \cdot [C^-(A,\sigma)_L] & \text{if } n \equiv 2 \bmod 4, \\ x^+ \cdot [C^-(A,\sigma)_L] + x^- \cdot [C^+(A,\sigma)_L] & \text{if } n \equiv 0 \bmod 4. \end{cases}$$

The proof is given in section 4.2.

1.4.6. *Outer type D_n.* Suppose the characteristic of F is not 2, and let $G = \mathbf{Spin}(A,\sigma)$ where A is a central simple F-algebra of degree $2n$ (with $n \geq 3$) and σ is an orthogonal involution on A. Assume the discriminant of σ is not trivial, hence the center of the Clifford algebra $C(A,\sigma)$ is a quadratic field extension Z of F. The center C of $\mathbf{Spin}(A,\sigma)$ embeds in $R_{Z/F}(\mathbf{G}_{\mathrm{m}})$ and we have

$$C = \begin{cases} \mu_{4[Z]} & \text{if } n \text{ is odd,} \\ R_{Z/F}(\mu_2) & \text{if } n \text{ is even.} \end{cases}$$

Theorem 1.15. *If n is odd, $I\big(\mathbf{Spin}(A,\sigma)\big)$ is generated by the invariant $\iota\colon H^1(\mu_{4[Z]}) \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ such that for every field extension L/F and every $x \in L^\times$, $y \in (Z \otimes_F L)^\times$ with $N_{Z\otimes L/L}(y) = x^4$,*

$$\iota_L\big((x,y)_4\big) = x \cdot [A_L] + N_{Z\otimes L/L}(z \cdot [C(A,\sigma)_{Z\otimes L}])$$

where $z \in (Z \otimes_F L)^\times$ is such that $yx^{-2} = z\overline{z}^{-1}$.

If n is even, the group $I\big(\mathbf{Spin}(A,\sigma)\big)$ is generated by the invariant $\iota\colon H^1\big(R_{Z/F}(\mu_2)\big) \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ such that for every field extension L/F and every $x \in (Z \otimes_F L)^\times$,

$$\iota_L\big((x)_2\big) = \begin{cases} N_{Z\otimes L/L}(x \cdot [C(A,\sigma)]) & \text{if } n \equiv 2 \bmod 4, \\ N_{Z\otimes L/L}(x \cdot [\overline{C(A,\sigma)}]) & \text{if } n \equiv 0 \bmod 4. \end{cases}$$

The proof is given in section 4.3.

Remark. In all the cases discussed above, it turns out that the Brauer classes which appear in the formulas for the invariants are Tits algebras associated with representations of the group (see [5, §27]). In fact, these formulas can be rewritten in a more compact form by making use of the Tits class of the group, see section 5.

1.5. **Applications.** Let $\overline{G} = G/C$ be the adjoint group corresponding to the simply connected absolutely simple group G over F. The cohomology sequence associated to

$$1 \to C \to G \to \overline{G} \to 1$$

yields an exact sequence

$$\overline{G}(F) \xrightarrow{\partial} H^1(F,C) \xrightarrow{i_*} H^1(F,G).$$

Proposition 1.16. *For every invariant $\iota \in I(G)$, the map ι_F vanishes on $\partial\big(\overline{G}(F)\big)$.*

Proof. This is clear, since $\iota_F \circ \partial$ factors through $i_* \circ \partial = 0$. □

For groups of type A_n, C_n (with n odd) or D_n, we thus get some restrictions on the image of ∂, which may be regarded as a kind of generalized multiplier map, see [5, §31]. In the rest of this section, we make this result explicit for the various types of groups.

1.5.1. *Inner type A_{n-1}.* With the notation of section 1.4.1, we have $G = \mathbf{SL}(A)$ and $\overline{G} = \mathbf{PGL}(A)$. Let $\mathrm{PGL}(A) = \mathbf{PGL}(A)(F)$ denote the group of rational points. The map

$$\partial\colon\ \mathrm{PGL}(A) \to F^\times/F^{\times n}$$

carries $g \cdot F^\times$ to $\big(\mathrm{Nrd}_A(g)\big)_n$, for $g \in A^\times$ (see [5, p. 424]), and Proposition 1.16 takes the following form:

Corollary 1.17. *For all $g \in \mathrm{GL}(A) = A^\times$,*

$$\mathrm{Nrd}_A(g)\cdot[A] = 0 \qquad \textit{in } H^3\big(F,\mathbb{Q}/\mathbb{Z}(2)\big).$$

This property is known, see [6].

1.5.2. *Outer type A_{n-1}.* With the notation of section 1.4.2, we have $G = \mathbf{SU}(B,\tau)$ and $\overline{G} = \mathbf{PGU}(B,\tau)$. For every similitude $g \in \mathrm{GU}(B,\tau)$ $(= \mathbf{GU}(B,\tau)(F))$, let $\mu(g) = \tau(g)g \in F^\times$ be the multiplier of g. The map

$$\partial\colon\ \mathrm{PGU}(B,\tau) \to H^1(F,\mu_{n[K]})$$

carries $g \cdot K^\times$ to $\big(\mu(g),\mathrm{Nrd}_B(g)\big)_n$ for $g \in \mathrm{GU}(B,\tau)$, see [5, p. 424].

If n is odd, Proposition 1.16 applied to the invariant of Theorem 1.10 yields

$$N_{K/F}(\mathrm{Nrd}_B(g)\cdot[B]) = 0 \qquad \text{for all } g \in \mathrm{GU}(B,\tau).$$

This also follows from Corollary 1.17.

Suppose n is even, $n = 2m$. Taking the reduced norm of each side of the equation $\mu(g) = \tau(g)g$, we see that

$$\mu(g)^{2m} = N_{K/F}\big(\mathrm{Nrd}_B(g)\big),$$

hence there exists $z \in K^\times$ such that $z\overline{z}^{-1} = \mu(g)^{-m}\,\mathrm{Nrd}_B(g)$. Applying Proposition 1.16 to the invariant of Theorem 1.11, we obtain:

Corollary 1.18. *For $g \in \mathrm{GU}(B,\tau)$,*

$$\mu(g)\cdot[D(B,\tau)] + N_{K/F}(z\cdot[B]) = 0 \qquad \textit{in } H^3\big(F,\mathbb{Q}/\mathbb{Z}(2)\big).$$

Example 1.19. Let (A,σ) be a central simple algebra with orthogonal involution of degree $n = 2m$. Let $\operatorname{disc}\sigma = dF^{\times 2}$ and consider $B = A \otimes_F F(\sqrt{t})$, where t is an indeterminate over F. Let τ be the unitary involution on B which restricts to σ on A and such that $\tau(\sqrt{t}) = -\sqrt{t}$. By [5, (10.33)], we have

$$[D(B,\tau)] = m[A] + [(t,d)_F].$$

Let $g \in \mathrm{GO}^+(A,\sigma) \subset \mathrm{GU}(B,\tau)$. We have $\mathrm{Nrd}_B(g) = \mathrm{Nrd}_A(g) = \mu(g)^m$, hence we may take $z = 1$ in Corollary 1.18 to get

$$\mu(g) \cdot \big(m[A] + [(t,d)_F]\big) = 0 \quad \text{in } H^3\big(F(t), \mathbb{Q}/\mathbb{Z}(2)\big).$$

If m is even, then $m[A] = 0$; if m is odd, then $m[A] = [A]$ and since $\mu(g)^m = \mathrm{Nrd}_A(g)$ it follows that $m\mu(g) \cdot [A] = 0$. Thus in each case we have

$$\mu(g) \cdot [(t,d)_F] = 0,$$

and since t is an indeterminate over F it follows that $(\mu(g), d)_F$ is split.

Suppose next $g \in \mathrm{GO}^-(A,\sigma) \subset \mathrm{GU}(B,\tau)$. Then $\mathrm{Nrd}_B(g) = -\mu(g)^m$, hence we may take $z = \sqrt{t}$ in Corollary 1.18. We thus obtain

$$\mu(g) \cdot \big(m[A] + [(t,d)_F]\big) + (-t) \cdot [A] = 0. \tag{12}$$

If m is even we have $m[A] = 0$, and the equation $-\mu(g)^m = \mathrm{Nrd}_A(g)$ shows that $(-1) \cdot [A] = 0$. Since t is an indeterminate, (12) yields $[A] = [(\mu(g), d)_F]$ by comparing residues at t.

If m is odd, the equation $-\mu(g)^m = \mathrm{Nrd}_A(g)$ yields $\mu(g) \cdot m[A] = (-1) \cdot [A]$, and (12) again yields $[A] = [(\mu(g), d)_F]$.

In conclusion, we have proved:

$$[(\mu(g), d)_F] = \begin{cases} 0 & \text{if } g \in \mathrm{GO}^+(A,\sigma), \\ [A] & \text{if } g \in \mathrm{GO}^-(A,\sigma). \end{cases}$$

This generalizes a theorem of Dieudonné on the multipliers of similitudes of quadratic forms. This generalization was first observed in [7, Theorem A] (see also [5, §13C]).

1.5.3. *Type D_n.* Let A be a central simple algebra of degree $2n$ over a field F of characteristic different from 2, and let σ be an orthogonal involution on A. Let Z be the center of the Clifford algebra $C(A,\sigma)$ and $\underline{\sigma}$ be the canonical involution on $C(A,\sigma)$. (We allow the possibility that $Z \simeq F \times F$.) For $G = \mathbf{Spin}(A,\sigma)$, we have $\overline{G} = \mathbf{PGO}^+(A,\sigma)$. We denote by $\mathrm{PGO}^+(A,\sigma)$ the group of rational points of $\mathbf{PGO}^+(A,\sigma)$. An extended Clifford group $\Omega(A,\sigma)$ and a map $\chi' \colon \Omega(A,\sigma) \to \mathrm{PGO}^+(A,\sigma)$ which is an analogue of the vector representation $\mathrm{Spin}(A,\sigma) \to \mathrm{O}^+(A,\sigma)$ are defined in [5, §13B].

Suppose n is even. The map

$$\partial\colon\ \mathrm{PGO}^+(A,\sigma) \to H^1\big(F, R_{Z/F}(\mu_2)\big) \simeq Z^\times/Z^{\times 2}$$

is described in [5, (13.32)] (where it is denoted by S). For $g \in \mathrm{GO}^+(A,\sigma)$, $\partial(g) = \underline{\mu}(\omega) \cdot Z^{\times 2}$ where $\omega \in \Omega(A,\sigma)$ is such that $\chi'(\omega) = g$, and $\underline{\mu}(\omega) = \underline{\sigma}(\omega)\omega \in Z^\times$. Proposition 1.16 takes the following form, with the invariant of Theorem 1.14 or 1.15:

Corollary 1.20. *For $g \in \mathrm{GO}^+(A,\sigma)$,*

$$N_{Z/F}\big(\partial(g) \cdot [C(A,\sigma)]\big) = 0 \qquad \text{if } n \equiv 2 \bmod 4,$$
$$N_{Z/F}\big(\partial(g) \cdot [\overline{C(A,\sigma)}]\big) = 0 \qquad \text{if } n \equiv 0 \bmod 4.$$

In particular, if $g \in \mathrm{O}^+(A,\sigma)(F)$, then $\partial(g)$ is the spinor norm $\mathrm{Sn}(g)$, see [5, (13.33)]. Since

$$N_{Z/F}\big([C(A,\sigma)]\big) = N_{Z/F}\big([\overline{C(A,\sigma)}]\big) = [A]$$

we obtain

$$\mathrm{Sn}(g) \cdot [A] = 0 \qquad \text{for } g \in \mathrm{O}^+(A,\sigma).$$

This also follows from the fact that spinor norms are reduced norms up to squares, see [8, §6].

Suppose n is odd. The map

$$\partial\colon\ \mathrm{PGO}^+(A,\sigma)\to H^1(F,\mu_{4[Z]})$$

is described in [5, (13.35)]. For $g\in\mathrm{GO}^+(A,\sigma)$,

$$\partial(g)=(\underline{\mu}(\omega),\underline{\mu}(\omega)^2z\overline{z}^{-1})_4$$

where $\omega\in\Omega(A,\sigma)$ is such that $\chi'(\omega)=g$ and $z\in Z^\times$ is such that $z^{-1}\omega^2$ is an element of the Clifford group mapped onto $\mu(g)^{-1}g^2$ by the vector representation. Proposition 1.16, applied with the invariant of Theorem 1.14 or 1.15, thus takes the form

Corollary 1.21. *For $g\in\mathrm{GO}^+(A,\sigma)$,*

$$\underline{\mu}(\omega)\cdot[A]+N_{Z/F}(z\cdot[C(A,\sigma)])=0.$$

If $g\in\mathrm{O}^+(A,\sigma)$, then we may choose ω in the Clifford group, hence we may choose $z=1$. Moreover, [5, (13.36)] shows that $\underline{\mu}(\omega)$ is then the spinor norm $\mathrm{Sn}(g)$. Thus, as in the preceding case,

$$\mathrm{Sn}(g)\cdot[A]=0\qquad\text{for } g\in\mathrm{O}^+(A,\sigma).$$

2. Invariants of quasi-trivial tori

In this section, we give a proof of Theorem 1.1. Recall that A is an étale algebra over an arbitrary field F. For brevity, we denote by

$$T^A=\mathbf{GL}(A)=R_{A/F}(\mathbf{G}_\mathrm{m})$$

the torus of invertible elements in A. Clearly, for any two étale F-algebras A, B,

$$T^{A\times B}=T^A\times T^B.$$

Moreover, the group homomorphism α^A defined in section 1.1 satisfies $\alpha^{A\times B}=\alpha^A\oplus\alpha^B$.

2.1. Cycle modules and Chow groups.

Let M be a cycle module over F, X a scheme over F (separated, of finite type). In [10, Sec. 5] the group of classes of cycles $A^p(X,M_d)$ with coefficients in M is defined as the homology group of a complex

$$\bigoplus_{x\in X^{p-1}}M_{d-p+1}\big(F(x)\big)\to\bigoplus_{x\in X^{p}}M_{d-p}\big(F(x)\big)\to\bigoplus_{x\in X^{p+1}}M_{d-p-1}\big(F(x)\big).$$

For an open subscheme $U\subset X$ there is a localization exact sequence associated with the pair (U,X):

$$0\to A^0(X,M_d)\to A^0(U,M_d)\xrightarrow{\partial}A^0(Z,M_{d-1})\to A^1(X,M_d)\to\dots$$

where $Z=X\setminus U$ is of codimension 1.

The Chow groups are contravariant with respect to morphisms of smooth schemes (the corresponding homomorphisms of the Chow groups are called the *inverse images*).

Let X be a smooth scheme over F. The structure morphism $i:X\to\mathrm{Spec}(F)$ induces

$$i^*:M_d(F)=A^0(\mathrm{Spec}(F),M_d)\to A^0(X,M_d).$$

Any point $p:\mathrm{Spec}(F)\to X$ gives

$$p^*:A^0(X,M_d)\to A^0(\mathrm{Spec}(F),M_d)=M_d(F).$$

Clearly, $p^*\circ i^*=\mathrm{Id}$. Thus we have a decomposition

$$A^0(X,M_d)=M_d(F)\oplus\overline{A}^0(X,M_d),$$

where

$$\overline{A}^0(X, M_d) = \ker(p^*).$$

If X is an algebraic group we will consider such a decomposition with respect to the unit p of the group.

The homotopy invariance theorem [10, Prop. 2.2] states that the canonical homomorphism

$$M_d(F) \to A^0(\mathbf{A}^1, M_d)$$

is an isomorphism and

$$A^1(\mathbf{A}^1, M_d) = 0.$$

Example 2.1. The localization exact sequence for the pair $(\mathbf{G}_{\mathrm{m}}, \mathbf{A}^1)$,

$$0 \to A^0(\mathbf{A}^1, M_d) \to A^0(\mathbf{G}_{\mathrm{m}}, M_d) \xrightarrow{\partial} A^0(\operatorname{Spec} F, M_{d-1}) \to A^1(\mathbf{A}^1, M_d)$$

and the homotopy invariance property show that the boundary homomorphism ∂ induces an isomorphism

$$\overline{A}^0(\mathbf{G}_{\mathrm{m}}, M_d) \simeq M_{d-1}(F).$$

In fact, this isomorphism is the inverse to α^F. Indeed, by the multiplicative property, it is sufficient to check the property in the case $d = 1$ and $M = K$ the Milnor's K-theory. But in this case the property is evident.

The boundary homomorphisms ∂ have a nice functorial property.

Proposition 2.2. *(*[10, Prop. 4.4]*) Let $h : Y \to Y'$ be a flat morphism of schemes of constant relative dimension, $Z' \subset Y'$ a closed subscheme, $U' = Y' \setminus Z'$. Set $Z = h^{-1}(Z')$, $U = h^{-1}(U')$. Then the following diagram is commutative*

$$\begin{array}{ccc} A^0(U', M_d) & \xrightarrow{\partial'} & A^0(Z', M_{d-1}) \\ \downarrow & & \downarrow \\ A^0(U, M_d) & \xrightarrow{\partial} & A^0(Z, M_{d-1}) \end{array}$$

(here the vertical homomorphisms are the inverse images and the horizontal ones are the boundary maps associated to the pairs (U', Y') and (U, Y)).

2.2. **The scheme S^A.** We consider the torus T^A as an open subscheme in the affine space $\mathbf{A}(A)$ of the algebra A. Let Z^A be the complementary closed subscheme in $\mathbf{A}(A)$. The purpose of this section is to define an open subscheme $S^A \subset Z^A$.

Consider first the split case $A = F^n$. Then $T^A = {\mathbf{G}_{\mathrm{m}}}^n$ is an open subscheme in $\mathbf{A}(A) = \mathbf{A}^n$ and the closed subscheme $Z^A \subset \mathbf{A}^n$ is given by the equation $X_1 X_2 \dots X_n = 0$. Denote by S^A the open subscheme in Z^A consisting of n-tuples $(x_1, x_2, \dots, x_n)$ such that exactly one of the x_i is zero. In other words, S^A is obtained from Z^A by removing all pairwise intersections of the hyperplanes in $\mathbf{A}^n$ given by $X_i = 0$. Equivalently, S^A is the smooth locus of Z^A. Clearly, S^A is the disjoint union of n copies of the torus ${\mathbf{G}_{\mathrm{m}}}^{n-1}$. We will consider S^A as a torus over $\operatorname{Spec}(A)$.

Let us give an algebraic description of S^A in the split case. We have $Z^A = \operatorname{Spec} C$ where

$$C = F[X_1, X_2, \dots, X_n]/(X_1 X_2 \cdots X_n).$$

Let s be the class in C of the following polynomial:

$$\sum_{i=1}^{n} \prod_{j \neq i} X_1 \cdots \widehat{X}_j \cdots X_n.$$

We set

$$S^A = \operatorname{Spec} C_s$$

(C_s is the localization of C by s). Thus S^A is a principal open subscheme in Z^A. Let C^i be the Laurent polynomial ring

$$F[X_1^{\pm 1}, \dots, \widehat{X}_i^{\pm 1}, \dots, X_n^{\pm 1}].$$

Denote g_i the ring homomorphism

$$C_s \to C^i, \quad f(X_1, \dots, X_i, \dots, X_n) \mapsto f(X_1, \dots, 0, \dots, X_n).$$

Then the collection (g_i) defines an isomorphism

$$C_s \simeq \prod_{i=1}^{n} C^i,$$

i.e. S^A is indeed a disjoint union of n copies of an $(n-1)$-dimensional split torus, so that S^A is a scheme over $A = F^n$. The unit elements of all tori $\operatorname{Spec} C^i$ give rise to the unit element of S^A, a morphism $\operatorname{Spec} A \to S^A$.

In general, when the étale algebra A of dimension n is not necessarily split, we have an action of the absolute Galois group Γ on the set of indeterminates X_i above by permutations. (Étale F-algebras correspond to finite Γ-sets and we identify the set corresponding to A with the set of indeterminates.) We have

$$T^A = \operatorname{Spec}\big(F_{\mathrm{sep}}[X_1^{\pm}, X_2^{\pm}, \dots, X_n^{\pm}]\big)^{\Gamma},$$

$$\mathbf{A}(A) = \operatorname{Spec}\big(F_{\mathrm{sep}}[X_1, X_2, \dots, X_n]\big)^{\Gamma},$$

$$Z^A = \operatorname{Spec}\big(F_{\mathrm{sep}}[X_1, X_2, \dots, X_n]/(X_1 X_2 \dots X_n)\big)^{\Gamma} = \operatorname{Spec}\big(C_{\mathrm{sep}}\big)^{\Gamma}.$$

The element s is Γ-invariant. We define

$$S^A = \operatorname{Spec}(C_{\mathrm{sep}})_s^{\Gamma}.$$

Also,

$$S^A = \operatorname{Spec}\big(\prod_{i=1}^{n} (C^i)_{\mathrm{sep}}\big)^{\Gamma}$$

where Γ acts naturally by permutations on the factors of the product in such a way that

$$(F^n)^{\Gamma} = A.$$

Thus, A is a subalgebra in $(C_{\mathrm{sep}})_s^{\Gamma}$ and hence the scheme S^A is equipped with a structure morphism

$$S^A \to \operatorname{Spec} A$$

of a scheme over A. As above, there is the unit point

$$\operatorname{Spec} A \to S^A.$$

Hence we have a canonical presentation

$$A^0(S^A, M_q) = M_q(A) \oplus \overline{A}^0(S^A, M_q). \tag{13}$$

If A and B are two étale F-algebras, then

$$S^{A \times B} = S^A \times T^B + T^A \times S^B$$

(here $+$ stands for the disjoint union).

2.3. **Definition of β^A.** We define in this section a homomorphism

$$\beta^A\colon\ \operatorname{Inv}^d(T^A,M)\to M_{d-1}(A)$$

and prove later that β^A is the inverse of α^A. The homomorphism β^A is a composition of three homomorphisms. The first is the embedding

$$\beta_1^A\colon\ \operatorname{Inv}^d(T^A,M)\hookrightarrow \overline{A}^0(T^A,M_d)$$

[9, Theorem 2.3, Lemma 1.9] given by evaluation of invariants at the generic point of T^A.

The second homomorphism is the restriction of the connecting homomorphism associated to the pair $\big(T^A,\mathbf{A}(A)\big)$,

$$\beta_2^A\colon\ \overline{A}^0(T^A,M_d)\xrightarrow{\partial} A^0(Z^A,M_{d-1}).$$

Lemma 2.3. *β_2^A is injective.*

Proof. Consider the localization exact sequence for the pair $\big(T^A,\mathbf{A}(A)\big)$:

$$A^0(\mathbf{A}(A),M_d)\to A^0(T^A,M_d)\xrightarrow{\partial} A^0(Z^A,M_{d-1}).$$

By homotopy invariance, the first term is canonically isomorphic to $M_d(F)$ and hence the first homomorphism in the sequence is the canonical embedding of $M_d(F)$ into $A^0(T^A,M_d)$, whence the result. $\square$

The third homomorphism is the restriction (inverse image) to the open subscheme $S^A\subset Z^A$,

$$\beta_3^A\colon\ A^0(Z^A,M_{d-1})\to A^0(S^A,M_{d-1}).$$

The exactness of the localization sequence for the pair (S^A,Z^A) implies that β_3^A is also injective.

We then set

$$\overline{\beta}^A=\beta_3^A\circ\beta_2^A\circ\beta_1^A\colon\ \operatorname{Inv}^d(T^A,M)\to A^0(S^A,M_{d-1}),$$

and we proceed to prove that the image of this map is in $M_{d-1}(A)$. The map β^A may then be defined as $\overline{\beta}^A$ viewed as a map $\operatorname{Inv}^d(T^A,M)\to M_{d-1}(A)$.

We make some preliminary observations on the map $\overline{\beta}^A$. First, it is clear that $\overline{\beta}^A$ commutes with base field change and norms under finite field extensions. Next, suppose A and B are étale F-algebras. Since $S^{A\times B}=S^A\times T^B+T^A\times S^B$, we have

$$A^0(S^{A\times B},M_{d-1})=A^0(S^A\times T^B,M_{d-1})\oplus A^0(T^A\times S^B,M_{d-1}). \tag{14}$$

The inverse images with respect to the projections $S^A\times T^B\to S^A$ and $T^A\times S^B\to S^B$ define the right vertical map in the diagram

$$\begin{array}{ccc}\operatorname{Inv}^d(T^A,M)\oplus\operatorname{Inv}^d(T^B,M) & \xrightarrow{\overline{\beta}^A\oplus\overline{\beta}^B} & A^0(S^A,M_{d-1})\oplus A^0(S^B,M_{d-1})\\ \Big\| & & \big\downarrow\\ \operatorname{Inv}^d(T^{A\times B},M) & \xrightarrow{\overline{\beta}^{A\times B}} & A^0(S^{A\times B},M_{d-1}).\end{array} \tag{15}$$

Lemma 2.4. *Diagram* (15) *is commutative.*

Proof. The proof is quite technical. Since the roles of A and B are symmetric, it suffices to prove commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Inv}^d(T^A, M) & \xrightarrow{\overline{\beta}^A} & A^0(S^A, M_{d-1}) \\ \downarrow & & \downarrow \\ \mathrm{Inv}^d(T^{A\times B}, M) & \xrightarrow{\overline{\beta}^{A\times B}} & A^0(S^{A\times B}, M_{d-1}) \end{array}$$

where the vertical homomorphisms are induced by the projection $A \times B \to A$.

The homomorphism $\overline{\beta}^A$ is defined as the composition of three homomorphisms, so we need to prove commutativity of a few diagrams. The first diagram is

$$\begin{array}{ccc} \mathrm{Inv}^d(T^A, M) & \xrightarrow{\beta_1^A} & A^0(T^A, M_d) \\ \downarrow & & \downarrow \\ \mathrm{Inv}^d(T^{A\times B}, M) & \xrightarrow{\beta_1^{A\times B}} & A^0(T^{A\times B}, M_d). \end{array}$$

The horizontal homomorphisms β_1 are given by the values at the generic points. The right vertical homomorphism, the inverse image homomorphism, is induced by the inclusion of function fields at the generic points [10, Lemma 12.8]. The commutativity readily follows.

The next diagram is

$$\begin{array}{ccc} A^0(T^A, M_d) & \xrightarrow{\beta_3^A \circ \beta_2^A} & A^0(S^A, M_{d-1}) \\ \downarrow & & \downarrow \\ A^0(T^{A\times B}, M_d) & \xrightarrow{\beta_3^{A\times B} \circ \beta_2^{A\times B}} & A^0(S^{A\times B}, M_{d-1}). \end{array}$$

Here the right vertical homomorphism is induced by the projection $S^A \times T^B \to S^A$ and the decomposition (14).

First, we represent the composition $\beta_3 \circ \beta_2$ as the boundary map for a certain pair. Denote $\mathbf{A}(A)'$ the open subset

$$\mathbf{A}(A) \setminus (Z^A \setminus S^A)$$

in $\mathbf{A}(A)$. Then T^A is an open subscheme in $\mathbf{A}(A)'$ and $\mathbf{A}(A)' \setminus T^A = S^A$. By Proposition 2.2, applied to the inclusion $h : \mathbf{A}(A)' \to \mathbf{A}(A)$ and closed subscheme $Z^A \subset \mathbf{A}(A)$ we get that the composition $\beta_3^A \circ \beta_2^A$ coincides with the boundary map ∂^A associated to the pair $(S^A, \mathbf{A}(A)')$. Similarly, the composition $\beta_3^{A\times B} \circ \beta_2^{A\times B}$ coincides with the boundary map $\partial^{A\times B}$ associated to the pair $(S^{A\times B}, \mathbf{A}(A\times B)')$.

Applying again Proposition 2.2 to the projection $h : \mathbf{A}(A)' \times T^B \to \mathbf{A}(A)'$ and the closed subscheme $S^A \subset \mathbf{A}(A)'$ we get the commutative diagram

$$\begin{array}{ccc} A^0(T^A, M_d) & \xrightarrow{\partial^A} & A^0(S^A, M_{d-1}) \\ \downarrow & & \downarrow \\ A^0(T^{A\times B}, M_d) & \xrightarrow{\partial} & A^0(S^A \times T^B, M_{d-1}) \end{array}$$

where ∂^A (resp. ∂) is the boundary map with respect to the pair $(S^A, \mathbf{A}(A)')$ (resp. $(S^A \times T^B, \mathbf{A}(A)' \times T^B)$).

Again, by Proposition 2.2 applied to the open embedding $h : \mathbf{A}(A)' \times T^B \to \mathbf{A}(A \times B)'$ and the closed subscheme $S^{A\times B} \subset \mathbf{A}(A \times B)'$ we get the commutative diagram:

$$\begin{array}{ccc} A^0(T^{A\times B}, M_d) & \xrightarrow{\partial^{A\times B}} & A^0(S^{A\times B}, M_{d-1}) \\ \| & & \downarrow \\ A^0(T^{A\times B}, M_d) & \xrightarrow{\partial} & A^0(S^A \times T^B, M_{d-1}). \end{array}$$

The combination of the last two diagrams gives the commutativity we need. □

Let $A = L_1 \times \cdots \times L_k$ be the decomposition of an étale F-algebra A into a product of fields. The *height* of A is the maximum of the degrees $[L_i : F]$. Thus, the height of A is 1 if and only if A splits. The following proposition gives the final step in the definition of β^A. It will be proved by induction on the height of A.

Proposition 2.5. *For any étale F-algebra A,*

$$\overline{\beta}^A\big(\mathrm{Inv}^d(T^A, M)\big) \subset M_{d-1}(A).$$

Proof. Consider the composition

$$\gamma^A\colon\ \mathrm{Inv}^d(T^A, M) \xrightarrow{\overline{\beta}^A} A^0(S^A, M_{d-1}) \to \overline{A}^0(S^A, M_{d-1}),$$

where the second homomorphism is the natural projection. To prove the proposition, it suffices to show that $\gamma^A = 0$. Arguing by induction on the height of A and using Lemma 2.4, we may assume $A = L$ is a field. If $L = F$, then $S^A = \operatorname{Spec} F$, $M_{d-1}(A) = A^0(S^A, M_{d-1})$, and the claim is clear.

If $L \neq F$, we extend scalars to L and use the fact that $A \otimes_F L = A \times A'$ for some algebra A', so the height of $A \otimes_F L$ is less than the height of A. By induction, the bottom homomorphism in the commutative diagram

$$\begin{array}{ccc} \mathrm{Inv}^d(T^A, M) & \xrightarrow{\gamma^A} & \overline{A}^0(S^A, M_{d-1}) \\ \downarrow & & \downarrow \\ \mathrm{Inv}^d\big((T^A)_L, M\big) & \xrightarrow{(\gamma^A)_L} & \overline{A}^0\big((S^A)_L, M_{d-1}\big) \end{array}$$

is trivial. It remains to notice that the right vertical homomorphism is injective: this is because $(S^A)_L$ is a union of two varieties, one of which splits off a copy of S^A since $A \otimes_F L = A \times A'$. □

2.4. **Proof of $\beta^A = (\alpha^A)^{-1}$.** Since β^A is injective, as it is the composition of three injective maps, it suffices to prove that $\beta^A \circ \alpha^A = \mathrm{Id}$. We prove this by induction on the height of A. By Lemma 2.4, we may assume that $A = L$ is a field. If $L = F$ then by the multiplicative property it is sufficient to consider the case where $d = 1$ and $M = K$ is Milnor's K-theory. In that case the statement is obvious.

Since the height of the étale algebra $A \otimes_F L$ over L is less than the height of A, the composition in question for the algebra $A \otimes_F L$ is the identity by induction. The homomorphisms α and β commute with the norms for the field extension L/F, and the norm homomorphism

$$M_{d-1}(A \otimes_F L) \to M_{d-1}(A)$$

is surjective, since $A \otimes_F L = A \times A'$ splits off a copy of A. The statement follows.

3. Invariants of torsors under roots of unity

In this section, we prove Corollaries 1.2, 1.4, 1.5 and 1.6. The basic tool is the following easy lemma, whose proof is left to the reader:

Lemma 3.1. *Let $P \xrightarrow{i} Q \xrightarrow{j} R$ be natural transformations of functors from Fields_F to Groups, and let M be a cycle module over F. If for every field extension L/F the sequence*

$$P(L) \xrightarrow{i_L} Q(L) \xrightarrow{j_L} R(L) \to 1$$

is exact, then the following sequence is exact for every $d \geq 0$:

$$0 \to \mathrm{Inv}^d(R, M) \xrightarrow{j^*} \mathrm{Inv}^d(Q, M) \xrightarrow{i^*} \mathrm{Inv}^d(P, M).$$

3.1. **Proof of Corollary 1.2.** The natural transformation φ of (1) fits in the following sequence

$$\mathbf{G}_{\mathrm{m}} \xrightarrow{n} \mathbf{G}_{\mathrm{m}} \xrightarrow{\varphi} H^1(\mu_n),$$

to which Lemma 3.1 may be applied. We thus obtain the exact sequence at the top of the following diagram:

$$\begin{tikzcd}
0 \to \mathrm{Inv}^d\big(H^1(\mu_n), M\big) \arrow[r, "\varphi^*"] & \mathrm{Inv}^d(\mathbf{G}_{\mathrm{m}}, M) \arrow[r, "n^*"] & \mathrm{Inv}^d(\mathbf{G}_{\mathrm{m}}, M) \\
& M_{d-1}(F) \arrow[u, "\alpha^F"] \arrow[r, "n"] & M_{d-1}(F). \arrow[u, "\alpha^F"']
\end{tikzcd}$$

It is easily verified that the diagram commutes. Since the vertical maps are isomorphisms by Theorem 1.1, Corollary 1.2 follows.

3.2. **Proof of Corollary 1.4.** The proof is completely similar to the preceding one. We apply Lemma 3.1 to the sequence of natural transformations

$$R_{K/F}(\mathbf{G}_{\mathrm{m}}) \xrightarrow{n} R_{K/F}(\mathbf{G}_{\mathrm{m}}) \xrightarrow{\varphi} H^1\big(R_{K/F}(\mu_n)\big),$$

and obtain the exact sequence at the top of the following commutative diagram, whose vertical maps are isomorphisms:

$$\begin{tikzcd}
0 \to \mathrm{Inv}^d\Big(H^1\big(R_{K/F}(\mu_n)\big), M\Big) \arrow[r, "\varphi^*"] & \mathrm{Inv}^d\big(R_{K/F}(\mathbf{G}_{\mathrm{m}}), M\big) \arrow[r, "n^*"] & \mathrm{Inv}^d\big(R_{K/F}(\mathbf{G}_{\mathrm{m}}), M\big) \\
& M_{d-1}(K) \arrow[u, "\alpha^K"] \arrow[r, "n"] & M_{d-1}(K). \arrow[u, "\alpha^K"']
\end{tikzcd}$$

3.3. **Proof of Corollary 1.5.** From the exact sequence (6), we derive the sequence of natural transformations

$$R_{K/F}(\mathbf{G}_{\mathrm{m}}) \xrightarrow{g_1} R_{K/F}(\mathbf{G}_{\mathrm{m}}) \xrightarrow{\varphi} H^1(\mu_{n[K]})$$

to which Lemma 3.1 may be applied. It remains to identify the map ℓ which makes the following diagram commute:

$$\begin{tikzcd}
0 \to \mathrm{Inv}^d\big(H^1(\mu_{n[K]}), M\big) \arrow[r, "\varphi^*"] & \mathrm{Inv}^d\big(R_{K/F}(\mathbf{G}_{\mathrm{m}}), M\big) \arrow[r, "g_1^*"] & \mathrm{Inv}^d\big(R_{K/F}(\mathbf{G}_{\mathrm{m}}), M\big) \\
& M_{d-1}(K) \arrow[u, "\alpha^K"] \arrow[r, "\ell"] & M_{d-1}(K). \arrow[u, "\alpha^K"']
\end{tikzcd}$$

Let $L \in \mathit{Fields}_F$. For $t \in (K \otimes_F L)^\times$, we have

$$(g_1)_L(t) = t^n N_{K\otimes L/L}(t)^{-m}.$$

Therefore, for $u \in M_{d-1}(K)$,

$$g_1^*\big(\alpha^K(u)\big)\colon\ t \mapsto nN_{K\otimes L/L}(t\cdot u_{K\otimes L}) - mN_{K\otimes L/L}(N_{K\otimes L/L}(t)\cdot u_{K\otimes L}).$$

By the projection formula, we have (writing simply N for $N_{K\otimes L/L}$)

$$N(N(t)\cdot u_{K\otimes L}) = N(t)\cdot N(u_{K\otimes L}) = N\big(t\cdot N(u_{K\otimes L})_{K\otimes L}\big).$$

Moreover, $N(u_{K\otimes L})_{K\otimes L} = u_{K\otimes L} + \overline{u}_{K\otimes L}$. Therefore, the image of t under $g_1^*\big(\alpha^K(u)\big)$ can be written as

$$N\big(t\cdot(nu_{K\otimes L} - m(u_{K\otimes L}+\overline{u}_{K\otimes L}))\big) = N\big(t\cdot((m+1)u_{K\otimes L} - m\overline{u}_{K\otimes L})\big).$$

It follows that the map ℓ is given by

$$\ell(u) = (m+1)u - m\overline{u}.$$

To finish the proof, observe that if $u \in \ker\ell$, then

$$(m+1)u = m\overline{u}$$

hence, taking the image of each side under $^-$,

$$(m+1)\overline{u} = mu.$$

These two equations are equivalent to $nu = u+\overline{u} = 0$. Thus, for every invariant $\iota\colon\ H^1(\mu_{n[K]}) \to M_d$, there is a uniquely determined element $u \in M_{d-1}(K)$ satisfying $nu = u+\overline{u} = 0$ such that for every field extension L/F and every $t \in L^\times$,

$$\varphi^*(\iota)_L(t) = N_{K\otimes L/L}(t\cdot u_{K\otimes L}).$$

Now, by definition we have

$$\varphi^*(\iota)_L(t) = \iota_L\big(\varphi_L(t)\big).$$

If $(x,y) \in P(L)$, i.e., if $x \in L^\times$ and $y \in (K\otimes L)^\times$ are such that $N_{K\otimes L/L}(y) = x^n$, then $(x,y)_n = \varphi_L(yx^{-m})$ (see equation (7)), hence

$$\iota_L\big((x,y)_n\big) = \varphi^*(\iota)_L(yx^{-m}) = N_{K\otimes L/L}(y\cdot u_{K\otimes L}) - mx\cdot N_{K/F}(u)_L.$$

The proof of Corollary 1.5 is thus complete.

3.4. **Proof of Corollary 1.6.** When n is even, $n = 2m$, we apply Lemma 3.1 to the following sequence of natural transformations derived from the exact sequence (8):

$$\mathbf{G}_{\mathrm{m}} \times R_{K/F}(\mathbf{G}_{\mathrm{m}}) \xrightarrow{g_2} \mathbf{G}_{\mathrm{m}} \times R_{K/F}(\mathbf{G}_{\mathrm{m}}) \xrightarrow{\varphi} H^1(\mu_{n[K]}).$$

We thus obtain the exact sequence

(16)
$$0 \to \mathrm{Inv}^d(H^1(\mu_{n[K]}), M) \xrightarrow{\varphi^*} \mathrm{Inv}^d(\mathbf{G}_{\mathrm{m}} \times R_{K/F}(\mathbf{G}_{\mathrm{m}}), M) \xrightarrow{g_2^*} \mathrm{Inv}^d(\mathbf{G}_{\mathrm{m}} \times R_{K/F}(\mathbf{G}_{\mathrm{m}}), M).$$

Let ℓ be the endomorphism of $M_{d-1}(F\times K) = M_{d-1}(F)\times M_{d-1}(K)$ defined by

$$\ell(u,v) = (N_{K/F}(v), u_K + mv).$$

Computation shows that the following diagram commutes:

$$\begin{array}{ccc} M_{d-1}(F\times K) & \xrightarrow{\ell} & M_{d-1}(F\times K) \\ {\scriptstyle \alpha^{F\times K}}\big\downarrow & & \big\downarrow{\scriptstyle \alpha^{F\times K}} \\ \mathrm{Inv}^d(\mathbf{G}_{\mathrm{m}}\times R_{K/F}(\mathbf{G}_{\mathrm{m}}),M) & \xrightarrow{g_2^*} & \mathrm{Inv}^d(\mathbf{G}_{\mathrm{m}}\times R_{K/F}(\mathbf{G}_{\mathrm{m}}),M). \end{array}$$

Therefore, the exact sequence (16) shows that for every invariant $\iota\colon H^1(\mu_{n[K]}) \to M_d$ there exist $u \in M_{d-1}(F)$, $v \in M_{d-1}(K)$ satisfying $N_{K/F}(v) = 0$ and $u_K + mv = 0$ such that for $L \in \mathsf{Fields}_F$ and $x \in L^\times$, $z \in (K\otimes L)^\times$,

$$\iota_L\big((x, x^m z\overline{z}^{-1})_n\big) = x\cdot u_L + N_{K\otimes L/L}(z\cdot v_{K\otimes L}).$$

This completes the proof of Corollary 1.6.

4. Rost invariants

In this section, we prove Theorems 1.11, 1.14 and 1.15. In each case, the idea is to reduce the situation by scalar extension to a case where the Rost invariant has been computed. Indeed, its functorial property implies that the Rost invariant is preserved under scalar extension, hence $I(G)$ is mapped by scalar extension onto $I(G_L)$, for every absolutely simple, simply connected linear algebraic group over F and every field extension L/F.

4.1. **Proof of Theorem 1.11.** We use the notation of section 1.4.2. Let ι be a generator of $I\big(\mathbf{SU}(B,\tau)\big)$. Corollary 1.6 shows that there exist Brauer classes $X \in \mathrm{Br}(F)$ and $Y \in \mathrm{Br}(K)$ such that for every field extension L/F and every $x \in L^\times$, $y \in (K\otimes_F L)^\times$ such that $N_{K\otimes L/L}(y) = x^n$,

$$\iota_L\big((x,y)_n\big) = x\cdot X_L + N_{K\otimes L/L}(z\cdot Y_{K\otimes L}),$$

where $z \in (K\otimes_F L)^\times$ is such that $yx^{-n/2} = z\overline{z}^{-1}$. We have to show that $X = [D(B,\tau)]$ and $Y = [B]$ for a suitable choice of ι.

Assume first that B is split, $B = \mathrm{End}_K V$ for some n-dimensional K-vector space V, and τ is the adjoint involution with respect to some hermitian form h on V. Let $q\colon V \to F$ be the trace form of h, defined by $q(v) = h(v,v)$. The description of the Rost invariant in [5, (31.44)] shows that

$$\iota_L\big((x,y)_n\big) = e_3(\langle x\rangle q_L - q_L) \in H^3(L,\mu_2),$$

where e_3 is the Arason invariant of quadratic forms. We have

$$e_3(\langle x,-1\rangle\cdot q_L) = (x)_2 \cup e_2(q_L) \qquad \text{in } H^3(L,\mu_2)$$

where e_2 is the Witt–Clifford invariant. On the other hand, by [11, p. 350[2]], $e_2(q)$ is the Brauer class of the quaternion algebra $(K, (-1)^{n/2}\det h)$, which is also the Brauer class of the discriminant algebra $D(B,\tau)$, by [5, (10.35)]. Theorem 1.11 is thus proved in the case where B is split.

To reduce the general situation to this case, we extend scalars to the function field E of the transfer of the Severi–Brauer variety of B from K to F. Since the Rost invariant is preserved under scalar extension, and since Theorem 1.11 is proved in the case where B is split, we must have

$$X_E = [D(B,\tau)_E] \qquad (\text{and } Y_{K\otimes E} = 0).$$

Now, the map $\mathrm{Br}(F) \to \mathrm{Br}(E)$ is injective, by [7, Corollary 2.12], hence $X = [D(B,\tau)]$.

[2]The formula for the Witt–Clifford invariant of q in [11, p. 350] misses the $(-1)^{n/2}$ factor.

To determine Y, we extend scalars to K. For every field extension L/K, we have $L \otimes_F K = L \times L$, and the restriction of the functor $H^1(\mu_{n[K]})$ to Fields_K is naturally equivalent to the restriction of $H^1(\mu_n)$, under the natural transformation which maps $(x, y)_n$ to $(y_1)_n$, for $x \in L^\times$, $y = (y_1, y_2) \in (K \otimes_F L)^\times = L^\times \times L^\times$ such that $x^n = y_1 y_2$. On the other hand, we have $B \otimes_L L = B_L \times \overline{B}_L$ (where $B_L = B \otimes_K L$) and $\mathbf{SU}(B \otimes_F L, \tau \otimes \mathrm{Id}) \simeq \mathbf{SL}(B_L)$, and we may use the description of $I\big(\mathbf{SL}(B_L)\big)$ in Theorem 1.9. For a suitable choice of the generator ι, we may thus assume that for every $L \in \mathit{Fields}_K$, x, y_1, $y_2 \in L^\times$ such that $x^n = y_1 y_2$,

$$x \cdot [D(B,\tau)_L] + N_{L\times L/L}(z_1 \cdot Y_L \times z_2 \cdot \overline{Y}_L) = y_1 \cdot [B_L]$$

where $(z_1, z_2) \in L^\times \times L^\times$ is such that $(y_1 x^{-n/2}, y_2 x^{-n/2}) = (z_1 z_2^{-1}, z_2 z_1^{-1})$. Given $z_1 \in L^\times$, we may set $z_2 = x = 1$ and $y_1 = z_1$, $y_2 = z_1^{-1}$, and the equality above yields

$$z_1 \cdot Y_L = z_1 \cdot [B_L].$$

So, the invariants $\alpha^K(Y)$, $\alpha^K([B])$: $\mathbf{G}_\mathrm{m} \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ coincide, and by Theorem 1.1 it follows that $Y = [B]$.

4.2. **Proof of Theorem 1.14.** We use the same notation as in section 1.4.5.

4.2.1. *Suppose n is odd.* The center of $\mathbf{Spin}(A, \sigma)$ is then μ_4, and, by Corollary 1.2, for any generator ι of $I\big(\mathbf{Spin}(A,\sigma)\big)$, there exists $X \in \mathrm{Br}(F)$ such that for $L \in \mathit{Fields}_F$ and $x \in L^\times$,

$$\iota_L\big((x)_4\big) = x \cdot X_L.$$

We have to show that $X = [C^+(A,\sigma)]$ for a suitable choice of ι.

Consider first the case where A is split, $A = \mathrm{End}_F V$ for some $2n$-dimensional F-vector space V and σ is the adjoint involution of some quadratic form q on V of trivial discriminant. From the description of the Rost invariant of $\mathbf{Spin}(V, q)$ in [5, (31.42)], it follows that for $L \in \mathit{Fields}_F$ and $x \in L^\times$,

$$\iota_L\big((x)_4\big) = e_3(\langle x\rangle q_L - q_L) = (x)_2 \cup e_2(q)_L$$

where, as in the preceding section, e_3 denotes the Arason invariant and e_2 the Witt–Clifford invariant. Since q has trivial discriminant, the Clifford algebra of q is Brauer-equivalent to any of the two factors of its even Clifford algebra, so

$$e_2(q) = [C^+(\mathrm{End}_F V, \sigma)]$$

and the theorem is proved in this particular case.

The general case is reduced to the case where A is split by scalar extension to the fuction field E of the Severi–Brauer variety of A. Thus, $X_E = [C^+(A,\sigma)_E]$. Since the only nontrivial element in the kernel of the scalar extension map $\mathrm{Br}(F) \to \mathrm{Br}(E)$ is $[A]$, it follows that

$$X = [C^+(A,\sigma)] \qquad \text{or} \qquad X = [C^+(A,\sigma)] + [A].$$

However, we have $[A] = 2[C^+(A,\sigma)]$, so

$$[C^+(A,\sigma)] + [A] = 3[C^+(A,\sigma)] \qquad (= [C^-(A,\sigma)]).$$

Therefore, substituting $3\iota = -\iota$ for ι if necessary, we may always assume $X = [C^+(A,\sigma)]$.

4.2.2. *Suppose n is even.* The center of $\mathbf{Spin}(A,\sigma)$ is then $\mu_2\times\mu_2$. By Corollary 1.3, for any generator ι of $I\big(\mathbf{Spin}(A,\sigma)\big)$ there exist X^+, $X^-\in\mathrm{Br}(F)$ satisfying $2X^+=2X^-=0$ such that for $L\in\mathit{Fields}_F$ and $(x^+,x^-)\in L^\times\times L^\times$,

$$\iota_L\big((x^+)_2,(x^-)_2\big)=x^+\cdot X_L^+ + x^-\cdot X_L^-.$$

We have to show that $X^\pm=[C^\pm(A,\sigma)]$ if $n\equiv 2 \bmod 4$ and $X^\pm=[C^\mp(A,\sigma)]$ if $n\equiv 0\bmod 4$, for a suitable choice of ι.

If A is split, $A=\mathrm{End}_F V$ for some $2n$-dimensional F-vector space V and σ is the adjoint involution of some quadratic form q on V with trivial discriminant, then we may use the explicit description of the Rost invariant of $\mathbf{Spin}(q)$ in [5, (31.42)]. We thus obtain

$$\iota_L\big((x^+)_2,(x^-)_2\big)=e_3(\langle x^+x^-\rangle q_L-q_L)=(x^+x^-)_2\cup e_2(q)_L.$$

As in the preceding section, we have

$$e_2(q)=[C^+(A,\sigma)]=[C^-(A,\sigma)],$$

so the proof is complete in the case where A is split.

For the rest of this section, we assume A is not split. The next lemma yields a first relation between X^+ and X^-.

Lemma 4.1. $X^++X^-=[A]$.

Proof. Consider the exact sequence

$$1\to\mu_2\to\mathbf{Spin}(A,\sigma)\xrightarrow{\chi}\mathbf{O}^+(A,\sigma)\to 1.$$

The kernel μ_2 of the vector representation χ is diagonally embedded in the center $\mu_2\times\mu_2$ of $\mathbf{Spin}(A,\sigma)$, and the Rost invariant on the image of $H^1(F,\mu_2)$ is known from [5, p. 441]: it is the cup-product with $[A]$, see the appendix. Therefore, for $L\in\mathit{Fields}_F$ and $x\in L^\times$,

$$\iota_L\big((x)_2,(x)_2\big)=x\cdot[A_L],$$

hence

$$x\cdot(X_L^++X_L^-)=x\cdot[A_L].$$

Thus, the invariants $\alpha^F(X^++X^-)$ and $\alpha^F([A])\colon\ \mathbf{G}_{\mathrm{m}}\to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ coincide, and Theorem 1.1 shows that $X^++X^-=[A]$. $\square$

The rest of the proof proceeds by reduction to the case where (A,σ) is hyperbolic. In this case, one of the Clifford factors $C_\pm(A,\sigma)$ is split, by [5, (8.31)]. The factors are not both split, since their tensor product is Brauer-equivalent to A by [5, (9.14)].

Lemma 4.2. *Suppose (A,σ) is hyperbolic of degree $2n=4m$ and $C^+(A,\sigma)$ is split. The image of the connecting map $\mathrm{PGO}^+(A,\sigma)\to H^1(F,\mu_2)\times H^1(F,\mu_2)$ in the cohomology sequence associated with*

$$1\to\mu_2\times\mu_2\to\mathbf{Spin}(A,\sigma)\to\mathbf{PGO}^+(A,\sigma)\to 1$$

contains $\big((\lambda^m)_2,(\lambda^{m+1})_2\big)$ for all $\lambda\in F^\times$.

Proof. Since (A,σ) is hyperbolic, there is an idempotent $e\in A$ such that $\sigma(e)=1-e$. As in [5, §8.E], let $\rho(e)=c(eAe)^m\subset C(A,\sigma)$, where $c\colon A\to C(A,\sigma)$ is the canonical map. The set $\rho(e)$ is a 1-dimensional F-vector space, and the left multiplication map $C(A,\sigma)\to\mathrm{End}_F\big(C(A,\sigma)\rho(e)\big)$ is onto with nontrivial kernel. This map therefore factors through an isomorphism

$$C^+(A,\sigma)\xrightarrow{\sim}\mathrm{End}_F\big(C(A,\sigma)\rho(e)\big)\tag{17}$$

(since $C^+(A,\sigma)$ is split and $C^-(A,\sigma)$ is not).

Now, fix $\lambda \in F^\times$ and let $g = e\lambda + (1-e)$. Computation shows that $\sigma(g)g = \lambda$, hence $g \in \mathrm{GO}(A,\sigma)$. We claim that the induced automorphism $C(g)$ of $C(A,\sigma)$ restricts to the identity on the center of $C(A,\sigma)$ and to multiplication by λ^m on $\rho(e)$. It suffices to prove the claim after scalar extension to a splitting field of A. We may thus assume that $A = \mathrm{End}_F V$ for some F-vector space V, that σ is adjoint to a hyperbolic quadratic form q on V, and that e is the projection onto a totally isotropic subspace $U \subset V$, parallel to some totally isotropic complement $W \subset V$. Let $u_1, \ldots, u_n$ be a basis of U and $w_1, \ldots, w_n$ be the basis of W such that $q(u_i + w_j) = \delta_{ij}$. Then $\rho(e) = u_1 \ldots u_n F$ and the center of $C(A,\sigma)$ is spanned by 1 and $(u_1+w_1)(u_1-w_1)\cdots(u_n-w_n)$. The automorphism $C(g)$ of $C(A,\sigma) = C_0(V,q)$ maps $v_1 \cdot v_2$ to $\lambda^{-1}g(v_1)\cdot g(v_2)$, and g restricts to multiplication by λ on U and to the identity on W. The claim then follows from a straightforward computation.

It follows from the claim that g is a direct similitude, and that $C(g)$ restricts to an F-linear map $g_*\colon C(A,\sigma)\rho(e) \to C(A,\sigma)\rho(e)$. Let $\omega^+ \in C^+(A,\sigma)$ be the preimage of g_* under the isomorphism (17), i.e., an element such that $(\omega^+,0) \in C^+(A,\sigma)\times C^-(A,\sigma) = C(A,\sigma)$ satisfies $(\omega^+,0)\cdot\xi = g_*(\xi)$ for all $\xi \in C(A,\sigma)\rho(e)$. Since $C(g)$ is an automorphism, we have

$$(\omega^+ u(\omega^+)^{-1},0)\cdot\xi = \big(C(g)(u),0\big)\cdot\xi \qquad \text{for } u \in C^+(A,\sigma),\ \xi \in C(A,\sigma)\rho(e),$$

hence $C(g) = \mathrm{Int}(\omega^+,\omega^-)$ for some $\omega^- \in C^-(A,\sigma)$. By definition, the image of $gF^\times \in \mathrm{PGO}^+(A,\sigma)$ under the connecting map to $H^1(F,\mu_2)\times H^1(F,\mu_2)$ is $\big((\underline{\sigma}(\omega^+)\omega^+)_2, (\underline{\sigma}(\omega^-)\omega^-)_2\big)$, where $\underline{\sigma}$ is the canonical involution on $C(A,\sigma)$. The restriction of $\underline{\sigma}$ to $C^+(A,\sigma)$ corresponds under (17) to the adjoint involution with respect to the bilinear form

$$b\colon C(A,\sigma)\rho(e)\times C(A,\sigma)\rho(e) \to \rho(e)$$

defined by $b(\xi,\eta) = \underline{\sigma}(\xi)\eta$, see [3, p. 334]. Since g_* is the restriction of an automorphism of $C(A,\sigma)$ and since it restricts to multiplication by λ^m on $\rho(e)$, it is a similitude of b with multiplier λ^m, hence $\underline{\sigma}(\omega^+)\omega^+ = \lambda^m$. On the other hand, Proposition (13.33) of [5] yields

$$\underline{\sigma}(\omega^+)\omega^+\cdot\underline{\sigma}(\omega^-)\omega^- \equiv \sigma(g)g \mod F^{\times 2},$$

hence $\big(\underline{\sigma}(\omega^-)\omega^-\big)_2 = (\lambda^{m+1})_2$. □

Corollary 4.3. *If (A,σ) is hyperbolic of degree $2n = 4m$ and $C^+(A,\sigma)$ is split, then $X^+ = 0$ if m is odd and $X^- = 0$ if m is even.*

Proof. For every $L \in \mathit{Fields}_F$, the map ι_L vanishes on the image of $\mathrm{PGO}^+(A_L,\sigma)$, since this image becomes trivial in $H^1\big(L,\mathbf{Spin}(A,\sigma)\big)$. From Lemma 4.2, it follows that for all $\lambda \in L^\times$,

$$\lambda^m\cdot X_L^+ + \lambda^{m+1}\cdot X_L^- = \lambda\cdot(mX_L^+ + (m+1)X_L^-) = 0.$$

Therefore, the invariant $\alpha^F(mX^+ + (m+1)X^-)\colon \mathbf{G}_{\mathrm{m}} \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ is trivial, and it follows from Theorem 1.1 that $mX^+ + (m+1)X^- = 0$. Since $2X^+ = 2X^- = 0$, this equation implies that $X^+ = 0$ if m is odd and $X^- = 0$ if m is even. □

We may now conclude the proof of Theorem 1.14. The scheme of isotropic ideals of reduced dimension n in (A,σ) has two irreducible components V^+, V^-, and the Brauer kernel of the scalar extension map to $F(V^\pm)$ is generated by $[C^\pm(A,\sigma)]$, see [7, Corollary 2.11]. The preceding corollary therefore yields

$$\begin{aligned} &X^+ = 0 \text{ or } X^+ = [C^+(A,\sigma)] &&\text{if } m \text{ is odd}\\ &X^- = 0 \text{ or } X^- = [C^+(A,\sigma)] &&\text{if } m \text{ is even.}\end{aligned}$$

Similarly, interchanging $+$ and $-$, we have

$$X^- = 0 \text{ or } X^- = [C^-(A,\sigma)] \qquad \text{if } m \text{ is odd}$$
$$X^+ = 0 \text{ or } X^+ = [C^-(A,\sigma)] \qquad \text{if } m \text{ is even.}$$

Taking into account the relation between X^+ and X^- in Lemma 4.1, the only solution for X^+, X^- is $X^+ = [C^+(A,\sigma)]$, $X^- = [C^-(A,\sigma)]$ if m is odd and $X^+ = [C^-(A,\sigma)]$, $X^- = [C^+(A,\sigma)]$ if m is even. The proof is thus complete.

4.3. **Proof of Theorem 1.15.** We use the notation of section 1.4.6.

4.3.1. *Suppose n is odd.* Let ι be a generator of $I\big(\mathbf{Spin}(A,\sigma)\big)$. Corollary 1.6 shows that there exist Brauer classes $X \in \operatorname{Br}(F)$ and $Y \in \operatorname{Br}(Z)$ such that for $L \in \mathsf{Fields}_F$ and $x \in L^\times$, $y \in (Z \otimes_F L)^\times$ such that $N_{Z\otimes L/L}(y) = x^4$,

$$\iota_L\big((x,y)_4\big) = x \cdot X_L + N_{Z\otimes L/L}(z \cdot Y_{Z\otimes L}),$$

where $z \in (Z \otimes_F L)^\times$ is such that $yx^{-2} = z\overline{z}^{-1}$. We have to show that $X = [A]$ and $Y = [C(A,\sigma)]$ for a suitable choice of ι.

The inclusion $\mu_2 \hookrightarrow \mu_{4[Z]}$ induces a map $H^1(L,\mu_2) \to H^1(L,\mu_{4[Z]})$ which carries $(x)_2$ to $(x,x^2)_4$ for $x \in L^\times$ (see [5, p. 444]). Since the Rost invariant on the image of $H^1(L,\mu_2)$ is multiplication by $[A]$ (see the appendix), we get

$$\iota_L\big((x,x^2)_4\big) = x \cdot X_L = x \cdot [A_L]$$

for $x \in L^\times$, hence $X = [A]$.

To determine Y, we extend scalars to Z to reduce to the inner case, and argue as in section 4.1. Observe that

$$C(A_Z,\sigma_Z) = C(A,\sigma) \otimes_F Z = C(A,\sigma) \times \overline{C(A,\sigma)},$$

so $C^+(A_Z,\sigma_Z) = C(A,\sigma)$. If ι is mapped by scalar extension on the generator described in Theorem 1.14, then for $L \in \mathsf{Fields}_Z$, x, y_1, $y_2 \in Z^\times$ such that $x^4 = y_1y_2$, we have

$$x \cdot [A_L] + N_{L\times L/L}(z_1 \cdot Y_L \times z_2 \cdot \overline{Y}_L) = y_1 \cdot [C(A,\sigma)_L]$$

where $(z_1,z_2) \in Z^\times \times Z^\times$ is such that $(y_1x^{-2}, y_2x^{-2}) = (z_1z_2^{-1}, z_2z_1^{-1})$. Given $z_1 \in Z^\times$, we may set $z_2 = x = 1$ and $y_1 = z_1$, $y_2 = z_1^{-1}$, and the equality above yields

$$z_1 \cdot Y_L = z_1 \cdot [C(A,\sigma)_L].$$

Therefore, $Y = [C(A,\sigma)]$.

4.3.2. *Suppose n is even.* Let ι be a generator of $I\big(\mathbf{Spin}(A,\sigma)\big)$. By Corollary 1.4, there exists a Brauer class $X \in \operatorname{Br}(Z)$ such that for $L \in \mathsf{Fields}_F$ and $x \in (Z \otimes_F L)^\times$,

$$\iota_L\big((x)_2\big) = N_{Z\otimes L/L}(x \cdot X_{Z\otimes L}).$$

We have to show that $X = [C(A,\sigma)]$ if $n \equiv 2 \bmod 4$ and $X = [\overline{C(A,\sigma)}]$ if $n \equiv 0 \bmod 4$, for a suitable choice of ι.

As in the preceding case, we extend scalars to Z to reduce to the inner case, and we observe that

$$C(A_Z,\sigma_Z) = C(A,\sigma) \otimes_F Z = C(A,\sigma) \times \overline{C(A,\sigma)}.$$

For $L \in \mathsf{Fields}_Z$, we have $Z \otimes_F L = L \times L$ under a map which carries $z \otimes \ell$ to $(z\ell, \overline{z}\ell)$. Therefore, the formula above for ι_L yields for x_1, $x_2 \in L^\times$ (for a suitable choice of ι)

$$\iota_L\big((x_1)_2,(x_2)_2\big) = N_{L\times L/L}(x_1 \cdot X_L \times x_2 \cdot \overline{X}_L) = x_1 \cdot X_L + x_2 \cdot \overline{X}_L.$$

Comparing with Theorem 1.14, we obtain

$$x_1 \cdot X_L + x_2 \cdot \overline{X}_L = \begin{cases} x_1 \cdot [C(A,\sigma)_L] + x_2 \cdot [\overline{C(A,\sigma)}_L] & \text{if } n \equiv 2 \bmod 4, \\ x_1 \cdot [\overline{C(A,\sigma)}_L] + x_2 \cdot [C(A,\sigma)_L] & \text{if } n \equiv 0 \bmod 4. \end{cases}$$

Therefore, the invariant $\alpha^Z(X)$ coincides with $\alpha^Z\big([C(A,\sigma)]\big)$ if $n \equiv 2 \bmod 4$, with $\alpha^Z\big([\overline{C(A,\sigma)}]\big)$ if $n \equiv 0 \bmod 4$, and Theorem 1.1 completes the proof.

5. Tits classes

Let G be an absolutely simple, simply connected linear algebraic group over an arbitrary field F whose characteristic is not a special prime of G, so that the center C of G is a smooth algebraic group scheme. The Tits class t_G is defined in [5, p. 426] as follows: consider the exact sequence

$$1 \to C \to G \to \overline{G} \to 1,$$

where $\overline{G} = G/C$ is the adjoint group corresponding to G, and the connecting map

$$\delta\colon\ H^1(F,\overline{G}) \to H^2(F,C)$$

in the corresponding exact sequence. The set $H^1(F,\overline{G})$ classifies the inner forms of G; it contains an element ν_G corresponding to the unique quasi-split inner form of G. We let

$$t_G = -\delta(\nu_G) \in H^2(F,C).$$

The Tits class t_G is explicitly determined for various groups G in [5, pp. 427–428]. Consider for instance $G = \mathbf{Spin}(A,\sigma)$ where σ is an orthogonal involution on a central simple F-algebra A of degree $2n$ with n even, and let Z be the center of the Clifford algebra $C(A,\sigma)$. Then $C = \mu_2 \times \mu_2$ if $\operatorname{disc}\sigma = 1$ and $C = R_{Z/F}(\mu_2)$ if $\operatorname{disc}\sigma \neq 1$, and the Tits class is related to the Clifford algebra as follows:

$$t_G = \begin{cases} \big([C^+(A,\sigma)],[C^-(A,\sigma)]\big) \in H^2(F,\mu_2\times\mu_2) & \text{if } \operatorname{disc}\sigma = 1, \\ [C(A,\sigma)] \in H^2\big(F,R_{Z/F}(\mu_2)\big) = H^2(Z,\mu_2) & \text{if } \operatorname{disc}\sigma \neq 1. \end{cases}$$

Fix an isomorphism $Z \otimes_F F_{\mathrm{sep}} \simeq F_{\mathrm{sep}} \times F_{\mathrm{sep}}$, so that $C(F_{\mathrm{sep}}) = \mu_2(F_{\mathrm{sep}}) \times \mu_2(F_{\mathrm{sep}})$ (with a nontrivial Galois action if $\operatorname{disc}\sigma \neq 1$), and define a pairing $C(F_{\mathrm{sep}}) \times C(F_{\mathrm{sep}}) \to \mathbb{Q}/\mathbb{Z}(2)$ as follows:

$$\big((x_1,x_2),(y_1,y_2)\big) \mapsto \begin{cases} x_1y_1 + x_2y_2 & \text{if } n \equiv 2 \bmod 4, \\ x_1y_2 + x_2y_1 & \text{if } n \equiv 0 \bmod 4. \end{cases}$$

The following proposition is clear:

Proposition 5.1. *The generator of $I\big(\mathbf{Spin}(A,\sigma)\big)$ in Theorems 1.14 and 1.15 can be written in the form*

$$\iota_L(\xi) = \xi \cup t_G$$

for $\xi \in H^1(L,C)$, where the cup-product is calculated for the pairing above.

In the rest of this section, we show that analogous results hold when n is odd, and also for G of type A_{n-1} with n even. In each case, the center has the form $\mu_{2m[K]}$ for some quadratic field extension K/F and some integer m. We may consider the restriction map

$$\mathrm{res}\colon\ H^2(F,\mu_{2m[K]}) \to H^2(K,\mu_{2m})$$

and the map λ_* induced by the m-th power map $\lambda(x) = x^m$,

$$\lambda_*\colon\ H^2(F,\mu_{2m[K]}) \to H^2(F,\mu_2).$$

The following proposition was pointed out to us by M. Rost (see also [1, Proposition 2.10]):

Proposition 5.2. *The map* $(\lambda_*, \mathrm{res})\colon H^2(F, \mu_{2m[K]}) \to H^2(F, \mu_2) \times H^2(K, \mu_{2m})$ *is injective. Its image consists of the pairs* (ξ, η) *such that* $\mathrm{res}(\xi) = \lambda_*(\eta)$ *and* $\mathrm{cor}(\eta) = 0$.

Proof. Since $H^1\big(F, \mathbf{G}_{\mathrm{m}} \times R_{K/F}(\mathbf{G}_{\mathrm{m}})\big) = 1$, the cohomology sequence associated to (8) yields the exact sequence

$$1 \to H^2(F, \mu_{2m[K]}) \xrightarrow{(\lambda_*, \mathrm{res})} \mathrm{Br}(F) \times \mathrm{Br}(K) \xrightarrow{(g_2)_*} \mathrm{Br}(F) \times \mathrm{Br}(K).$$

The proposition follows. □

Corollary 5.3. *For every invariant* $\iota\colon H^1(\mu_{2m[K]}) \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$*, there exists a unique element* $\theta \in H^2(F, \mu_{2m[K]})$ *such that for every* $L \in \mathsf{Fields}_F$*,* $\varphi \in H^1(L, \mu_{2m[K]})$*,*

$$\iota_L(\varphi) = \varphi \cup \theta_L.$$

Proof. Corollary 1.6 shows that for every invariant $\iota\colon H^1(\mu_{2m[K]}) \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ there exist uniquely determined elements $u \in H^2\big(F, \mathbb{Q}/\mathbb{Z}(1)\big)$, $v \in H^2\big(K, \mathbb{Q}/\mathbb{Z}(1)\big)$ satisfying $u_K + mv = 0$ and $N_{K/F}(v) = 0$ such that for $L \in \mathsf{Fields}_F$ and $x \in L^\times$, $y \in (K \otimes_F L)^\times$ with $N_{K\otimes L/L}(y) = x^{2m}$,

$$\iota_L\big((x, y)_{2m}\big) = x \cdot u_L + N_{K\otimes L/L}(z \cdot v_{K\otimes L}),$$

where $z \in (K \otimes_F L)^\times$ is such that $y = x^m z\overline{z}^{-1}$. Taking the norm of $u_K + mv = 0$, we get $2u = 0$ since $N_{K/F}(v) = 0$, hence $2mv = 0$. Therefore, we may represent u by an element $\xi \in H^2(F, \mu_2)$ and v by an element $\eta \in H^2(K, \mu_{2m})$ such that $\mathrm{cor}(\eta) = 0$ and $\mathrm{res}(\xi) + \lambda_*(\eta) = 0$, where $\lambda_*\colon H^2(K, \mu_{2m}) \to H^2(K, \mu_2)$ is induced by the m-th power map. Proposition 5.2 yields a unique element $\theta \in H^2(F, \mu_{2m[K]})$ such that $\mathrm{res}(\theta) = \eta$ and $\lambda_*(\theta) = \xi$, and computation shows that for x, y, z as above, $x \cdot u_L + N_{K\otimes L/L}(z \cdot v_{K\otimes L})$ is represented by the cup-product $(x, y)_{2m} \cup \theta$ for the canonical pairing

$$\mu_{2m[K]}(F_{\mathrm{sep}}) \times \mu_{2m[K]}(F_{\mathrm{sep}}) \to \mu_{2m[K]}(F_{\mathrm{sep}})^{\otimes 2} = \mu_{2m}(F_{\mathrm{sep}})^{\otimes 2} \hookrightarrow \mathbb{Q}/\mathbb{Z}(2).$$

(Observe that $(x, y)_{2m} = (x, x^m)_{2m}(1, z\overline{z}^{-1})_{2m}$ and

$$(x, x^m)_{2m} = j_*(x)_2, \qquad (1, z\overline{z}^{-1})_{2m} = \mathrm{cor}'(z)_{2m},$$

where j_* is induced by the inclusion $\mu_2 \hookrightarrow \mu_{2m[K]}$ and $\mathrm{cor}'\colon H^1(K, \mu_{2m}) \to H^1(F, \mu_{2m[K]})$ is the corestriction map.) □

Corollary 5.4. *With the same notation as in Theorem 1.11,*

$$\iota_L(\varphi) = \varphi \cup t_{\mathbf{SU}(B,\tau)}$$

for every $L \in \mathsf{Fields}_F$ *and* $\varphi \in H^1(L, \mu_{n[K]})$.

Proof. In view of Theorem 1.11 and the proof of Corollary 5.3, it suffices to see that

$$\mathrm{res}(t_{\mathbf{SU}(B,\tau)}) = [B] \qquad \text{and} \qquad \lambda_*(t_{\mathbf{SU}(B,\tau)}) = [D(B, \tau)].$$

This is shown in [5, (31.8)]. □

Corollary 5.5. *Use the same notation as in Theorem 1.15, and assume* n *is odd. Then*

$$\iota_L(\varphi) = \varphi \cup t_{\mathbf{Spin}(A,\sigma)}$$

for $L \in \mathsf{Fields}_F$ *and* $\varphi \in H^1(L, \mu_{4[Z]})$.

Proof. It suffices to see that $\mathrm{res}(t_{\mathbf{Spin}(A,\sigma)}) = [C(A, \sigma)]$ and $\lambda_*(t_{\mathbf{Spin}(A,\sigma)}) = [A]$. This is proved in [5, (31.11)]. □

Appendix: the Rost invariant on the kernel of the vector representation

Let A be a central simple algebra of even degree over a field F of characteristic different from 2, and let σ be an orthogonal involution on A. Consider the exact sequence

$$1 \to \mu_2 \xrightarrow{i} \mathbf{Spin}(A,\sigma) \xrightarrow{\chi} \mathbf{O}^+(A,\sigma) \to 1$$

where χ is the vector representation, and let $\rho\colon H^1\big(\mathbf{Spin}(A,\sigma)\big) \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ be the Rost invariant.

The following proposition, stated without proof in [5, p. 441], is used in the proofs of Theorems 1.14 and 1.15:

Proposition. *For $L \in \mathsf{Fields}_F$ and $x \in L^\times$,*

$$\rho\big(i_*(x)_2\big) = x \cdot [A_L].$$

Proof. The composition $\rho \circ i_*\colon H^1(\mu_2) \to H^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)$ is an invariant of $H^1(\mu_2)$. By Corollary 1.2, there is a Brauer class X satisfying $2X = 0$ such that for $L \in \mathsf{Fields}_F$ and $x \in L^\times$,

$$\rho\big(i_*(x)_2\big) = x \cdot X_L.$$

We have to show that $X = [A]$.

If A is split, then the Rost invariant of a torsor in $H^1\big(\mathbf{Spin}(A,\sigma)\big)$ only depends on its image in $H^1\big(\mathbf{O}^+(A,\sigma)\big)$, by [5, p. 437]. Therefore, $\rho \circ i_* = 0$, and $X = 0$ in this case.

For the rest of the proof, we may thus assume A is not split. Let $\underline{\sigma}$ be the canonical involution on the Clifford algebra $C(A,\sigma)$, and let $\Gamma(A,\sigma) \subset C(A,\sigma)^\times$ be the Clifford group. Let t be an indeterminate over F. The closed subscheme $V_t \subset \Gamma(A_{F(t)},\sigma)$ defined by the equation

$$\underline{\sigma}(v)v = t$$

is a torsor under $\mathbf{Spin}(A_{F(t)},\sigma)$ which represents $i_*(t)_2 \in H^1\big(F(t), \mathbf{Spin}(A,\sigma)\big)$. Therefore, by [2, Theorem B.11], $\rho\big(i_*(t)_2\big) = t \cdot X_{F(t)}$ generates the kernel of the scalar extension map

$$\mathrm{res}\colon H^3\big(F(t), \mathbb{Q}/\mathbb{Z}(2)\big) \to H^3\big(F(t)(V_t), \mathbb{Q}/\mathbb{Z}(2)\big).$$

Since t is a spinor norm over $F(t)(V_t)$, and since spinor norms are reduced norms up to squares (see [8, §6]), we have $t \cdot [A_{F(t)}] \in \ker \mathrm{res}$. On the other hand, $t \cdot [A_{F(t)}] \neq 0$ since A is not split, hence

$$t \cdot [A_{F(t)}] = t \cdot X_{F(t)}.$$

Taking residues at t, we obtain $[A] = X$. □

References

[1] J.-L. Colliot-Thélène, P. Gille, R. Parimala, Homogeneous spaces over fraction fields of strictly local two-dimensional rings, preprint.

[2] H. Esnault, B. Kahn, M. Levine, E. Viehweg, The Arason invariant and mod 2 algebraic cycles, *J. Amer. Math. Soc.* **11** (1998), 73–118.

[3] R.S. Garibaldi, Clifford algebras of hyperbolic involutions, *Math. Z.* **236** (2001), 321–349.

[4] P. Gille, Invariants cohomologiques de Rost en caractéristique positive, *K-theory* **21** (2000), 57–100.

[5] M.-A. Knus, A.S. Merkurjev, M. Rost, J.-P. Tignol, *The Book of Involutions*, Amer. Math. Soc. Coll. Pub. 44, AMS, Providence, RI, 1998.

[6] A.S. Merkurjev, Certain K-cohomology groups of Severi-Brauer varieties. *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, 319–331, Proc. Sympos. Pure Math., 58, Part 2, Amer. Math. Soc., Providence, RI, 1995.

[7] A.S. Merkurjev, J.-P. Tignol, The multipliers of similitudes and the Brauer group of homogeneous varieties, *J. reine angew. Math.* **461** (1995), 13–47.

[8] A.S. Merkur′ev, The norm principle for algebraic groups. (Russian) *Algebra i Analiz* **7** (1995), no. 2, 77–105; *translation in St. Petersburg Math. J.* **7** (1996), no. 2, 243–264.

[9] A.S. Merkurjev, Invariants of algebraic groups, *J. reine angew. Math.* **508** (1999), 127–156.
[10] M. Rost, Chow groups with coefficients, *Documenta Math.* **1** (1996), 319–393 (electronic).
[11] W. Scharlau, *Quadratic and Hermitian Forms*, Grund. math. Wiss. 270, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1985.
[12] A.A. Suslin, K-theory and K-cohomology of certain group varieties, *Algebraic K-theory* (A.A. Suslin, ed.), Adv. Soviet Math. **4**, Amer. Math. Soc., Providence, RI, 1991, pp. 53–74.

Department of Mathematics, University of California, Los Angeles, California 90095–1555
E-mail address: merkurev@math.ucla.edu

School of Mathematics, Tata Institute of Fundamental Research, Mumbai 400005, India
E-mail address: parimala@math.tifr.res.in

Institut de Mathématique Pure et Appliquée, Université catholique de Louvain, 1348 Louvain-la-Neuve, Belgium
E-mail address: tignol@agel.ucl.ac.be